Quasi-varieties of presheaves

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Abstract. In analogy with the varietal case, we give an abstract characterization of those categories occurring as regular epireflective subcategories of presheaf categories such that the inclusion functor preserves small sums.

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The aim of this short note is to add one step to the nice parallelism between presheaf categories and algebraic categories.

Presheaf categories can be abstractly characterized as exact and extensive categories with a small set of indecomposable, regular projective regular generators (Bunge [1]) (here extensive means with disjoint and universal small sums, and indecomposable means that each morphism from a generator to a small sum of generators factors through one of the canonical injections). Bearing in mind this characterization, Giraud's axioms for Grothendieck toposes become transparent: drop the assumptions which are not stable under localization (observe that the extensivity assumption, which is a little bit redundant in Bunge's theorem, is essential in Giraud's theorem because generators are no longer indecomposable).

On the algebraic side of the world, we have Lawvere's theorem which states that a category is equivalent to an algebraic one if and only if it is exact, and has a finitely presentable regular projective regular generator [6]. Lawvere's theorem is the non-additive extension of Gabriel-Mitchell's characterization of module categories. Localizations of module categories are exactly Grothendieck categories (Gabriel-Popescu [3]), and the non-additive version of Gabriel-Popescu's theorem states that exact categories with a regular generator, and with exact filtered colimits are the localizations of algebraic categories [8].

But in the algebraic case something more is known. Weakening the exactness condition in Lawvere's axioms, one recaptures the so-called quasi-varieties, i.e. regular epireflective subcategories of algebraic categories such that the inclusion functor preserves filtered colimits (see [7] for more details). It is now natural to come back to presheaf categories, and wonder which categories are characterized by weakening the exactness assumption in Bunge's theorem. The answer is provided in the proposition below, but first we need some preliminaries.

Recall that the full inclusion of the 2-category of exact categories and exact functors into the 2-category of regular categories has a left biadjoint, the exact completion of a regular category (see [2]). We denote by

$$\gamma\colon \mathbb{B} \to \mathbb{B}_{EX}$$

the counit of this biadjunction. It is an exact, full and faithful functor. Moreover, each object of $\mathbb{B}_{EX}$ is a regular quotient of an object of $\mathbb{B}$.

The first two points of the next lemma are borrowed from [7]. The third one is the infinitary extension of theorem 2.3 in [5] and we omit the proof.

Lemma: *Let $\mathbb{B}$ be a regular category and $\gamma\colon \mathbb{B} \to \mathbb{B}_{EX}$ its exact completion;*

1. *the functor γ preserves regular projective objects and small sums of regular projectives;*
2. *$\mathbb{B}$ is regular epireflective in $\mathbb{B}_{EX}$ iff it has coequalizers of equivalence relations;*
3. *$\mathbb{B}$ is extensive iff $\mathbb{B}_{EX}$ is extensive and γ preserves sums.*

Proposition: *A category $\mathbb{B}$ is equivalent to a regular epireflective subcategory of a presheaf category such that the inclusion functor preserves small sums if and only if it is a regular and extensive category with coequalizers of equivalence relations, and has a small set of indecomposable regular projective regular generators.*

Proof: A regular epireflection of an exact category is regular and has coequalizers of equivalence relations. Moreover the reflector preserves regular projective objects (because the full inclusion preserves regular epimorphisms), so that the reflection of the representable presheaves is a set of regular projective regular generators. Now let C and $\{C_j\}_J$ be representable presheaves and write r for the reflector and i for the inclusion. Consider a morphism

$$f\colon r(C) \to \coprod_J r(C_j)$$

embed it into the presheaf category and cover the objects

$$i(r(C)) \quad \text{and} \quad i(\coprod r(C_j)) \simeq i(r(\coprod C_j))$$

with the units η_C and $\eta_{\coprod C_j}$. Since C is regular projective and units are regular epis, we obtain a factorization

$$g\colon C \to \coprod_J C_j$$

of $i(f) \cdot \eta_C$ through $\eta_{\coprod C_j}$. Since representable presheaves are indecomposable, g factors through an injection, and the reflection of this factorization gives a factorization for f. As far as the extensivity of $\mathbb{B}$ is concerned, it follows immediately from that of the presheaf category because $\mathbb{B}$ is closed under sums and (finite) limits.
Conversely, since the regular category $\mathbb{B}$ has coequalizers of equivalence relations, it is regular epireflective in its exact completion $\mathbb{B}_{EX}$ (point 2. of the lemma). The set of indecomposable regular projective regular generators of $\mathbb{B}$ maintains these properties in $\mathbb{B}_{EX}$. This easily follows from point 1. of the lemma, and using that each object of $\mathbb{B}_{EX}$ is a regular quotient of an object of $\mathbb{B}$. Finally, by point 3. of the lemma, $\mathbb{B}_{EX}$ is extensive, and $\gamma\colon \mathbb{B} \to \mathbb{B}_{EX}$ preserves small sums. The category $\mathbb{B}_{EX}$ now satisfies all the conditions to be equivalent to a presheaf category and the proof is complete. $\square$

A simple example of the situation of the previous proposition is provided by the category of separated presheaves over the non-empty open sets of a topological space. One has to exclude the empty open to prove that the inclusion functor preserves sums. The regular epireflector is built up as in the usual case.

This fact holds more in general, as suggested to me by the referee. Let $\mathcal{L}$ be a Grothendieck topology on a small category $\mathbb{C}$ and consider the category $\mathcal{L}$-Sep of separated presheaves, which is regular epireflective in the presheaf category $[\mathbb{C}^{op}, \mathrm{Set}]$. If, for each object C in $\mathbb{C}$ and for each presheaf R in $\mathcal{L}(C)$, there is an object D in $\mathbb{C}$ such that $R(D)$ is non empty, then the inclusion of $\mathcal{L}$-Sep in $[\mathbb{C}^{op}, \mathrm{Set}]$ preserves sums.

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