On categorical crossed modules

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Abstract

The well-known notion of crossed module of groups is raised in this paper to the categorical level supported by the theory of categorical groups. We construct the cokernel of a categorical crossed module and we establish the universal property of this categorical group. We also prove a suitable 2-dimensional version of the kernel-cokernel lemma for a diagram of categorical crossed modules. We then study derivations with coefficients in categorical crossed modules and show the existence of a categorical crossed module given by inner derivations. This allows us to define the low-dimensional cohomology categorical groups and, finally, these invariants are connected by a six-term 2-exact sequence obtained by using the kernel-cokernel lemma.

Introduction

Crossed modules of groups were introduced by J.H.C. Whitehead [39]; they have been shown to be relevant both in topological and algebraic contexts. They provided algebraic models for connected 2-types [26,28] and allowed to develop, as adequate coefficients, a non-abelian cohomology of groups [12,25]. It is convenient and sensible to regard crossed modules of groups as 2-dimensional versions of groups (c.f. [30]). They correspond, in fact, to strict categorical groups, that is, strict monoidal groupoids where each object is invertible up to isomorphism. Categorical groups have been studied in the last twenty-five years by several authors from different points of view (algebraic models for homotopy types [4,6], cohomology [8,36], extensions [1,2,31], derivations [18,19], ring theory [15,37]). All of these investigations have clarified the relevance of

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this 2-dimensional point of view. Nevertheless, with respect to cohomology, the reader could wonder why to study cohomology of non-necessarily strict categorical groups, since the strict case has been already studied by several authors (see [9,12,20,21,30] and the references therein). We want to make this fact clear underlining the relevance of some approaches made in this direction. Thus, in order to give an interpretation of Hattori cohomology in dimension three (see [22]) Ulbrich developed a group cohomology for Picard (i.e., symmetric) categorical groups, providing in this way a remarkable example of cohomology in the non-strict case. Also, we should emphasize Breen's work [2] about the Schreier theory for categorical groups by means of a cohomology set able of codifying equivalence classes of extensions of a group by a categorical group. The above results led to the development carried out in [6], where, via an adequate nerve notion of a (braided) categorical group, it is stated how the cohomology set of categorical groups, there studied, allows to codify sets of homotopy classes of continuous maps between spaces which are algebraically modelled by (braided) categorical groups. This kind of interpretation could be carried out thanks to the fact that the non-strict algebraic model associated to a space, whose simplices actually have a nice geometrical description, is easier to handle than the strict one. Together with this topological interpretation, we point out that these cohomology sets classify equivalence classes of extensions of categorical groups [7]. From a 2-dimensional categorical point of view, the reader can also find in Remark 6.7 other reasons why the study of categorical group cohomology is worth.

In the context of categorical groups, the notion of crossed module was suggested by L. Breen in [2]. It was given in a precise form (although in the restricted case of the codomain of the crossed module being discrete) by P. Carrasco and J. Martínez in [8], in order to obtain an interpretation of the 4-th Ulbrich's cohomology group [36]. (Recently the definition by Carrasco and Martínez has been further generalized by Turaev assuming that the domain of the crossed module is just a monoidal category [35].)

In the papers quoted above, a lot of interesting and relevant examples are discussed. In our opinion, they justify a general theory of *categorical crossed modules*, which is the aim of this paper.

After the preliminaries, Section 2 is devoted to present definitions of categorical precrossed and crossed modules and to state some basic examples.

A way to think about a crossed module of groups $\delta : H \to G$ is as a morphism δ of groups which believes that the codomain G is abelian. In fact, the image of δ is a normal subgroup of G, so that the cokernel $G/Im(\delta)$ of δ can be constructed as in the abelian case. This is relevant in non-abelian cohomology of groups, where the first cohomology group of a group G with coefficients in the crossed module δ is defined as the cokernel of the crossed module given by inner derivations [27,20]. This intuition on crossed modules of groups can be exploited also at the level of categorical groups. In fact, in Section 3 we associate to a categorical crossed module $\mathbf{T} : \mathbb{H} \to \mathbb{G}$ a new categorical group $\mathbb{G}/\langle \mathbb{H}, \mathbf{T} \rangle$, which we call the *quotient categorical group*, and we justify our terminology establishing its universal property. As in the case of groups, the notion of categorical crossed module subsumes the notion of normal sub-categorical group. To test our definition, we show that, in the 2-category of categorical groups, normal sub-categorical groups correspond to kernels and quotients correspond to essentially surjective morphisms. Also, for a fixed categorical group $\mathbb{G}$, normal sub-categorical groups of $\mathbb{G}$ and quotients of $\mathbb{G}$ correspond each other. In Section 4, we establish a higher dimensional version of the kernel-cokernel lemma: We associate a six-term 2-exact sequence of categorical groups to a convenient diagram of categorical crossed modules. In next two sections we apply the machinery developed for categorical crossed modules, to define low-dimensional cohomology categorical groups, following a parallel process to that made in [20,27,38], where the Whitehead group of regular derivations was the basic ingredient to develop low-dimensional non-abelian cohomology of groups. Thus, we introduce derivations with coefficients in categorical crossed modules and we study the monoidal structure inherited by the groupoid of derivations $Der(\mathbb{G},\mathbb{H})$ whenever $\mathbb{H}$ is a categorical $\mathbb{G}$-crossed module (c.f. [17] in the particular case of $\mathbb{G}$ being discrete and [19] in the case of $\mathbb{H}$ being a $\mathbb{G}$-module). Then, we define the Whitehead categorical group of regular derivations, $Der^*(\mathbb{G},\mathbb{H})$, as the Picard categorical group associated to this monoidal groupoid. This categorical group actually coherently acts on $\mathbb{H}$. In fact, the functor $\mathbb{H} \longrightarrow Der^*(\mathbb{G},\mathbb{H})$ given by inner derivations, provides our most relevant example of categorical crossed module that allows us, in last section, to define the low-dimensional cohomology categorical groups of a categorical group $\mathbb{G}$ with coefficients in a $\mathbb{G}$-categorical crossed module $\mathbb{H}$, as the kernel and the quotient of the categorical crossed module of inner derivations. Further, we apply the kernel-cokernel lemma to get a six-term 2-exact sequence for the cohomology categorical groups from a short exact sequence of categorical $\mathbb{G}$-crossed modules. This sequence generalizes and unifies several similar exact sequences connecting cohomology sets, groups, groupoids and categorical groups (c.f. [3,13,14,18–21,27,32]).

1 Preliminaries

A categorical group $\mathbb{G} = (\mathbb{G}, \otimes, a, I, l, r)$ (see [33,2,31,16] for general background) is a monoidal groupoid such that each object is invertible, up to isomorphism, with respect to the tensor product. This means that, for each object X, there is an object X^* and an arrow $m_X : I \to X \otimes X^*$. It is then possible to choose an arrow $n_X : X^* \otimes X \to I$ in such a way that (X, X^*, m_X, n_X) is a duality. Moreover, one can choose $I^* = I$. When it is irrelevant, we will omit the associativity isomorphism "a" of any categorical group $\mathbb{G}$ and we will write "*can*" for any canonical structural isomorphism of $\mathbb{G}$.
Categorical groups are the objects of a 2-category $\mathcal{CG}$, whose arrows are

monoidal functors $\mathbf{T} = (T, \mu)$ and whose 2-cells are monoidal natural transformations. A categorical group is said to be braided (symmetric) (see [23]) if it is braided (symmetric) as a monoidal category. If $\mathbb{G}$ is a categorical group, we write $\pi_0\mathbb{G}$ for the group (abelian if $\mathbb{G}$ is braided) of isomorphism classes of objects, and $\pi_1\mathbb{G}$ for the abelian group $\mathbb{G}(I, I)$ of automorphisms of the unit object. If G is a group, we can see it as a discrete categorical group, which we write $G[0]$. If G is an abelian group, we can see it also as a categorical group with only one object, which we write $G[1]$.
Fix a categorical group $\mathbb{G}$. A $\mathbb{G}$-*categorical group* [16] consists of a categorical group $\mathbb{H}$ together with a morphism of categorical groups (a $\mathbb{G}$-*action*) $(\mathcal{F}, \mu) : \mathbb{G} \to \mathcal{E}q(\mathbb{H})$ from $\mathbb{G}$ to the categorical group of monoidal *autoequivalences*, $\mathcal{E}q(\mathbb{H})$, of $\mathbb{H}$ [2]. Equivalently, we have a functor

$$ac : \mathbb{G} \times \mathbb{H} \to \mathbb{H} \quad , \quad (X, A) \mapsto ac(X, A) = {}^{X}A$$

together with two natural isomorphisms

$$\psi_{X,A,B} : {}^{X}(A \otimes B) \to {}^{X}A \otimes {}^{X}B, \quad \Phi_{X,Y,A} : {}^{(X \otimes Y)}A \to {}^{X}({}^{Y}A)$$

satisfying the coherence conditions, [16,19]. Note that a canonical morphism $\phi_{0,A} : {}^{I}A \to A$ can be then deduced from them.

Let $\mathbb{H}$ and $\mathbb{H}'$ be $\mathbb{G}$-categorical groups. A morphism $(\mathbf{T}, \varphi) : \mathbb{H} \to \mathbb{H}'$ consists of a categorical group morphism $\mathbf{T} = (T, \mu) : \mathbb{H} \to \mathbb{H}'$ and a natural transformation φ

$$\begin{tikzcd} \mathbb{G} \times \mathbb{H} \arrow[r, "ac"] \arrow[d, "Id \times T"'] \arrow[dr, phantom, "\varphi\Downarrow"] & \mathbb{H} \arrow[d, "T"] \\ \mathbb{G} \times \mathbb{H}' \arrow[r, "ac"'] & \mathbb{H}' \end{tikzcd}$$

compatible with ψ, ϕ and ϕ_0 in the sense of [16]. $\mathbb{G}$-categorical groups and morphisms of $\mathbb{G}$-categorical groups are the objects and 1-cells of a 2-category, denoted by $\mathbb{G} - \mathcal{C}G$, where a 2-cell $\alpha : (\mathbf{T}, \varphi) \Rightarrow (\mathbf{T}', \varphi') : \mathbb{H} \to \mathbb{H}'$ is a 2-cell $\alpha : \mathbf{T} \Rightarrow \mathbf{T}'$ in $\mathcal{C}G$ satisfying the corresponding compatibility condition with φ and φ'.
If $\mathbb{H}$ is a $\mathbb{G}$-categorical group, then the categorical group of autoequivalences $\mathcal{E}q(\mathbb{H})$ also is a $\mathbb{G}$-categorical group under the *diagonal action*, ${}^{X}(T, \mu) = ({}^{X}T, {}^{X}\mu)$, where, for any $A \in \mathbb{H}$, $({}^{X}T)(A) = {}^{X}T({}^{X^*}A)$ and , for any $A, B \in \mathbb{H}$, $({}^{X}\mu)_{A,B} = \psi_{X,T({}^{X^*}A),T({}^{X^*}B)} \cdot {}^{X}\mu_{{}^{X^*}A,\, {}^{X^*}B} \cdot {}^{X}T(\psi_{X^*,A,B})$.

Besides, considering the morphism which defines the $\mathbb{G}$-action on $\mathbb{H}$, $(\mathcal{F}, \mu) : \mathbb{G} \to \mathcal{E}q(\mathbb{H})$, we have a morphism in $\mathbb{G} - \mathcal{CG}$, $((\mathcal{F}, \mu), \varphi_{\mathcal{F}}) : \mathbb{G} \to \mathcal{E}q(\mathbb{H})$, where $(\varphi_{\mathcal{F}})_{X,Y} : \mathcal{F}({}^{X}Y) \Rightarrow {}^{X}\mathcal{F}(Y)$ is given, for any $A \in \mathbb{H}$, by $\Big((\varphi_{\mathcal{F}})_{X,Y}\Big)_A = \phi_{X,Y,{}^{X^*}A} \cdot \phi_{X \otimes Y, X^*, A} : {}^{X \otimes Y \otimes X^*}A \to {}^{X}({}^{Y}({}^{X^*}A))$.

Furthermore, considering now the morphism $\mathbf{i} = (i, \mu_i) : \mathbb{H} \to \mathcal{E}q(\mathbb{H})$ given by conjugation, we also have a morphism in $\mathbb{G} - \mathcal{CG}$, $(\mathbf{i}, \varphi_i) : \mathbb{H} \to \mathcal{E}q(\mathbb{H})$, where

$(\varphi_i)_{X,A} : i({}^XA) \Rightarrow {}^Xi(A)$ is given, for any $B \in \mathbb{H}$, by the composition

$$\Big((\varphi_i)_{X,A}\Big)_B = \psi^{-1}_{X,A\otimes\, X^*B,A^*} \cdot (\psi^{-1}_{X,A,\, X^*B} \otimes 1) \cdot (1 \otimes \phi_{X,X^*,B} \otimes can) \cdot (1 \otimes \phi^{-1}_{0,B} \otimes 1).$$

2 Categorical $\mathbb{G}$-crossed modules.

In this section we define the notion of categorical $\mathbb{G}$-crossed modules and give some examples. Fix a categorical group $\mathbb{G}$ and consider it as a $\mathbb{G}$-categorical group by conjugation. We first give the following

Definition 2.1 *The 2-category of categorical $\mathbb{G}$-precrossed modules is the slice 2-category $\mathbb{G} - \mathcal{C}G/\mathbb{G}$.*

More explicitly, a categorical $\mathbb{G}$-precrossed module is a pair $\langle \mathbb{H}, (\mathbf{T}, \varphi)\rangle$ where $(\mathbf{T}, \varphi) : \mathbb{H} \to \mathbb{G}$ is a morphism in $\mathbb{G} - \mathcal{C}G$, thus we have a picture

$$\begin{tikzcd} \mathbb{G} \times \mathbb{H} \arrow[r, "ac"] \arrow[d, "Id\times T"'] \arrow[dr, phantom, "\varphi\Downarrow"] & \mathbb{H} \arrow[d, "T"] \\ \mathbb{G} \times \mathbb{G} \arrow[r, "\text{conj}"'] & \mathbb{G} \end{tikzcd} .$$

which means that, for any objects $X \in \mathbb{G}$ and $A \in \mathbb{H}$, there is a natural family of isomorphisms

$$\varphi = \varphi_{X,A} : T({}^XA) \to X \otimes TA \otimes X^*$$

satisfying the corresponding compatibility conditions with the natural isomorphisms of the $\mathbb{G}$-action on $\mathbb{H}$. Observe that the family $\varphi_{X,A}$ of natural isomorphisms corresponds to a family of natural isomorphisms

$$\nu = \nu_{X,A} : T({}^XA) \otimes X \longrightarrow X \otimes T(A) \ ,$$

and now, using this family $\nu = \nu_{X,A}$, we give the following equivalent definition of categorical $\mathbb{G}$-precrossed module:

Definition 2.2 *Let $\mathbb{G}$ be a categorical group. A categorical $\mathbb{G}$-precrossed module is a triple $\langle \mathbb{H}, \mathbf{T}, \nu\rangle$, where $\mathbb{H}$ is a $\mathbb{G}$-categorical group, $\mathbf{T} = (T, \mu) : \mathbb{H} \to \mathbb{G}$ is a morphism of categorical groups and*

$$\nu = \Big(\nu_{X,A} : T({}^XA) \otimes X \longrightarrow X \otimes T(A)\Big)_{(X,A)\in\mathbb{G}\times\mathbb{H}}$$

is a family of natural isomorphisms in $\mathbb{G}$, making commutative the following diagrams (which are the translations, in terms of the family $\nu = \nu_{X,A}$, of the coherence conditions that φ satisfies) :

(pcr1)

$$\begin{tikzcd}
T({}^X({}^YA)) \otimes X \otimes Y \arrow[r, "T(\phi^{-1})\otimes 1\otimes 1"] \arrow[d, "\nu\otimes 1"'] & T({}^{(X\otimes Y)}A) \otimes X \otimes Y \arrow[d, "\nu"] \\
X \otimes T({}^YA) \otimes Y \arrow[r, "1\otimes\nu"'] & X \otimes Y \otimes T(A)\ ,
\end{tikzcd}$$

(pcr2)

$$\begin{tikzcd}
T({}^XA) \otimes T({}^XB) \otimes X \arrow[r, "1\otimes\nu"] \arrow[d, "can"'] & T({}^XA) \otimes X \otimes T(B) \arrow[d, "\nu\otimes 1"] \\
T({}^XA \otimes {}^XB) \otimes X \arrow[d, "T(\psi^{-1})\otimes 1"'] & X \otimes T(A) \otimes T(B) \arrow[d, "can"] \\
T({}^X(A\otimes B)) \otimes X \arrow[r, "\nu"'] & X \otimes T(A\otimes B)\ ,
\end{tikzcd}$$

Now, a morphism of categorical $\mathbb{G}$-precrossed modules is a triple

$$(\mathbf{F}, \eta, \alpha) : \langle \mathbb{H}, \mathbf{T}, \nu \rangle \to \langle \mathbb{H}', \mathbf{T}', \nu' \rangle$$

with $(\mathbf{F}, \eta) : \mathbb{H} \to \mathbb{H}'$ a morphism in $\mathbb{G} - \mathcal{C}G$ and $\alpha : T \Rightarrow T'F$ a 2-cell in $\mathcal{C}G$ such that, for any $X \in \mathbb{G}$ and $A \in \mathbb{H}$, the following diagram is commutative (which corresponds to the coherence condition for $\alpha : T \Rightarrow T'F$ being a 2-cell in $\mathbb{G} - \mathcal{C}G$):

$$\begin{tikzcd}
T({}^XA) \otimes X \arrow[rr, "\nu"] \arrow[d, "\alpha\otimes 1"'] & & X \otimes T(A) \arrow[d, "1\otimes\alpha"] \\
T'F({}^XA) \otimes X \arrow[dr, "T'(\eta)\otimes 1"'] & & X \otimes T'F(A) \\
& T'({}^XF(A)) \otimes X\ . \arrow[ur, "\nu'"'] &
\end{tikzcd}$$

Finally, a 2-cell $\beta : (\mathbf{F}, \eta, \alpha) \Rightarrow (\mathbf{F}', \eta', \alpha')$ is a 2-cell $\beta : (\mathbf{F}, \eta) \Rightarrow (\mathbf{F}', \eta') : \mathbb{H} \to \mathbb{H}'$ in $\mathbb{G} - \mathcal{C}G$, such that for any $A \in \mathbb{H}$, the diagram

$$\begin{tikzcd}
T(A) \arrow[r, "\alpha_A"] \arrow[dr, "\alpha'_A"'] & T'F(A) \arrow[d, "T'(\beta_A)"] \\
& T'F'(A)
\end{tikzcd}$$

is commutative (which corresponds to the condition for $\beta : (\mathbf{F}, \eta) \Rightarrow (\mathbf{F}', \eta')$ being a 2-cell in the slice 2-category $\mathbb{G} - \mathcal{C}G/\mathbb{G}$.

As an easy example, note that giving a $\mathbb{G}$-precrossed module structure on the identity morphism $1_{\mathbb{G}} : \mathbb{G} \to \mathbb{G}$, of the trivial $\mathbb{G}$-categorical group $\mathbb{G}$, is the same as giving a braiding c on $\mathbb{G}$ via the formula $c_{X,A} = \nu^{-1}_{X,A} : X \otimes A \to A \otimes X$. We will see more examples after defining categorical $\mathbb{G}$-crossed module. As for precrossed modules, we next give the following short definition:

Definition 2.3 *A categorical $\mathbb{G}$-crossed module is given by a categorical $\mathbb{G}$-precrossed module $\langle \mathbb{H}, (\mathbf{T}, \varphi) \rangle$ together with a 2-cell in $\mathbb{G} - \mathcal{C}G$*

$$\begin{tikzcd} \mathbb{H} \arrow[rr, "(\mathbf{i},\varphi_i)"] \arrow[dr, "(\mathbf{T},\varphi)"'] & \arrow[d, phantom, "\varepsilon\Downarrow"] & \mathcal{E}q(\mathbb{H}) \\ & \mathbb{G} \arrow[ur, "(\digamma,\varphi_\digamma)"'] & \end{tikzcd} ,$$

where $(\digamma, \varphi_\digamma) : \mathbb{G} \to \mathcal{E}q(\mathbb{H})$ and $(\mathbf{i}, \varphi_i) : \mathbb{H} \to \mathcal{E}q(\mathbb{H})$ are the morphism of $\mathbb{G}$-categorical groups remarked in the Preliminaries. The following compatibility condition between φ and ε have to be satisfied:

$$\begin{tikzcd} \mathbb{H}\times\mathbb{H} \arrow[r, "conj"] \arrow[d, "T\times Id"'] \arrow[dr, phantom, "\bar{\varepsilon}\Downarrow"] & \mathbb{H} \arrow[d, "Id"] \\ \mathbb{G}\times\mathbb{H} \arrow[r, "ac"] \arrow[d, "Id\times T"'] \arrow[dr, phantom, "\varphi\Downarrow"] & \mathbb{H} \arrow[d, "T"] \\ \mathbb{G}\times\mathbb{G} \arrow[r, "conj"'] & \mathbb{G} \end{tikzcd} = \begin{tikzcd} \mathbb{H}\times\mathbb{H} \arrow[r, "conj"] \arrow[d, "T\times T"'] \arrow[dr, phantom, "can\Downarrow"] & \mathbb{H} \arrow[d, "T"] \\ \mathbb{G}\times\mathbb{G} \arrow[r, "conj"'] & \mathbb{G}. \end{tikzcd}$$

(where, for any $A, B \in \mathbb{H}$, $\bar{\varepsilon}_{A,B} = (\varepsilon_A)_B$)

We refer to φ as the precrossed structure of the categorical crossed module and to ε as its crossed structure, even if for a categorical $\mathbb{G}$-precrossed module to be crossed is a property and not a structure (see Remark 3.1 below).
In order to unpack the above definition we proceed in the same way we did for categorical precrossed modules. The picture

$$\begin{tikzcd} \mathbb{H}\times\mathbb{H} \arrow[r, "\mathrm{conj}"] \arrow[d, "T\times Id"'] \arrow[dr, phantom, "\bar{\varepsilon}\Downarrow"] & \mathbb{H} \arrow[d, "Id"] \\ \mathbb{G}\times\mathbb{H} \arrow[r, "ac"] & \mathbb{H} \end{tikzcd}$$

means that there is a natural family of isomorphisms in $\mathbb{H}$

$$\bar{\varepsilon} = \bar{\varepsilon}_{A,B} : A \otimes B \otimes A^* \longrightarrow {}^{T(A)}B$$

and then, there is also a corresponding family of natural isomorphisms in $\mathbb{H}$

$$\chi = \chi_{A,B} : {}^{TA}B \otimes A \longrightarrow A \otimes B$$

through which we obtain the following equivalent definition of categorical $\mathbb{G}$-crossed module:

Definition 2.4 *A categorical $\mathbb{G}$-crossed module consists of a 4-tuple $\langle \mathbb{H}, \mathbf{T}, \nu, \chi \rangle$, where $\langle \mathbb{H}, \mathbf{T}, \nu \rangle$ is a categorical $\mathbb{G}$-precrossed module as in definition 2.2, and*

$$\chi = \left(\chi_{A,B} : {}^{TA}B \otimes A \longrightarrow A \otimes B \right)_{(A,B)\in\mathbb{H}}$$

is a family of natural isomorphisms in $\mathbb{H}$ such that the following diagrams (which are the translations in terms of χ of the fact that ε is a 2-cell in $\mathbb{G}-\mathcal{C}G$) are commutative:

(cr1)

$$\begin{tikzcd}
{}^{T(A\otimes B)}C\otimes A\otimes B \arrow[rr, "\chi"] \arrow[d, "can"'] & & A\otimes B\otimes C \\
{}^{(TA\otimes TB)}C\otimes A\otimes B \arrow[r, "\phi\otimes 1"'] & {}^{TA}({}^{TB}C)\otimes A\otimes B \arrow[r, "\chi\otimes 1"'] & A\otimes {}^{TB}C\otimes B \arrow[u, "1\otimes\chi"']
\end{tikzcd}$$

(cr2)

$$\begin{tikzcd}
{}^{TA}(B\otimes C)\otimes A \arrow[r, "\chi"] \arrow[d, "\psi\otimes 1"'] & A\otimes B\otimes C \\
{}^{TA}B\otimes {}^{TA}C\otimes A \arrow[r, "1\otimes\chi"'] & {}^{TA}B\otimes A\otimes C \arrow[u, "\chi\otimes 1"']
\end{tikzcd}$$

(cr3)

$$\begin{tikzcd}
{}^{X}({}^{TA}B\otimes A) \arrow[rr, "{}^{X}\chi"] \arrow[d, "\psi"'] & & {}^{X}(A\otimes B) \arrow[d, "\psi"] \\
{}^{X}({}^{TA}B)\otimes {}^{X}A \arrow[d, "\phi^{-1}\otimes 1"'] & & {}^{X}A\otimes {}^{X}B \\
{}^{(X\otimes TA)}B\otimes {}^{X}A \arrow[r, "\nu^{-1}{}_{B}\otimes 1"'] & {}^{(T({}^{X}A)\otimes X)}B\otimes {}^{X}A \arrow[r, "\phi\otimes 1"'] & {}^{T({}^{X}A)}({}^{X}B)\otimes {}^{X}A \arrow[u, "\chi"']
\end{tikzcd}$$

and such that the following diagram (expressing the compatibility between the precrossed structure and the crossed structure) is also commutative:

(cr4)

$$\begin{tikzcd}
T({}^{TA}B\otimes A) \arrow[r, "T(\chi)"] \arrow[d, "can"'] & T(A\otimes B) \arrow[d, "can"] \\
T({}^{TA}B)\otimes T(A) \arrow[r, "\nu"'] & T(A)\otimes T(B)
\end{tikzcd} .$$

Remark 2.5

If $T : H \to G$ is a precrossed module of groups and the morphism T is injective, then the precrossed module actually is a crossed module. A trace of this property remains in the case of categorical groups. Indeed, if the functor-part $T : \mathbb{H} \to \mathbb{G}$ of a categorical $\mathbb{G}$-precrossed module is faithful, then there is at most one structure of $\mathbb{G}$-crossed module compatible with the precrossed structure. If, moreover, $T : \mathbb{H} \to \mathbb{G}$ is full, then there is exactly one structure of $\mathbb{G}$-crossed module on the categorical $\mathbb{G}$-precrossed module $\langle \mathbb{H}, (\mathbf{T}, \varphi)\rangle$.

We define the 2-category $\mathbb{G}-cross$ of categorical $\mathbb{G}$-crossed modules as the sub-2-category of the 2-category of categorical $\mathbb{G}$-precrossed modules whose objects are the categorical $\mathbb{G}$-crossed modules. An arrow $(\mathbf{F},\eta,\alpha) : \langle \mathbb{H},\mathbf{T},\nu,\chi\rangle \to \langle \mathbb{H}',\mathbf{T}',\nu',\chi'\rangle$ is an arrow between the underlying categorical $\mathbb{G}$-precrossed modules such that, for any $A,B \in \mathbb{H}$, the following diagram (expressing a compatibility condition between the natural isomorphisms χ and χ') is commutative:

$$
\begin{tikzcd}
F({}^{T(A)}B \otimes A) \arrow[rr, "F(\chi)"] \arrow[d, "can"'] & & F(A \otimes B) \arrow[d, "can"] \\
F({}^{T(A)}B) \otimes F(A) \arrow[d, "\eta\otimes 1"'] & & F(A) \otimes F(B) \\
{}^{T(A)}F(B) \otimes F(A) \arrow[rr, "\alpha_{F(B)}\otimes 1"] & & {}^{T'F(A)}F(B) \otimes F(B) \arrow[u, "\chi'"']
\end{tikzcd} .
$$

Finally $\mathbb{G}-cross$ is full on 2-cells.

Example 2.6 i) Any crossed module of groups $H \stackrel{\delta}{\to} G$ is a categorical crossed module when both G and H are seen as discrete categorical groups.

ii) Let $(N \stackrel{\delta}{\to} O \stackrel{d}{\to} G, \{-,-\})$ be a 2-crossed module in the sense of Conduché, [11]. Then, following [8], it has associated a categorical $G[0]$-crossed module $\langle \mathbb{G}(\delta), d, id, \chi\rangle$, where $\mathbb{G}(\delta)$ is the strict categorical group associated to the crossed module δ, ν is the identity and $\chi_{x,y} = (\{x,y\}, {}^{d(x)}y{+}x)$, for all $x,y \in O$.

iii) In [7] a $\mathbb{G}$-module is defined as a braided categorical group $(\mathbb{A},c)$ provided with a $\mathbb{G}$-action such that $c_{{}^X A, {}^X B} \cdot \psi_{X,A,B} = \psi_{X,B,A} \cdot {}^X c_{A,B}$, for any $X \in \mathbb{G}$ and $A,B \in \mathbb{H}$. If $(\mathbb{A},c)$ is a $\mathbb{G}$-module, the zero-morphism $\mathbf{0} : \mathbb{A} \to \mathbb{G}$ is a categorical $\mathbb{G}$-crossed module where, for any $A,B \in \mathbb{A}$, $\chi_{A,B} :{}^I B \otimes A \to A \otimes B$ is given by the braiding $c_{B,A}$, up to composition with the obvious canonical isomorphism.

iv) Consider a morphism $\mathbf{T} : \mathbb{H} \to \mathbb{G}$ of categorical groups and look at $\mathbb{H}$ as a trivial $\mathbb{G}$-categorical group (i.e., via the second projection $p_2 : \mathbb{G}\times\mathbb{H} \to \mathbb{H}$). If $\mathbb{G}$ is braided, then we get a precrossed structure by $\nu_{X,A} = c^{-1}_{X,TA} : TA \otimes X \to X \otimes TA$. Now to give a crossed structure is the same as giving a braiding on $\mathbb{H}$ via the formula $\chi_{A,B} = c_{B,A} : B \otimes A \to A \otimes B$ and the compatibility between ν and χ precisely means that T preserves the braiding.

v) Let $\mathbf{L} : \mathbb{G} \to \mathbb{K}$ be a morphism in $\mathcal{C}G$ and $Ker\mathbf{L} \xrightarrow{e_{\mathbf{L}}} \mathbb{G} \xrightarrow{L} \mathbb{K}$, $\epsilon_L : L\, e_{\mathbf{L}} \Rightarrow 0$, its kernel (see [24]). The categorical group $Ker\mathbf{L}$ is a $\mathbb{G}$-categorical group with action given by ${}^X(A, \epsilon_A : L(A) \to I) = (X \otimes A \otimes X^*, {}^X\epsilon_A : L(X \otimes A \otimes X^*) \to I)$ where

$${}^X\epsilon_A : L(X\otimes A\otimes X^*) \xrightarrow{can} L(X)\otimes L(A)\otimes L(X)^* \xrightarrow{1\otimes\epsilon_A\otimes 1} L(X)\otimes I\otimes L(X)^* \xrightarrow{can} I$$

and with natural isomorphisms ψ, ϕ, ϕ_0 induced by $(X \otimes Y)^* \simeq Y^* \otimes X^*$, $A \simeq I \otimes A \otimes I^*$ and $I \simeq X^* \otimes X$.

Moreover, the morphism $e_{\mathbf{L}} : Ker\mathbf{L} \to \mathbb{G}$, $e_{\mathbf{L}}(f : (A, \epsilon_A) \to (B, \epsilon_B)) = f : A \to B$, is a categorical $\mathbb{G}$-crossed module, with both natural isomorphisms ν and χ given by canonical isomorphisms.

vi) Let $H' \stackrel{\delta'}{\to} G'$ be a normal subcrossed module of a crossed module of groups $H \stackrel{\delta}{\to} G$ (see [30]). Then the inclusion induces a homomorphism $\mathbb{G}(\delta') \to \mathbb{G}(\delta)$ which defines a categorical $\mathbb{G}(\delta)$-crossed module, where both ν and χ are identities and the action is given by conjugation on objects and on arrows.

vii) For any categorical group $\mathbb{H}$, the conjugation homomorphism $\mathbb{H} \to \mathcal{E}q(\mathbb{H})$ provides a categorical $\mathcal{E}q(\mathbb{H})$-crossed module, where the action corresponds to the identity morphism in $\mathcal{E}q(\mathbb{H})$, and ν and χ are canonical.

viii) In Example 3.10 we will show that any *central extension* of categorical groups gives rise to a categorical crossed module.

ix) If $\mathbf{T} : \mathbb{H} \to \mathbb{G}$ is a categorical crossed module then $\pi_0(\mathbb{H} \to \mathbb{G})$ and $\pi_1(\mathbb{H} \to \mathbb{G})$ are crossed modules of groups.

It is well known that the kernel of a crossed module of groups is an abelian group. In our context of categorical groups this fact translates into the following:

Proposition 2.7 *Let $\langle \mathbb{H}, \mathbf{T}, \nu, \chi \rangle$ be a categorical $\mathbb{G}$-crossed module. Then $Ker\mathbf{T}$ can be equipped with a braiding.*

Proof: Let (A, ϵ_A) and (B, ϵ_B) be in the kernel; the braiding is given by

$$B \otimes A \xrightarrow{\phi_{0,B} \otimes 1} {}^{I}B \otimes A \xrightarrow{\epsilon_A^{-1} B \otimes 1} {}^{T(A)}B \otimes A \xrightarrow{\chi_{A,B}} A \otimes B.$$

To check that the previous arrow is a morphism in the kernel, use conditions (pcr3) and (cr4). The coherence conditions for the braiding follow from (cr1) and (cr2). ∎

Remark 2.8 In the previous proof observe that the construction of the braiding does not use ϵ_B. This means that $e_{\mathbf{T}} : Ker\mathbf{T} \to \mathbb{H}$ factors through the center of $\mathbb{H}$ [23].

3 The quotient categorical group.

We start with the construction of the quotient categorical groupoid of a morphism of categorical groups and, then, we see that the structure natural isomorphisms of a categorical crossed module allow us to obtain a monoidal structure in the quotient.

Consider a morphism of categorical groups $\mathbf{T} = (T, \mu) : \mathbb{H} \to \mathbb{G}$. The quotient pointed groupoid $\mathbb{G}/\langle \mathbb{H}, \mathbf{T} \rangle$ is defined in the following way:

- Objects: those of $\mathbb{G}$;
- Premorphisms: pairs $(A, f) : X \to Y$, with $A \in \mathbb{H}$ and $f : X \to T(A) \otimes Y$;
- Morphisms: classes of premorphisms $[A, f] : X \multimap Y$, where two pairs (A, f) and (A', f') are equivalent if there is $a : A \to A'$ in $\mathbb{H}$ such that $f' = (T(a) \otimes 1_Y) f$

Given two morphisms $[A, f] : X \multimap Y$, $[B, g] : Y \multimap Z$ we define their composition by $[A \otimes B, ?] : X \multimap Z$, with arrow-part

$$? :\ X \xrightarrow{f} T(A) \otimes Y \xrightarrow{1 \otimes g} T(A) \otimes T(B) \otimes Z \xrightarrow{can} T(A \otimes B) \otimes Z.$$

For an object X the identity $[I, ?] : X \multimap X$ has arrow-part $X \overset{can}{\simeq} T(I) \otimes X$. Note tha $\mathbb{G}/\langle \mathbb{H}, \mathbf{T} \rangle$ is indeed a groupoid (pointed by I) where the inverse of $[A, f] : X \multimap Y$ is $[A^*, ?] : Y \multimap X$ with arrow-part $Y \overset{can}{\to} T(A^*) \otimes T(A) \otimes Y \overset{1 \otimes f^{-1}}{\longrightarrow} T(A^*) \otimes X$. Let us point out that $\mathbb{G}/\langle \mathbb{H}, \mathbf{T} \rangle$ is the classifying groupoid of a bigroupoid having as 2-cells arrows $a : A \to A'$ compatible with f and f' as before.

Suppose now we have a categorical $\mathbb{G}$-crossed module $\langle \mathbb{H}, \mathbf{T} : \mathbb{H} \to \mathbb{G}, \nu, \chi \rangle$ as defined in Section 2. Then we can define a tensor product on $\mathbb{G}/\langle \mathbb{H}, \mathbf{T} \rangle$ in the following way: given two morphisms $[A, f] : X \multimap Y$, $[B, g] : H \multimap K$ their tensor product is $[A \otimes {}^Y B, ?] : X \otimes H \multimap Y \otimes K$ with arrow-part

$$\begin{tikzcd}
X \otimes H \arrow[r, "f \otimes g"] & T(A) \otimes Y \otimes T(B) \otimes K \arrow[r, "1 \otimes \nu^{-1} \otimes 1"] & T(A) \otimes T({}^Y B) \otimes Y \otimes K \arrow[d, "can"] \\
& & T(A \otimes {}^Y B) \otimes Y \otimes K
\end{tikzcd}$$

Let us just point out that the natural isomorphism χ, and its compatibility with ν, are needed to prove the bifunctoriality of this tensor product. To complete the monoidal structure of $\mathbb{G}/\langle \mathbb{H}, \mathbf{T} \rangle$, we use the essentially surjective functor $P_T : \mathbb{G} \to \mathbb{G}/\langle \mathbb{H}, \mathbf{T} \rangle$ $P_T(f : X \to Y) = [I, ?] : X \multimap Y$, with arrow-part $X \overset{f}{\to} Y \overset{can}{\simeq} T(I) \otimes Y$. Now, as unit and associativity constraints in $\mathbb{G}/\langle \mathbb{H}, \mathbf{T} \rangle$, we take the constraints in $\mathbb{G}$. It is long but essentially straightforward to check that $\mathbb{G}/\langle\mathbb{H},\mathbf{T}\rangle$ is a categorical group and P_T is a monoidal functor. Moreover, there is a 2-cell in $\mathcal{C}G$

$$\begin{tikzcd} & \mathbb{G} \arrow[dr, "P_T"] & \\ \mathbb{H} \arrow[ur, "T"] \arrow[rr, "0"'] \arrow[rr, phantom, shift left=3, "\pi_T\Downarrow"] & & \mathbb{G}/\langle\mathbb{H},\mathbf{T}\rangle \end{tikzcd}$$

(where 0 is the morphism sending each arrow into the identity of the unit object) defined by $(\pi_T)_A = [A,?] : T(A) \multimap I$, with arrow-part $T(A) \overset{can}{\simeq} T(A)\otimes I$.

Remark 3.1 Observe that if a categorical $\mathbb{G}$-precrossed module $\langle\mathbb{H},\mathbf{T},\nu\rangle$ has two different crossed structures χ and χ', the quotient categorical groups we obtain using χ and χ' are equal. This is why we consider the crossed structure as a property.

Example 3.2 i) Let $\delta : H \to G$ be a crossed module of groups. As in Example 2.6 i) we can look at it as a categorical crossed module. Its quotient $G[0]/\langle H[0],\delta\rangle$ is the strict (but not discrete) categorical group $\mathbb{G}(\delta)$ corresponding to δ in the biequivalence between crossed modules of groups and categorical groups (see [5,34]). Note that $\pi_0(\mathbb{G}(\delta)) = Coker(\delta)$ and $\pi_1(\mathbb{G}(\delta)) = Ker(\delta)$.

ii) If $(d : \mathbb{H} \to G, \chi)$ is a categorical G-crossed module as in [8], then $G/\langle\mathbb{H}, d\rangle = G/\langle\pi_0(\mathbb{H})[0], \pi_0(d)\rangle = \mathbb{G}(\pi_0(d))$.

iii) If $\mathbf{T} : \mathbb{A} \to \mathbb{B}$ is a morphism of symmetric categorical groups, then $\mathbb{B}/\langle\mathbb{A},\mathbf{T}\rangle$ is the cokernel of $\mathbf{T}$ studied in [37].

iv) Let $\mathbb{H}$ be a categorical group and consider the "inner automorphism" categorical crossed module $\mathbf{i} : \mathbb{H} \to \mathcal{E}q(\mathbb{H})$ as in Preliminaries. Its quotient is equivalent to the categorical group $\mathcal{O}ut(\mathbb{H})$ defined in [10] and used to study obstruction theory for extensions of categorical groups. Note also that $\mathcal{O}ut(\mathbb{H})$ is the classifying category of the bicategory used in [31] to classify extensions of categorical groups.

We next deal with *The universal property* of this quotient.
The previous construction $(\mathbb{G}/\langle\mathbb{H},\mathbf{T}\rangle, P_T : \mathbb{G} \to \mathbb{G}/\langle\mathbb{H},\mathbf{T}\rangle, \pi_T : P_T T \Leftarrow 0)$ is universal with respect to triples in $\mathcal{C}G$, $(\mathbb{F}, G : \mathbb{G} \to \mathbb{F}, \delta : GT \Rightarrow 0)$ satisfying

the following condition: For any $X \in \mathbb{G}$ and $A \in \mathbb{H}$ the diagram

$$\begin{tikzcd} G(X) \otimes I \arrow[d, "can"'] & G(X) \otimes GT(A) \arrow[l, "1\otimes\delta_A"'] \\ I \otimes G(X) & G(X \otimes T(A)) \arrow[u, "can"'] \\ GT({}^{X}A) \otimes G(X) \arrow[u, "\delta_{X_A}\otimes 1"] & G(T({}^{X}A) \otimes X) \arrow[l, "can"] \arrow[u, "G(\nu_{X,A})"'] \end{tikzcd} \tag{1}$$

is commutative. In fact, the word *universal* means here two different things.

1) $\mathbb{G}/\langle\mathbb{H}, \mathbf{T}\rangle$ is a standard homotopy cokernel: for each triple $(\mathbb{F}, G : \mathbb{G} \to \mathbb{F}, \delta : GT \Rightarrow 0)$ in $\mathcal{C}G$, satisfying condition (1) there is a unique morphism

$$G' : \mathbb{G}/\langle\mathbb{H}, \mathbf{T}\rangle \to \mathbb{F}$$

in $\mathcal{C}G$ such that $G'P_T = G$ and $G'\pi_T = \delta$.

2) $\mathbb{G}/\langle\mathbb{H}, \mathbf{T}\rangle$ is a bilimit: for each triple $(\mathbb{F}, G : \mathbb{G} \to \mathbb{F}, \delta : GT \Rightarrow 0)$ in $\mathcal{C}G$, satisfying condition (1) there are $G' : \mathbb{G}/\langle\mathbb{H}, \mathbf{T}\rangle \to \mathbb{F}$ and $\delta' : G'P_T \Rightarrow 0$ in $\mathcal{C}G$ making commutative the following diagram

$$\begin{tikzcd} G'P_TT \arrow[r, Rightarrow, "\delta' T"] \arrow[d, Rightarrow, "G'\pi_T"'] & GT \arrow[d, Rightarrow, "\delta"] \\ G'0 \arrow[r, Rightarrow] & 0 \end{tikzcd}$$

Moreover, if $G'' : \mathbb{G}/\langle\mathbb{H}, \mathbf{T}\rangle \to \mathbb{F}, \delta'' : G''P_T \Rightarrow 0$ are in $\mathcal{C}G$ and make commutative an analogous diagram as above, then there is a unique $\lambda : G' \Rightarrow G''$ in $\mathcal{C}G$ such that the following diagram commutes

$$\begin{tikzcd} G'P_T \arrow[rr, Rightarrow, "\lambda P_T"] \arrow[dr, Rightarrow, "\delta'"'] & & G''P_T \arrow[dl, Rightarrow, "\delta''"] \\ & G & \end{tikzcd}$$

Observe that the first universal property characterizes $\mathbb{G}/\langle\mathbb{H}, \mathbf{T}\rangle$ up to isomorphism, whereas the second one characterizes it up to equivalence.
The proof of the uniqueness, in both the universal properties, is based on the following lemma.

Lemma 3.3 *Let* $[A, f] \colon X \multimap Y$ *be an arrow in* $\mathbb{G}/\langle\mathbb{H}, \mathbf{T}\rangle$. *The following diagram commutes*

$$\begin{tikzcd} X \arrow[r, "{[A,f]}", "\circ" marking] \arrow[d, "P_T(f)"', "\circ" marking] & Y \arrow[d, "can", "\circ" marking] \\ T(A) \otimes Y \arrow[r, "(\pi_T)_A \otimes 1"', "\circ" marking] & I \otimes Y \end{tikzcd}$$

Now, as far as the first universal property is concerned, define $G' : \mathbb{G}/\langle \mathbb{H}, \mathbf{T} \rangle \to \mathbb{F}$ by

$$G'[A,f] : \, G(X) \xrightarrow{G(f)} G(T(A) \otimes Y) \overset{can}{\simeq} G(T(A)) \otimes G(Y) \xrightarrow{\delta_A \otimes 1} I \otimes G(Y) \overset{can}{\simeq} G(Y)$$

and use condition (1) to check that the monoidal structure of G', which is that of G, is natural with respect to the arrows of $\mathbb{G}/\langle \mathbb{H}, \mathbf{T} \rangle$. As far as the second universal property is concerned, just take δ' to be the identity and define λ via the formula $\lambda_X = (\delta''_X)^{-1} : G'(X) = G(X) \to G''(X)$.

Remark 3.4 The fact that to make the functor $G' : \mathbb{G}/\langle \mathbb{H}, \mathbf{T} \rangle \to \mathbb{F}$ monoidal we need a condition (1) relating δ only to ν (and not to χ) is not a surprise. The fact that G' is monoidal or not depends only on the definition of the tensor product in $\mathbb{G}/\langle \mathbb{H}, \mathbf{T} \rangle$ and not on its functoriality, and the definition of the tensor in $\mathbb{G}/\langle \mathbb{H}, \mathbf{T} \rangle$ only uses ν (whereas χ is needed to make this tensor a functor).

Example 3.5 Consider the categorical $\mathbb{G}$-crossed module structure associated to a morphism $\mathbf{T} : \mathbb{H} \to \mathbb{G}$ of braided categorical groups, as in Example 2.6 iv), and the corresponding quotient categorical group. Consider also $\mathbb{F}, G$ and δ in $\mathcal{C}G$ as in the following diagram

$$\begin{tikzcd} \mathbb{H} \arrow[r, "T"] \arrow[dr, "0"'] & \mathbb{G} \arrow[r, "P_T"] \arrow[d, "G"] \arrow[dl, phantom, "\delta\Downarrow"] & \mathbb{G}/\langle \mathbb{H}, \mathbf{T} \rangle \\ & \mathbb{F} & \end{tikzcd}$$

Condition (1) is satisfied if $\mathbb{F}$ is braided and G is compatible with the braiding. (Recall that, as pointed out in [37], $\mathbb{G}/\langle \mathbb{H}, \mathbf{T} \rangle$ in general is not braided. Indeed, to prove that the braiding of $\mathbb{G}$ is natural in $\mathbb{G}/\langle \mathbb{H}, \mathbf{T} \rangle$, one needs that the braiding is a symmetry.)

In Example 2.6 v) we saw that the kernel of a morphism of categorical groups is a categorical crossed module. In the following proposition we consider the kernel of the "projection" $P_T : \mathbb{G} \to \mathbb{G}/\langle \mathbb{H}, \mathbf{T} \rangle$:

Proposition 3.6 *Consider a categorical $\mathbb{G}$-crossed module $\mathbf{T} : \mathbb{H} \to \mathbb{G}$ and the factorization $\mathbf{T}'$ of $\mathbf{T}$ through the kernel of P_T*

$$\begin{tikzcd} \mathbb{H} \arrow[r, "T"] \arrow[dr, "T'"'] & \mathbb{G} \arrow[r, "P_T"] & \mathbb{G}/\langle \mathbb{H}, \mathbf{T} \rangle \\ & Ker P_T \arrow[u, "e_{P_T}"'] & \end{tikzcd}$$

The functor T' is a morphism of categorical $\mathbb{G}$-crossed modules. Moreover, it is full and essentially surjective on objects.

The previous proposition means that the sequence

$$
\begin{tikzcd}
\mathbb{H} \arrow[r, "T"] \arrow[rr, bend left, "0"] & \mathbb{G} \arrow[r, "P_T"] \arrow[u, phantom, "\pi_T \Uparrow"] & \mathbb{G}/\langle \mathbb{H}, \mathbf{T} \rangle
\end{tikzcd}
$$

is 2-exact (see Definition 4.4 below). More important, it means that T' is an equivalence if and only if T is faithful. Therefore, we can give the following definition.

Definition 3.7 *A normal sub-categorical group of a categorical group* $\mathbb{G}$ *is a categorical* $\mathbb{G}$*-crossed module* $\mathbf{T} : \mathbb{H} \to \mathbb{G}$ *with* T *faithful.*

Proposition 3.8 *Let* $\mathbf{L} : \mathbb{G} \to \mathbb{K}$ *be a morphism in* $\mathcal{C}G$ *and consider its kernel* $Ker\mathbf{L} \xrightarrow{e_{\mathbf{L}}} \mathbb{G} \xrightarrow{L} \mathbb{K}$, $\epsilon_{_L} : L\, e_{\mathbf{L}} \Rightarrow 0$. *Consider also the normal sub-categorical group* $\langle Ker\mathbf{L}, e_{\mathbf{L}} \rangle$ *of* $\mathbb{G}$, *the corresponding quotient categorical group and the factorization of* L *through the quotient:*

$$
\begin{tikzcd}
Ker\mathbf{L} \arrow[r, "e_{\mathbf{L}}"] & \mathbb{G} \arrow[r, "L"] \arrow[d, "P_{e_{\mathbf{L}}}"'] & \mathbb{K} \\
& \mathbb{G}/\langle Ker\mathbf{L}, e_{\mathbf{L}} \rangle \arrow[ur, "L'"'] &
\end{tikzcd}
$$

Then L' *is a full and faithful functor.*

In the previous proposition, the factorization L' exists because the condition (1) in the universal property of the quotient is verified when $\delta = \epsilon_{_L}$.
The previous proposition means that $L : \mathbb{G} \to \mathbb{K}$ is essentially surjective on objects if and only if $L' : \mathbb{G}/\langle Ker\mathbf{L}, e_{\mathbf{L}} \rangle \to \mathbb{K}$ is an equivalence. In other words, quotients in the the 2-category $\mathcal{C}G$ are, up to equivalence, precisely the essentially surjective morphisms.

Remark 3.9 Let us recall that the image of a precrossed module of groups is a normal subgroup of the codomain. The situation for categorical groups is similar. If, in Proposition 3.8, $\mathbb{G}$ is a $\mathbb{K}$-categorical group and $\mathbf{L}$ is equipped with the structure of categorical $\mathbb{K}$-precrossed module, then $\mathbf{L}'$ inherits such structure and $P_{e_{\mathbf{L}}}$ is a morphism of categorical $\mathbb{K}$-precrossed modules. In fact, following Remark 2.5, $\mathbf{L}'$ is a categorical $\mathbb{K}$-crossed module. (In other words the full image of a categorical precrossed module and the not full image of a categorical crossed module are normal sub-categorical groups.) Let us just describe the action $\mathbb{K} \times \mathbb{G}/\langle Ker\mathbf{L}, e_{\mathbf{L}} \rangle \to \mathbb{G}/\langle Ker\mathbf{L}, e_{\mathbf{L}} \rangle$.
On objects, it is given by the action of $\mathbb{K}$ over $\mathbb{G}$. Now consider an arrow $[(N, \epsilon_{_N}), f] : G_1 \multimap G_2$ in the quotient and an object $K \in \mathbb{K}$. We need an arrow ${}^K G_1 \multimap {}^K G_2$. Since L' is full and faithful, it suffices to define an arrow

$L'({}^{K}G_1) \longrightarrow L'({}^{K}G_2)$ in $\mathbb{K}$. This is given by the following composition:

$$\begin{tikzcd} L({}^{K}G_1) \arrow[r, "\varphi_{K,G_1}"] \arrow[d, dotted] & K \otimes L(G_1) \otimes K^* \arrow[r, "1\otimes L(f)\otimes 1"] & K \otimes L(N \otimes G_2) \otimes K^* \arrow[d, "can"] \\ L({}^{K}G_2) & K \otimes L(G_2) \otimes K^* \arrow[l, "\varphi^{-1}_{K,G_2}"] & K \otimes L(N) \otimes L(G_2) \otimes K^* \arrow[l, "1\otimes\epsilon_N\otimes 1"] \end{tikzcd}$$

Example 3.10 An important example of crossed module of groups is given by a central extension. Recall that a central extension of groups is defined as a surjective morphism $H \xrightarrow{\delta} G$ such that the kernel of δ is contained in the center of H. To define the action of G on H (well-defined because of the centrality), you have to choose, for given $g \in G$, an element $x \in H$ such that $\delta(x) = g$ and you put ${}^{g}h = xhx^{-1}$. Looking for a generalization to categorical groups, let us reformulate the definition of central extension in such a way we can avoid the choice of the element x. Since δ is surjective and the center of H is the kernel of the inner automorphism $i : H \to Aut(H)$, to give a central extension is equivalent to give a surjective morphism δ together with a (necessarily unique) morphism $\alpha : G \to Aut(H)$ such that $\alpha\delta = i$. With the previous discussion in mind, we define a central extension of categorical groups to be an essentially surjective morphism $\mathbf{T} : \mathbb{H} \to \mathbb{G}$ together with an action $ac : \mathbb{G} \to \mathcal{E}q(\mathbb{H})$ such that $ac \cdot T = i : \mathbb{H} \to \mathcal{E}q(\mathbb{H})$.
Now we provide a categorical $\mathbb{G}$-crossed module structure on $\mathbf{T}$: The action of $\mathbb{G}$ on $\mathbb{H}$ is given by ac. By Proposition 3.8, the comparison morphism $\mathbf{T}' : \mathbb{H}/\langle Ker\mathbf{T}, e_{\mathbf{T}}\rangle \to \mathbb{G}$ is an equivalence, so that we get an action of $\mathbb{H}/\langle Ker\mathbf{T}, e_{\mathbf{T}}\rangle$ on $\mathbb{H}$. Because of the identity $ac \cdot T = i$, such an action must be given, on objects, by conjugation. Finally, the precrossed structure and the crossed structure are given by the canonical isomorphism $X \otimes A \otimes X^* \otimes A \to X \otimes A$ in $\mathbb{H}/\langle Ker\mathbf{T}, e_{\mathbf{T}}\rangle$ (for the precrossed structure) and in $\mathbb{H}$ (for the crossed structure).

4 The kernel-cokernel lemma

The aim of this section is to obtain an analogous to the classical kernel-cokernel lemma (see [29]). For it, we first extend the definitions given in Section 2, considering categorical precrossed modules based on different categorical groups:

Definition 4.1 *The 2-category of categorical precrossed modules "PreCross", has as objects the categorical precrossed modules. Given two categorical precrossed modules* $\langle \mathbb{H}, \mathbf{T} : \mathbb{H} \to \mathbb{G}, \nu\rangle$ *and* $\langle \mathbb{H}', \mathbf{T}' : \mathbb{H}' \to \mathbb{G}', \nu'\rangle$*, a morphism between them consists of a 4-tuple* $(\mathbf{F}, \mathbf{G}, \eta, \alpha)$*, as in the following diagram*

$$\begin{tikzcd}
\mathbb{G}\times\mathbb{H} \arrow[r, "ac"] \arrow[d, "G\times F"'] \arrow[dr, phantom, "\eta\Downarrow"] & \mathbb{H} \arrow[r, "T"] \arrow[d, "F"] \arrow[dr, phantom, "\alpha\Downarrow"] & \mathbb{G} \arrow[d, "G"] \\
\mathbb{G}'\times\mathbb{H}' \arrow[r, "ac"'] & \mathbb{H}' \arrow[r, "T'"'] & \mathbb{G}'
\end{tikzcd}$$

where $\mathbf{F}:\mathbb{H}\to\mathbb{H}'$, $\mathbf{G}:\mathbb{G}\to\mathbb{G}'$ *and* $\alpha : GT \Rightarrow T'F$ *are in* $\mathcal{C}G$. $(\mathbf{F},\eta):\mathbb{H}\to\mathbb{H}'$ *is a morphism in* $\mathbb{G}-\mathcal{C}G$, *considering* $\mathbb{H}'$ *a* $\mathbb{G}$*-categorical group via* $G:\mathbb{G}\to\mathbb{G}'$ *(see the Preliminaries section). In addition, for any* $X\in\mathbb{G}$ *and* $A\in\mathbb{H}$, *the following diagram has to be commutative*

$$\begin{tikzcd}
G(T({}^{X}A)\otimes X) \arrow[rr, "G(\nu_{X,A})"] \arrow[d, "can"'] & & G(X\otimes T(A)) \arrow[d, "can"] \\
GT({}^{X}A)\otimes G(X) \arrow[d, "\alpha_{{}^{X}A}\otimes 1"'] & & G(X)\otimes GT(A) \arrow[d, "1\otimes\alpha_A"] \\
T'F({}^{X}A)\otimes G(X) \arrow[dr, "T'(\eta_{X,A})\otimes 1"'] & & G(X)\otimes T'F(A) \\
& T'({}^{G(X)}F(A))\otimes G(X) \arrow[ur, "\nu'_{G(X),F(A)}"'] &
\end{tikzcd} \tag{2}$$

Given two parallel morphisms $(\mathbf{F},\mathbf{G},\eta,\alpha)$ *and* $(\mathbf{F}',\mathbf{G}',\eta',\alpha')$, *a 2-cell is a pair* $(\beta,\lambda):(\mathbf{F},\mathbf{G},\eta,\alpha)\Rightarrow(\mathbf{F}',\mathbf{G}',\eta',\alpha')$ *where* $\beta:\mathbf{F}\Rightarrow\mathbf{F}'$ *and* $\lambda:\mathbf{G}\Rightarrow\mathbf{G}'$ *are 2-cells in* $\mathcal{C}G$ *such that, for any* $X\in\mathbb{G}$, $A\in\mathbb{H}$, *the following diagrams are commutative*

$$\begin{tikzcd}
F({}^{X}A) \arrow[rr, "\eta_{X,A}"] \arrow[d, "\beta_{{}^{X}A}"'] & & {}^{G(X)}F(A) \arrow[d, "\lambda_X{}_{F(A)}"] \\
F'({}^{X}A) \arrow[dr, "\eta'_{X,A}"'] & & {}^{G'(X)}F(A) \arrow[dl, "{}^{G'(X)}\beta_A"] \\
& {}^{G'(X)}F'(A) &
\end{tikzcd} \tag{3}$$

$$\begin{tikzcd}
GT(A) \arrow[r, "\alpha_A"] \arrow[d, "\lambda_{T(A)}"'] & T'F(A) \arrow[d, "T'(\beta_A)"] \\
G'T(A) \arrow[r, "\alpha'_A"'] & T'F'(A)
\end{tikzcd} \tag{4}$$

Remark 4.2 i) In the previous definition, consider the natural isomorphisms $\varphi=\varphi_{X,A} : T({}^{X}A)\to X\otimes T(A)\otimes X^*$ in $\mathbb{G}$, and $\varphi'=\varphi'_{X',A'} : T({}^{X'}A')\to X'\otimes T(A')\otimes X'^*$ in $\mathbb{G}'$, obtained from ν and ν' respectively, such that $(\mathbf{T},\varphi):\mathbb{H}\to\mathbb{G}$ is a morphism in $\mathbb{G}-\mathcal{C}G$ and $(\mathbf{T}',\varphi'):\mathbb{H}'\to\mathbb{G}'$ is a morphism in $\mathbb{G}'-\mathcal{C}G$ (see Section 2). Then the compatibility condition (2) means that $\alpha:(\mathbf{GT},can\varphi)\Rightarrow(\mathbf{T'F},\varphi'T'\eta):\mathbb{H}\to\mathbb{G}'$ is a 2-cell in $\mathbb{G}-\mathcal{C}G$, where the $\mathbb{G}$-action over $\mathbb{G}'$ is that obtained via $\mathbf{G}:\mathbb{G}\to\mathbb{G}'$.

ii) The compatibility condition (3) means that $\beta : (\mathbf{F}, \bar{\eta}) \Rightarrow (\mathbf{F}', \eta') : \mathbb{H} \to \mathbb{H}'$ is a 2-cell in $\mathbb{G} - \mathcal{CG}$, considering in $\mathbb{H}'$ the action given via $\mathbf{G}'$ and where $\bar{\eta}_{X,A} : F({}^{X}A) \to {}^{G'(X)}F(A)$ is given by $\bar{\eta}_{X,A} = ^{\lambda_X} F(A) \cdot \eta_{X,A}$. Finally the compatibility condition (4) means that (β, λ) is a 2-cell in the 2-category of morphisms $\mathcal{CG}^{\to}$.

iii) If, in the previous definition, we take $\mathbf{G}, \mathbf{G}'$ and λ to be identities, we get the 2-category $\mathbb{G} - \mathcal{CG}/\mathbb{G}$ of categorical $\mathbb{G}$-precrossed modules defined in Section 2.

iv) Given two categorical precrossed modules there is a "zero-morphism" between them taking $\mathbf{F}$ and $\mathbf{G}$ as the zero-morphism, and where α, η are given by canonical isomorphisms.

The consideration of the 2-category *PreCross* is justified by the following proposition

Proposition 4.3 *Consider two categorical precrossed modules* $\langle \mathbb{H}, \mathbf{T} : \mathbb{H} \to \mathbb{G}, \nu \rangle$ *and* $\langle \mathbb{H}', \mathbf{T}' : \mathbb{H}' \to \mathbb{G}', \nu' \rangle$*, and a morphism between them* $(\mathbf{F}, \mathbf{G}, \eta, \alpha)$*. Assume that the categorical precrossed modules are in fact crossed modules. Then the triple* $(\mathbf{F}, \alpha, \mathbf{G})$ *extends to a morphism between the quotient categorical groups*

$$\widehat{\mathbf{G}} : \mathbb{G}/\langle \mathbb{H}, \mathbf{T} \rangle \to \mathbb{G}'/\langle \mathbb{H}', \mathbf{T}' \rangle$$

(that is, there is a monoidal functor $\widehat{\mathbf{G}}$ *and a monoidal natural transformation* $g : \widehat{G}P_T \Rightarrow P_{T'}G$ *compatible with* α, π_T *and* $\pi_{T'}$*).*

Proof: Consider the natural transformation

$$P_{T'}GT \xRightarrow{P_{T'}\alpha} P_{T'}T'F \xRightarrow{\pi_{T'}F} 0F \Longrightarrow 0$$

Using the compatibility condition on (η, α), one can check that this natural transformation satisfies condition (1) in the universal property of $\mathbb{G}/\langle \mathbb{H}, \mathbf{T} \rangle$. ▌

We now recall two definitions needed for establishing the kernel-cokernel sequence (c.f. [24,31]). Consider the following diagram in $\mathcal{CG}$

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "0"] \arrow[dr, "G"'] & \arrow[d, phantom, "\beta\Uparrow"] & \mathbb{C} \\ & \mathbb{B} \arrow[ur, "G'"'] & \end{tikzcd}$$

Definition 4.4 *We say that the triple* $(\mathbf{G}, \beta, \mathbf{G}')$ *as in the previous diagram is 2-exact if the factorization of* G *through the kernel of* G' *is a full and essentially surjective functor.*
We say that the triple $(\mathbf{G}, \beta, \mathbf{G}')$ *as in the previous diagram is an extension if it is 2-exact,* G *is faithful and* G' *is essentially surjective; or, equivalently, if*

the factorization of G through the kernel of G' is an equivalence and, moreover, G' is essentially surjective.

Consider three categorical crossed modules $\langle \mathbb{H}, \mathbf{T} : \mathbb{H} \to \mathbb{G}, \nu, \chi \rangle$, $\langle \mathbb{H}', \mathbf{T}' : \mathbb{H}' \to \mathbb{G}', \nu', \chi' \rangle$ and $\langle \mathbb{H}'', \mathbf{T}'' : \mathbb{H}'' \to \mathbb{G}'', \nu'', \chi'' \rangle$ and two morphisms of categorical precrossed modules, $(\mathbf{F}, \mathbf{G}, \eta, \alpha) : \langle \mathbb{H}, \mathbf{T}, \nu \rangle\rangle \to \langle \mathbb{H}', \mathbf{T}', \nu' \rangle\rangle$ and $(\mathbf{F}', \mathbf{G}', \eta', \alpha') : \langle \mathbb{H}', \mathbf{T}', \nu' \rangle\rangle \to \langle \mathbb{H}'', \mathbf{T}'', \nu'' \rangle\rangle$. Consider also a 2-cell $(\beta : \mathbf{F}'\mathbf{F} \Rightarrow 0, \lambda : \mathbf{G}'\mathbf{G} \Rightarrow 0)$ from the composite morphism to the zero-morphism. Using the universal property of the kernel and Proposition 4.3, we get the following diagram in $\mathcal{C}G$

$$
\begin{tikzcd}
\mathrm{Ker}\mathbf{T} \arrow[r, "\widehat{F}"] \arrow[d, "e_T"'] \arrow[rr, bend left, "0", "\widehat{\beta}\Uparrow"'] \arrow[dr, phantom, "\overset{f}{\Rightarrow}"] & \mathrm{Ker}\mathbf{T}' \arrow[r, "\widehat{F'}"] \arrow[d, "e_{T'}"'] \arrow[dr, phantom, "\overset{f'}{\Rightarrow}"] & \mathrm{Ker}\mathbf{T}'' \arrow[d, "e_{T''}"] \\
\mathbb{H} \arrow[r, "F"] \arrow[d, "T"'] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & \mathbb{H}' \arrow[r, "F'"] \arrow[d, "T'"'] \arrow[dr, phantom, "\overset{\alpha'}{\Rightarrow}"] & \mathbb{H}'' \arrow[d, "T''"] \\
\mathbb{G} \arrow[r, "G"'] \arrow[d, "P_T"'] \arrow[dr, phantom, "\overset{g}{\Rightarrow}"] & \mathbb{G}' \arrow[r, "G'"'] \arrow[d, "P_{T'}"'] \arrow[dr, phantom, "\overset{g'}{\Rightarrow}"] & \mathbb{G}'' \arrow[d, "P_{T''}"] \\
\mathbb{G}/\langle \mathbb{H}, \mathbf{T} \rangle \arrow[r, "\widehat{G}"'] \arrow[rr, bend right, "0"', "\widehat{\lambda}\Downarrow"] & \mathbb{G}'/\langle \mathbb{H}', \mathbf{T}' \rangle \arrow[r, "\widehat{G'}"'] & \mathbb{G}''/\langle \mathbb{H}'', \mathbf{T}'' \rangle
\end{tikzcd}
$$

Using the previous notation, we get the following facts.

Lemma 4.5 *(1) If the triple $(\mathbf{F}, \beta, \mathbf{F}')$ is 2-exact and the morphism $\mathbf{G}$ is faithful, then the triple $(\widehat{\mathbf{F}}, \widehat{\beta}, \widehat{\mathbf{F}'})$ is 2-exact.*

(2) If the triple $(\mathbf{G}, \lambda, \mathbf{G}')$ is 2-exact and the morphism $\mathbf{F}'$ is essentially surjective, then the triple $(\widehat{\mathbf{G}}, \widehat{\lambda}, \widehat{\mathbf{G}'})$ is 2-exact.

Proposition 4.6 *If the triples $(\mathbf{F}, \beta, \mathbf{F}')$ and $(\mathbf{G}, \lambda, \mathbf{G}')$ are extensions, then there are a morphism $\mathbf{D}$ and two 2-cells Σ, Ψ in $\mathcal{C}G$ such that the following sequence is 2-exact in $Ker\mathbf{T}'$, $Ker\mathbf{T}''$, $\mathbb{G}/\langle \mathbb{H}, \mathbf{T} \rangle$ and $\mathbb{G}'/\langle \mathbb{H}', \mathbf{T}' \rangle$*

$$
\begin{tikzcd}
Ker\mathbf{T} \arrow[r, "\widehat{\mathbf{F}}"] \arrow[rr, bend right, "0"', "\widehat{\beta}\Downarrow"] & Ker\mathbf{T}' \arrow[r, "\widehat{\mathbf{F}'}"] \arrow[rr, bend left, "0", "\Sigma\Uparrow"'] & Ker\mathbf{T}'' \arrow[r, "\mathbf{D}"] \arrow[rr, bend right, "0"', "\Psi\Downarrow"] & \mathbb{G}/\langle \mathbb{H}, \mathbf{T} \rangle \arrow[r, "\widehat{\mathbf{G}}"] \arrow[rr, bend left, "0", "\widehat{\lambda}\Uparrow"'] & \mathbb{G}'/\langle \mathbb{H}', \mathbf{T}' \rangle \arrow[r, "\widehat{\mathbf{G}'}"] & \mathbb{G}''/\langle \mathbb{H}'', \mathbf{T}'' \rangle
\end{tikzcd}
$$

Moreover, $\widehat{\mathbf{F}}$ is faithful and $\widehat{\mathbf{G}'}$ is essentially surjective.

Proof: Let us just describe the functor

$$D : Ker\mathbf{T}'' \to \mathbb{G}/\langle \mathbb{H}, \mathbf{T}\rangle$$

Observe that, since F is faithful and $(\mathbf{F}, \beta, \mathbf{F}')$ is 2-exact, $\mathbf{F} : \mathbb{H} \to \mathbb{H}'$ inherits from the kernel of $\mathbf{F}'$ a structure of categorical $\mathbb{H}'$-crossed module. Moreover, since F' is essentially surjective, up to equivalence $\mathbb{H}''$ is the quotient cat-group $\mathbb{H}'/\langle \mathbb{H}, \mathbf{F}\rangle$ and $\mathbf{F}'$ is the projection P_F. Observe also that, since G is faithful and $(\mathbf{G}, \lambda, \mathbf{G}')$ is 2-exact, $\mathbb{G}$ is equivalent to the kernel of $\mathbf{G}'$ and $\mathbf{G}$ is the injection $e_{\mathbf{G}'}$. We use these descriptions of $\mathbb{H}''$ and $\mathbb{G}$ to construct the functor D.

An object in $Ker\mathbf{T}''$ is a pair $(B \in \mathbb{H}', b : T''(B) \to I)$. We get an object in $Ker\mathbf{G}'$, $(T'(B), G'(T'(B)) \overset{\alpha'_B}{\to} T''(P_F(B)) = T''(B) \overset{b}{\to} I)$, and we define

$$D(B, b) = P_T\Big(T'(B), b\alpha'_{B}\Big)$$

An arrow in $Ker\mathbf{T}''$ is $[A, x] : (B_1, b_1 : T''(B_1) \to I) \to (B_2, b_2 : T''(B_2) \to I)$, with $[A, x] : B_1 \multimap B_2$ a morphism in $\mathbb{H}'/\langle \mathbb{H}, \mathbf{F}\rangle$ (with representative $x : B_1 \to F(A) \otimes B_2$) such that the following diagram commutes

$$\begin{tikzcd}
T''(B_1) \arrow[rr, "T''[A,x]"] \arrow[dr, "b_1"'] & & T''(B_2) \arrow[dl, "b_2"] \\
& I &
\end{tikzcd}$$

We define

$$D[A, x] = [A, ?] : \ (P_T(T'(B_1)), b_1\alpha'_{B_1}) \multimap (P_T(T'(B_2)), b_2\alpha'_{B_2})$$

where the arrow part must be an arrow $? : (T'(B_1), b_1\alpha'_{B_1}) \to T(A) \otimes (T'(B_2), b_2\alpha'_{B_2})$ in $Ker\mathbf{G}'$, that is an arrow $? : T'(B_1) \to e_{\mathbf{G}'}(T(A)) \otimes T'(B_2)$ in $\mathbb{G}'$ making commutative the following diagram

$$\begin{tikzcd}
G'(T'(B_1)) \arrow[rrr, "G'(?)"] \arrow[d, "\alpha'_{B_1}"'] & & & G'(e_{\mathbf{G}'}(T(A)) \otimes T'(B_2)) \arrow[d, "can"] \\
T''(B_1) \arrow[d, "b_1"'] & & & G'(e_{\mathbf{G}'}(T(A))) \otimes G'(T'(B_2)) \arrow[d, "\epsilon_{G'}(T(A)) \otimes \alpha'_{B_2}"] \\
I & T''(B_2) \arrow[l, "b_2"] & & I \otimes T''(B_2) \arrow[ll, "can"]
\end{tikzcd}$$

For this, we take

$$? : \ T'(B_1) \xrightarrow{T'(x)} T'(F(A) \otimes B_2) \overset{can}{\simeq} T'(F(A)) \otimes T'(B_2) \xrightarrow{\alpha_A^{-1} \otimes 1} e_{\mathbf{G}'}(T(A)) \otimes T'(B_2)$$

and, to check that it is an arrow in the kernel of $\mathbf{G}'$, one uses that (β, λ) is a 2-cell in the 2-category of morphisms $\mathcal{C}G^{\rightarrow}$ (see ii) in Remark 4.2).
In this way, we have defined $D : Ker\mathbf{T}'' \rightarrow \mathbb{G}/\langle \mathbb{H}, \mathbf{T} \rangle$. It is easy to verify that it is well-defined and that it is a functor. To prove that its (obvious) monoidal structure is natural with respect to the arrows of $Ker\mathbf{T}''$, one uses the compatibility condition between the precrossed structure and the crossed structure of $T' : \mathbb{H}' \rightarrow \mathbb{G}'$ (see condition *(cr4)* of the definition of categorical crossed module in Section 2). ∎

Remark 4.7 Observe that in the previous proposition, the assumption that $\mathbf{G}'$ is essentially surjective is needed only to get that $\widehat{\mathbf{G}'}$ slso is essentially surjective.

For the sake of generality, let us point out that, to establish the results of Section 3 and Section 4 we do not need condition *(cr3)* in the definition of categorical crossed module.

5 The "inner derivations" categorical crossed module

We first recall the notion of derivation of categorical groups given in [18] (see also [17,19]: Let a $\mathbb{G}$-categorical group $\mathbb{H}$ be given, a *derivation* from $\mathbb{G}$ into $\mathbb{H}$ is a functor $D : \mathbb{G} \rightarrow \mathbb{H}$ together with a family of natural isomorphisms $\beta = \beta_{X,Y} : D(X \otimes Y) \rightarrow D(X) \otimes {}^{X}D(Y), X, Y \in \mathbb{G}$, verifying a coherence condition with respect to the canonical isomorphisms of the action.

Given two derivations $(D, \beta), (D', \beta') : \mathbb{G} \rightarrow \mathbb{H}$, a *morphism* from (D, β) to (D', β') consists of a natural transformation $\epsilon : D \rightarrow D'$ compatible with β and β' in the sense the reader can easily write.
The vertical composition of natural transformations determines a composition for morphisms between derivations so that we can consider the category $Der(\mathbb{G}, \mathbb{H})$ of derivations from $\mathbb{G}$ into $\mathbb{H}$, which is actually a groupoid. This groupoid is pointed by the trivial derivation, that is, the pair (D_0, β_0) where D_0 is the constant functor with value the unit object $I \in \mathbb{H}$ and β_0 is given by canonicals.
If $\mathbb{G} = G[0]$ and $\mathbb{H} = H[0]$ are the discrete categorical groups associated to groups G and H, then $Der(\mathbb{G}, \mathbb{H})$ is the discrete groupoid associated to the set $Der(G, H)$ of derivations from G into the G-group H. Now in [38], Whitehead shows that $Der(G, H)$ is a monoid provided that H is a G-crossed module. We will first show an analogous result for categorical groups. That is, if $(\mathbb{H}, \mathbf{T}, \nu, \chi)$ is a categorical $\mathbb{G}$-crossed module, then the groupoid $Der(\mathbb{G}, \mathbb{H})$ has a natural monoidal structure, which is inherited from the $\mathbb{G}$-crossed module structure in $\mathbb{H}$. When $\mathbb{G}$ is discrete, this fact was already observed in [18]. We first show the following:

Lemma 5.1 *If* $(\mathbb{H}, \mathbf{T}, \nu, \chi)$ *is a categorical* $\mathbb{G}$*-crossed module, then there are functors of pointed groupoids:*

$$\sigma : Der(\mathbb{G}, \mathbb{H}) \longrightarrow \mathcal{E}nd_{\mathcal{CG}}(\mathbb{G}) \quad \textit{and} \quad \theta : Der(\mathbb{G}, \mathbb{H}) \longrightarrow \mathcal{E}nd_{\mathcal{CG}}(\mathbb{H}).$$

Proof: The functor $\sigma : Der(\mathbb{G}, \mathbb{H}) \longrightarrow \mathcal{E}nd_{\mathcal{CG}}(\mathbb{G})$ is defined on objects $(D, \beta) \in Der(\mathbb{G}, \mathbb{H})$ by $\sigma(D, \beta) = (\sigma_{_D}, \mu_{\sigma_D})$ where, for any $X \in \mathbb{G}$ and for any arrow f in $\mathbb{G}$,

$$\sigma_{_D}(X) = T(D(X)) \otimes X \quad \text{ and } \quad \sigma_{_D}(f) = T(D(f)) \otimes f\ .$$

Besides, for any $X, Y \in \mathbb{G}$, $(\mu_{\sigma_D})_{X,Y} : TD(X \otimes Y) \otimes X \otimes Y \to TD(X) \otimes X \otimes TD(Y) \otimes Y$ is the composition $(\mu_{\sigma_D})_{X,Y} = (1 \otimes \nu_{X,D(Y)} \otimes 1) \cdot ((\mu_{T})_{D(X),{}^{X}D(Y)} \otimes 1) \cdot (T(\beta_{X,Y}) \otimes 1)$. On arrows $\epsilon : (D, \beta) \to (D', \beta')$, $\sigma(\epsilon)$ is given, for any $X \in \mathbb{G}$, by $\sigma(\epsilon)_X = T(\epsilon_X) \otimes 1 : TD(X) \otimes X \to TD'(X) \otimes X$.
As far as the functor $\theta : Der(\mathbb{G}, \mathbb{H}) \longrightarrow \mathcal{E}nd_{\mathcal{CG}}(\mathbb{H})$ is concerned, it is defined on objects $(D, \beta) \in Der(\mathbb{G}, \mathbb{H})$ by $\theta(D, \beta) = (\theta_{_D}, \mu_{\theta_D})$ where, for any object $A \in \mathbb{H}$ and for any arrow $u \in \mathbb{H}$,

$$\theta_{_D}(A) = DT(A) \otimes A \quad \text{and} \quad \theta_{_D}(u) = DT(u) \otimes u\ .$$

Besides, for any $A, B \in \mathbb{H}$, $(\mu_{\theta_D})_{A,B} : DT(A \otimes B) \otimes A \otimes B \to DT(A) \otimes A \otimes DT(B) \otimes B$ is the composition $(\mu_{\theta_D})_{A,B} = (1 \otimes \chi_{A,DT(B)} \otimes 1) \cdot (\beta_{T(A),T(B)} \otimes 1) \cdot (D((\mu_T)_{A,B}) \otimes 1)$. On arrows $\epsilon : (D, \beta) \to (D', \beta')$, $\theta(\epsilon)$ is given, for any $A \in \mathbb{H}$, by $(\theta(\epsilon))_A = \epsilon_{T(A)} \otimes 1 : DT(A) \otimes A \to D'T(A) \otimes A$. ▮

Both groupoids $\mathcal{E}nd_{\mathcal{CG}}(\mathbb{G})$ and $\mathcal{E}nd_{\mathcal{CG}}(\mathbb{H})$ are monoidal grupoids where the tensor functor is given by composition of endomorphisms. Now, using the endomorphism $\sigma_{_D}$ defined in the above lemma, we can establish the following:

Theorem 5.2 *If* $(\mathbb{H}, \mathbf{T}, \nu, \chi)$ *is a categorical* $\mathbb{G}$*-crossed module, then there is a natural monoidal structure on* $Der(\mathbb{G}, \mathbb{H})$ *such that* $\sigma : Der(\mathbb{G}, \mathbb{H}) \to \mathcal{E}nd_{\mathcal{CG}}(\mathbb{G})$ *and* $\theta : Der(\mathbb{G}, \mathbb{H}) \to \mathcal{E}nd_{\mathcal{CG}}(\mathbb{H})$ *are monoidal functors.*

Proof: The tensor functor for $Der(\mathbb{G}, \mathbb{H})$ is given, on objects, by $(D_1, \beta_1) \otimes (D_2, \beta_2) = (D_1 \otimes D_2, \beta_1 \otimes \beta_2)$ where, for any $X \in \mathbb{G}$,

$$(D_1 \otimes D_2)(X) = D_1(\sigma_{_{D_2}}(X)) \otimes D_2(X) = D_1(TD_2(X) \otimes X) \otimes D_2(X)$$

and, for any $X, Y \in \mathbb{G}$, $(\beta_1 \otimes \beta_2)_{X,Y}$ is given by the composition:

$$
\begin{array}{c}
(D_1\otimes D_2)(X\otimes Y)=D_1\,\sigma_{D_2}(X\otimes Y)\otimes D_2(X\otimes Y)\quad .\\
\big\downarrow{\scriptstyle can}\\
D_1(\sigma_{D_2}(X)\otimes\sigma_{D_2}(Y))\otimes D_2(X\otimes Y)\\
\big\downarrow{\scriptstyle \beta_1\otimes\beta_2}\\
D_1\,\sigma_{D_2}(X)\otimes\ {}^{\sigma_{D_2}(X)}D_1\,\sigma_{D_2}(Y)\otimes D_2(X)\otimes\ {}^{X}D_2(Y)\\
\big\downarrow{\scriptstyle can}\\
D_1\,\sigma_{D_2}(X)\otimes\ {}^{TD_2(X)}[{}^{X}D_1\,\sigma_{D_2}(Y)]\otimes D_2(X)\otimes\ {}^{X}D_2(Y)\\
\big\downarrow{\scriptstyle 1\otimes\chi\otimes 1}\\
D_1\,\sigma_{D_2}(X)\otimes D_2(X)\otimes\ {}^{X}D_1\,\sigma_{D_2}(Y)\otimes\ {}^{X}D_2(Y)\\
\big\downarrow{\scriptstyle can}\\
D_1\,\sigma_{D_2}(X)\otimes D_2(X)\otimes\ {}^{X}[D_1\,\sigma_{D_2}(Y)\otimes D_2(Y)]\\
\big\|\\
(D_1\otimes D_2)(X)\otimes\ {}^{X}(D_1\otimes D_2)(Y)
\end{array}
$$

It is straightforward to prove $(D_1\otimes D_2,\beta_1\otimes\beta_2)\in Der(\mathbb{G},\mathbb{H})$. As for arrows $\epsilon_1:(D_1,\beta_1)\to(D_1',\beta_1')$ and $\epsilon_2:(D_2,\beta_2)\to(D_2',\beta_2')$, we define, for any $X\in\mathbb{G}$, $(\epsilon_1\otimes\epsilon_2)_X=[D_1'(T((\epsilon_2)_X)\otimes 1)\otimes(\epsilon_2)_X]\cdot[(\epsilon_1)_{TD_2(X)\otimes X}\otimes 1]$ and again it is straightforward to see that $\epsilon_1\otimes\epsilon_2$ is a morphism of derivations.
This tensor functor defines a monoidal structure on the groupoid $Der(\mathbb{G},\mathbb{H})$ where the associativity canonical isomorphism is defined using the associativity morphisms in $\mathbb{G}$ and $\mathbb{H}$ together with the isomorphism μ of the monoidal structure of $\mathbf{T}$. The unit object is the trivial derivation (D_0,β_0) and the right and left unit constraints are also defined by using canonical isomorphisms of $\mathbb{G}$, $\mathbb{H}$ and $\mathbf{T}$ (see [18] for the particular case of $\mathbb{G}$ being discrete).
Finally, the monoidal structure of the functor σ, $(\mu_\sigma)_{D_1,D_2}:\sigma_{D_1\otimes D_2}\to\sigma_{D_1}\sigma_{D_2}$ is given, for any $X\in\mathbb{G}$, by $(\mu_\sigma)_X=\mu_{D_1(TD_2(X)\otimes X),D_2(X)}\otimes 1$, and the corresponding one for θ, $(\mu_\theta)_{D_1,D_2}:\theta_{D_1\otimes D_2}\to\theta_{D_1}\theta_{D_2}$ is given, for any $A\in\mathbb{H}$, by $(\mu_\theta)_A=D_1(\mu^{-1}_{D_2T(A),A})\otimes 1$.
The required verifications of all these facts are straightforward. For instance, the coherence condition for μ_θ means that, for derivations (D_i,β_i), $i=1,2,3$

the following diagram has to be commutative:

$$
\begin{tikzcd}
 & \theta_{D_1}(\theta_{D_2}\theta_{D_3}) & \\
(\theta_{D_1}\theta_{D_2})\theta_{D_3} \arrow[ur, equal] & & \theta_{D_1}\theta_{D_2\otimes D_3} \arrow[ul, "\theta_{D_1}\cdot(\mu_\theta)_{D_2,D_3}"'] \\
\theta_{D_1\otimes D_2}\theta_{D_3} \arrow[u, "(\mu_\theta)_{D_1,D_2}\cdot\theta_{D_3}"] & & \theta_{D_1\otimes(D_2\otimes D_3)} \arrow[u, "(\mu_\theta)_{D_1,D_2\otimes D_3}"'] \\
 & \theta_{(D_1\otimes D_2)\otimes D_3} \arrow[ul, "(\mu_\theta)_{D_1\otimes D_2,D_3}"] \arrow[ur, "\theta_a"'] &
\end{tikzcd} .
$$

It is easy to see that the commutativity of this diagram follows from the commutativity, for any $A \in \mathbb{H}$, of the following one:

$$
\begin{tikzcd}
 & \mathbf{1} \arrow[dl, "u"'] \arrow[dr, "f"] \arrow[d, phantom, "(\mathbf{I})"] & \\
\mathbf{2} \arrow[r, "w"] \arrow[dr, "v"'] & \mathbf{3} \arrow[d, "t"] \arrow[dl, phantom, "(\mathbf{II})"] \arrow[dr, phantom, "(\mathbf{III})"] & \mathbf{4} \arrow[l, "h"'] \arrow[d, "g"] \\
 & \mathbf{5} & \mathbf{6} \arrow[l, "k"']
\end{tikzcd}
\tag{5}
$$

where:

$$\mathbf{1} = T\Big(D_2T\big(D_3T(A)\otimes A\big)\otimes D_3T(A)\otimes A\Big),$$

$$\mathbf{2} = TD_2T\Big(D_3T(A)\otimes A\Big)\otimes T\Big(D_3T(A)\otimes A\Big),$$

$$\mathbf{3} = TD_2\Big(TD_3T(A)\otimes T(A)\Big)\otimes T\Big(D_3T(A)\otimes A\Big),$$

$$\mathbf{4} = T\Big(D_2\big(TD_3T(A)\otimes T(A)\big)\otimes D_3T(A)\otimes A\Big),$$

$$\mathbf{5} = TD_2\Big(TD_3T(A)\otimes T(A)\Big)\otimes TD_3T(A)\otimes T(A),$$

$$\mathbf{6} = T\Big(D_2\big(TD_3T(A)\otimes T(A)\big)\otimes D_3T(A)\Big)\otimes T(A),$$

and

$u = \mu_{D_2T(D_3T(A)\otimes A), D_3T(A)\otimes A}$, $v = TD_2(\mu_{D_3T(A),A}) \otimes \mu_{D_3T(A),A}$, $t = 1 \otimes \mu_{D_3T(A),A}$

$f = T(D_2\mu_{D_3T(A),A} \otimes 1)$, $g = \mu_{D_2(TD_3T(A)\otimes T(A))\otimes D_3T(A),A}$, $h = \mu_{D_2(TD_3T(A)\otimes T(A)), D_3T(A)\otimes A}$

$k = \mu_{D_2(TD_3T(A)\otimes T(A)), D_3T(A)} \otimes 1$, $w = TD_2(\mu_{D_3T(A),A}) \otimes 1$.

Now, diagram (5) is commutative because (**I**) is commutative by naturality of μ, (**II**) is commutative by bifunctoriality of $\otimes$ in $\mathbb{G}$ and (**III**) is commutative due to the coherence condition for μ (omitting the associativity isomorphisms). ∎

Let us recall now [37] that, for any monoidal category $\mathcal{C}$, the *Picard categorical group* $\mathcal{P}(\mathcal{C})$ of $\mathcal{C}$ is the subcategory of $\mathcal{C}$ given by invertible objects and isomorphisms between them. Clearly, $\mathcal{P}(\mathcal{C})$ is a categorical group and any monoidal functor $F : \mathcal{C} \to \mathcal{D}$ restricts to a homomorphism of categorical groups $\mathcal{P}(F) : \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{D})$. In this way, there is a 2-functor $\mathcal{P}$ from the 2-category of monoidal categories to $\mathcal{CG}$.

Definition 5.3 *For any categorical group $\mathbb{G}$ and any categorical $\mathbb{G}$-crossed module $(\mathbb{H}, \mathbf{T}, \nu, \chi)$ we define the Whitehead categorical group of derivations $Der^*(\mathbb{G}, \mathbb{H})$ as the Picard categorical group, $\mathcal{P}(Der(\mathbb{G}, \mathbb{H}))$, of the monoidal category $Der(\mathbb{G}, \mathbb{H})$ introduced in Theorem 5.2.*

Let us observe that this definition gives a 2-functor

$$Der^*(\mathbb{G}, -) : \mathbb{G} - cross \xrightarrow{Der(\mathbb{G},-)} \text{Mon. Cat.} \xrightarrow{\mathcal{P}} \mathcal{CG}$$

which is actually left 2-exact in the sense asserted in the following proposition (c.f. [19]).

Proposition 5.4 *Let $(\mathbf{F}, \eta, \alpha) : (\mathbb{H}, \mathbf{T}, \nu, \chi) \to (\mathbb{H}', \mathbf{T}', \nu', \chi')$ be a morphism of categorical $\mathbb{G}$-crossed modules. Consider the categorical $\mathbb{G}$-crossed module structure in the kernel $Ker\mathbf{F}$ inherited via $\mathbf{T}$ (see Example 2.6 v)). Then the categorical group $Der^*(\mathbb{G}, Ker\mathbf{F})$ is isomorphic to the kernel of the induced homomorphism $\mathbf{F}_* : Der^*(\mathbb{G}, \mathbb{H}) \longrightarrow Der^*(\mathbb{G}, \mathbb{H}'')$. In particular the sequence*

$$Der^*(\mathbb{G}, Ker\mathbf{F}) \xrightarrow{\mathbf{j}_*} Der^*(\mathbb{G}, \mathbb{H}) \xrightarrow{\mathbf{F}_*} Der^*(\mathbb{G}, \mathbb{H}'')$$

is 2-exact.

Proof: This follows from the fact that $Der(\mathbb{G}, -)$ is representable by the semidirect product $\mathbb{H} \rtimes \mathbb{G}$. When $\mathbb{G}$ is discrete this is proved in [17] and the proof still works for any categorical group $\mathbb{G}$. ∎

We remark that, when $\mathbb{G} = G[0]$ and $\mathbb{H} = H[0]$, the categorical group $Der^*(\mathbb{G}, \mathbb{H})$ is exactly the discrete one associated to the Whitehead group $Der^*(G, H)$ of the regular derivations from G into H [38,20]. Also, note that

if $(\mathbb{A}, c)$ is a $\mathbb{G}$-module then $Der^*(\mathbb{G}, \mathbb{A}) = Der(\mathbb{G}, \mathbb{A})$, where the latter is the categorical group of derivations studied in [18].
In the general case, if we apply the 2-functor $\mathcal{P}$ to the monoidal functor $\theta : Der(\mathbb{G}, \mathbb{H}) \to \mathcal{E}nd_{\mathcal{CG}}(\mathbb{H})$ (see Theorem 5.2), we obtain a homomorphism of categorical groups, also denoted by θ, $\theta : Der^*(\mathbb{G}, \mathbb{H}) \longrightarrow \mathcal{E}q(\mathbb{H})$ which defines in $\mathbb{H}$ a structure of $Der^*(\mathbb{G}, \mathbb{H})$-categorical group. Explicitly, this structure is given, for any $(D, \beta) \in Der^*(\mathbb{G}, \mathbb{H})$ and $A \in \mathbb{H}$, by

$$^{(D,\beta)}A = \theta_{D}(A) = DT(A) \otimes A \ . \tag{6}$$

Using Theorem 5.2 we obtain the following characterization of the objects of $Der^*(\mathbb{G}, \mathbb{H})$, that is, of the invertible derivations, whose proof is left to the reader (c.f. [18]).

Proposition 5.5 *Let $\mathbb{G}$ be a categorical group and $(\mathbb{H}, \mathbf{T}, \nu, \chi)$ a categorical $\mathbb{G}$-crossed module. Then, the following statements on a derivation (D, β) are equivalent: a) $(D, \beta) \in Der^*(\mathbb{G}, \mathbb{H})$, b) $\sigma_{D} \in \mathcal{E}q(\mathbb{G})$, c) $\theta_{D} \in \mathcal{E}q(\mathbb{H})$.*

If G is a group and H is a G-crossed module, then the morphism $H \to Der^*(G, H)$ from H to the Whitehead group of regular derivations $Der^*(G, H)$, given by inner derivations, is a crossed module of groups (see [27]). This fact translates also to our context:
Suppose a categorical $\mathbb{G}$-crossed module $(\mathbb{H}, \mathbf{T}, \nu, \chi)$ be given. Any object $A \in \mathbb{H}$ defines a *inner derivation* $(D_{A}, \beta_{A}) : \mathbb{G} \to \mathbb{H}$ given, for any $X \in \mathbb{G}$, by $D_{A}(X) = A \otimes {}^{X}A^*$ and where $(\beta_{A})_{X,Y}$ is a composition of canonical isomorphisms (see [18,19]). It is easy to see that $(D_{A}, \beta_{A}) \in Der^*(\mathbb{G}, \mathbb{H})$ and we have:

Proposition 5.6 *The functor $\overline{T} : \mathbb{H} \longrightarrow Der^*(\mathbb{G}, \mathbb{H})$ given by inner derivations, $\overline{T}(A) = (D_{A}, \beta_{A})$, defines a homomorphism of categorical groups.*

Proof: The natural isomorphisms $\bar{\mu}_{A,B} : \overline{T}(A \otimes B) \longrightarrow \overline{T}(A) \otimes \overline{T}(B)$ are given, for any $X \in \mathbb{G}$, by the following composition:

$$\begin{array}{c}
D_{A\otimes B}(X) = A \otimes B \otimes {}^{X}(A \otimes B)^* \\
\Big\downarrow{\scriptstyle can} \\
A \otimes D_{B}(X) \otimes {}^{X}A^* \\
\Big\downarrow{\scriptstyle 1\otimes\chi^{-1}_{D_B(X),\,{}^{X}A^*}} \\
A \otimes {}^{TD_B(X)}({}^{X}A^*) \otimes D_{B}(X) \\
\Big\downarrow{\scriptstyle can} \\
(D_{A} \otimes D_{B})(X) = A \otimes {}^{(TD_B(X)\otimes X)}A^* \otimes D_{B}(X) \ .
\end{array}$$

The required coherence condition for $\bar{\mu}$ follows from the ones of the canonical isomorphisms involved in the definition as well as those χ satisfies. ▮

And, as in the case of groups, we have:

Proposition 5.7 *Let $(\mathbb{H}, \mathbf{T}, \nu, \chi)$ be a categorical $\mathbb{G}$-crossed module. For any $(D,\beta) \in Der^*(\mathbb{G},\mathbb{H})$ and $A, B \in \mathbb{H}$, there are natural isomorphisms*

$$\overline{\nu}_{(D,\beta),A} : \overline{T}(\,{}^{(D,\beta)}A) \otimes (D,\beta) \longrightarrow (D,\beta) \otimes \overline{T}(A), \quad \overline{\chi}_{A,B} : {}^{\overline{T}(A)}B \otimes A \longrightarrow A \otimes B$$

such that $(\mathbb{H}, \overline{\mathbf{T}}, \overline{\nu}, \overline{\chi})$ is a categorical $Der^*(\mathbb{G},\mathbb{H})$-crossed module, where the action of $Der^*(\mathbb{G},\mathbb{H})$ on $\mathbb{H}$ is given in (6).

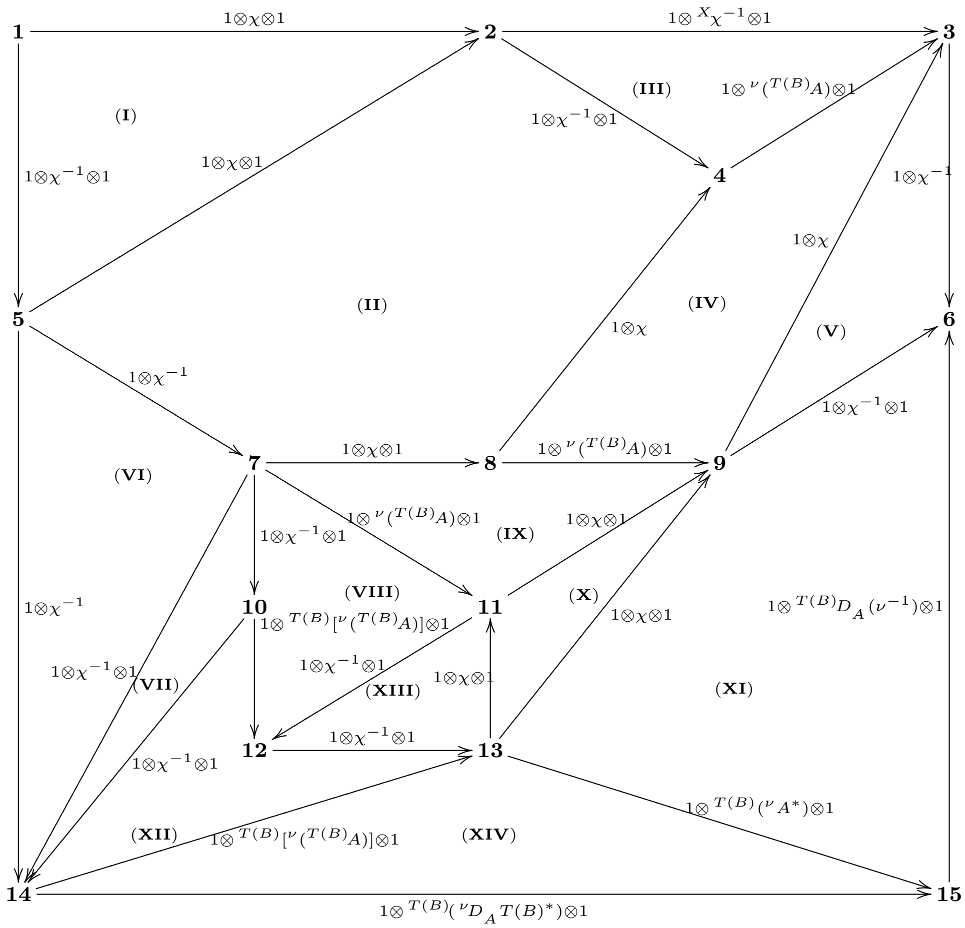
Proof: For any $X \in \mathbb{G}$, $(\overline{\nu}_{(D,\beta),A})_X : (D_{DT(A)\otimes A} \otimes D)(X) \longrightarrow (D \otimes D_A)(X)$ is given by the following composition:

$$\begin{array}{c}
DT(A) \otimes A \otimes {}^{(TD(X)\otimes X)}(DT(A)\otimes A)^* \otimes D(X) \\
\Big\downarrow {\scriptstyle 1\otimes\chi_{D(X),\, {}^X(DT(A)\otimes A)^*}} \\
DT(A) \otimes A \otimes D(X) \otimes {}^XA^* \otimes {}^XDT(A)^* \\
\Big\downarrow {\scriptstyle 1\otimes\chi^{-1}_{A,D(X)}\otimes 1} \\
DT(A) \otimes {}^{T(A)}D(X) \otimes A \otimes {}^XA^* \otimes {}^XDT(A)^* \\
\Big\downarrow {\scriptstyle 1\otimes\chi^{-1}_{D_A(X),\, {}^XDT(A)^*}} \\
DT(A) \otimes {}^{T(A)}D(X) \otimes {}^{(TD_A(X)\otimes X)}DT(A)^* \otimes D_A(X) \\
\Big\downarrow {\scriptstyle can} \\
DT(A) \otimes {}^{T(A)}[D(X) \otimes {}^{(T({}^XA^*)\otimes X)}DT(A)^*] \otimes D_A(X) \\
\Big\downarrow {\scriptstyle 1\otimes {}^{T(A)}[1\otimes {}^{\nu_{X,A^*}}(DT(A)^*)]\otimes 1} \\
DT(A) \otimes {}^{T(A)}[D(X) \otimes {}^{(X\otimes T(A^*))}DT(A)^*] \otimes D_A(X) \\
\Big\downarrow {\scriptstyle can} \\
DT(A) \otimes {}^{T(A)}[D(X) \otimes {}^XDT(A^*)] \otimes D_A(X) \\
\Big\downarrow {\scriptstyle 1\otimes {}^{T(A)}\beta^{-1}_{X,T(A^*)}\otimes 1} \\
DT(A) \otimes {}^{T(A)}D(X\otimes T(A^*)) \otimes D_A(X) \\
\Big\downarrow {\scriptstyle 1\otimes {}^{T(A)}D(\nu^{-1}_{X,A^*})\otimes 1} \\
DT(A) \otimes {}^{T(A)}D(T({}^XA^*)\otimes X) \otimes D_A(X) \\
\Big\downarrow {\scriptstyle \beta^{-1}_{T(A),T({}^XA^*)\otimes X}\otimes 1} \\
D(T(A)\otimes T({}^XA^*)\otimes X) \otimes D_A(X) \\
\Big\downarrow {\scriptstyle can} \\
D(TD_A(X)\otimes X) \otimes D_A(X)\ .
\end{array}$$

For any $A, B \in \mathbb{H}$, $\overline{\chi}_{A,B}$ is given by the following diagram:

$$\begin{tikzcd}
{}^{\overline{T}(A)}B \otimes A \arrow[rr, "\overline{\chi}_{A,B}"] \arrow[d, equal] & & A \otimes B \\
A \otimes {}^{T(B)}A^* \otimes B \otimes A \arrow[r, "1\otimes\chi_{B,A^*}\otimes 1"'] & A \otimes B \otimes A^* \otimes A \arrow[ru, "can"'] &
\end{tikzcd}$$

We will only write down the proof of condition *(cr4)* and all the other conditions are proved in a similar way. To prove *(cr4)* is equivalent to verify the commutativity of the following diagram, where we have omitted all the canonical isomorphisms:



being:

$$\mathbf{1} = D_A T(B) \otimes B \otimes A \otimes {}^X\Big(D_A T(B) \otimes B \otimes A\Big)^*,$$
$$\mathbf{2} = A \otimes B \otimes {}^X\Big(D_A T(B) \otimes B \otimes A\Big)^*,$$
$$\mathbf{3} = A \otimes B \otimes {}^X(A \otimes B)^*,$$

$$\mathbf{4} = A \otimes B \otimes {}^{X}A^{*} \otimes {}^{T({}^{X}B^{*})}\Big({}^{(X \otimes T(B))}A\Big) \otimes {}^{X}B^{*} \otimes {}^{X}A^{*},$$
$$\mathbf{5} = D_{A}T(B) \otimes {}^{T(B)}D_{A}(X) \otimes B \otimes {}^{X}\Big(D_{A}T(B) \otimes B\Big)^{*},$$
$$\mathbf{6} = A \otimes {}^{TD_B(X)}({}^{X}A^{*}) \otimes D_{B}(X),$$
$$\mathbf{7} = A \otimes {}^{(T(B)\otimes X)}A^{*} \otimes B \otimes {}^{T({}^{X}B^{*})}\Big({}^{(X \otimes T(B))}A \otimes {}^{X}A^{*}\Big) \otimes {}^{X}B^{*},$$
$$\mathbf{8} = A \otimes B \otimes {}^{X}A^{*} \otimes {}^{T({}^{X}B^{*})}\Big({}^{(X \otimes T(B))}A \otimes {}^{X}A^{*}\Big) \otimes {}^{X}B^{*},$$
$$\mathbf{9} = A \otimes B \otimes {}^{X}A^{*} \otimes {}^{(X \otimes T(B^{*}))}\Big({}^{T(B)}A\Big) \otimes {}^{T({}^{X}B^{*})}({}^{X}A^{*}) \otimes {}^{X}B^{*},$$
$$\mathbf{10} = A \otimes {}^{(T(B)\otimes X)}A^{*} \otimes {}^{(TD_B(X)\otimes X)}\Big({}^{T(B)}A\Big) \otimes B \otimes {}^{(T({}^{X}B^{*})\otimes X)}A^{*} \otimes {}^{X}B^{*},$$
$$\mathbf{11} = A \otimes {}^{(T(B)\otimes X)}A^{*} \otimes B \otimes {}^{(X \otimes T(B^{*}))}\Big({}^{T(B)}A\Big) \otimes {}^{T({}^{X}B^{*})}({}^{X}A^{*}) \otimes {}^{X}B^{*},$$
$$\mathbf{12} = A \otimes {}^{(T(B)\otimes X)}A^{*} \otimes {}^{(T(B)\otimes X)}A \otimes B \otimes {}^{(T({}^{X}B^{*})\otimes X)}A^{*} \otimes {}^{X}B^{*},$$
$$\mathbf{13} = A \otimes {}^{(T(B)\otimes X)}A^{*} \otimes {}^{(T(B)\otimes X)}A \otimes {}^{(T(B)\otimes T({}^{X}B^{*}))}\Big({}^{X}A^{*}\Big) \otimes D_{B}(X),$$
$$\mathbf{14} = D_{A}T(B) \otimes {}^{T(B)}D_{A}(X) \otimes {}^{TD_B(X)}\Big({}^{X}D_{A}T(B)^{*}\Big) \otimes D_{B}(X),$$
$$\mathbf{15} = D_{A}T(B) \otimes {}^{T(B)}D_{A}(X) \otimes {}^{T(B)}\Big({}^{(X\otimes T(B^{*}))}D_{A}T(B)^{*}\Big) \otimes D_{B}(X).$$

Now, diagrams (**I**), (**VII**), (**X**) and (**XIII**) are commutative by applying condition *(cr2)* of the given $\mathbb{G}$-crossed module; (**II**) is commutative by bifunctoriality of $\otimes$ and by (cr2); (**III**) is commutative by *(cr3)*; (**IV**) and (**V**) are commutative by *(cr1)*; (**VI**) and (**IX**) are commutative by bifunctoriality of $\otimes$; (**VIII**) and (**XII**) are commutative by naturality of χ; (**XI**) is obviously commutative and finally (**XIV**) is commutative by naturality of the canonical isomorphism ψ of the action of $\mathbb{G}$ on $\mathbb{H}$ and by bifunctoriality of $\otimes$. ∎

This categorical crossed module, given by inner derivations, provides the key to develop below a low-dimensional cohomology for categorical groups with coefficients in categorical crossed modules.

6 Cohomology with coefficients in categorical crossed modules.

We will define cohomology categorical groups at dimensions 0 and 1. Let us first remember (c.f. [20,27]) that if G is a group and H is a G- crossed module, then the cohomology groups $H^0(G,H)$ and $H^1(G,H)$ are, respectively, the kernel and the cokernel of the group homomorphism $H \to Der(G,H)$, given by inner derivations.

Now consider a categorical $\mathbb{G}$-crossed module $\langle \mathbb{H}, \mathbf{T}, \nu, \chi \rangle$, and let $\langle \mathbb{H}, \overline{\mathbf{T}} : \mathbb{H} \to Der^{*}\mathbb{G}, \mathbb{H}), \overline{\nu}, \overline{\chi} \rangle$ be the categorical $Der^{*}(\mathbb{G},\mathbb{H})$-crossed module we have obtained in the previous section. Then taking into account what we have recalled for groups, we give the following definition:

Definition 6.1 *Let $\mathbb{G}$ be a categorical group and $\langle \mathbb{H}, \mathbf{T}, \nu, \chi \rangle$ a categorical $\mathbb{G}$-crossed module. Then zero-th and first cohomology categorical groups of $\mathbb{G}$ with coefficients in $\langle \mathbb{H}, \mathbf{T}, \nu, \chi \rangle$, are defined by*

$$\mathcal{H}^0(\mathbb{G}, \mathbb{H}) = Ker(\overline{\mathbf{T}} : \mathbb{H} \to Der^*(\mathbb{G}, \mathbb{H}))$$

$$\mathcal{H}^1(\mathbb{G}, \mathbb{H}) = Der^*(\mathbb{G}, \mathbb{H})/\langle \mathbb{H}, \overline{\mathbf{T}} \rangle$$

where the second one is the quotient categorical group built in Section 3 for the categorical crossed module $\langle \mathbb{H}, \overline{\mathbf{T}}, \overline{\nu}, \overline{\chi} \rangle$.

Both definitions are functorial. Indeed, the existence of a 2-functor

$$\mathcal{H}^0(\mathbb{G}, -) : \mathbb{G} - cross \longrightarrow \mathcal{C}G$$

is consequence of the fact that the kernel construction is 2-functorial. For the 2-functoriality of $\mathcal{H}^1$ we first prove the following lemma:

Lemma 6.2 *Let $(\mathbf{F}, \eta, \alpha) : \langle \mathbb{H}, \mathbf{T}, \nu, \chi \rangle \longrightarrow \langle \mathbb{H}', \mathbf{T}', \nu', \chi' \rangle$ be a morphism of categorical $\mathbb{G}$-crossed modules, then it extends to a morphism in PreCross between the associated inner derivations categorical crossed modules:*

$$(\mathbf{F}, \mathbf{F}_*, \eta_*, \alpha_*) : \langle \mathbb{H}, \overline{\mathbf{T}}, \overline{\nu}, \overline{\chi} \rangle \longrightarrow \langle \mathbb{H}', \overline{\mathbf{T}}', \overline{\nu}, \overline{\chi} \rangle$$

Furthermore, if $\gamma : (\mathbf{F}, \eta, \alpha) \Rightarrow (\mathbf{F}', \eta', \alpha')$ is a 2-cell in $\mathbb{G} - cross$, it induces a 2-cell in PreCross, $(\gamma, \gamma_*) : (\mathbf{F}, \mathbf{F}_*, \eta_*, \alpha_*) \Rightarrow (\mathbf{F}', \mathbf{F}'_*, \eta'_*, \alpha'_*)$.

Proof: We first recall that the functor $F_* : Der^*(\mathbb{G}, \mathbb{H}) \to Der^*(\mathbb{G}, \mathbb{H}')$ sends a derivation $(D, \beta) \in Der^*(\mathbb{G}, \mathbb{H})$ to $F_*(D, \beta) = (FD, F_*\beta)$, where, for any $X, Y \in \mathbb{G}$, $(F_*)_{X,Y} = (1 \otimes \eta_{X,D(Y)}) \cdot F(\beta_{X,Y})$. The natural isomorphism $\alpha_* : F_*\overline{T} \Rightarrow \overline{T}'F$ applies any object $A \in \mathbb{H}$ to the morphism of derivations $(\alpha_*)_A : (FD_A, F_*\beta_A) \Rightarrow (D_{F(A)}, \beta_{F(A)})$ which, for any $X \in \mathbb{G}$, is given by the commutativity of the following diagram

$$\begin{tikzcd}
F(A \otimes {}^X A^*) \arrow[r, "((\alpha_*)_A)_X"] \arrow[d, "can"'] & F(A) \otimes {}^X F(A)^* \arrow[d, "can"] \\
F(A) \otimes F({}^X A^*) \arrow[r, "1 \otimes \eta_{X,A^*}"'] & F(A) \otimes {}^X F(A^*)
\end{tikzcd}$$

For any $(D, \beta) \in Der^*(\mathbb{G}, \mathbb{H})$ and $A \in \mathbb{H}$, the natural isomorphism $((\eta)_*)_{(D,\beta),A}$: $F({}^{(D,\beta)}A) \to {}^{(FD,F_*\beta)}F(A)$ is the composition

$$F(DT(A) \otimes A) \overset{can}{\simeq} FDT(A) \otimes F(A) \xrightarrow{FD(\alpha_A) \otimes 1} FDT'F(A) \otimes F(A)$$

The verification that $(\mathbf{F}, \mathbf{F}_*, \eta_*, \alpha_*)$ is in fact a morphism of categorical pre-crossed modules is straightforward. We only point out that the commutativity of diagram (2) follows from the compatibility condition between the natural isomorphisms χ, χ' (see Section 2).

Finally the 2-cell $\gamma_* : \mathbf{F}_* \Rightarrow \mathbf{F}'_* : Der^*(\mathbb{G}, \mathbb{H}) \to Der^*(\mathbb{G}, \mathbb{H}')$ is given, for any $(D, \beta) \in Der^*(\mathbb{G}, \mathbb{H})$ and $X \in \mathbb{G}$, by $((\gamma_*)_{(D,\beta)})_X = \gamma_{D(X)} : FD(X) \to F'D(X)$. ∎

Then, by Proposition 4.3 we have a 2-functor:

$$\mathcal{H}^1(\mathbb{G}, -) : \mathbb{G} - cross \longrightarrow \mathcal{CG}$$

which applies a morphism $(\mathbf{F}, \eta, \alpha) : \langle \mathbb{H}, \mathbf{T}, \nu, \chi \rangle \longrightarrow \langle \mathbb{H}', \mathbf{T}', \nu', \chi' \rangle$ to the morphism of categorical groups $\widehat{\mathbf{F}_*} : \mathcal{H}^1(\mathbb{G}, \mathbb{H}) \longrightarrow \mathcal{H}^1(\mathbb{G}, \mathbb{H}')$.

Remark 6.3 The categorical group $\mathcal{H}^0(\mathbb{G}, \mathbb{H})$ is equivalent to the categorical group of $\mathbb{G}$-*invariant objects* $\mathbb{H}^{\mathbb{G}}$ constructed in [18,19] as follows: A $\mathbb{G}$-invariant object of $\mathbb{H}$ consists of a pair (A, φ_A), where $A \in \mathbb{H}$ and $\varphi_A = \left(\varphi_A^X : {}^X A \to A\right)_{X \in \mathbb{G}}$ is a family of natural isomorphisms in $\mathbb{H}$ such that $\varphi_A^{X \otimes Y} = \varphi_A^X \, {}^X\varphi_A^Y \phi_{X,Y,A}$, for any $X, Y \in \mathbb{G}$. An arrow $u : (A, \varphi_A) \to (B, \varphi_B)$ is an arrow $u : A \to B$ in $\mathbb{H}$ such that $u\varphi_A^X = \varphi_B^X \, {}^X u$, for any $X \in \mathbb{G}$.

Example 6.4 i) If (H, T, ν, χ) is a discrete categorical G-crossed module, i.e. it is induced by a crossed module of groups, then $\pi_0(\mathcal{H}^i(G, H))$, $i = 0, 1$, are the cohomology groups of G with coefficients in H as defined in [27].

ii) Let $\mathbb{A}$ be a $\mathbb{G}$-module and consider the categorical $\mathbb{G}$-crossed module $\mathbf{0} : \mathbb{A} \to \mathbb{G}$ (see Example 2.6 iii)). Then the cohomology categorical group $\mathcal{H}^0(\mathbb{G}, \mathbb{A})$ and $\mathcal{H}^1(\mathbb{G}, \mathbb{A})$ coincide with those defined in [18]. Therefore we have, when $\mathbb{G} = G[0]$ and $\mathbb{A}$ is symmetric,

$$\pi_1(\mathcal{H}^0(\mathbb{G}, \mathbb{A})) = \pi_1(\mathbb{A})^G = H^0_{Ulb}(G, \mathbb{A})$$

$$\pi_0(\mathcal{H}^1(\mathbb{G}, \mathbb{A})) = H^2_{Ulb}(G, \mathbb{A})$$

and

$$\pi_0(\mathcal{H}^0(\mathbb{G}, \mathbb{A})) = \pi_1(\mathcal{H}^1(\mathbb{G}, \mathbb{A})) = H^1_{Ulb}(G, \mathbb{A})$$

where $H^i_{Ulb}(G, \mathbb{A})$, $i = 0, 1, 2$, are the cohomology groups defined by Ulbrich in [36].

Definition 6.5 *Consider three categorical $\mathbb{G}$-crossed modules $\langle \mathbb{H}, \mathbf{T} : \mathbb{H} \to \mathbb{G}, \nu, \chi \rangle$, $\langle \mathbb{H}', \mathbf{T}' : \mathbb{H}' \to \mathbb{G}, \nu', \chi' \rangle$ and $\langle \mathbb{H}'', \mathbf{T}'' : \mathbb{H}'' \to \mathbb{G}, \nu'', \chi'' \rangle$ and two composable morphisms of categorical $\mathbb{G}$-crossed modules $(\mathbf{F}', \eta', \alpha') : \langle \mathbb{H}', \mathbf{T}' : \mathbb{H}', \nu', \chi' \rangle \to \langle \mathbb{H}, \mathbf{T}, \nu, \chi \rangle$ and $(\mathbf{F}, \eta, \alpha) : \langle \mathbb{H}, \mathbf{T}, \nu, \chi \rangle \to \langle \mathbb{H}'', \mathbf{T}'', \nu'', \chi'' \rangle$. Con-*

sider also a 2-cell β from the composite morphism to the zero morphism

$$
\begin{tikzcd}
\mathbb{H}' \arrow[r, "F'"] \arrow[dr, "T'"'] \arrow[rr, bend left=40, "0"] \arrow[rr, phantom, bend left=20, "\beta\Uparrow"] & \mathbb{H} \arrow[r, "F"] \arrow[d, "T"] \arrow[dl, phantom, "\overset{\alpha'}{\Rightarrow}"] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & \mathbb{H}'' \arrow[dl, "T''"] \\
& \mathbb{G} &
\end{tikzcd}
$$

This sequence is called a short exact sequence of categorical $\mathbb{G}$-crossed modules if the triple $(\mathbf{F}', \beta, \mathbf{F})$ is an extension in the sense of Definition 4.4.

Now we obtain the main result of this section.

Proposition 6.6 *For a short exact sequence of categorical $\mathbb{G}$-crossed modules, as defined above, there is a natural induced 2-exact sequence of categorical groups*

$$
\begin{tikzcd}
\mathcal{H}^0(\mathbb{G},\mathbb{H}') \arrow[r, "\widehat{\mathbf{F}'}"] & \mathcal{H}^0(\mathbb{G},\mathbb{H}) \arrow[r, "\widehat{\mathbf{F}}"] & \mathcal{H}^0(\mathbb{G},\mathbb{H}'') \arrow[dl, "\Delta"'] & \\
& \mathcal{H}^1(\mathbb{G},\mathbb{H}') \arrow[r, "\widehat{\mathbf{F}'_*}"] & \mathcal{H}^1(\mathbb{G},\mathbb{H}) \arrow[r, "\widehat{\mathbf{F}_*}"] & \mathcal{H}^1(\mathbb{G},\mathbb{H}'')
\end{tikzcd}
$$

Moreover the functor $\widehat{F'}$ is faithful.

Proof: By lemma 6.2 we have a diagram of arrows and 2-cells in *PreCross*

$$
\begin{tikzcd}
\mathbb{H}' \arrow[r, "F'"] \arrow[d, "\overline{T}'"'] \arrow[rr, bend left=20, "0"] \arrow[rr, phantom, bend left=10, "\beta\Uparrow"] \arrow[dr, phantom, "\overset{\alpha'_*}{\Rightarrow}"] & \mathbb{H} \arrow[r, "F"] \arrow[d, "\overline{T}"] \arrow[dr, phantom, "\overset{\alpha_*}{\Rightarrow}"] & \mathbb{H}'' \arrow[d, "\overline{T}''"] \\
Der^*(\mathbb{G},\mathbb{H}') \arrow[r, "F'_*"'] \arrow[rr, bend right=20, "0"'] \arrow[rr, phantom, bend right=10, "\beta_*\Downarrow"] & Der^*(\mathbb{G},\mathbb{H}) \arrow[r, "F_*"'] & Der^*(\mathbb{G},\mathbb{H}'')
\end{tikzcd}
$$

By Proposition 5.4, the triple $(\mathbf{F'}_*, \beta_*, \mathbf{F}_*)$ is 2-exact and $\widehat{F'}$ is faithful. The existence of the six-term 2-exact sequence follows now from Proposition 4.6 and Remark 4.7. ▮

Remark 6.7 We want to end making again the differences between the approach in this paper and the approaches in other papers on crossed modules clear. For it, let us point out that we focus our attention in the 2-dimensional aspects of categorical groups (or crossed modules) instead of their non-strict (or strict) structure. The point is that most of the existing literature considers crossed modules as the objects of a *category* whereas we consider categorical

groups as the objects of a *2-category*. This completely changes the theory: Homological Algebra is, in some sense, the study of classes of morphisms (monos, epis) and then of those limits we use to describe them (kernels, cokernels). Now, kernels and cokernels in the category of crossed modules have definitely no universal property in the 2-category of crossed modules, and viceversa, kernels and cokernels (in the sense of bilimits) computed in the 2-category of crossed modules have no universal property in the underlying category of crossed modules. For instance, the kernel in the category of crossed modules of a morphism of crossed modules $(f_1, f_0) : (\delta : H \to G) \to (\delta' : H' \to G')$ is the induced $Ker(f_1) \to Ker(f_0)$, whereas, in the 2-category of crossed modules, the objects of the kernel are the elements of the pullback $G \times_{G'} H'$. This last construction could seem quite artificial, but if you consider the associated diagram of categorical groups $\mathbb{G}(\delta) \to \mathbb{G}(\delta')$, the kernel is simply the homotopy fibre over the unit object and therefore, the objects are precisely the elements of the above pullback.
Another example where to see the difference between the categorical and the 2-categorical theory is the classification of split extensions. Thus, split extensions of a crossed module δ by a crossed module δ' (using kernels in the category of crossed modules) are classified by morphisms of crossed modules from δ to the actor $Act(\delta')$, where the latter is the crossed module defined by Norrie in [30]. Passing to the associated categorical groups, this means that split extensions of $\mathbb{G}(\delta)$ by $\mathbb{G}(\delta')$ are classified by morphisms from $\mathbb{G}(\delta)$ to the categorical group of monoidal automorphisms of $\mathbb{G}(\delta')$. On the contrary if we define split extensions of crossed modules using kernel in the 2-categorical sense, then split extensions of $\mathbb{G}(\delta)$ by $\mathbb{G}(\delta')$ correspond to categorical group morphisms from $\mathbb{G}(\delta)$ to $\mathcal{E}q(\mathbb{G}(\delta'))$, where the last one is the categorical group of monoidal autoequivalences (see [2,16]).

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