What are Sifted Colimits?

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Dedicated to Dominique Bourn at the occasion of his sixtieth birthday

Abstract

Sifted colimits are “almost” just the combination of filtered colimits and reflexive coequalizers. We present affirmative results for all categories $A$ with finite colimits: $A$ has sifted colimits iff it has filtered colimits and reflexive coequalizers. A functor with domain $A$ preserves sifted colimits iff it preserves filtered colimits and reflexive coequalizers. And the free completion of $A$ under sifted colimits is obtained in two steps: first a completion $Rec A$ of $A$ under reflexive coequalizers and then a completion of $Rec A$ under filtered colimits. However, none of the statements is true for general categories $A$.

Introduction

Sifted colimits play for the doctrine of finite products precisely the role which filtered colimit play for the doctrine of finite limits. Recall that a small category $D$ which is filtered has the property that $D$-colimits commute with finite limits in $Set$. The converse is less well known (but trivial to prove using representable functors as diagrams): if $D$-colimits commute with finite limits in $Set$, then $D$ is filtered. Now sifted categories are defined as those small categories $D$ such that $D$-colimits commute with finite products in $Set$. They were first studied (without any name) in the classical lecture notes

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of P. Gabriel and F. Ulmer [7] who proved that $\mathcal{D}$ is sifted iff the diagonal $\Delta: \mathcal{D} \to \mathcal{D} \times \mathcal{D}$ is a final functor; this nicely corresponds to the fact that $\mathcal{D}$ is filtered iff the diagonals $\Delta: \mathcal{D} \to \mathcal{D}^J$ are final for all finite graphs $J$.

Sifted colimits are colimits whose schemes are sifted categories; they were studied (independently of [7]) by C. Lair [11] who called them “tamisante”, later P. T. Johnstone suggested the translation to “sifted”. Besides filtered colimits, prime examples of sifted colimits are reflexive coequalizers, that is, coequalizers of parallel pairs of epimorphisms with a joint splitting.

Sifted colimits are of major importance in general algebra. Recall that an algebraic theory (in the sense of F. W. Lawvere [12]) is a small category $\mathcal{T}$ with finite products and an algebra for $\mathcal{T}$ is a functor $A: \mathcal{T} \to \text{Set}$ preserving finite products. The category $\text{Alg}\mathcal{T}$ of algebras is a full subcategory of the functor category $\text{Set}^\mathcal{T}$. Now, let us denote by $\text{Sind}\mathcal{A}$ the free completion of a category $\mathcal{A}$ under sifted colimits (resembling the name $\text{Ind}\mathcal{A}$ for Grothendieck’s completion under filtered colimits). Then for every algebraic theory $\mathcal{T}$ the category of algebras is just the above completion of $\mathcal{T}^{op}$:

$$\text{Alg}\mathcal{T} = \text{Sind}\mathcal{T}^{op}$$

see [3]. And algebraic functors, that is functors between algebraic categories induced by morphisms of algebraic theories, are precisely the functors preserving limits and sifted colimits, see [2].

The aim of the present paper is to prove that the answer to the question in the title is: “essentially just filtered colimits and reflexive coequalizers”. There are three aspects of that answer:

(1) A category $\mathcal{A}$ has sifted colimits iff it has filtered colimits and reflexive coequalizers.

This is trivial in case $\mathcal{A}$ has finite coproducts: reflexive coequalizers then imply finite colimits. Thus, filtered colimits imply cocompleteness of $\mathcal{A}$. For general categories $\mathcal{A}$ (1) is not true, see Example 2.6 below.

(2) If $\text{Rec}\mathcal{A}$ is the free completion of a small category $\mathcal{A}$ under reflexive coequalizers, then the free completions of $\text{Rec}\mathcal{A}$ under filtered colimits is the free completion of $\mathcal{A}$ under sifted colimits. Shortly: $\text{Sind}\mathcal{A} = \text{Ind}(\text{Rec}\mathcal{A})$.

This is true if $\mathcal{A}$ has finite coproducts, see [3]: for the algebraic theory $\mathcal{T} = \mathcal{A}^{op}$ we have $\text{Alg}\mathcal{T} = \text{Sind}\mathcal{A}$, whereas the full subcategory $(\text{Alg}\mathcal{T})_{fp}$ of all finitely presentable algebras is equal to $\text{Rec}\mathcal{A}$. The equation $\text{Alg}\mathcal{T} = \text{Ind}(\text{Alg}\mathcal{T})_{fp}$ is trivial. Again, for general categories (2) is not true, see 3.6 below.

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(3) A functor \( F : \mathcal{A} \to \mathcal{B} \) preserves sifted colimits iff it preserves filtered colimits and reflexive coequalizers.

This was independently proved for cocomplete categories \( \mathcal{A} \) by A. Joyal (his proof even works for quasicategories, see [9]) and by S. Lack (see [10]).

We show how this statement quickly follows from (2) in case both \( \mathcal{A} \) and \( \mathcal{B} \) are cocomplete categories. And we also present a (less elegant, but quite elementary) proof of the above statement whenever \( \mathcal{A} \) has at least finite colimits. But you cannot prove much more: for general categories \( \mathcal{A}, \mathcal{B} \) the result is not true, see Example 4.2 below.

We are preparing a manuscript on algebraic theories [4] and the material of Chapter 7 is closely related to the present paper.

\section{Existence of Sifted Colimits}

As mentioned in the Introduction, a small category \( \mathcal{D} \) is called sifted iff \( \mathcal{D} \)-colimits commute in \( \text{Set} \) with finite products. That is, given a diagram

\[
\mathcal{D} \times \mathcal{J} \to \text{Set}
\]

where \( \mathcal{J} \) is a finite discrete category, then the canonical morphism

\[
\colim_{\mathcal{D}} \left( \prod_{\mathcal{J}} D(d, j) \right) \to \prod_{\mathcal{J}} \left( \colim_{\mathcal{D}} D(d, j) \right)
\]

is an isomorphism.

Colimits of diagrams over sifted categories are called sifted colimits.

\subsection{Remark}

(i) As proved by P. Gabriel and F. Ulmer [7], a small, nonempty category \( \mathcal{D} \) is sifted if and only if the diagonal functor \( \Delta : \mathcal{D} \to \mathcal{D} \times \mathcal{D} \) is final. This means that for every pair of objects \( A, B \) of \( \mathcal{D} \) the category \( (A, B) \downarrow \Delta \) of cospans on \( A, B \) is connected. That is:

(a) a cospan \( A \to X \leftarrow B \) exists, and

(b) every pair of cospans on \( A, B \) is connected by a zig-zag of cospans.

This characterization was later re-discovered by C. Lair [11] (see also [3]).

(ii) P. Gabriel and F. Ulmer [7] also proved that a small category \( \mathcal{D} \) is sifted if and only if \( \mathcal{D} \) is final in its free completion \( \text{Fam} \mathcal{D} \) under finite
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coproducts. This was later rediscovered (and generalized to quasi-categories) by A. Joyal [9] and we have learnt it from him. In fact, (a) and (b) above clearly imply the same property for finite families of objects too. This is precisely the finality of $D \to \text{Fam} D$.

(iii) Every small category with finite coproducts is sifted. This immediately follows from (i).

2.2 Lemma. For every algebraic theory $\mathcal{T}$, the algebras, that is, the finite-product preserving functors $A: \mathcal{T} \to \text{Set}$, are precisely the sifted colimits of representables in $\text{Set}^T$.

Proof. If $A$ is a sifted colimit of representables, then $A$ preserves finite products because representables do, and finite products commute with sifted colimits in $\text{Set}$.

If $A$ preserves finite products, then the category $\text{El} A$ of elements of $A$ clearly has coproducts. Recall that its objects are pairs $(X, x)$ with $X \in \text{obj} \mathcal{T}$ and $x \in A$ and its morphisms $f: (X, x) \to (Y, y)$ are those $f: Y \to X$ with $Af(y) = x$. By 2.1 (iii), the diagram of representables $\phi_A: \text{El} A \to \text{Set}^T$, $(X, x) \mapsto T(X, -)$, is sifted, and $A = \text{colim} \phi_A$. \hfill $\square$

2.3 Example. ([3]) Reflexive coequalizers are sifted colimits. That is, the category $\mathcal{D}$ given by the graph

\[
\begin{array}{c}
P \\
\cdot \hspace{1cm} a_1 \\
\downarrow \hspace{1cm} \cdot a_2 \\
Q 
\end{array}
\]

and the equations

$a_1 \cdot d = \text{id}_B = a_2 \cdot d$

is sifted. This follows from the characterization of sifted colimits mentioned in the introduction. We present a full proof here because we are going to use it again below. Let us add that this fact was was already realized by Y. Diers [6] but remained unnoticed. Another proof is given in [14], Lemma 1.2.3.

In fact, suppose that

\[
\begin{array}{ccc}
A & \xrightarrow{a_1} & B \\
\cdot a_2 \downarrow & c & \cdot \downarrow C
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
A' & \xrightarrow{a'_1} & B' \\
\cdot a'_2 \downarrow & c' & \cdot \downarrow C'
\end{array}
\]

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are reflexive coequalizers in $\text{Set}$. We can assume, without loss of generality, that $c$ is the canonical function of the quotient $C = B/\sim$ modulo the equivalence relation described as follows: two elements $x, y \in B$ are equivalent iff there exists a zig-zag

\[
\begin{align*}
A: & \quad z_1 \leftarrow a_{i_1} \leftarrow x \\
B: & \quad y \\
& \quad \downarrow a_{i_2} \quad \downarrow a_{i_3} \quad \downarrow a_{i_4} \quad \cdots \quad \downarrow a_{i_{2k-1}} \quad \downarrow a_{i_{2k}}
\end{align*}
\]

where $i_1, i_2, \ldots, i_{2k}$ are 1 or 2. For reflexive pairs $a_1, a_2$ the zig-zags can always be chosen to have the following form

\[
\begin{align*}
A: & \quad \leftarrow z_1 \leftarrow a_1 \\
B: & \quad y \\
& \quad \downarrow a_2 \quad \downarrow a_2 \quad \downarrow a_1 \quad \cdots \quad \downarrow a_2 \quad \downarrow a_2
\end{align*}
\]

where for the elements $z_{2i}$ of $A$ we use $a_1, a_2$ and for the elements $z_{2i+1}$ we use $a_2, a_1$. In fact, let $d: B \to A$ be a joint splitting of $a_1, a_2$. Thus given a zig-zag, say,

\[
\begin{align*}
& \quad a_2 \quad \downarrow a_1 \\
& \quad \downarrow a_2 \quad \downarrow a_1
\end{align*}
\]

we can modify it as follows: put $z_1 = d(x)$ and $z_2 = z$ to get

\[
\begin{align*}
& \quad a_1 \quad \downarrow a_2 \quad \downarrow a_2 \quad \downarrow a_1 \\
& \quad \downarrow a_1 \quad \downarrow a_1 \quad \downarrow a_1
\end{align*}
\]

Moreover, the length $2k$ of the zig-zag $(\ast)$ can be prolonged to $2k + 2$ or $2k + 4$ etc. by using $d$. Analogously, we can assume $C' = B'/\sim'$ where $\sim'$ is the equivalence relation given by zig-zags of $a'_1$ and $a'_2$ of the above form $(\ast)$. Now we form the parallel pair

\[
A \times A' \xrightarrow{a_1 \times a'_1} B \times B'
\]

and obtain its coequalizer by the zig-zag equivalence $\approx$ on $B \times B'$. Given $(x, x') \approx (y, y')$ in $B \times B'$, we obviously have zig-zags both for $x \sim y$ and for...
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$x' \sim y'$ (use projections of the given zig-zag). But also the other way round: whenever $x \sim y$ and $x' \sim y'$, then we choose the two zig-zags so that they both have the above type $(\ast)$ and have the same lengths. They create an obvious zig-zag for $(x, x') \approx (y, y')$. From this it follows that the map

$$A \times A' \xrightarrow{a_1 \times a_1'} B \times B' \xrightarrow{c \times c'} (B / \sim) \times (B' / \sim)$$

is a coequalizer, as required.

2.4 Example. By merging two copies of reflexive pairs we also obtain a sifted category $\mathcal{D}$: let $\mathcal{D}$ be given by the graph

$$\begin{array}{c}
A \\
\downarrow d \\
\downarrow a_2 \\
B \\
\downarrow d' \\
\downarrow a_2' \\
A'
\end{array}$$

and the equations making both parallel pairs reflexive:

$$a_i \cdot d = \text{id}_B = a_i' \cdot d' \quad \text{for} \quad i = 1, 2$$

The proof that $\mathcal{D}$ is sifted is completely analogous to the proof of Example 2.3: we verify that colimits over $\mathcal{D}$ in $\text{Set}$ commute with finite products. Assume that the above graph depicts sets $A, B$ and $A'$ and functions between them. Then a colimit can be described as the canonical function $c: B \to C = B / \sim$ where two elements $x, y \in B$ are equivalent iff they are connected by a zig-zag formed by $a_1, a_2, a_1'$ and $a_2'$. Since the two pairs are reflexive, the length of the zig-zag can be arbitrarily prolonged. And the type can be chosen to be

$$\begin{array}{c}
a_1 \\
a_2 \\
a_1' \\
a_2' \\
a_1'' \\
\vdots \\
a_1'''
\end{array}$$

From that it is easy to derive that $\mathcal{D}$ is sifted.

2.5 Observation. For categories $\mathcal{A}$ with finite coproducts the following conditions are equivalent:

(i) $\mathcal{A}$ has sifted colimits,
(ii) \( \mathcal{A} \) has filtered colimits and reflexive coequalizers
and
(iii) \( \mathcal{A} \) is cocomplete.

This follows from the fact that finite coproducts and reflexive coequalizers imply finite colimits. In fact, the classical construction of a finite colimit via a coequalizer between finite coproducts (see [13]) only uses reflexive coequalizers.

2.6 Example. A category \( \mathcal{A} \) which does not have sifted colimits although it has both filtered colimits and reflexive coequalizers: \( \mathcal{A} \) is the free completion of \( \mathcal{D} \) from 2.4 under filtered colimits and reflexive coequalizers. We claim that \( \mathcal{A} \) is obtained from \( \mathcal{D} \) by simply adding the coequalizer \( c \) of \( a_1, a_2 \) and the coequalizers \( c' \) of \( a'_1, a'_2 \). That is, we consider the graph

\[
\begin{array}{c}
\mathcal{A} \\
\downarrow \quad \downarrow \\
\mathcal{A}' \\
\downarrow \quad \downarrow \\
\mathcal{B} \\
\downarrow \quad \downarrow \\
\mathcal{C} \\
\downarrow \\
\mathcal{C}'
\end{array}
\]

and the equations
\[
c \cdot a_1 = c \cdot a_2 \quad \text{and} \quad c' \cdot a'_1 = c' \cdot a'_2.
\]

In fact, the category \( \mathcal{A} \) is clearly finite. Therefore, its only filtered diagrams are its idempotents:
\[
e_i = d \cdot a_i \quad \text{and} \quad e'_i = d' \cdot a'_i \quad (i = 1, 2).
\]

We claim that \( a_1 \) is the colimit of \( e_1 \). In fact, \( a_1 \cdot e_1 = a_1 \), and given a morphism \( f \) with
\[
f \cdot e_1 = f,
\]
then we see that \( f \cdot d \cdot a_1 = f \), consequently, \( f \) factorizes through \( a_1 \). Since \( a_1 \) is an epimorphism, this factorization is unique. Analogously for \( e_2, e'_1 \) and \( e'_2 \). Thus, \( \mathcal{A} \) has filtered colimits. And it has reflexive coequalizers because its only reflexive pairs of distinct morphisms are \( a_1, a_2 \) whose coequalizer is \( c \), and \( a'_1, a'_2 \) whose coequalizer is \( c' \).

It is obvious that the (sifted) embedding \( D : \mathcal{D} \to \mathcal{A} \) does not have a colimit.
3 Free Completion Sind

3.1 Notation. ([3]) A free completion of a category $\mathcal{A}$ under sifted colimits is denoted by

$$E_{\text{Sind}} : \mathcal{A} \rightarrow \text{Sind} \mathcal{A}$$

Explicitly: $\text{Sind} \mathcal{A}$ has sifted colimits and for every functor $F : \mathcal{A} \rightarrow \mathcal{B}$ where $\mathcal{B}$ has sifted colimits there exists an essentially unique functor $F^* : \text{Sind} \mathcal{A} \rightarrow \mathcal{B}$ with $F \cong F^* \cdot E_{\text{Sind}}$ preserving sifted colimits.

3.2 Example. (see [3]) For every algebraic theory $\mathcal{T}$ the category of $\mathcal{T}$-algebras is the free completion of $\mathcal{T}^{\text{op}}$ under sifted colimits:

$$\text{Sind} \mathcal{T}^{\text{op}} = \text{Alg} \mathcal{T}$$

with $E_{\text{Sind}}$ given by representables.

This is easy to prove: $\text{Alg} \mathcal{T}$ is cocomplete, and given $F : \mathcal{A} \rightarrow \mathcal{B}$ where $\mathcal{B}$ has sifted colimits we have essentially no choice for $F^*$ then defining it on object $A$ by

$$F^* A = \text{colim}(F \cdot \phi_A),$$

see 2.2. The definition of $F^*$ on morphisms is evident. This clearly defines a functor $F^* : \text{Alg} \mathcal{T} \rightarrow \mathcal{B}$ with $F \cong F^* \cdot E_{\text{Sind}}$. It remains to prove that $F^*$ preserves sifted colimits. For every object $B$ in $\mathcal{B}$ consider the functor

$$\mathcal{B}(F^* -, B) : (\text{Alg} \mathcal{T})^{\text{op}} \rightarrow \text{Set}$$

which assigns to every algebra $A$ all cocones of $F \cdot \phi_A$ with codomain $B$. These are precisely the natural transformations from $A$ to $\mathcal{B}(F -, B)$. Thus, since the inclusion functor $I : \text{Alg} \mathcal{T} \rightarrow \text{Set}^\mathcal{T}$ preserves sifted colimits (see 2.2), we obtain an isomorphism

$$\mathcal{B}(F^* A, B) \cong \text{Set}^\mathcal{T}(IA, \mathcal{B}(F -, B))$$

natural in $A$ and $B$. Since $I$ preserves sifted colimits, so does $\text{Set}^\mathcal{T}(I -, G)$ for every $G$, and the case $G = \mathcal{B}(F -, B)$ shows that $\mathcal{B}(F^* -, B)$ preserves sifted colimits for every $B \in \mathcal{B}$. Consequently, $F^*$ preserves sifted colimits.

3.3 Remark. Grothendieck’s free completion $\text{Ind} \mathcal{A}$ under filtered colimits (see [8]) has a particularly simple description for small categories $\mathcal{A}$ with finite colimits: $\text{Ind} \mathcal{A}$ is the full subcategory of $\text{Set}^{\mathcal{A}^{\text{op}}}$ of all functors preserving...
finite limits. Moreover, the inclusion of $\mathcal{A}$ into $\text{Ind}\mathcal{A}$ preserves finite colimits (see [7]).

The above example yields the complete analogy for small categories $\mathcal{A}$ with finite coproducts: $\text{Sind}\mathcal{A}$ is the full subcategory of $\text{Set}^{\mathcal{A}^{op}}$ of all functors preserving finite products.

3.4 Notation. We denote by $E_{\text{Rec}} : \mathcal{A} \to \text{Rec}\mathcal{A}$ the free completion under reflexive coequalizers. A description of $\text{Rec}\mathcal{A}$ in the special case where $\mathcal{A}$ has finite coproducts, due to A. Pitts, can be found in [5]. The following result was given in [3]. Since the proof there is somewhat incomplete we present a short proof here. A proof in a completely general situation was given in [1].

3.5 Proposition. For every algebraic theory $\mathcal{T}$ the category of finitely presentable $\mathcal{T}$-algebras is the free completion of $\mathcal{T}^{op}$ under reflexive coequalizers:

$$\text{Rec}\mathcal{T}^{op} = (\text{Alg}\mathcal{T})_{fp}$$

with $E_{\text{Rec}}$ given by representables. Moreover,

$$\text{Sind}\mathcal{T} = \text{Ind}\text{Rec}\mathcal{T}.$$ 

Proof. Following [5], finite coproduct preserving functors

$$\mathcal{T}^{op} \to \text{Set}^{op}$$

correspond to finite colimit preserving functors

$$\text{Rec}\mathcal{T}^{op} \to \text{Set}^{op}.$$ 

Consequently, $\mathcal{T}$-algebras correspond to finite limit preserving functors

$$(\text{Rec}\mathcal{T}^{op})^{op} \to \text{Set}.$$ 

Following 3.2 and 3.3,

$$\text{Sind}\mathcal{T} = \text{Ind}\text{Rec}\mathcal{T}.$$ 

Consequently

$$\text{Rec}\mathcal{T}^{op} = (\text{Alg}\mathcal{T})_{fp}.$$ 

3.6 Remark. An example of a category $\mathcal{A}$ with $\text{Sind}\mathcal{A} \neq \text{Ind}(\text{Rec}\mathcal{A})$ is given in [3]. The category $\mathcal{A}$ from 2.6 provides another example. In fact, since $\mathcal{A}$ is sifted, $\text{Sind}\mathcal{A}$ has a terminal object. But $\text{Rec}\mathcal{A}$ is obtained from $\mathcal{A}$, by freely adding the coequalizers of $a_1, a_2$ and $a'_1, a'_2$. It is easy to see that $\text{Ind}(\text{Rec}\mathcal{A})$ does not have a terminal object.

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4 Preservation of Sifted Colimits

In the Introduction we mentioned that the fact that functors preserve sifted colimits iff they preserve filtered colimits and reflexive coequalizers has been proved by several authors independently. Here we prove this result. A simpler proof in case of cocomplete categories can be found in the Appendix.

4.1 Theorem. A functor $F: \mathcal{A} \to \mathcal{B}$ with $\mathcal{A}$ finitely cocomplete preserves sifted colimits iff it preserves filtered colimits and reflexive coequalizers.

Proof. Given a sifted diagram $D: \mathcal{D} \to \mathcal{A}$ with a colimit in $\mathcal{A}$, we prove that $F \cdot D$ has colimit $F(\text{colim } D)$ in $\mathcal{D}$.

Recall from 2.1(ii) that $D: \mathcal{D} \to \text{Fam}\mathcal{D}$ is final, thus, $D$ has the same colimit as its extension $\overrightarrow{D}: \text{Fam}\mathcal{D} \to \mathcal{A}$ preserving finite coproducts. Therefore, without loss of generality we can assume that $\mathcal{D}$ has finite coproducts and $D$ preserves them (if not, substitute $\overrightarrow{D}$ for $D$). Recall also the construction of finite colimits via finite coproducts and coequalizers from [13]: given a graph $M$ and a functor $F: M \to \mathcal{A}$ form coproducts

$$\coprod_i F(i)$$

indexed by objects $i$ of $M$ and with injections

$$\alpha_i : F(i) \to \coprod_i F(i).$$

Analogously, we form coproducts

$$\coprod_{f: i \to i'} F(i)$$

indexed by morphisms $f$ of $M$ and with injections

$$\beta_f : F(i) \to \coprod_{f: i \to i'} F(i).$$

Consider morphisms

$$a, b : \coprod_{f: i \to i'} F(i) \to \coprod_i F(i)$$
such that \( a \cdot \beta_f = \alpha_i \) and \( b \cdot \beta_f = \alpha_i' \cdot Ff \) for each morphism \( f : i \to i' \) in \( M \).

If \( q : \coprod_i F(i) \to Q \) is the coequalizer of \( a \) and \( b \), then \( Q_M = \text{colim} \ F \) with the colimit cocone \( q_M \cdot \alpha_i \).

We now prove the theorem:

1. For every finite reflexive subgraph \( M \) of \( D \) we form coproducts in \( D \)

\[
i_M = \coprod_i \quad j_M = \coprod_{f : i \to i'} i
\]

and morphisms

\[
a_M, b_M : j_M \to i_M
\]

analogous to those considered above. Since \( D \) preserve the two coproducts, we have \( a = Da_M \) and \( b = Db_M \) and the colimit \( Q_M \) of the domain restriction \( D/M \) of \( D \) on \( M \) is given by the coequalizer

\[
Dj_M \xrightarrow{Da_M} Di_M \xrightarrow{q_M} Q_M = \text{colim} \ D/M
\]

Since the graph \( M \) is reflexive, \( a_M, b_M \) is a reflexive pair, this, so is \( Da_M, Db_M \).

Let \( M \) be the directed family of all finite reflexive subgraphs of \( D \).

2. Let \( k_i : Di \to K \) \((i \in \text{obj} D)\) be a colimit of \( D \), then we prove that \( (Fk_i) \) is a colimit of \( FD \). We express \( D \) as the directed union of all \( D/M \) for \( M \in \text{M} \) and for each \( M \in \text{M} \) we see that

\[
k_{i_M} \cdot Da_M = k_{j_M} = k_{i_M} \cdot Db_M
\]

from which we derive that \( k_{i_M} \) factors through the coequalizer

\[
k_{i_M} = r_M \cdot q_M \quad \text{for some} \quad r_M : Q_M \to K.
\]

Then \( K \) is the filtered colimit of all \( Q_M \) with the colimit cocone \((r_M)_{M \in \text{M}}\) (since every colimit is a filtered colimits of all finite subcolimits). We conclude that

1. \( FK \) is a colimit of \( FQ_M \) with the cocone \( Fr_M \) \((M \in \text{M})\),

and

2. for every \( M \in \text{M} \) the coequalizer of \( FDA_M \) and \( FDb_M \) is \( Fq_M \).
(3) Given a cocone

\[ x_i : FD_i \to X \quad (i \in \text{obj} \mathcal{D}) \]

of \( FD \), we are to find a factorization through \((Fk_i)\). Analogously to (1) above we have, for every \( M \in \mathcal{M} \)

\[ x_{iM} : FDA_M = x_{jM} = x_{iM} : FDb_M \]

thus, there exists a unique

\[ y_M : FQ_M \to C \quad \text{with} \quad x_{iM} = y_M : Fq_M. \quad (3) \]

These morphisms form a cocone of the filtered diagram of all \( FQ_M \)'s: in fact, the connecting morphisms

\[ q_{M,M'} : Q_M \to Q_{M'} \quad (M, M' \in \mathcal{M}, \ M \subseteq M') \]

are defined by the commutative squares

\[
\begin{array}{ccc}
D_i M & \xrightarrow{Di MM'} & D_i M' \\
\downarrow q_M & & \downarrow q_{M'} \\
Q_M & \xrightarrow{Q M M'} & Q_{M'}
\end{array}
\]

where \( i_{MM'} : i_M \to i_{M'} \) is the coproduct injection in \( \mathcal{D} \). The desired equality

\[ y_M = y_{M'} : Fq_{MM'} \]

easily follows since, by (ii), \( Fq_M \) is an epimorphism:
Consequently, we obtain the unique
\[ y : FK \to X \quad \text{with} \quad y \cdot Fr_M = y_M. \]
This is the desired factorization: for every \( i \in I \) we have
\[ y \cdot Fk_i = x_i. \]
In fact, consider the singleton subgraph \( M \) with one object \( i \) and its identity morphism. Obviously
\[ i_M = i \quad \text{and} \quad q_M = \text{id}, \quad \text{thus}, \quad r_M = k_i \]
which yields by (3)
\[ y_M \cdot Fk_i = y_M = y_M \cdot Fq_M = x_{i_M} = x_i. \]
The uniqueness is clear: since each \( Fq_M \) is an epimorphism, from (2) we see that \( (Fr_M \cdot Fq_M) \) is collectively epic, and then (1) implies that \( (Fk_i) \) is collectively epic.

\section*{4.2 Example.}
A functor \( F \) which

(1) does not preserve sifted colimits

but

(2) preserves filtered colimits and reflexive coequalizers

can be constructed as follows.

By adding to the category \( A \) of 2.6 a terminal object \( T \) we obtain a category \( A' \) in which the sifted diagram \( D : D \to A \) has colimit
\[ \text{colim} \, D = T. \]
Let \( B \) be the category obtained from \( A' \) by adding a new terminal object \( S \). The functor \( F : A' \to B \) with \( F(T) = S \) which is the identity map on objects and morphisms of \( A \) does not preserve sifted colimits because \( \text{colim} \, F \cdot D = T \) but \( F(\text{colim} \, D) = S \). It is easy to verify that \( F \) preserves filtered colimits and reflexive coequalizers.
5 Appendix: A simple proof of theorem 4.1

Here we present a simpler and more elegant proof of Theorem 4.1 in case $\mathcal{A}$ and $\mathcal{B}$ are cocomplete categories:

Let $F: \mathcal{A} \to \mathcal{B}$ preserve filtered colimits and reflexive coequalizers where $\mathcal{A}$ and $\mathcal{B}$ are cocomplete. For every sifted diagram $D: \mathcal{D} \to \mathcal{A}$ choose a small full subcategory $U: \mathcal{C} \hookrightarrow \mathcal{A}$ containing the image of $D$ and closed in $\mathcal{A}$ under finite coproducts. Consider the following diagram

$$
\begin{array}{c}
\text{Sind } \mathcal{C} \\
\downarrow \downarrow \\
\text{Rec } \mathcal{C} \\
U \\
\uparrow \uparrow \\
\text{Ind } \mathcal{C} \\
\end{array}
\quad
\begin{array}{c}
(F \cdot U)^* \\
\downarrow \\
F \\
\end{array}
\quad
\begin{array}{c}
\text{Sind } \mathcal{B} \\
\downarrow \downarrow \\
\text{Rec } \mathcal{B} \\
\mathcal{B} \\
\end{array}
$$

where $U^*$ and $(F \cdot U)^*$ are the extensions of $U$ and $F \cdot U$, respectively, preserving sifted colimits, and $U'$ is the extension of $U$ preserving reflexive coequalizers. Since by 3.5 we have $E_{\text{Sind}} = E_{\text{Ind}} \cdot E_{\text{Rec}}$, it follows that

$$(F \cdot U)^* \cdot E_{\text{Ind}} \cdot E_{\text{Rec}} \cong (F \cdot U)^* \cdot E_{\text{Sind}} \cong F \cdot U \cong F \cdot U' \cdot E_{\text{Rec}}$$

The functor $E_{\text{Ind}}$ preserves reflexive coequalizers by 3.3, and so do the functors $F, U'$ and $(F \cdot U)^*$. Thus, the universal property of $E_{\text{Rec}}$ yields

$$(F \cdot U)^* \cdot E_{\text{Ind}} \cong F \cdot U'$$

Since

$$U^* \cdot E_{\text{Ind}} \cdot E_{\text{Rec}} \cong U^* \cdot E_{\text{Sind}} \cong U \cong U' \cdot E_{\text{Rec}}$$

and, once again, the functors $U^*, E_{\text{Ind}}$ and $U'$ preserve reflexive coequalizers, we have

$$U^* \cdot E_{\text{Ind}} \cong U'$$

Finally, since

$$F \cdot U^* \cdot E_{\text{Ind}} \cong F \cdot U' \cong (F \cdot U)^* \cdot E_{\text{Ind}}$$

and the functors $F, U^*$ and $(F \cdot U)^*$ preserve filtered colimits, we have

$$F \cdot U^* \cong (F \cdot U)^*$$

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We are ready to prove \( F(\text{colim } D) \cong \text{colim}(F \cdot D) \). Let 

\[ D' : \mathcal{D} \to \mathcal{C} \quad \text{with} \quad D = U \cdot D' \]

be the codomain restriction of \( D \). Then 

\[ F \cdot D = F \cdot U \cdot D' \cong (F \cdot U)^* \cdot E_{Snd} \cdot D' \]

implies, since \((F \cdot U)^*\) preserves the sifted colimit of \( E_{Snd} \cdot D'\), that 

\[ \text{colim}(F \cdot D) \cong (F \cdot U^*)(\text{colim}(E_{Snd} \cdot D')) \]

From \((F \cdot U)^* \cong F \cdot U^*\) and the fact that \( U^* \) also preserves the sifted colimit of \( E_{Snd} \cdot D' \) we derive 

\[ \text{colim}(F \cdot D) \cong F(\text{colim}(U^* \cdot E_{Snd} \cdot D')) \cong F(\text{colim } D) \]

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