WHAT ARE SIFTED COLIMITS?

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Dedicated to Dominique Bourn on the occasion of his sixtieth birthday

Abstract. Sifted colimits, important for algebraic theories, are "almost" just the combination of filtered colimits and reflexive coequalizers. For example, given a finitely cocomplete category $\mathcal{A}$, then a functor with domain $\mathcal{A}$ preserves sifted colimits iff it preserves filtered colimits and reflexive coequalizers. But for general categories $\mathcal{A}$ that statement is not true: we provide a counter-example.

Introduction

Sifted colimits play for the doctrine of finite products precisely the role which filtered colimits play for the doctrine of finite limits. Recall that a small category $\mathcal{D}$ which is filtered has the property that $\mathcal{D}$-colimits commute with finite limits in *Set*. The converse is less well known (but trivial to prove using representable functors as diagrams): if $\mathcal{D}$-colimits commute with finite limits in *Set*, then $\mathcal{D}$ is filtered. Now sifted categories are defined as those small categories $\mathcal{D}$ such that $\mathcal{D}$-colimits commute with finite products in *Set*. They were first studied (without any name) in the classical lecture notes of P. Gabriel and F. Ulmer [6] who proved that $\mathcal{D}$ is sifted iff the diagonal $\Delta\colon \mathcal{D} \to \mathcal{D} \times \mathcal{D}$ is a final functor; this nicely corresponds to the fact that $\mathcal{D}$ is filtered iff the diagonals $\Delta\colon \mathcal{D} \to \mathcal{D}^{\mathcal{J}}$ are final for all finite graphs $\mathcal{J}$. Sifted colimits are colimits whose schemes are sifted categories; they were studied (independently of [6]) by C. Lair [9] who called them "tamisante", later P. T. Johnstone suggested the translation to "sifted". Besides filtered colimits, prime examples of sifted colimits are reflexive coequalizers, that is, coequalizers of parallel pairs of epimorphisms with a joint splitting.

Sifted colimits are of major importance in general algebra. Recall that an algebraic theory (in the sense of F. W. Lawvere [10]) is a small category $\mathcal{T}$ with finite products and an algebra for $\mathcal{T}$ is a functor $A : \mathcal{T} \to Set$ preserving finite products. The category $Alg\,\mathcal{T}$ of algebras is a full subcategory of the functor category $Set^{\mathcal{T}}$. Now, let us denote by $Sind\,\mathcal{A}$ the free completion of a category $\mathcal{A}$ under sifted colimits (resembling the name $Ind\,\mathcal{A}$ for Grothendieck's completion under filtered colimits, see [4]). Then for every algebraic theory $\mathcal{T}$ the category of algebras is just the above completion of $\mathcal{T}^{op}$:

$$Alg\,\mathcal{T} = Sind\,\mathcal{T}^{op}$$

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see [2]. And algebraic functors, that is functors between algebraic categories induced by morphisms of algebraic theories, are precisely the functors preserving limits and sifted colimits, see [1].

The aim of our paper is to discuss the slogan

"filtered colimits = filtered colimits + reflexive coequalizers."

This could mean the existence:

A category $\mathcal{A}$ has sifted colimits iff it has filtered colimits and reflexive coequalizers.

Or the preservation:

A functor $F : \mathcal{A} \to \mathcal{B}$ preserves sifted colimits iff it preserves filtered colimits and reflexive coequalizers.

Unfortunately, none of these two statements holds in general, as we demonstrate by counter-examples. However, both statements are true whenever $\mathcal{A}$ is finitely cocomplete. Whereas the first one is trivial, since filtered colimits imply cocompleteness, the latter one concerning preservation is not. Let us mention that this result, assuming $\mathcal{A}$ is cocomplete, was proved by A. Joyal (his proof even works for quasicategories, see [7]) and by S. Lack (see [8]). There proofs are different, and more elegant than our proof below, however, for our proof we only assume the existence of finite colimits. (Another proof assuming cocompleteness is presented in [3].)

Let us also remark that there is another interpretation of the above slogan: the free completion of a category $\mathcal{A}$ under sifted colimits can be constructed as a free completion of $Ind\mathcal{A}$ under reflexive coequalizers. This is true if $\mathcal{A}$ has finite coproducts and false in general, see [2].

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1. Existence of Sifted Colimits

As mentioned in the Introduction, a small category $\mathcal{D}$ is called sifted iff $\mathcal{D}$-colimits commute in *Set* with finite products. That is, given a diagram

$$\mathcal{D} \times \mathcal{J} \to Set$$

where $\mathcal{J}$ is a finite discrete category, then the canonical morphism

$$\underset{\mathcal{D}}{colim}(\prod_{\mathcal{J}} D(d,j) \to \prod_{\mathcal{J}}(\underset{\mathcal{D}}{colim} D(d,j)$$

is an isomorphism.

Colimits of diagrams over sifted categories are called sifted colimits.

Remark 1.1

(i) As proved by P. Gabriel and F. Ulmer [6], a small, nonempty category $\mathcal{D}$ is sifted if and only if the diagonal functor $\Delta\colon \mathcal{D} \to \mathcal{D} \times \mathcal{D}$ is final. This means that for every pair of objects A, B of $\mathcal{D}$ the category $(A, B) \downarrow \Delta$ of cospans on A, B is connected. That is:

  (a) a cospan $A \to X \leftarrow B$ exists, and

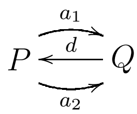
  (b) every pair of cospans on A, B is connected by a zig-zag of cospans.

  This characterization was later re-discovered by C. Lair [9].

(ii) P. Gabriel and F. Ulmer [6] also proved that a small category $\mathcal{D}$ is sifted if and only if $\mathcal{D}$ is final in its free completion $\operatorname{Fam}\mathcal{D}$ under finite coproducts. In fact, (a) and (b) above clearly imply the same property for finite families of objects too. This is precisely the finality of $\mathcal{D} \to \operatorname{Fam}\mathcal{D}$.

(iii) Every small category with finite coproducts is sifted. This immediately follows from (i).

Example 1.2 ([2]) Reflexive coequalizers are sifted colimits. That is, the category $\mathcal{D}$ given by the graph



and the equations

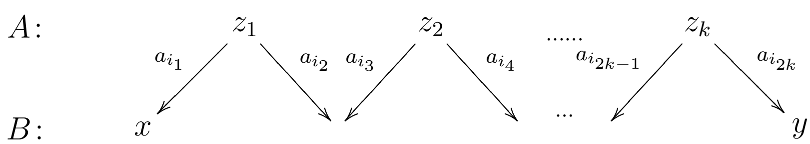
$$a_1 \cdot d = \mathrm{id}_B = a_2 \cdot d$$

is sifted. This follows from the characterization of sifted colimits mentioned in the Introduction. We present a full proof here because we are going to use it again below. Let us add that this fact was already realized by Y. Diers [5] but remained unnoticed. Another proof is given in [12], Lemma 1.2.3.

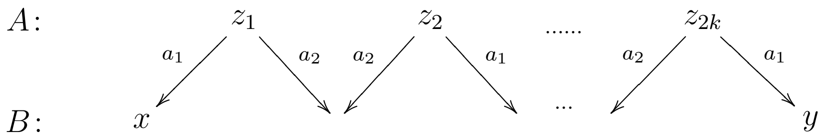
In fact, suppose that

$$A \overset{a_1}{\underset{a_2}{\rightrightarrows}} B \xrightarrow{\ c\ } C \quad \text{and} \quad A' \overset{a_1'}{\underset{a_2'}{\rightrightarrows}} B' \xrightarrow{\ c'\ } C'$$

are reflexive coequalizers in *Set*. We can assume, without loss of generality, that c is the canonical function of the quotient $C = B/\sim$ modulo the equivalence relation described as follows: two elements $x, y \in B$ are equivalent iff there exists a zig-zag

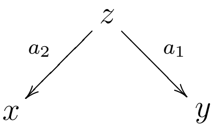


where $i_1, i_2, \ldots, i_{2k}$ are 1 or 2. For reflexive pairs a_1, a_2 these zig-zags can always be chosen to have the following form

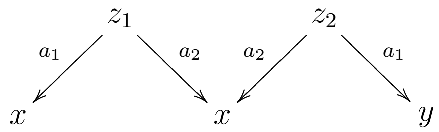


(∗)

where for the elements z_{2i} of A we use a_1, a_2 and for the elements z_{2i+1} we use a_2, a_1. In fact, let $d\colon B \to A$ be a joint splitting of a_1, a_2. Thus given a zig-zag, say,



we can modify it as follows: put $z_1 = d(x)$ and $z_2 = z$ to get



Moreover, the length $2k$ of the zig-zag $(*)$ can be prolonged to $2k+2$ or $2k+4$ etc. by using d. Analogously, we can assume $C' = B'/\sim'$ where $\sim'$ is the equivalence relation given by zig-zags of a_1' and a_2' of the above form $(*)$. Now we form the parallel pair

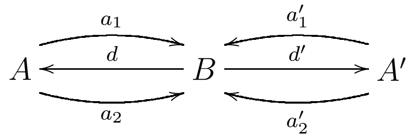
$$A \times A' \underset{a_2 \times a_2'}{\overset{a_1 \times a_1'}{\rightrightarrows}} B \times B'$$

and obtain its coequalizer by the zig-zag equivalence $\approx$ on $B \times B'$. Given $(x, x') \approx (y, y')$ in $B \times B'$, we obviously have zig-zags both for $x \sim y$ and for $x' \sim' y'$ (use projections of the given zig-zag). But also the other way round: whenever $x \sim y$ and $x' \sim' y'$, then we choose the two zig-zags so that they both have the above type $(*)$ and have the same lengths. They create an obvious zig-zag for $(x, x') \approx (y, y')$. From this it follows that the map

$$A \times A' \underset{a_2 \times a_2'}{\overset{a_1 \times a_1'}{\rightrightarrows}} B \times B' \xrightarrow{\ c \times c'\ } (B/\sim) \times (B'/\sim')$$

is a coequalizer, as required.

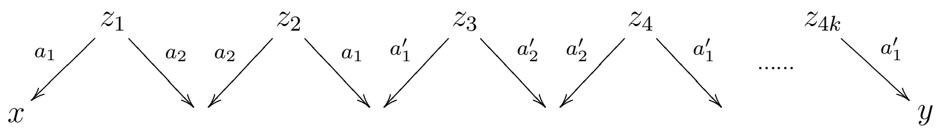
Example 1.3 By merging two copies of reflexive pairs we also obtain a sifted category $\mathcal{D}$: let $\mathcal{D}$ be given by the graph



and the equations making both parallel pairs reflexive:

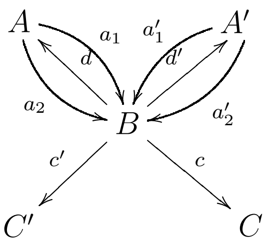
$$a_i \cdot d = \mathrm{id}_B = a'_i \cdot d' \quad \text{for} \quad i = 1, 2$$

The proof that $\mathcal{D}$ is sifted is completely analogous to the proof of Example 1.2: we verify that colimits over $\mathcal{D}$ in *Set* commute with finite products. Assume that the above graph depicts sets A, B and A' and functions between them. Then a colimit can be described as the canonical function $c\colon B \to C = B/\sim$ where two elements $x, y \in B$ are equivalent iff they are connected by a zig-zag formed by a_1, a_2, a'_1 and a'_2. Since the two pairs are reflexive, the length of the zig-zag can be arbitrarily prolonged. And the type can be chosen to be



From that it is easy to derive that $\mathcal{D}$ is sifted.

Example 1.4 A category $\mathcal{A}$ which does not have sifted colimits although it has both filtered colimits and reflexive coequalizers: $\mathcal{A}$ is the free completion of $\mathcal{D}$ from 1.3 under filtered colimits and reflexive coequalizers. We claim that $\mathcal{A}$ is obtained from $\mathcal{D}$ by simply adding the coequalizer c of a_1, a_2 and the coequalizer c' of a'_1, a'_2. That is, we consider the graph



and the equations

$$c \cdot a_1 = c \cdot a_2 \qquad\qquad c' \cdot a'_1 = c' \cdot a'_2.$$

In fact, the category $\mathcal{A}$ is clearly finite. Therefore, its only filtered diagrams are its idempotents:

$$e_i = d \cdot a_i \quad \text{and} \quad e'_i = d' \cdot a'_i \qquad (i = 1, 2)\,.$$

We claim that a_1 is the colimit of e_1. In fact, $a_1 \cdot e_1 = a_1$, and given a morphism f with

$$f \cdot e_1 = f\,,$$

then we see that $f \cdot d \cdot a_1 = f$, consequently, f factorizes through a_1. Since a_1 is an epimorphism, this factorization is unique. Analogously for e_2, e'_1 and e'_2. Thus, $\mathcal{A}$ has filtered colimits. And it has reflexive coequalizers because its only reflexive pairs of distinct morphisms are a_1, a_2 whose coequalizer is c, and a'_1, a'_2 whose coequalizer is c'. It is obvious that the (sifted) embedding $D : \mathcal{D} \to \mathcal{A}$ does not have a colimit.

2. Preservation of Sifted Colimits

Theorem 2.1 *A functor $F\colon \mathcal{A} \to \mathcal{B}$ with $\mathcal{A}$ finitely cocomplete preserves sifted colimits iff it preserves filtered colimits and reflexive coequalizers.*

Proof. Given a sifted diagram $D\colon \mathcal{D} \to \mathcal{A}$ with a colimit in $\mathcal{A}$, we prove that $F \cdot D$ has colimit $F(\mathit{colim}\, D)$ in $\mathcal{D}$.

Recall from 1.1(ii) that $D\colon \mathcal{D} \to \mathit{Fam}\mathcal{D}$ is final, thus, D has the same colimit as its extension $\overline{D}\colon \mathit{Fam}\mathcal{D} \to \mathcal{A}$ preserving finite coproducts. Therefore, without loss of generality we can assume that $\mathcal{D}$ has finite coproducts and D preserves them (if not, substitute $\overline{D}$ for D). Recall also the construction of finite colimits via finite coproducts and coequalizers from [11]: given a finite graph M and a functor $F : M \to \mathcal{A}$ we form coproducts

$$\coprod_i F(i)$$

indexed by objects i of M and with injections.

$$\alpha_i : F(i) \to \coprod_i F(i).$$

Analogously, we form coproducts

$$\coprod_{f:i\to i'} F(i)$$

indexed by morphisms f of M and with injections

$$\beta_f : F(i) \to \coprod_{f:i\to i'} F(i).$$

Consider morphisms

$$a, b : \coprod_{f:i\to i'} F(i) \to \coprod_i F(i)$$

such that $a \cdot \beta_f = \alpha_i$ and $b \cdot \beta_f = \alpha_{i'} \cdot Ff$ for each morphism $f : i \to i'$ in M. If $q : \coprod_i F(i) \to Q$ is the coequalizer of a and b, then $Q = \mathit{colim}\, F$ with the colimit cocone $q \cdot \alpha_i$.

We now prove the theorem:

(1) For every finite reflexive subgraph M of $\mathcal{D}$ we form coproducts in $\mathcal{D}$

$$i_M = \coprod_i i \qquad j_M = \coprod_{f:i\to i'} i$$

and morphisms

$$a_M, b_M : j_M \to i_M$$

analogous to those considered above. Since D preserves the two coproducts, we have $a = Da_M$ and $b = Db_M$ and the colimit Q_M of the domain restriction D/M of D on M is given by the coequalizer

$$Dj_M \overset{Da_M}{\underset{Db_M}{\rightrightarrows}} Di_M \xrightarrow{\;q_M\;} Q_M = colim\, D/M$$

Since the graph M is reflexive, a_M, b_M is a reflexive pair, thus, so is Da_M, Db_M. Let $\mathcal{M}$ be the directed family of all finite reflexive subgraphs of $\mathcal{D}$.

(2) Let $k_i\colon Di \to K$ $(i \in obj\,\mathcal{D})$ be a colimit of D, then we prove that (Fk_i) is a colimit of FD. We express D as the directed union of all D/M for $M \in \mathcal{M}$ and for each $M \in \mathcal{M}$ we see that

$$k_{i_M} \cdot Da_M = k_{j_M} = k_{i_M} \cdot Db_M \tag{1}$$

from which we derive that k_{i_M} factors through the coequalizer

$$k_{i_M} = r_M \cdot q_M \quad \text{for some} \quad r_M\colon Q_M \to K\,. \tag{2}$$

Then K is the filtered colimit of all Q_M with the colimit cocone $(r_M)_{M\in\mathcal{M}}$ (since every colimit is a filtered colimits of all finite subcolimits). We conclude that

(i) FK is a colimit of FQ_M with the cocone Fr_M $(M \in \mathcal{M})$,

and

(ii) for every $M \in \mathcal{M}$ the coequalizer of FDa_M and FDb_M is Fq_M.

(3) Given a cocone

$$x_i\colon FD_i \to X \quad (i \in obj\,\mathcal{D})$$

of FD, we are to find a factorization through (Fk_i). Analogously to (1) above we have, for every $M \in \mathcal{M}$

$$x_{i_M} \cdot FDa_M = x_{j_M} = x_{i_M} \cdot FDb_M$$

thus, there exists a unique

$$y_M\colon FQ_M \to C \quad \text{with} \quad x_{i_M} = y_M \cdot Fq_M\,. \tag{3}$$

These morphisms form a cocone of the filtered diagram of all FQ_M's: in fact, the connecting morphisms

$$q_{M,M'}\colon Q_M \to Q_{M'} \qquad (M, M' \in \mathcal{M},\ M \subseteq M')$$

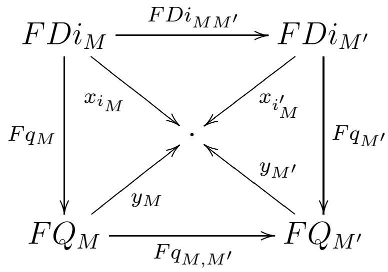
are defined by the commutative squares

$$\begin{array}{ccc} Di_M & \xrightarrow{\;Di_{MM'}\;} & Di_{M'} \\ {\scriptstyle q_M}\downarrow & & \downarrow{\scriptstyle q_{M'}} \\ Q_M & \xrightarrow[\;q_{M,M'}\;]{} & Q_{M'} \end{array}$$

where $i_{MM'}\colon i_M \to i_{M'}$ is the coproduct injection in $\mathcal{D}$. The desired equality

$$y_M = y_{M'} \cdot Fq_{MM'}$$

easily follows since, by (ii), Fq_M is an epimorphism:



Consequently, we obtain the unique

$$y\colon FK \to X \quad \text{with} \quad y \cdot Fr_M = y_M\,.$$

This is the desired factorization: for every $i \in I$ we have

$$y \cdot Fk_i = x_i.$$

In fact, consider the singleton subgraph M with one object i and its identity morphism. Obviously

$$i_M = i \quad \text{and} \quad q_M = \mathrm{id}\,, \quad \text{thus,} \quad r_M = k_i$$

which yields by (3)

$$y_M \cdot Fk_i = y_M = y_M \cdot Fq_M = x_{i_M} = x_i\,.$$

The uniqueness is clear: since each Fq_M is an epimorpism, from (2) we see that $(Fr_M \cdot Fq_M)$ is collectively epic, and then (1) implies that (Fk_i) is collectively epic. ∎

Example 2.2 A functor F which

(1) does not preserve sifted colimits

but

(2) preserves filtered colimits and reflexive coequalizers

can be constructed as follows.

By adding to the category $\mathcal{A}$ of 1.4 a terminal object T we obtain a category $\mathcal{A}'$ in which the sifted diagram $D\colon \mathcal{D} \to \mathcal{A}$ has colimit

$$\mathit{colim}\, D = T\,.$$

Let $\mathcal{B}$ be the category obtained from $\mathcal{A}'$ by adding a new terminal object S. The functor $F\colon \mathcal{A}' \to \mathcal{B}$ with $F(T) = S$ which is the identity map on objects and morphisms of $\mathcal{A}$ does not preserve sifted colimits because $\mathit{colim}\, F \cdot D = T$ but $F(\mathit{colim}\, D) = S$. It is easy to verify that F preserves filtered colimits and reflexive coequalizers.

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