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Left covering functors

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Introduction

The notion of exact category is one of the most interesting notion studied in category theory. In fact, several important mathematical situations can be axiomatized in categorical terms as exact categories satisfying some typical axioms. Let us clarify this fact with some examples.

The categorical setting for homological algebra is given by abelian categories; an abelian category is an exact category satisfying moreover the typical axiom which holds in module categories, i.e. the set of homomorphisms between two objects is an abelian group and composition is a group homomorphism.

The categorical generalization of a space, or, better, of the category of sheaves on a space, is given by the notion of Grothendieck topos; a Grothendieck topos is an exact category with a set of generators satisfying moreover typical axioms which are true for sheaves: sums are disjoint and universal.

The categorical counterpart of universal algebra is given by monads over $\mathcal{SET}$; a monadic category over $\mathcal{SET}$ is an exact category satisfying moreover some extra-assumptions saying that there exist free algebras and each algebra is a quotient of a free one.

Another important example is given by elementary topo, i.e. the categorical universes in which to develop mathematics; of course, an elementary topos is an exact category.

Now, after a little bit of publicity to the notion of exact category, let us recall the definition:
a category is exact if

i) is left exact

ii) each arrow can be factored as a regular epi followed by a mono and regular epis are pullback stable

iii) equivalence relations are effective

A category satisfying i) and ii) is also called regular.

(By the way, a category which is exact in the sense of the above definition is sometimes called "Barr-exact". This is to avoid confusion with other, not equivalent, notions of exact category used, for example, in algebraic K-theory. The reference to Barr is due to the fact that one of the first, maybe the first, work in which our definition can be found is Barr's contribution in [2]. Nevertheless, this definition is older, at least as Tierney's theorem characterizing abelian categories. For a general introduction to regular and exact categories, the reader can refer to [8]).

Looking at the previous definition, the question naturally arising is: given a left exact category $\mathbb{C}$, can one "complete $\mathbb{C}$ in the best possible way" to obtain a regular category or an exact category?

Let us start with the problem of the exact completion. More exactly, the question is: given a left exact category $\mathbb{C}$, do there exist an exact category $\mathbb{C}_{\mathrm{ex}}$ and a left exact functor $\Gamma\colon \mathbb{C} \longrightarrow \mathbb{C}_{\mathrm{ex}}$ which are universal? Here universal means that, for each exact category $\mathbb{B}$, composing with Γ induces an equivalence between the category of exact functors from $\mathbb{C}_{\mathrm{ex}}$ to $\mathbb{B}$ and the category of left exact functors from $\mathbb{C}$ to $\mathbb{B}$. (Recall that a functor between regular or exact categories is exact when it is left exact and preserves regular epis).

The answer is affirmative: in fact, following a suggestion of A. Joyal, A. Carboni and R. Celia Magno in [15] have given an explicit construction of the free exact category $\Gamma\colon \mathbb{C} \longrightarrow \mathbb{C}_{\mathrm{ex}}$ over a left exact one.

As usual, one of the points of interest in performing this free construction is that if one wants to study the exact functors between two exact categories $\mathbb{A}$ and $\mathbb{B}$ and $\mathbb{A}$ is free over a certain left exact base $\mathbb{C}$ (that is $\mathbb{A}$ is equivalent to $\mathbb{C}_{\mathrm{ex}}$), then one can equivalently look at the left exact functors from $\mathbb{C}$ to $\mathbb{B}$, which can be considerably easier.

Of course, to make this, one must be able to recognize when an exact category is free, that is one needs a characterization of free exact categories.

In fact, Carboni and Celia Magno give the following elegant characterization theorem:
an exact category $\mathbb{A}$ is free if and only if the two following conditions are satisfied ("projective" means "regular projective"):

1) $\mathbb{A}$ has enough projectives (that is each object is quotient of a projective one)

2) the full subcategory $P(\mathbb{A})$ of projective objects is closed in $\mathbb{A}$ under finite limits.

When this is the case, $\mathbb{A}$ is free over $P(\mathbb{A})$.

Some interesting applications of those results have been found, for example, in the study of the so called Effective topos (see [14]) and for the construction of the free abelian category over an additive one (cf. [1] and [15]). However, there exist important examples of exact categories which satisfy the first condition of the characterizing theorem but not the second one.

The two crucial examples are the following:
the category of algebras for a monad over $\mathcal{SET}$ is exact and has enough projectives, because free algebras are projective and each algebra is quotient of a free one;

each presheaf on a small category is quotient of a sum of representable presheaves which are projective objects in the exact category of presheaves.

The question naturally arising is then whether an exact category with enough projectives is free in some sense; this leads to look more carefully at the full subcategory of projective objects of a category with enough projectives.

We have said that, if the whole category $\mathbb{A}$ is left exact, the full subcategory $P(\mathbb{A})$ is not necessarily so; nevertheless in $P(\mathbb{A})$ remains a trace of the left exactness of $\mathbb{A}$.

Consider, for example, two projective objects P_1 and P_2 and their product

$$P_1 \xleftarrow{\pi_1} P_1 \times P_2 \xrightarrow{\pi_2} P_2$$

in $\mathbb{A}$: there are no reasons why $P_1 \times P_2$ is projective but, as the projective are "enough", one can consider a projective cover of $P_1 \times P_2$, that is a projective object P and a regular epimorphism $q \colon P \longrightarrow P_1 \times P_2$; now the diagram

$$P_1 \xleftarrow{\pi_1} P_1 \times P_2 \xleftarrow{q} P \xrightarrow{q} P_1 \times P_2 \xrightarrow{\pi_2} P_2$$

is a "weak" product of P_1 and P_2 in $P(\mathbb{A})$. In fact, given a projective object Q with two arrows $P_1 \longleftarrow Q \longrightarrow P_2$, there exists a factorization $Q \longrightarrow P$ which, in general, is not unique.

This kind of argument indicates that if $\mathbb{A}$ is a left exact category (in particular an exact one) with enough projectives, then the full subcategory of $P(\mathbb{A})$ of projective objects is weakly left exact.

The next important step is then to observe that the construction of an exact category $\mathbb{C}_{\mathrm{ex}}$ from a left exact one can be carried out even if the base $\mathbb{C}$ is only weakly left exact.

Now the problem is: given a weakly left exact category $\mathbb{C}$ and its exact completion $\mathbb{C}_{\mathrm{ex}}$, what kind of functors with domain $\mathbb{C}$ classifies the exact functors with domain $\mathbb{C}_{\mathrm{ex}}$ and exact codomain? That is, what functors $\mathbb{C} \longrightarrow \mathbb{B}$ (with $\mathbb{B}$ exact) correspond to exact functors $\mathbb{C}_{\mathrm{ex}} \longrightarrow \mathbb{B}$? Of course, if we want the embedding $\mathbb{C} \longrightarrow \mathbb{C}_{\mathrm{ex}}$ to be universal, this functor must be an admissible one (it must correspond to the identity of $\mathbb{C}_{\mathrm{ex}}$).

Now, taking into account that we would like to characterize free exact categories over weakly left exact categories as exact categories with enough projectives, it is clear that we have to look at the full inclusion of $P(\mathbb{A})$ in $\mathbb{A}$: the exact completion $\Gamma \colon \mathbb{C} \longrightarrow \mathbb{C}_{\mathrm{ex}}$ will be of the form $P(\mathbb{A}) \hookrightarrow \mathbb{A}$.

Keeping in mind the previous discussion on the weak left exactness of $P(\mathbb{A})$, we can axiomatize the behaviour of $P(\mathbb{A}) \hookrightarrow \mathbb{A}$ with respect to the weak limits of $P(\mathbb{A})$ with the following definition: consider a functor $F \colon \mathbb{C} \longrightarrow \mathbb{B}$ with $\mathbb{C}$ weakly left exact and $\mathbb{B}$ left exact; we say that F is left covering if the unique factorization between the image by F of a weak finite limit in $\mathbb{C}$ and the corresponding limit computed in $\mathbb{B}$ is a strong epimorphism.

Once this definition is reached, it becomes only a technical work to obtain the universal property of the embedding $\Gamma \colon \mathbb{C} \longrightarrow \mathbb{C}_{\mathrm{ex}}$: for each exact category $\mathbb{B}$, composing with Γ induces an equivalence between the category of exact functors from $\mathbb{C}_{\mathrm{ex}}$ to $\mathbb{B}$ and the category of left covering functors from $\mathbb{C}$ to $\mathbb{B}$.

Moreover, one can obtain the expected characterization theorem: an exact category is free on a weakly left exact one if and only if it has enough projectives. More exactly, if $\mathbb{A}$ is an exact category with enough projectives $P(\mathbb{A})$, then the unique exact extension $(P(\mathbb{A}))_{\mathrm{ex}} \longrightarrow \mathbb{A}$ of the full inclusion $P(\mathbb{A}) \hookrightarrow \mathbb{A}$ is an equivalence.

If also the domain is left exact, a left covering functor is exactly a left exact one, so that we obtain the case discussed in [15]; in this case the exact completion gives the left biadjoint to the inclusion of the 2-category of exact categories in the 2-category of left exact categories.

Unfortunately, this is no longer true for the exact completion of weakly left categories. There is no way to choose morphisms between weakly left exact categories so that the universal property of $\Gamma\colon \mathbb{C} \longrightarrow \mathbb{C}_{\mathrm{ex}}$ becomes part of an adjunction between exact categories and weakly left exact categories.

Now that we have studied the exact completion of a weakly left exact category, we can come back to the definition of an exact category and start again with the analogous problem of the regular completion of a weakly left exact category. Once again, the problem can be solved keeping in mind the left exact case, even if the definition of $\mathbb{C}_{\mathrm{reg}}$ needs some important modifications. The interesting functors turn out to be again the left covering functors.

Of course, one can study at first the regular completion of a weakly lex category and then obtain the exact completion using the fact that the forgetful functor from the 2-category of exact categories and exact functors to the 2-category of regular categories and exact functors has a left biadjoint. Nevertheless, for the applications it is crucial to have a simple one-step description of the exact completion.

Now that the whole story of the regular and exact completion of a weakly left exact category has been told, we can look at the applications of this theory.

First of all, a full understanding of the characterization of our major examples.

The category $\mathrm{EM}(\mathbb{T})$ of algebras for a monad $\mathbb{T}$ over $\mathcal{SET}$ is exact and the free algebras constitute enough projectives, so that, from the characterization of free exact categories, to characterize $\mathrm{EM}(\mathbb{T})$ it suffices to characterize the full subcategory $\mathrm{KL}(\mathbb{T})$ of free algebras. This last problem is only an easy exercise.

Moreover, the theory as far developed allows us to characterize some important classes of subcategories of monadic categories in terms of generators (more precisely, we can characterize reflections, epireflections and localizations of monadic categories over $\mathcal{SET}$).

Analogously, one can try to characterize the category of presheaves $\mathcal{SET}^{\mathcal{D}^{\mathrm{op}}}$ on a small category $\mathcal{D}$; here enough projectives are provided by sums of representable functors, so that one has only to characterize the full subcategory of these objects, that is the sum-completion $\mathrm{Fam}\mathcal{D}$ of $\mathcal{D}$.

Remaining in the context of Grothendieck topo, we use the universal property of $\mathcal{SET}^{\mathcal{D}^{\mathrm{op}}}$ as exact completion of $\mathrm{Fam}\mathcal{D}$ to give a direct proof of the equivalence between localizations of presheaf categories and "Giraud" topo.

Another interesting application of our results can be found in the study of geometric morphisms from a cocomplete (pre)topos to a presheaf category; the crucial point is to observe that a functor F from a small category $\mathcal{D}$ to a regular category with good sums $\mathbb{A}$ is filtering exactly when its sum-preserving extension F' from Fam$\mathcal{D}$ to $\mathbb{A}$ is left covering.

Throughout the second, the third and the fourth chapter, we give some other examples and applications; to do this, we are sometimes obliged to consider further developments of the theory. We leave the reader with some open problems naturally arising from the examples.

The idea of generalizing the exact completion from left exact bases to weakly lex bases was firstly conjectured by A. Carboni during his conference at the Category Theory 1991 in Montreal. The main results contained in this thesis has been presented by A. Carboni and myself in a talk at the 51st PSSL (Valenciennes, February 1993). Another ancestor of this work can be found in Freyd's work [20], where the problem of the reflection of an additive category in the category of abelian categories is studied.

To end, I must apologize with the reader for my notation: the composition of two arrows as

$$A \xrightarrow{f} B \xrightarrow{g} C$$

will be written $f \cdot g$. At least, my work has a common point with the mathematical production of the great mathematician Charles Ehresmann (and this is an honour for me): to stress the reader with unusual notations.

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Chapter 1

The exact completion

1.1 Weak limits and pseudo equivalence-relations

In this section we look at the existence of weak limits and we state a basic fact about pseudo equivalence-relations.

Definition 1.1.1 *Let $\mathcal{L}: \mathcal{D} \longrightarrow \mathbb{C}$ be a functor with $\mathcal{D}$ a small category; a weak limit of $\mathcal{L}$ is a cone $(\pi_D: L \longrightarrow \mathcal{L}D)_{D \in \mathcal{D}_0}$ in $\mathbb{C}$ such that for each other cone $(\tau_D: C \longrightarrow \mathcal{L}D)_{D \in \mathcal{D}_0}$ in $\mathbb{C}$ there exists a morphism $\tau: C \longrightarrow L$ with $\tau \circ \pi_D = \tau_D$ $\forall\ D \in \mathcal{D}_0$; we write wlim$\mathcal{L}$ for such a weak limit.*

So in the definition of a weak limit we require only the existence of a factorization and not its uniqueness (as in the "strong" case). A first consequence of this fact is that a functor can admit several not isomorphic weak limits; for example, in the category $\mathcal{SET}$, each not emtpy set is a weak terminal object.

Now we give some conditions for the existence of weak limits.

Proposition 1.1.2

1) *the existence of weak binary products and weak equalizers implies the existence of all weak not empty finite limits*

2) *the existence of weak pullbacks, of a weak terminal object T and of a weak product of T with itself implies the existence of weak finite products*

3) *the existence of small weak products and of weak equalizers of small families of parallel arrows implies the existence of all small weak limits.*

Proof:

1) the existence of weak pullbacks follows as in the strong case and in the same way binary implies finite. Now, if $\mathcal{L}: \mathcal{D} \longrightarrow \mathbb{C}$ is a functor defined on a finite category $\mathcal{D}$, let us consider a weak product indexed over all the objects $D \in \mathcal{D}_0$ with the corresponding projections

$$\pi_D: (\textstyle\prod_{D \in \mathcal{D}_0} \mathcal{L}D) \longrightarrow \mathcal{L}D$$

and, for each arrow $d\colon D \longrightarrow D'$, the two parallel arrows in $\mathbb{C}$

$$\prod_{D\in\mathcal{D}_0} \mathcal{L}D \underset{\pi_D \cdot \mathcal{L}d}{\overset{\pi_{D'}}{\rightrightarrows}} \mathcal{L}D'$$

and a weak equalizer of them

$$E_d \xrightarrow{e_d} \prod_{D\in\mathcal{D}_0} \mathcal{L}D \underset{\pi_D \cdot \mathcal{L}d}{\overset{\pi_{D'}}{\rightrightarrows}} \mathcal{L}D'$$

Now we can find a weak pullback $(E \xrightarrow{e_d'} E_d)_{d\in\mathcal{D}_1}$ over the diagram of convergent arrows $(E_d \xrightarrow{e_d} \prod_{D\in\mathcal{D}_0} \mathcal{L}D)_{d\in\mathcal{D}_1}$. It is easy to check that the cone

$$(E \xrightarrow{e_d'} E_d \xrightarrow{e_d} \prod_{D\in\mathcal{D}_0} \mathcal{L}D \xrightarrow{\pi_D} \mathcal{L}D)_{D\in\mathcal{D}_0}$$

is a weak limit of $\mathcal{L}$;

2) let $t_A\colon A \longrightarrow T$ and $t_B\colon B \longrightarrow T$ be two arrows into the weak terminal object T and let $T \xleftarrow{\pi_1} T\times T \xrightarrow{\pi_2} T$ be the required weak product. Consider the following weak pullbacks

$$\begin{array}{ccc}
\begin{tikzcd} T_1 \arrow[r, "t_1"] \arrow[d, "\pi_1'"'] & T\times T \arrow[d, "\pi_1"] \\ A \arrow[r, "t_A"'] & T \end{tikzcd}
&
\begin{tikzcd} T_2 \arrow[r, "t_2"] \arrow[d, "\pi_2'"'] & T\times T \arrow[d, "\pi_2"] \\ B \arrow[r, "t_B"'] & T \end{tikzcd}
&
\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & T_2 \arrow[d, "t_2"] \\ T_1 \arrow[r, "t_1"'] & T\times T \end{tikzcd}
\end{array}$$

It is easy to see that $A \xleftarrow{p_1\cdot\pi_1'} P \xrightarrow{p_2\cdot\pi_2'} B$ is a weak product of A and B;

3) analogous to 1). ■

Let us remark that in the proof of part 1) of the previous proposition it is not possible to work as in the strong case (cf. vol.1, ch.2 of [8]) because the family of projections $\prod_{d\in\mathcal{D}_1} \mathcal{L}(\text{codomain}(d)) \longrightarrow \mathcal{L}(\text{codomain}(d))$ is not in general a monomorphic family.

Definition 1.1.3 *We call a category $\mathbb{C}$ weakly lex (lex = left exact) if, for every functor $\mathcal{L}\colon \mathcal{D} \longrightarrow \mathbb{C}$ defined on a finite category $\mathcal{D}$, there exists a weak limit of $\mathcal{L}$.*

Now some points of terminology: recall that an object P of a category is called regular projective if, for each arrow $f\colon P \longrightarrow X$ and for each regular epi $q\colon Y \longrightarrow X$, there exists an arrow $f'\colon P \longrightarrow Y$ such that $f' \cdot q = f$. For brevity, we write projective instead of regular projective.

Definition 1.1.4 *Let $\mathbb{C}$ be a category and $\mathbb{P}$ a full subcategory of $\mathbb{C}$; we say that $\mathbb{P}$ is a projective cover of $\mathbb{C}$ if the two following conditions are satisfied:*

- *each object of $\mathbb{P}$ is projective in $\mathbb{C}$*
- *for each object X of $\mathbb{C}$ there exists a $\mathbb{P}$-cover of X, that is an object P of $\mathbb{P}$ and a regular epi $P \longrightarrow X$.*

Of course, a category $\mathbb{C}$ admits a projective cover if and only if it has enough projectives; we have given the previous definition because we will often have to distinguish between "all the projectives" and "enough projectives". Elsewhere a projective cover is called a resolving set of projectives (see, for example, [20]). The relation between two projective covers of a same category is stated in the following proposition. The terminology involved in the statement is explained all along the proof.

Proposition 1.1.5 *Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be two projective covers of a category $\mathbb{C}$; the splitting of idempotents of $\mathbb{P}_1$ is equivalent to those of $\mathbb{P}_2$ (so that, if $\mathbb{P}_1$ and $\mathbb{P}_2$ are small, they are Cauchy-equivalent); in particular, if the idempotents split in $\mathbb{C}$, the splitting of idempotents of $\mathbb{P}$ is equivalent to the full subcategory of all the projective objects of $\mathbb{C}$.*

Proof: Let us recall that the splitting of idempotents $SI(\mathbb{P}_1)$ of the category $\mathbb{P}_1$ is the category defined as follows:
objects: arrows $p\colon A \longrightarrow A$ with $A \in \mathbb{P}_1$ and $p \cdot p = p$
arrows: an arrow $x\colon (p\colon A \longrightarrow A) \longrightarrow (q\colon B \longrightarrow B)$ is an arrow $x\colon A \longrightarrow B$ such that the following diagram is commutative in each part

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "x"'] \arrow[dr, "x"] & A \arrow[d, "x"] \\ B \arrow[r, "q"'] & B \end{tikzcd}$$

identity: $p\colon (p\colon A \longrightarrow A) \longrightarrow (p\colon A \longrightarrow A)$
composition: the obvious one.

Given an arrow $x\colon (p\colon A \longrightarrow A) \longrightarrow (q\colon B \longrightarrow B)$ in $SI(\mathbb{P}_1)$, we can consider a $\mathbb{P}_2$-cover $r\colon A' \longrightarrow A$ of A and a $\mathbb{P}_2$-cover $m\colon B' \longrightarrow B$ of B.

Since A and B are projectives, there exist sections $s\colon A \longrightarrow A'$ and $n\colon B \longrightarrow B'$ (that is $s \cdot r = 1_A$ and $n \cdot m = 1_B$). It is easy to verify the commutativity of the following diagram

$$\begin{tikzcd} A' \arrow[r, "r \cdot p \cdot s"] \arrow[d, "r \cdot x \cdot n"'] \arrow[dr, "r \cdot x \cdot n"] & A' \arrow[d, "r \cdot x \cdot n"] \\ B' \arrow[r, "m \cdot q \cdot n"'] & B' \end{tikzcd}$$

This means that

$$r \cdot x \cdot n \colon (r \cdot p \cdot s \colon A' \longrightarrow A') \longrightarrow (m \cdot q \cdot n \colon B' \longrightarrow B')$$

is an arrow in $SI(\mathbb{P}_2)$.

If we think of choosing a distinguished $\mathbb{P}_2$-cover $r \colon A' \longrightarrow A$ (together with a distinguished section s) for each object A of $\mathbb{P}_1$, the previous construction gives us a functor $SI(\mathbb{P}_1) \longrightarrow SI(\mathbb{P}_2)$.

Analogously one can build up a functor $SI(\mathbb{P}_2) \longrightarrow SI(\mathbb{P}_1)$.

If A is in $\mathbb{P}_1$ and we have choosen as $\mathbb{P}_2$-cover $r \colon A' \longrightarrow A$ and as $\mathbb{P}_1$-cover $p \colon A'' \longrightarrow A'$ (with sections $s \colon A \longrightarrow A'$ and $\sigma \colon A' \longrightarrow A''$), we obtain a natural transformation from the composite $SI(\mathbb{P}_1) \longrightarrow SI(\mathbb{P}_2) \longrightarrow SI(\mathbb{P}_1)$ to the identity functor on $SI(\mathbb{P}_1)$ taking as component at $(p \colon A \longrightarrow A)$ the arrow

$$p \cdot r \cdot p \colon (p \cdot r \cdot p \cdot s \cdot \sigma \colon A'' \longrightarrow A'') \longrightarrow (p \colon A \longrightarrow A).$$

But this arrow is an isomorphism having as inverse the arrow

$$p \cdot s \cdot \sigma \colon (p \cdot A \longrightarrow A) \longrightarrow (p \cdot r \cdot p \cdot s \cdot \sigma \colon A'' \longrightarrow A'').$$

This shows that $SI(\mathbb{P}_1)$ is equivalent to $SI(\mathbb{P}_2)$.

To prove the remamining part of our statement, we need some general facts. Given an idempotent $p \colon A \longrightarrow A$ in a category $\mathbb{X}$, we say that p splits if there exist $e \colon A \longrightarrow E$ and $f \colon E \longrightarrow A$ such that $e \cdot f = p$ and $f \cdot e = 1_E$. Equivalently, p splits if there exists the equalizer $f \colon E \longrightarrow A$ of p and 1_A; if such an equalizer exists, then it is an absolute equalizer. We say that $\mathbb{X}$ is Cauchy-complete if in $\mathbb{X}$ all idempotents split. The category $SI(\mathbb{X})$ is then the "Cauchy-completion" of $\mathbb{X}$, that is

- $SI(\mathbb{X})$ is Cauchy-complete
- there exists a full and faithful functor $\mathcal{S} \colon \mathbb{X} \longrightarrow SI(\mathbb{X})$
- for each functor $F \colon \mathbb{X} \longrightarrow \mathbb{A}$, with $\mathbb{A}$ Cauchy-complete, there exists a unique (up to natural isomorphisms) functor $\hat{F} \colon SI(\mathbb{X}) \longrightarrow A$ such that $\mathcal{S} \cdot \hat{F} \simeq F$.

In fact, given an idempotent $x \colon (p \colon A \longrightarrow A) \longrightarrow (p \colon A \longrightarrow A)$ in $SI(\mathbb{X})$, it splits as

$$(p \colon A \longrightarrow A) \xrightarrow{x} (x \colon A \longrightarrow A) \xrightarrow{x} (p \colon A \longrightarrow A).$$

The functor

$$\mathbb{X} \longrightarrow SI(\mathbb{X})$$

sends $x \colon A \longrightarrow B$ on $x \colon (1_A \colon A \longrightarrow A) \longrightarrow (1_B \colon B \longrightarrow B)$. As far as the universal property of $\mathbb{X} \longrightarrow SI(\mathbb{X})$ is concerned, it suffices to observe that for each arrow $x \colon (p \colon A \longrightarrow A) \longrightarrow (q \colon B \longrightarrow B)$ in $SI(\mathbb{X})$, the following diagram is commutative and each line is an absolute equalizer

$$\begin{tikzcd}
(p\colon A \to A) \arrow[r, "p"] \arrow[d, "x"'] & (1_A\colon A \to A) \arrow[r, shift left, "p"] \arrow[r, shift right, "1_A"'] \arrow[d, "q"] & (1_A\colon A \to A) \arrow[d, "q"] \\
(q\colon B \to B) \arrow[r, "q"'] & (1_B\colon B \to B) \arrow[r, shift left, "q"] \arrow[r, shift right, "1_B"'] & (1_B\colon B \to B).
\end{tikzcd}$$

In particular, if in $\mathbb{X}$ idempotents split, then $SI(\mathbb{X})$ is (equivalent to) $\mathbb{X}$.

It is a well-known fact that, if $\mathbb{X}$ is small, $SI(\mathbb{X})$ is equivalent to the full subcategory $R(\mathbb{X})$ of $\mathcal{SET}^{\mathbb{X}^{\mathrm{op}}}$ spanned by retracts of representable functors; moreover, two small categories $\mathbb{X}$ and $\mathbb{Y}$ are Cauchy-equivalent (that is $\mathcal{SET}^{\mathbb{X}^{\mathrm{op}}}$ is equivalent to $\mathcal{SET}^{\mathbb{Y}^{\mathrm{op}}}$) if and only if $R(\mathbb{X})$ is equivalent to $R(\mathbb{Y})$ (cf. [8] vol. I chap.6 5, where also the name "Cauchy-completion" is exaplained).

Coming back to our statement, it remains to prove that if idempotents split in $\mathbb{C}$, then they split in the full subcategory of projective objects: this is the case because retracts of projective objects are projectives. ■

The relation between weak limits and projective covers is given in the following proposition.

Proposition 1.1.6 *Let $\mathcal{L}\colon \mathcal{D} \longrightarrow \mathbb{C}$ be a functor defined on a small category $\mathcal{D}$ and suppose that it can be factored as*

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "\mathcal{L}"] \arrow[dr, "\mathcal{L}'"'] & & \mathbb{C} \\
& \mathbb{P} \arrow[ur] &
\end{tikzcd}$$

where $\mathbb{P}$ is a projective cover of $\mathbb{C}$ and $\mathbb{P} \longrightarrow \mathbb{C}$ is the inclusion; if there exists wlim$\mathcal{L}$, then there exists also wlim$\mathcal{L}'$. In particular if $\mathbb{C}$ is weakly lex, the same holds for $\mathbb{P}$.

Proof: Let $(\pi_D\colon L \longrightarrow \mathcal{L}D)_{D \in \mathcal{D}_0}$ be a weak limit of $\mathcal{L}$ and consider a $\mathbb{P}$-cover $p\colon P \longrightarrow L$ of L; then $(p \cdot \pi_D\colon P \longrightarrow \mathcal{L}D)_{D \in \mathcal{D}_0}$ is a weak limit of $\mathcal{L}'$: in fact, if $(\tau_D\colon Q \longrightarrow \mathcal{L}D)_{D \in \mathcal{D}_0}$ is a cone on $\mathcal{L}$ with $Q \in \mathbb{P}$, then the factorization $\tau\colon Q \longrightarrow L$ can be lifted to a factorization $\tau\colon Q \longrightarrow P$ because Q is projective. ■

Let us recall another definition.

Definition 1.1.7

1) *in a category $\mathbb{C}$, a pseudo relation on an object X is a pair of parallel arrows $r_1, r_2\colon R \rightrightarrows X$; the pseudo relation is a relation if r_1 and r_2 are jointly monic;*

2) *the pseudo relation* $r_1, r_2 \colon R \rightrightarrows X$ *is*

- *reflective if there exists an arrow* $r_R \colon X \longrightarrow R$ *such that* $r_R \cdot r_1 = 1_X = r_R \cdot r_2$
- *symmetric if there exists an arrow* $s_R \colon R \longrightarrow R$ *such that* $s_R \cdot r_1 = r_2$ *and* $s_R \cdot r_2 = r_1$

3) *if*

$$\begin{tikzcd} P \arrow[r, "l_1"] \arrow[d, "l_2"'] & R \arrow[d, "r_2"] \\ R \arrow[r, "r_1"'] & X \end{tikzcd}$$

is a weak pullback, the pseudo relation $r_1, r_2 \colon R \rightrightarrows X$ *is transitive if there exists an arrow* $t_R \colon P \longrightarrow R$ *such that the following diagram is commutative in each part*

$$\begin{tikzcd} R \arrow[d, "r_1"'] & P \arrow[l, "l_1"'] \arrow[r, "l_2"] \arrow[d, "t_R"] & R \arrow[d, "r_2"] \\ X & R \arrow[l, "r_1"] \arrow[r, "r_2"'] & X \end{tikzcd}$$

Let us remark that the transitivity of a pseudo relation $r_1, r_2 \colon R \rightrightarrows X$ does not depend on the choice of the weak pullback of r_1 and r_2; in fact, if

$$\begin{tikzcd} \overline{P} \arrow[r, "\overline{l_1}"] \arrow[d, "\overline{l_2}"'] & R \arrow[d, "r_2"] \\ R \arrow[r, "r_1"'] & X \end{tikzcd}$$

is another weak pullback, the factorization $\overline{P} \longrightarrow P$ composed with the transitivity $t_R \colon P \longrightarrow R$ ensures that the pseudo relation is transitive also with respect to the second weak pullback. In particular, this tells us that, if $\mathbb{C}$ has pullbacks and the pair r_1, r_2 is jointly monic, our definition coincides with the usual one given in terms of (strong) pullback and so with the one given in terms of hom-set (cf. vol. II, ch. 2 of [8]). Of course a pseudo equivalence-relation is a pseudo relation which is, at the same time, reflective, symmetric and transitive.

Definition 1.1.8 *A category* $\mathbb{A}$ *is regular if*

1) *it is left exact*

2) *each arrow can be factored as a regular epi followed by a monomorphism*

3) regular epis are pullback stable

Recall that, if conditions 1) and 3) of the previous definition are satisfied, then condition 2) is equivalent to

2') each kernel pair has a coequalizer.

Proposition 1.1.9 *Let $\mathbb{A}$ be a regular category and $\mathbb{P}$ a projective cover of $\mathbb{A}$; if $r_1, r_2 \colon R \rightrightarrows X$ is a pseudo equivalence-relation in $\mathbb{P}$ and*

$$\begin{tikzcd}
R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_1"'] \arrow[dr] & X \\
& \underline{R} \arrow[u, shift left, "i_1"] \arrow[u, shift right, "i_2"']
\end{tikzcd}$$

is its regular epi-jointly monic factorization in $\mathbb{A}$, then $i_1, i_2 \colon \underline{R} \rightrightarrows X$ is an equivalence relation in $\mathbb{A}$. ∎

This proposition, together with proposition 1.1.6, constitutes the basic fact to understand the notion of left covering functor and its properties. Nevertheless, we do not give here a proof for this proposition, because it is a very particular case of a more general theorem which will be proved in section 1.4. Let us only remark that in general $r_1, r_2 \colon R \rightrightarrows X$ is not transitive in the whole category $\mathbb{A}$. In fact, the transitivity of $r_1, r_2 \colon R \rightrightarrows X$ in $\mathbb{P}$ means that there exists a transitivity $t_R \colon P \longrightarrow R$ defined on a $\mathbb{P}$-cover P of the pullback of r_1 and r_2. Such an object P is a weak pullback of r_1 and r_2 in $\mathbb{P}$ but not necessarily in $\mathbb{A}$, while the transitivity of $r_1, r_2 \colon R \rightrightarrows X$ in $\mathbb{A}$ requires the existence of a transitivity morphism defined on a weak pullback of r_1 and r_2 in $\mathbb{A}$.

1.2 The exact completion

This section is completely devoted to the construction of an exact category from a weakly lex one.

Definition 1.2.1 *A category $\mathbb{A}$ is exact if*

1) it is regular

2) each equivalence relation is effective (that is it is a kernel pair).

Definition 1.2.2 *Let $\mathbb{C}$ be a weakly lex category; we define a new category $\mathbb{C}_{ex}$ as follows:*

- *objects: an object of $\mathbb{C}_{ex}$ is a pseudo equivalence-relation in $\mathbb{C}$*

$$r_1, r_2 \colon R \rightrightarrows X$$

- *arrows: an arrow between two objects*

$$r_1, r_2 \colon R \rightrightarrows X \quad \text{and} \quad s_1, s_2 \colon S \rightrightarrows Y$$

of $\mathbb{C}_{\text{ex}}$ *is an equivalence class of pairs of compatible arrows* $(\overline{f}, f)$ *as in the following diagram*

$$\begin{tikzcd} R \arrow[r, "\overline{f}"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where the pair $(\overline{f}, f)$ *is said to be compatible if* $\overline{f} \cdot s_1 = r_1 \cdot f$ *and* $\overline{f} \cdot s_2 = r_2 \cdot f$*; such two pairs* $(\overline{f}, f)$ *and* $(\overline{g}, g)$ *are considered to be equivalent if there exists an arrow (a "homotopy")* $\Sigma \colon X \longrightarrow S$ *such that* $\Sigma \cdot s_1 = f$ *and* $\Sigma \cdot s_2 = g$*.*

The previous definition needs some comments. First of all, it is easy to see that the relation between compatible pairs is effectively an equivalence relation (to check each condition, use the corresponding condition of the pseudo relation $s_1, s_2 \colon S \rightrightarrows Y$) and that it is compatible with the composition in $\mathbb{C}$; so $\mathbb{C}_{\text{ex}}$ is a category with respect to the composition between equivalence classes defined as the componentwise composition of the corresponding representatives and the identity of $r_1, r_2 \colon R \rightrightarrows X$ given by the class of the pair of identities $(1_R, 1_X)$.

Let us also observe that, if $(\overline{f}, f)$ and $(\tilde{f}, f)$ are two compatible pairs of arrows between the same objects of $\mathbb{C}_{\text{ex}}$, then they are always equivalent (use f followed by the reflectivity of the codomain as homotopy); so it makes sense to write $[f]$ for the equivalence class of $(\overline{f}, f)$.

(Nevertheless, by abuse of language, we will often say that a double-commutative diagram

$$\begin{tikzcd} R \arrow[r, "\overline{f}"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is an arrow in $\mathbb{C}_{\text{ex}}$, forgetting the passage to the quotient.)

Observe also that no size conditions are requested on $\mathbb{C}$ to construct $\mathbb{C}_{\text{ex}}$ and that $\mathbb{C}_{\text{ex}}$ is small if $\mathbb{C}$ is small.

Before starting with the verification that the category $\mathbb{C}_{\text{ex}}$ is exact, let us look more informally at the definition of an arrow in $\mathbb{C}_{\text{ex}}$. Think of the category $\mathbb{C}$ as an exact category in which every object is projective (for example $\mathcal{SET}$) and think of $r_1, r_2 \colon R \rightrightarrows X$ and $s_1, s_2 \colon S \rightrightarrows Y$ as two equivalence relations. Then an arrow between $r_1, r_2 \colon R \rightrightarrows X$ and $s_1, s_2 \colon S \rightrightarrows Y$ as in the definition of $\mathbb{C}_{\text{ex}}$ is exactly an arrow from the quotient of X by R to the quotient of Y by S (we leave to the interested reader the easy but instructive verification of this last assertion).

Theorem 1.2.3 *Let $\mathbb{C}$ be a weakly lex category and let $\mathbb{C}_{\mathrm{ex}}$ be as in definition 1.2.2; then $\mathbb{C}_{\mathrm{ex}}$ is an exact category.*

We give here only the constructions of finite limits and regular epi-mono factorization in $\mathbb{C}_{\mathrm{ex}}$. We refer to chapter 4 (sections 6 and 7) for two detailed proofs of the theorem.

Proof: Step 1: $\mathbb{C}_{\mathrm{ex}}$ is a left exact category. Let T be a weak terminal object in $\mathbb{C}$ and consider a weak product $T \xleftarrow{\pi_1} T\times T \xrightarrow{\pi_2} T$. Then $\pi_1, \pi_2 \colon T\times T \rightrightarrows T$ is the terminal object in $\mathbb{C}_{\mathrm{ex}}$.

Let us consider now the following pair of arrows in $\mathbb{C}_{\mathrm{ex}}$

$$\begin{tikzcd}
R \arrow[r, "\overline{f}"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] & T \arrow[l, "\overline{g}"'] \arrow[d, shift right, "t_1"'] \arrow[d, shift left, "t_2"] \\
X \arrow[r, "f"'] & Y & Z \arrow[l, "g"]
\end{tikzcd}$$

In order to build up its pullback, consider a weak limit $(P; \underline{f}, \varphi, \underline{g})$ as in the following diagram

$$\begin{tikzcd}
X \arrow[d, "f"'] & P \arrow[l, "\underline{f}"'] \arrow[r, "\underline{g}"] \arrow[d, "\varphi"] & Z \arrow[d, "g"] \\
Y & S \arrow[l, "s_1"] \arrow[r, "s_2"'] & Y
\end{tikzcd}$$

and a weak limit $(E; \rho, e_1, e_2, \tau)$ as in the following diagram

$$\begin{tikzcd}
 & & & E \arrow[dlll, "\rho"'] \arrow[dl, "e_1"] \arrow[dr, "e_2"'] \arrow[drrr, "\tau"] & & & \\
R \arrow[d, "r_1"'] \arrow[drr, "r_2"] & & P \arrow[dll, "\underline{f}"] \arrow[drr, "\underline{g}"] & & P \arrow[dll, "\underline{f}"] \arrow[drr, "\underline{g}"] & & T \arrow[dll, "t_1"] \arrow[d, "t_2"] \\
X & & X & & Z & & Z
\end{tikzcd}$$

The required pullback in $\mathbb{C}_{\mathrm{ex}}$ is given by

$$\begin{tikzcd}
R \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & E \arrow[l, "\rho"'] \arrow[r, "\tau"] \arrow[d, shift right, "e_1"'] \arrow[d, shift left, "e_2"] & T \arrow[d, shift right, "t_1"'] \arrow[d, shift left, "t_2"] \\
X & P \arrow[l, "\underline{f}"] \arrow[r, "\underline{g}"'] & Z
\end{tikzcd}$$

We also need an explicit description for equalizers in $\mathbb{C}_{\text{ex}}$; for this, consider two parallel arrows in $\mathbb{C}_{\text{ex}}$

$$\begin{tikzcd}
R \arrow[r, shift left, "\overline{f}"] \arrow[r, shift right, "\overline{g}"'] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd}$$

In order to build up their equalizer, consider a weak limit $(E; e, \varphi)$ as in the following diagram

$$\begin{tikzcd}
 & E \arrow[dl, "e"'] \arrow[dr, "\varphi"] & \\
X \arrow[d, "f"'] \arrow[drr, "g" near start] & & S \arrow[d, "s_2"] \arrow[dll, "s_1"' near start] \\
Y & & Y
\end{tikzcd}$$

Consider again a weak limit $(\underline{R}; e_1, \overline{e}, e_2)$ as in the following diagram

$$\begin{tikzcd}
E \arrow[d, "e"'] & \underline{R} \arrow[l, "e_1"'] \arrow[r, "e_2"] \arrow[d, "\overline{e}"] & E \arrow[d, "e"] \\
X & R \arrow[l, "r_1"] \arrow[r, "r_2"'] & X
\end{tikzcd}$$

Then the equalizer in $\mathbb{C}_{\text{ex}}$ is given by

$$\begin{tikzcd}
\underline{R} \arrow[r, "\overline{e}"] \arrow[d, shift right, "e_1"'] \arrow[d, shift left, "e_2"] & R \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] \\
E \arrow[r, "e"'] & X
\end{tikzcd}$$

Step 2: $\mathbb{C}_{\text{ex}}$ has regular epi-mono factorization and regular epis are stable under pullbacks.

Given an arrow in $\mathbb{C}_{\text{ex}}$

$$\begin{tikzcd}
R \arrow[r, "\overline{f}"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

consider a weak limit $(I; i_1, i, i_2)$ as in the following diagram

$$\begin{tikzcd}
X \arrow[d, "f"'] & I \arrow[l, "i_1"'] \arrow[r, "i_2"] \arrow[d, "i"] & X \arrow[d, "f"] \\
Y & S \arrow[l, "s_1"] \arrow[r, "s_2"'] & Y
\end{tikzcd}$$

Since

$$\begin{tikzcd}
& R \arrow[dl, "r_1"'] \arrow[d, "\overline{f}"] \arrow[dr, "r_2"] & \\
X & S & X
\end{tikzcd}$$

is a cone on the diagram defining I, there exists $t: R \longrightarrow I$ such that, in particular, $t \cdot i_1 = r_1$ and $t \cdot i_2 = r_2$. The required factorization is given by

$$\begin{tikzcd}
R \arrow[r, "t"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & I \arrow[r, "i"] \arrow[d, shift right, "i_1"'] \arrow[d, shift left, "i_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\
X \arrow[r, "1_X"'] & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

As far as the previous factorization is concerned, one can prove (and it will be done in section 6 of chapter 4) that in $\mathbb{C}_{\text{ex}}$ regular epimorphisms are, up to isomorphisms, exactly the arrows of the form

$$\begin{tikzcd}
R \arrow[r, "\overline{f}"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\
X \arrow[r, "1_X"'] & X
\end{tikzcd}$$

■

1.3 Projective objects in the exact completion

In this section we study the embedding of a weakly lex category $\mathbb{C}$ in its exact completion $\mathbb{C}_{\text{ex}}$ defined in the previous section. We deduce from this study that the category $\mathbb{C}_{\text{ex}}$ has enough projectives, which will be the condition characterizing a free exact category.

Proposition 1.3.1 *Let $\mathbb{C}$ be a weakly lex category and $\mathbb{C}_{\text{ex}}$ its exact completion as defined in 1.2.2; there exists a functor*

$$\Gamma \colon \mathbb{C} \longrightarrow \mathbb{C}_{\text{ex}}$$

defined by

$$X \xrightarrow{f} Y \quad \rightsquigarrow \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, shift right, "1_X"'] \arrow[d, shift left, "1_X"] & Y \arrow[d, shift right, "1_Y"'] \arrow[d, shift left, "1_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

which is full and faithful and preserves monomorphic families.

Proof: The functoriality of Γ is obvious.
Γ is full: consider an arrow in $\mathbb{C}_{\text{ex}}$

$$\begin{tikzcd} X \arrow[r, "\overline{f}"] \arrow[d, shift right, "1_X"'] \arrow[d, shift left, "1_X"] & Y \arrow[d, shift right, "1_Y"'] \arrow[d, shift left, "1_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

The compatibility condition immediately implies that $\overline{f} = f$ so that $[f] = \Gamma(f)$.
Γ is faithful: if $\Gamma(f) = \Gamma(g)$ then there exists a homotopy $\Sigma \colon X \longrightarrow Y$ such that $\Sigma \cdot 1_Y = f$ and $\Sigma \cdot 1_Y = g$ so that $f = g$.
Γ preserves monomorphic families: as for the faithfulness. ■

Proposition 1.3.2 *Let $\Gamma \colon \mathbb{C} \longrightarrow \mathbb{C}_{\text{ex}}$ be as in the previous proposition; the image $\Gamma(\mathbb{C})$ generates $\mathbb{C}_{\text{ex}}$ via coequalizers (that is, if*

$$\begin{tikzcd} R \arrow[r, "\overline{f}"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is an arrow in $\mathbb{C}_{\mathrm{ex}}$, then in the following diagram in $\mathbb{C}_{\mathrm{ex}}$ the two horizontal lines are coequalizers and the last vertical arrow is the unique extension to the quotient

$$\begin{tikzcd}
\Gamma R \arrow[r, shift left, "\Gamma r_1"] \arrow[r, shift right, "\Gamma r_2"'] \arrow[d, "\Gamma \overline{f}"'] & \Gamma X \arrow[r, "{[r_R, 1_X]}"] \arrow[d, "\Gamma f"] & (R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X) \arrow[d, "{[f]}"] \\
\Gamma S \arrow[r, shift left, "\Gamma s_1"] \arrow[r, shift right, "\Gamma s_2"'] & \Gamma Y \arrow[r, "{[r_S, 1_Y]}"'] & (S \arrow[r, shift left, "s_1"] \arrow[r, shift right, "s_2"'] & Y) &).
\end{tikzcd}$$

Proof: Let us write explicitely the first line

$$\begin{tikzcd}
R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] \arrow[d, shift right, "1_R"'] \arrow[d, shift left, "1_R"] & X \arrow[r, "r_R"] \arrow[d, shift right, "1_X"'] \arrow[d, shift left, "1_X"] & R \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] \\
R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X \arrow[r, "1_X"'] & X
\end{tikzcd}$$

($r_R\colon X \longrightarrow R$ is the arrow realizing the reflexivity of $r_1, r_2\colon R \rightrightarrows X$); using as homotopy the identity $1_R\colon R \longrightarrow R$, one has that the arrow on the right coequalizes the two parallel arrows on the left; suppose now that the same is true for an arrow

$$\begin{tikzcd}
X \arrow[r, "\overline{g}"] \arrow[d, shift right, "1_X"'] \arrow[d, shift left, "1_X"] & T \arrow[d, shift right, "t_1"'] \arrow[d, shift left, "t_2"] \\
X \arrow[r, "g"'] & Z
\end{tikzcd}$$

so that there exists a homotopy $\Sigma\colon R \longrightarrow T$ such that $\Sigma \cdot t_1 = r_1 \cdot g$ and $\Sigma \cdot t_2 = r_2 \cdot g$; one can choose then as factorization the arrow

$$\begin{tikzcd}
R \arrow[r, "\Sigma"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & T \arrow[d, shift right, "t_1"'] \arrow[d, shift left, "t_2"] \\
X \arrow[r, "g"'] & Z
\end{tikzcd}$$

in fact the resulting diagram in $\mathbb{C}_{\mathrm{ex}}$ is obviously commutative; moreover, the factorization is necessarily unique because

$$\begin{tikzcd}
X \arrow[r, "r_R"] \arrow[d, shift right, "1_X"'] \arrow[d, shift left, "1_X"] & R \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] \\
X \arrow[r, "1_X"'] & X
\end{tikzcd}$$

is a regular epi (cf. Step 2 of theorem 1.2.3). In order to conclude our proof, it is enough to observe that the following square in $\mathbb{C}_{\text{ex}}$ is commutative

$$\begin{tikzcd}
\Gamma X \arrow[r, "{[r_R, 1_X]}"] \arrow[d, "\Gamma f"'] & (R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X) \arrow[d, "{[f]}"] \\
\Gamma Y \arrow[r, "{[r_S, 1_Y]}"'] & (S \arrow[r, shift left, "s_1"] \arrow[r, shift right, "s_2"'] & Y)
\end{tikzcd}$$

■

Proposition 1.3.3 *Consider the functor $\Gamma\colon \mathbb{C} \longrightarrow \mathbb{C}_{\text{ex}}$ defined in 1.3.1 and an object $Y \in \mathbb{C}$; ΓY is a projective object in $\mathbb{C}_{\text{ex}}$.*

Proof: Consider an arrow in $\mathbb{C}_{\text{ex}}$

$$\begin{tikzcd}
R \arrow[r, "\overline{f}"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & Y \arrow[d, shift right, "1_Y"'] \arrow[d, shift left, "1_Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

and its regular epi-mono factorization as in 1.2.3

$$\begin{tikzcd}
R \arrow[r] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & I \arrow[r] \arrow[d, shift right, "\overline{s_1}"'] \arrow[d, shift left, "\overline{s_2}"] & Y \arrow[d, shift right, "1_Y"'] \arrow[d, shift left, "1_Y"] \\
X \arrow[r, "1_X"'] & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

If the given arrow is a regular epi, then the monic part of its factorization has a left-inverse, that is there exists an arrow

$$\begin{tikzcd}
Y \arrow[r, "\overline{l}"] \arrow[d, shift right, "1_Y"'] \arrow[d, shift left, "1_Y"] & I \arrow[d, shift right, "\overline{s_1}"'] \arrow[d, shift left, "\overline{s_2}"] \\
Y \arrow[r, "l"'] & X
\end{tikzcd}$$

together with a homotopy $\Sigma \colon Y \longrightarrow Y$ such that $\Sigma \cdot 1_Y = l \cdot f$ and $\Sigma \cdot 1_Y = 1_Y$ so that, as a section of the given arrow, one can choose

$$\begin{tikzcd}
Y \arrow[r, "l \cdot r_R"] \arrow[d, shift right, "1_Y"'] \arrow[d, shift left, "1_Y"] & R \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] \\
Y \arrow[r, "l"'] & X
\end{tikzcd}$$

■

Corollary 1.3.4 *The image $\Gamma(\mathbb{C})$ of the functor $\Gamma \colon \mathbb{C} \longrightarrow \mathbb{C}_{ex}$ is a projective cover of $\mathbb{C}_{ex}$, so that $\mathbb{C}_{ex}$ has enough projectives.* ■

Let us remark that in general the functor $\Gamma \colon \mathbb{C} \longrightarrow \mathbb{C}_{ex}$ does not preserve weak limits of $\mathbb{C}$ (we will give an explicit counterexample in section 1.6); nevertheless, we can observe some useful facts.

Proposition 1.3.5 *Consider the functor $\Gamma \colon \mathbb{C} \longrightarrow \mathbb{C}_{ex}$*

1) Γ preserves all the strong finite limits which turn out to exist in $\mathbb{C}$

2) the corestriction of Γ to the full subcategory of projective objects of $\mathbb{C}_{ex}$ preserves the weak finite limits of $\mathbb{C}$.

Proof: Part 1) is a particular case of a more general property which will be proved in the next section; part 2) is routine using the characterization of projective objects in $\mathbb{C}_{ex}$ given in the following lemma.

Lemma 1.3.6 *An object $r_1, r_2 \colon R \rightrightarrows X$ of $\mathbb{C}_{ex}$ is projective if and only if it is contractible, that is if there exists an arrow $\Sigma \colon X \longrightarrow R$ such that $\Sigma \cdot r_1 = 1_X$ and $r_1 \cdot \Sigma \cdot r_2 = r_2 \cdot \Sigma \cdot r_2$.*

Proof: (if) : use $\Sigma \colon X \longrightarrow R$ as a homotopy to show that the two following arrows

$$\begin{tikzcd}
R \arrow[r, "r_1 \cdot \Sigma \cdot r_2"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & X \arrow[r, "r_R"] \arrow[d, shift right, "1_X"'] \arrow[d, shift left, "1_X"] & R \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] \\
X \arrow[r, "\Sigma \cdot r_2"'] & X \arrow[r, "1_X"'] & X
\end{tikzcd}$$

exhibit $r_1, r_2 \colon R \rightrightarrows X$ as a retract of $1_X, 1_X \colon X \rightrightarrows X$;
(only if) : if $r_1, r_2 \colon R \rightrightarrows X$ is projective, then there exists a section of the regular epi

$$\begin{tikzcd} X \arrow[r, "r_R"] \arrow[d, shift right, "1_X"'] \arrow[d, shift left, "1_X"] & R \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] \\ X \arrow[r, "1_X"'] & X \end{tikzcd}$$

that is an arrow

$$\begin{tikzcd} R \arrow[r, "\overline{s}"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & X \arrow[d, shift right, "1_X"'] \arrow[d, shift left, "1_X"] \\ X \arrow[r, "s"'] & X \end{tikzcd}$$

together with a homotopy $\Sigma\colon X \longrightarrow R$ such that $\Sigma \cdot r_1 = 1_X$ and $\Sigma \cdot r_2 = s$; from the second equation and the compatibility condition, one has $r_1 \cdot \Sigma \cdot r_2 = r_1 \cdot s = \overline{s} = r_2 \cdot s = r_2 \cdot \Sigma \cdot r_2$. ■

1.4 Left covering functors

Keeping in mind the proof of Proposition 1.1.6, we are ready to give our basic definition.

Definition 1.4.1 *Consider a functor $F\colon \mathbb{C} \longrightarrow \mathbb{A}$ with $\mathbb{C}$ weakly lex and $\mathbb{A}$ left exact; we say that F is left covering if, for all functors $\mathcal{L}\colon \mathcal{D} \longrightarrow \mathbb{C}$ defined on a finite category $\mathcal{D}$ and for all weak limits*

$$wlim\mathcal{L} = (\pi_D\colon L \longrightarrow \mathcal{L}D)_{D \in \mathcal{D}_0},$$

the canonical factorization $p\colon FL \longrightarrow \tilde{L}$ is a strong epimorphism. Here p is the unique arrow such that

$$\begin{tikzcd} FL \arrow[rr, "p"] \arrow[dr, "F\pi_D"'] & & \tilde{L} \arrow[dl, "\tilde{\pi}_D"] \\ & F(\mathcal{L}D) & \end{tikzcd}$$

is commutative for all $D \in \mathcal{D}_0$, where

$$(\tilde{\pi}_D\colon \tilde{L} \longrightarrow F(\mathcal{L}D))_{D \in \mathcal{D}_0} = lim\mathcal{L} \cdot F$$

As a first remark, observe that in the previous definition the second "for all" can be equivalently replaced by a "for one". In fact, if

$$(\pi_D\colon L \longrightarrow \mathcal{L}D)_{D \in \mathcal{D}_0} \quad \text{and} \quad (\pi_D'\colon L' \longrightarrow \mathcal{L}D)_{D \in \mathcal{D}_0}$$

are weak limits of $\mathcal{L}: \mathcal{D} \longrightarrow \mathbb{C}$, then there exists a factorization $t: L \longrightarrow L'$; now, if the factorization $p: FL \longrightarrow \tilde{L}$ is a strong epimorphism, even the factorization $p': FL' \longrightarrow \tilde{L}$ must be a strong epimorphism because the following diagram is commutative

$$\begin{tikzcd} FL \arrow[rr, "Ft"] \arrow[dr, "p"'] & & FL' \arrow[dl, "p'"] \\ & \tilde{L} & \end{tikzcd}$$

(to check this commutativity, we can compose with the $\tilde{\pi}_D$'s, because $\tilde{L}$ is the strong limit).

Proposition 1.4.2 *Let $F: \mathbb{C} \longrightarrow \mathbb{A}$ be a functor with $\mathbb{C}$ weakly lex and $\mathbb{A}$ a left exact category with strong epi-mono factorization; the following conditions are equivalent (notations of 1.4.1)*

1) *F is left covering with respect to a given finite diagram $\mathcal{L}: \mathcal{D} \longrightarrow \mathbb{C}$ in $\mathbb{C}$*

2) *the jointly monic part of the strong epi-jointly monic factorization of $(F\pi_D: FL \longrightarrow F(\mathcal{L}D))_{D \in \mathcal{D}_0}$ is the limit of $\mathcal{L} \cdot F$*

3) *there exists a weak limit $(\pi_D: L \longrightarrow \mathcal{L}D)_{D \in \mathcal{D}_0}$ of $\mathcal{L}$, a weak limit $(\hat{\pi}_D: \hat{L} \longrightarrow F(\mathcal{L}D))_{D \in \mathcal{D}_0}$ of $\mathcal{L} \cdot F$ and a strong epi $q: F(L) \longrightarrow \hat{L}$ such that $q \cdot \hat{\pi}_D = F(\pi_D)$ for each D in $\mathcal{D}_0$.*

Proof: The equivalence between 1) and 2) and the implication 1) $\Rightarrow$ 3) are obvious.

It remains to prove that 3) implies 1): if $(\tilde{\pi}_D: \tilde{L} \longrightarrow F(\mathcal{L}D))_{D \in \mathcal{D}_0}$ is the limit of $\mathcal{L} \cdot F$, then weak limits of $\mathcal{L} \cdot F$ are exactly coretracts of $\tilde{L}$ (and retraction and section commute with projections). As factorization $p: F(L) \longrightarrow \tilde{L}$ we can choose $F(L) \longrightarrow \hat{L} \longrightarrow \tilde{L}$; by assumption, the first component is a strong epi and the second one is a strong epi because it has a section. ∎

The interest of the previous proposition lies in the fact that condition 3) can be formally written when the codomain $\mathbb{A}$ is only weakly lex. Unfortunately, this condition is not stable by composition; an explicit counterexample will be given at the end of section 1.6. The stability of the notion of left covering functor will be discussed in propositions 1.4.7 and 1.4.8.

Let us look now at an important particular case; we say that a functor $F: \mathbb{C} \longrightarrow \mathbb{A}$ defined on a weakly lex category $\mathbb{C}$ is weakly lex if for each functor $\mathcal{L}: \mathcal{D} \longrightarrow \mathbb{C}$, with $\mathcal{D}$ finite, and for each (eqivalently, for one) weak limit $(\pi_D: L \longrightarrow \mathcal{L}D)_{D \in \mathcal{D}_0}$ of $\mathcal{L}$, one has that $(F(\pi_D): F(L) \longrightarrow F(\mathcal{L}D))_{D \in \mathcal{D}_0}$ is a weak limit of $\mathcal{L} \cdot F$.

Proposition 1.4.3 *Consider a functor $F: \mathbb{C} \longrightarrow \mathbb{A}$ with $\mathbb{C}$ weakly lex and $\mathbb{A}$ left exact; consider also the following conditions*

1) *F is left covering*

2) F is weakly lex

3) F is left exact.

One has that 2) implies 1); moreover, if $\mathbb{C}$ is left exact, the three conditions are equivalent.

Proof: To prove that 2) implies 1) and, if $\mathbb{C}$ is left exact, that 3) implies 2), it suffices to use once again the fact that, if there exists the strong limit, then weak limits are exactly the coretracts of the strong one. Now, assume $\mathbb{C}$ left exact and F left covering; let us start showing that F preserves the terminal object T of $\mathbb{C}$: by hypothesis, the unique arrow $q\colon FT \longrightarrow \tilde{T}$ (where $\tilde{T}$ is the terminal object of $\mathbb{A}$) is a strong epimorphism; in $\mathbb{C}$, one has that $T \xleftarrow{1_T} T \xrightarrow{1_T} T$ is the product of T with itself, so that the unique factorization $FT \longrightarrow FT \times FT$ is a (strong) epimorphism and then the two projections $FT \xleftarrow{\pi_1} FT \times FT \xrightarrow{\pi_2} FT$ are equal. But the pair π_1, π_2 is the kernel pair of q, so that q is a mono and then an iso.

To show that F preserves not empty finite limits, we need a lemma.

Lemma 1.4.4 *Let $F\colon \mathbb{C} \longrightarrow \mathbb{A}$ be a left covering functor; F preserves finite monomorphic families.*

Proof: (up to some minor modifications, it is the same argument used in 1.829 of [22])
A family of arrows $(f_i\colon A \longrightarrow A_i)_{i\in I}$ is monomorphic if and only if the following diagram is a limit

$$
\begin{tikzcd}
 & A \arrow[dl, "1_A"'] \arrow[d, "f_i"] \arrow[dr, "1_A"] & \\
A \arrow[r, "f_i"'] & A_i & A \arrow[l, "f_i"]
\end{tikzcd}
$$

(when the family is reduced to one single arrow $f\colon A \longrightarrow B$, one has the familiar argument that f is a mono if and only if

$$
\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "1_A"'] & A \arrow[d, "f"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

is a pullback); now apply F and consider the factorization $q\colon FA \longrightarrow M$ (where

$$\begin{tikzcd} & M \arrow[dl, "x"'] \arrow[d, "a_i"] \arrow[dr, "y"] & \\ FA \arrow[r, "Ff_i"'] & FA_i & FA \arrow[l, "Ff_i"] \end{tikzcd}$$

is the limit in $\mathbb{A}$); by hypothesis q is a strong epimorphism, but, by the commutativity of

$$\begin{tikzcd} FA \arrow[rr, "q"] \arrow[dr, "1_{FA}"'] & & M \arrow[dl, "x"] \\ & FA & \end{tikzcd}$$

q is also a monomorphism and so it is an isomorphism. This means that

$$\begin{tikzcd} & FA \arrow[dl, "1_{FA}"'] \arrow[d, "Ff_i"] \arrow[dr, "1_{FA}"] & \\ FA \arrow[r, "Ff_i"'] & FA_i & FA \arrow[l, "Ff_i"] \end{tikzcd}$$

is a limit and then the family $(Ff_i: FA \longrightarrow FA_i)_{i \in I}$ is monomorphic. ■

Let us come back to the proof of proposition 1.4.3: $F: \mathbb{C} \longrightarrow \mathbb{A}$ is a left covering functor between left exact categories and we have to prove that F preserves not empty finite limits. Consider a functor $\mathcal{L}: \mathcal{D} \longrightarrow \mathbb{C}$ defined on a not empty finite category $\mathcal{D}$; consider also $\lim \mathcal{L} = (\pi_D: L \longrightarrow \mathcal{L}D)_{D \in \mathcal{D}_0}$ and $\lim \mathcal{L} \cdot F = (\tilde{\pi}_D: \tilde{L} \longrightarrow F(\mathcal{L}D))_{D \in \mathcal{D}_0}$: the family $(\pi_D: L \longrightarrow \mathcal{L}D)_{D \in \mathcal{D}_0}$ is monomorphic so that also the family $(F\pi_D: FL \longrightarrow F(\mathcal{L}D))_{D \in \mathcal{D}_0}$ is monomorphic by 1.4.4; for all $D \in \mathcal{D}_0$, the following diagram is commutative

$$\begin{tikzcd} FL \arrow[rr, "p"] \arrow[dr, "F\pi_D"'] & & \tilde{L} \arrow[dl, "\tilde{\pi}_D"] \\ & F(\mathcal{L}D) & \end{tikzcd}$$

so that the unique factorization $p: FL \longrightarrow \tilde{L}$ is a monomorphism; but, by hypothesis, it is a strong epimorphism, so it is an isomorphism. ■

Let us remark that in the proof of proposition 1.4.3 we have established a more general fact: a left covering functor from a weakly lex category to a left exact one preserves all the finite limits which turn out to exist in the domain (i.e. being left covering is a kind of flatness).

In the next proposition we complete the comparison between left covering functors and weakly lex functors. We call strong projective an object which is projective with respect to strong epimorphisms.

Proposition 1.4.5 *Consider a functor $F: \mathbb{C} \longrightarrow \mathbb{A}$ with $\mathbb{C}$ weakly lex and $\mathbb{A}$ left exact; suppose that F factors through the full subcategory $P(\mathbb{A})$ of strong projective objects of $\mathbb{A}$ and call $F': \mathbb{C} \longrightarrow P(\mathbb{A})$ its corestriction;*

1) *if F is left covering, then F' is weakly lex*

2) *if $\mathbb{A}$ has enough strong projectives and F' is weakly lex, then F is left covering.*

Proof: 1) Let $\mathcal{D}$ be a finite category and consider

$$\mathcal{D} \xrightarrow{\mathcal{L}} \mathbb{C} \xrightarrow{F} \mathbb{A}$$

$\text{wlim}\mathcal{L} = (\pi_D: L \longrightarrow \mathcal{L}D)_{D \in \mathcal{D}_0}$ and $\lim \mathcal{L} \cdot F = (\tilde{\pi}_D: \tilde{L} \longrightarrow F(\mathcal{L}D))_{D \in \mathcal{D}_0}$; the factorization $p: FL \longrightarrow \tilde{L}$ is a strong epimorphism in $\mathbb{A}$, so that if P is the vertex of a cone over $\mathcal{L} \cdot F$ and P is in $P(\mathbb{A})$, then the unique factorization $P \longrightarrow \tilde{L}$ can be lifted to a factorization $P \longrightarrow FL$.
2) if $t: P \longrightarrow \tilde{L}$ is a strong epimorphism in $\mathbb{A}$ with $P \in P(\mathbb{A})$, then there exists a factorization $t': P \longrightarrow FL$ such that the following diagram is commutative for all $D \in \mathcal{D}_0$

$$\begin{tikzcd} FL \arrow[d, "F(\pi_D)"'] & P \arrow[l, "t'"'] \arrow[d, "t"] \\ F(\mathcal{L}D) & \tilde{L} \arrow[l, "\tilde{\pi}_D"] \end{tikzcd}$$

but the family $(\tilde{\pi}_D: \tilde{L} \longrightarrow F(\mathcal{L}D))_{D \in \mathcal{D}_0}$ is monomorphic, so that the following diagram is commutative

$$\begin{tikzcd} FL \arrow[dr, "p"'] & & P \arrow[ll, "t'"'] \arrow[dl, "t"] \\ & \tilde{L} & \end{tikzcd}$$

This implies that $p: FL \longrightarrow \tilde{L}$ is a strong epimorphism in $\mathbb{A}$. ∎

Corollary 1.4.6 *Let $F:\mathbb{C} \longrightarrow \mathbb{A}$ be a functor with $\mathbb{C}$ weakly lex; consider the three following conditions:*

1) *F is flat*

2) *F is weakly lex*

3) *F is left covering*

The two first conditions are equivalent; moreover, if the axiom of choice holds in $\mathbb{A}$ (that is each object is strong projective) and $\mathbb{A}$ is left exact, then the three conditions are equivalent.

Proof: the equivalence between 2) and 3) immediately follows from the previous proposition. Now let us make clear that in the present situation we call F flat if, for each functor $\mathcal{L}:\mathcal{D} \longrightarrow \mathbb{C}$ with $\mathcal{D}$ finite and for each cone $(\tilde{\pi}_D: \tilde{L} \longrightarrow F(\mathcal{L}D))_{D \in \mathcal{D}_0}$ over $\mathcal{L} \cdot F$ in $\mathbb{A}$, there exist a cone $(\pi_D: L \longrightarrow \mathcal{L}D)_{D \in \mathcal{D}_0}$ over $\mathcal{L}$ in $\mathbb{C}$ and a factorization $t: \tilde{L} \longrightarrow F(L)$. (Observe that this condition does not require the existence of weak limits in $\mathbb{C}$. It means that, for each $A \in \mathbb{A}$, the comma category (A, F) is filtering (cf. [6]).) To show that 2) implies 1), we can take as L a weak limit of $\mathcal{L}$ so that, by assumption, $F(L)$ is a weak limit of $\mathcal{L} \cdot F$ and then there exists a factorization from $\tilde{L}$ to $F(L)$. Conversely, by assumption, there exist a cone $(\pi'_D: L' \longrightarrow \mathcal{L}D)_{D \in \mathcal{D}_0}$ in $\mathbb{C}$ and a factorization $t': \tilde{L} \longrightarrow F(L')$. But, if L is a weak limit of $\mathcal{L}$, there exist a factorization $h: L' \longrightarrow L$. Now $t' \cdot Fh$ gives us a factorization from $\tilde{L}$ to $F(L)$, that is $F(L)$ is a weak limit of $\mathcal{L} \cdot F$. ∎

In the two next propositions, we establish the stability of the notion of left covering functor; we leave the easy proofs to the reader.

Proposition 1.4.7 *Let $\mathbb{B} \xrightarrow{G} \mathbb{C} \xrightarrow{F} \mathbb{A}$ be two functors with $\mathbb{B}$ and $\mathbb{C}$ weakly lex and $\mathbb{A}$ left exact; if G is weakly lex and F is left covering, then the composition $G \cdot F: \mathbb{B} \longrightarrow \mathbb{A}$ is left covering.* ∎

Proposition 1.4.8 *Let $\mathbb{C} \xrightarrow{F} \mathbb{A} \xrightarrow{G} \mathbb{B}$ be two functors with $\mathbb{C}$ weakly lex and $\mathbb{A}$ and $\mathbb{B}$ left exact; if F is left covering, G is left exact and moreover G sends strong epimorphisms into strong epimorphisms, then the composition $F \cdot G: \mathbb{C} \longrightarrow \mathbb{B}$ is left covering.* ∎

Let us recall that in a regular category, the strong epimorphisms coincide with the regular epimorphisms. Now we are ready to state the most important property of left covering functors.

Theorem 1.4.9 *Consider a left covering functor $F: \mathbb{C} \longrightarrow \mathbb{A}$ with $\mathbb{C}$ a weakly lex category and $\mathbb{A}$ a regular one; let $r_1, r_2: R \rightrightarrows X$ be a pseudo equivalence-relation in $\mathbb{C}$ and consider the regular epi-jointly monic factorization of its image by F*

$$\begin{tikzcd}
FR \arrow[r, shift left, "Fr_1"] \arrow[r, shift right, "Fr_2"'] \arrow[dr, "p"'] & FX \\
 & \underline{R} \arrow[u, shift left, "i_1"] \arrow[u, shift right, "i_2"']
\end{tikzcd}$$

Then $i_1, i_2 \colon \underline{R} \rightrightarrows FX$ *is an equivalence relation in* $\mathbb{A}$.

Observe that this theorem is the announced generalization of proposition 1.1.9; in fact the full inclusion of a projective cover $\mathbb{P}$ in a regular category $\mathbb{A}$ is obviously a left covering functor (cf. the proof of proposition 1.1.6). In particular, the embedding $\Gamma \colon \mathbb{C} \longrightarrow \mathbb{C}_{\mathrm{ex}}$ described in 1.3.1 is a left covering functor (cf. corollary 1.3.4).

Proof: Let us write the previous diagram in the following form

$$\begin{tikzcd}
FR \arrow[rr, "\langle Fr_1, Fr_2 \rangle"] \arrow[dr, "p"'] & & FX \times FX \\
 & \underline{R} \arrow[ur, "\langle i_1, i_2 \rangle"'] &
\end{tikzcd}$$

The reflexivity and the symmetry of $\langle i_1, i_2 \rangle$ are easy to be shown; in fact, they are equational conditions, so that they are preserved by each functor. If $r_R \colon X \longrightarrow R$ is the reflexivity of $r_1, r_2 \colon R \rightrightarrows X$, then $Fr_R \colon FX \longrightarrow FR$ is the reflexivity of $Fr_1, Fr_2 \colon FR \rightrightarrows FX$ and so $Fr_R \cdot p \colon FX \longrightarrow \underline{R}$ is the reflexivity of $i_1, i_2 \colon \underline{R} \rightrightarrows FX$. If $s_R \colon R \longrightarrow R$ is the symmetry of $r_1, r_2 \colon R \rightrightarrows X$, then $Fs_R \colon FR \longrightarrow FR$ makes commutative the following diagram

$$\begin{tikzcd}
FR \arrow[r, "p"] \arrow[d, "Fs_R"'] & \underline{R} \arrow[r, "\langle i_1, i_2 \rangle"] & FX \times FX \arrow[d, "\tau"] \\
FR \arrow[r, "p"'] & \underline{R} \arrow[r, "\langle i_1, i_2 \rangle"'] & FX \times FX
\end{tikzcd}$$

where τ is the twisting isomorphism. So, by the naturality of the regular epi-mono factorization, there exists an arrow $\underline{R} \longrightarrow \underline{R}$ making commutative the right-hand part of the previous diagram.

For the transitivity, consider a weak pullback

$$\begin{tikzcd} P \arrow[r, "l_1"] \arrow[d, "l_2"'] & R \arrow[d, "r_2"] \\ R \arrow[r, "r_1"'] & X \end{tikzcd}$$

and the transitivity morphism $t_R \colon P \longrightarrow R$ (that is $t_R \cdot r_1 = l_1 \cdot r_1$ and $t_R \cdot r_2 = l_2 \cdot r_2$).

Consider now the following diagram in $\mathbb{A}$, in which both squares are pullbacks and p is a regular epi; by associativity of pullbacks, v is a regular epi.

$$\begin{tikzcd} S \arrow[rr, "j_1"] \arrow[dr, "v"] \arrow[dd, "j_2"'] & & FR \arrow[d, "p"] \\ & Q \arrow[r, "d_1"] \arrow[d, "d_2"'] & \underline{R} \arrow[d, "i_2"] \\ FR \arrow[r, "p"'] & \underline{R} \arrow[r, "i_1"'] & FX \end{tikzcd}$$

Using now the fact that the functor $F \colon \mathbb{C} \longrightarrow \mathbb{A}$ is left covering, we have that the factorization $q \colon FP \longrightarrow S$ such that $q \cdot j_1 = Fl_1$ and $q \cdot j_2 = Fl_2$ is a regular epi.

A diagram chasing show now that the following diagram is commutative

$$\begin{tikzcd} FP \arrow[r, "q \cdot v"] \arrow[d, "Ft_R"'] & Q \arrow[r, "\langle d_1 \cdot i_1, d_2 \cdot i_2 \rangle"] & FX \times FX \arrow[d, "1_{FX \times FX}"] \\ FR \arrow[r, "p"'] & \underline{R} \arrow[r, "\langle i_1, i_2 \rangle"'] & FX \times FX \end{tikzcd}$$

But $q \cdot v$ is a regular epi and $\langle i_1, i_2 \rangle$ is a mono, so that there exists an arrow $Q \longrightarrow \underline{R}$ making commutative the right-hand part of the previous diagram. This shows that $i_1, i_2 \colon \underline{R} \rightrightarrows FX$ is transitive. ■

The next step is crucial to make handly the notion of left covering functor.

Proposition 1.4.10 *Consider a functor $F \colon \mathbb{C} \longrightarrow \mathbb{A}$ with $\mathbb{C}$ weakly lex and $\mathbb{A}$ regular; if F is left covering with respect to binary products, equalizers of pairs of parallel arrows and terminal object, then it is left covering.*

We need a lemma

Lemma 1.4.11 *Let $\mathbb{A}$ be a regular category;*

1) *in the following commutative diagram, if f_1 and f_2 are regular epis, then the unique factorization $f_1 \times f_2$ is a regular epi*

$$\begin{tikzcd}
A_1 \arrow[d, "f_1"'] & A_1 \times A_2 \arrow[l, "\pi_{A_1}"'] \arrow[r, "\pi_{A_2}"] \arrow[d, "f_1 \times f_2"] & A_2 \arrow[d, "f_2"] \\
B_1 & B_1 \times B_2 \arrow[l, "\pi_{B_1}"] \arrow[r, "\pi_{B_2}"'] & B_2
\end{tikzcd}$$

2) *in the following commutative diagram, where the two horizontal lines are equalizers, if f_1 is a regular epi and f_2 a mono, then the unique factorization f is a regular epi*

$$\begin{tikzcd}
E \arrow[r, "e"] \arrow[d, "f"'] & A_1 \arrow[r, shift left, "h"] \arrow[r, shift right, "g"'] \arrow[d, "f_1"'] & A_2 \arrow[d, "f_2"] \\
L \arrow[r, "l"'] & B_1 \arrow[r, shift left, "m"] \arrow[r, shift right, "n"'] & B_2
\end{tikzcd}$$

3) *if the following diagram is commutative, f_1 and f_2 are regular epis and f is a mono, then the unique factorization from the pullback of a_1 and a_2 to the pullback of b_1 and b_2 is a regular epi*

$$\begin{tikzcd}
A_1 \arrow[r, "a_1"] \arrow[d, "f_1"'] & A \arrow[d, "f"'] & A_2 \arrow[l, "a_2"'] \arrow[d, "f_2"] \\
B_1 \arrow[r, "b_1"'] & B & B_2 \arrow[l, "b_2"]
\end{tikzcd}$$

Proof: 1) $f_1 \times f_2$ can be decomposed in the following way

$$A_1 \times A_2 \xrightarrow{f_1 \times 1_{A_2}} B_1 \times A_2 \xrightarrow{1_{B_1} \times f_2} B_1 \times B_2$$

but (in every category) the following diagram is a pullback

$$\begin{tikzcd}
A_1 \times A_2 \arrow[r, "\pi_{A_1}"] \arrow[d, "f_1 \times 1_{A_2}"'] & A_1 \arrow[d, "f_1"] \\
B_1 \times A_2 \arrow[r, "\pi_{B_1}"'] & B_1
\end{tikzcd}$$

so that $f_1 \times 1_{A_2}$ is a regular epi if f_1 is a regular epi; in the same way $1_{B_1} \times f_2$ is a regular epi and so also $f_1 \times f_2$ is a regular epi
2) if f_2 is a mono and l is the equalizer of m and n, then (in every category) the pullback of l along f_1 is the equalizer of h and g; this means that f is the pullback of f_1 along l and then f is a regular epi
3) using the usual construction of pullbacks via products and equalizers, this point follows from the two previous ones. ■

We leave to the reader the straightforward generalization of the previous lemma from the case of binary products, equalizers and pullbacks to the n-ary case.

We come back now to proposition 1.4.10; we divide its proof into three steps.

Step 1: consider a functor $F\colon \mathbb{C} \longrightarrow \mathbb{A}$ as in proposition 1.4.10; it is left covering with respect to n-ary products and n-ary equalizers.

Proof: We limit ourselves to the case n = 3;
products: consider three objects A, B and C in $\mathbb{C}$; $(A \times B) \times C$ is a weak product of A, B and C so that the unique factorization $F((A \times B) \times C) \longrightarrow FA \times FB \times FC$ can be decomposed as follows

$$p \cdot (q \times 1_{FC}) : \ F((A \times B) \times C) \longrightarrow F(A \times B) \times FC \longrightarrow (FA \times FB) \times FC$$

where p and q are the obvious factorizations. By assumption, p and q are regular epis, so that (cf. Lemma 1.4.11) also $q \times 1_{FC}$ and $p \cdot (q \times 1_{FC})$ are regular epis.
equalizers: consider three parallel arrows in $\mathbb{C}$

$$a\colon A \longrightarrow B \ , \ b\colon A \longrightarrow B \ , \ c\colon A \longrightarrow B$$

and two weak equalizers

$$E \xrightarrow{e} A \overset{b}{\underset{c}{\rightrightarrows}} B \qquad E' \xrightarrow{e'} E \overset{e \cdot a}{\underset{e \cdot b}{\rightrightarrows}} B$$

so that $e' \cdot e\colon E' \longrightarrow E$ is a weak equalizer of a, b and c.

Consider the equalizer in $\mathbb{A}$

$$L \xrightarrow{l} FA \overset{Fb}{\underset{Fc}{\rightrightarrows}} FB$$

by assumption the unique arrow $p\colon FE \longrightarrow L$ such that $p \cdot l = Fe$ is a regular epi; considering the following diagram (where the two horizontal lines are equalizers)

$$\begin{tikzcd}
S \arrow[r, "s"] \arrow[d, "q"'] & FE \arrow[r, shift left, "F(e \cdot a)"] \arrow[r, shift right, "F(e \cdot b)"'] \arrow[d, "p"'] & FB \arrow[d, "1_{FB}"] \\
L' \arrow[r, "l'"'] & L \arrow[r, shift left, "l \cdot Fa"] \arrow[r, shift right, "l \cdot Fb"'] & FB
\end{tikzcd}$$

by lemma 1.4.11, the unique factorization $q: S \longrightarrow L'$ is a regular epi; once again by assumption the unique arrow $t: FE' \longrightarrow S$ such that $t \cdot s = Fe'$ is a regular epi. But $l' \cdot l: L' \longrightarrow FA$ is the equalizer of Fa, Fb and Fc; the required factorization $FE' \longrightarrow L'$ is given by $t \cdot q$ (in fact $t \cdot q \cdot l' \cdot l = t \cdot s \cdot p \cdot l = Fe' \cdot Fe = F(e' \cdot e)$) and so it is a regular epi.

Step 2: consider a functor $F: \mathbb{C} \longrightarrow \mathbb{A}$ as in proposition 1.4.10; it is left covering with respect to n-ary pullbacks.

Proof: once again, we limit ourselves to the case n = 3. Given three convergent arrows in $\mathbb{C}$

$$a: A \longrightarrow D\ ,\ b: B \longrightarrow D\ ,\ c: C \longrightarrow D\ ,$$

consider a weak product

$$\begin{tikzcd} & A \times B \times C \arrow[dl, "\pi_A"'] \arrow[d, "\pi_B"] \arrow[dr, "\pi_C"] & \\ A & B & C \end{tikzcd}$$

and a weak equalizer $e: E \longrightarrow A \times B \times C$ of $\pi_A \cdot a, \pi_B \cdot b, \pi_C \cdot c$. A weak pullback of a, b and c is then

$$\begin{tikzcd} & A \arrow[dr, "a"] & \\ E \arrow[ur, "e \cdot \pi_A"] \arrow[r, "e \cdot \pi_B"] \arrow[dr, "e \cdot \pi_C"'] & B \arrow[r, "b"] & D \\ & C \arrow[ur, "c"'] & \end{tikzcd}$$

Consider now the following diagram in $\mathbb{A}$ where $s: S \longrightarrow F(A \times B \times C)$ is the equalizer of $F(\pi_A \cdot a)$, $F(\pi_B \cdot b)$, $F(\pi_C \cdot c)$ and $l: L \longrightarrow FA \times FB \times FC$ is the equalizer of $\pi_{FA} \cdot Fa$, $\pi_{FB} \cdot Fb$, $\pi_{FC} \cdot c$

$$\begin{tikzcd} S \arrow[r, "s"] \arrow[d, "q"'] & F(A \times B \times C) \arrow[d, "p"] \\ L \arrow[r, "l"'] & FA \times FB \times FC \end{tikzcd}$$

By step 1 p is a regular epi, so that by Lemma 1.4.11 q is a regular epi; again by step 1 the unique arrow $t: FE \longrightarrow S$ such that $t \cdot s = Fe$ is a regular epi. But the bottom equalizer is the pullback in $\mathbb{A}$ of Fa, Fb and Fc; so the desired factorization $FE \longrightarrow L$ is given by the regular epi $t \cdot q$ (in fact $t \cdot q \cdot l \cdot \pi_{FA} = t \cdot s \cdot p \cdot \pi_{FA} = Fe \cdot F\pi_A = F(e \cdot \pi_A)$).

Step 3: consider a functor $F: \mathbb{C} \longrightarrow \mathbb{A}$ as in proposition 1.4.10; it is left covering.

Proof: Consider a functor $\mathcal{L}: \mathcal{D} \longrightarrow \mathbb{C}$ defined on a finite category $\mathcal{D}$. Let us recall the construction of a weak limit of $\mathcal{L}$ proposed in proposition 1.1.2.

Consider a weak product

$$\pi_D: \prod_{D \in \mathcal{D}_0} \mathcal{L}D \longrightarrow \mathcal{L}D$$

Then, for each arrow $d: D \longrightarrow D'$ in $\mathcal{D}$, consider a weak equalizer

$$E_d \xrightarrow{e_d} \prod_{D \in \mathcal{D}_0} \mathcal{L}D \underset{\pi_{D'}}{\overset{\pi_D \cdot \mathcal{L}d}{\rightrightarrows}} \mathcal{L}D'$$

Consider finally a weak pullback $(e_d': E \longrightarrow E_d)_{d \in \mathcal{D}_1}$ of the finite family $(e_d: E_d \longrightarrow \prod_{D \in \mathcal{D}_0} \mathcal{L}D)_{d \in \mathcal{D}_1}$ of convergent arrows. Then

$$\mathrm{wlim}\mathcal{L} = (e_d' \cdot e_d \cdot \pi_D: E \longrightarrow \mathcal{L}D)_{D \in \mathcal{D}_0}.$$

Performing at each step the corresponding strong limit in $\mathbb{A}$, one can construct in the same way

$$\lim\mathcal{L} \cdot F = (l_d' \cdot l_d \cdot \tilde{\pi}_D: L \longrightarrow F(\mathcal{L}D))_{D \in \mathcal{D}_0}$$

Observe that for each $d \in \mathcal{D}_1$, l_d and l_d' are monomorphisms.

We have to prove that the unique arrow $\tau: FE \longrightarrow L$ such that, for all $D \in \mathcal{D}_0$, $\tau \cdot l_d' \cdot l_d \cdot \tilde{\pi}_D = F(e_d' \cdot e_d \cdot \pi_D)$ is a regular epi.

Let us call $l_d' \cdot l_d = l: L \longrightarrow \prod_{D \in \mathcal{D}_0} F(\mathcal{L}D)$ and observe that this mono is, by construction, the limit on the diagram

$$\prod_{D \in \mathcal{D}_0} F(\mathcal{L}D) \underset{\tilde{\pi}_{D'}}{\overset{\tilde{\pi}_D \cdot F(\mathcal{L}d)}{\rightrightarrows}} F(\mathcal{L}D')$$

(think of a pair of parallel arrows for each $d \in \mathcal{D}_1$).

Consider now the following pullback

$$\begin{tikzcd}
P \arrow[r, "l'"] \arrow[d, "p'"'] & F(\prod_{D \in \mathcal{D}_0} \mathcal{L}D) \arrow[d, "p"] \\
L \arrow[r, "l"'] & \prod_{D \in \mathcal{D}_0} F(\mathcal{L}D)
\end{tikzcd}$$

where p is the unique arrow such that $p \cdot \tilde{\pi}_D = F(\pi_D)$ for each $D \in \mathcal{D}_0$.

An easy diagram chasing shows that $l' \colon P \longrightarrow F(\prod_{D \in \mathcal{D}_0} \mathcal{L}D)$ is the limit on the diagram

$$F(\textstyle\prod_{D \in \mathcal{D}_0} \mathcal{L}D) \underset{F(\pi_{D'})}{\overset{F(\pi_D) \cdot F(\mathcal{L}d)}{\rightrightarrows}} F(\mathcal{L}D')$$

(think of a pair of parallel arrows for each $d \in \mathcal{D}_1$). But this limit can be performed also in two steps, that is first taking, for each $d \in \mathcal{D}_1$, the equalizer

$$S_d \xrightarrow{s_d} F(\textstyle\prod_{D \in \mathcal{D}_0} \mathcal{L}D) \underset{F(\pi_{D'})}{\overset{F(\pi_D) \cdot F(\mathcal{L}d)}{\rightrightarrows}} F(\mathcal{L}D')$$

and then taking the pullback $(s_d^{'} \colon P \longrightarrow S_d)_{d \in \mathcal{D}_1}$ of the family of convergent arrows $(s_d \colon S_d \longrightarrow F(\prod_{D \in \mathcal{D}_0} \mathcal{L}D))_{d \in \mathcal{D}_1}$ (in particular, one has $s_d^{'} \cdot s_d = l'$). Observe that, by assumption, the unique arrow $q_d \colon FE_d \longrightarrow S_d$ such that $q_d \cdot s_d = Fe_d$ is a regular epi; if one consider now the pullback $(j_d \colon Q \longrightarrow FE_d)_{d \in \mathcal{D}_1}$ of the family of convergent arrows

$$(Fe_d \colon FE_d \longrightarrow F(\prod_{D \in \mathcal{D}_0} \mathcal{L}D))_{d \in \mathcal{D}_1}$$

by lemma 1.4.11 it follows that the unique arrow $q \colon Q \longrightarrow P$ such that $q \cdot s_d^{'} = j_d \cdot q_d$ for each $d \in \mathcal{D}_1$ is a regular epi. Moreover, the unique arrow $t \colon FE \longrightarrow Q$ such that $t \cdot j_d = Fe_d^{'}$ for each $d \in \mathcal{D}_1$ is, by step 2, a regular epi.

Now we can consider the composition

$$FE \xrightarrow{t} Q \xrightarrow{q} P \xrightarrow{p'} L$$

where each factor is a regular epi (p' is a regular epi because it is the pullback of p which is a regular epi by step 1). To finish, it remains to prove that the regular epi $t \cdot q \cdot p'$ is the required factorization; in fact

$$\begin{aligned} t \cdot q \cdot p' \cdot l_d^{'} \cdot l_d \cdot \tilde{\pi}_D &= t \cdot q \cdot p' \cdot l \cdot \tilde{\pi}_D \\ &= t \cdot q \cdot l' \cdot p \cdot \tilde{\pi}_D \\ &= t \cdot q \cdot l' \cdot F(\pi_D), \end{aligned}$$

On the other side

$$\begin{aligned} F(e_d^{'} \cdot e_d \cdot \pi_D) &= Fe_d^{'} \cdot Fe_d \cdot F\pi_D \\ &= t \cdot j_d \cdot Fe_d \cdot F\pi_D \\ &= t \cdot j_d \cdot q_d \cdot s_d \cdot F\pi_D \\ &= t \cdot q \cdot s_d^{'} \cdot s_d \cdot F\pi_D \\ &= t \cdot q \cdot l' \cdot F\pi_D. \end{aligned}$$

∎

Let us make a last remark about the notion of left covering functor. Consider a left exact functor $F\colon \mathbb{C} \longrightarrow \mathbb{A}$ and a finite diagram $\mathcal{L}\colon \mathcal{D} \longrightarrow \mathbb{C}$ in $\mathbb{C}$. The strong epimorphism $p\colon F(\lim\mathcal{L}) \longrightarrow \lim\mathcal{L} \cdot F$ (which, of course, in this case is an isomorphism) can be seen as the quotient in $\mathbb{A}$ of an equivalence relation on $F(\lim\mathcal{L})$ which is the image by F of an equivalence relation on $\lim\mathcal{L}$ (take, for example, the diagonal on $\lim\mathcal{L}$).

This idea can be recovered also for left covering functors (at least when the canonical factorization $F(\lim\mathcal{L}) \longrightarrow \lim\mathcal{L} \cdot F$ is a regular epi).

For this, consider a pair of arrows $x_1, x_2\overline{X} \rightrightarrows \mathrm{wlim}\mathcal{L}$ weakly universal with respect to the conditions $x_1 \cdot \pi_D = x_2 \cdot \pi_D$ for each D (the π_D's are the projections of $\mathrm{wlim}\mathcal{L}$).

Such a pair is a pseudo equivalence-relation (this will be proved in section 3.2).

Now consider the family of arrows $(F(\pi_D\colon F(\mathrm{wlim}\mathcal{L}) \longrightarrow F(\mathcal{L}D))_{D \in \mathcal{D}_0}$ and its factorization $\langle F(\pi_D)\rangle\colon F(\mathrm{wlim}\mathcal{L}) \longrightarrow \pi_{\mathcal{D}_0} F(\mathcal{L}D)$.

Its kernel pair $k_1, k_2\colon K \rightrightarrows F(\mathrm{wlim}\mathcal{L})$ is universal with respect to the conditions $k_1 \cdot F(\pi_D) = k_2 \cdot F(\pi_D)$ for each $D \in \mathcal{D}_0$. By assumption on F, there exists a regular epi $q\colon F(\overline{X}) \longrightarrow K$ such that $q \cdot k_1 = F(x_1)$ and $q \cdot k_2 = F(x_2)$. Consider now the canonical factorization $p\colon F(\mathrm{wlim}\mathcal{L}) \longrightarrow \lim\mathcal{L} \cdot F$, so that $p \cdot \tilde{\pi}_D = F(\pi_D)$ for each $D \in \mathcal{D}_0$ (the $\tilde{\pi}_D$'s are the projections of $\lim\mathcal{L}F$). Since (k_1, k_2) is the kernel pair of $\langle F(\pi_D)\rangle$ and the $\tilde{\pi}_D$'s are a monomorphic family, (k_1, k_2) is also the kernel pair of p. Since p is, by assumption, a regular epimorphism, it is the coequalizer of k_1 and k_2 and then of $(F(x_1)$ and $F(x_2)$ because q is an epimorphism

$$
\begin{tikzcd}
F(\overline{X}) \arrow[r, shift left, "F(x_1)"] \arrow[r, shift right, "F(x_2)"'] \arrow[dr, "q"] & F(\mathrm{wlim}\mathcal{L}) \arrow[r, "p"] \arrow[dr, "F(\pi_D)"] & \lim\mathcal{L} \cdot F \arrow[d, "\tilde{\pi}_D"] \\
 & K \arrow[u, shift left, "k_1"] \arrow[u, shift right, "k_2"'] & F(\mathcal{L}D)
\end{tikzcd}
$$

1.5 The universality of the exact completion

We devote this section to show that the embedding $\Gamma\colon \mathbb{C} \longrightarrow \mathbb{C}_{\mathrm{ex}}$ of a weakly lex category in its exact completion is universal. A more economical, but not self-contained, proof is given in chapter 4 (section 7).

Definition 1.5.1 *A functor between two regular categories is exact when it is left exact and preserves regular epis.*

Theorem 1.5.2 *Let $\mathbb{C}$ be a weakly lex category and $\mathbb{A}$ an exact one; consider the exact category $\mathbb{C}_{\mathrm{ex}}$ and the functor $\Gamma\colon \mathbb{C} \longrightarrow \mathbb{C}_{\mathrm{ex}}$ described in 1.2.2 and*

1.3.1. Composing with Γ induces an equivalence

$$\Gamma_{ex}\colon Ex(\mathbb{C}_{ex}, \mathbb{A}) \longrightarrow Lco(\mathbb{C}, \mathbb{A})$$

between the category of exact functors from $\mathbb{C}_{ex}$ to $\mathbb{A}$ and the category of left covering functors from $\mathbb{C}$ to $\mathbb{A}$.

From the fact that the embedding $\Gamma\colon \mathbb{C} \longrightarrow \mathbb{C}_{\text{ex}}$ is itself a left covering functor (and using the stability stated in 1.4.8), one has immediately the following corollary

Corollary 1.5.3 *Let $\mathbb{C}$ be a weakly lex category; the exact completion $\Gamma\colon \mathbb{C} \longrightarrow \mathbb{C}_{ex}$ is uniquely determined (up to equivalences) by the previous universal property.* ■

Proof: Proof of 1.5.2: for the sake of brevity, we write $N(f)$ for the kernel pair of a morphism f.

The more difficult part consists in proving that, given a left covering functor $F\colon \mathbb{C} \longrightarrow \mathbb{A}$, there exists a unique (up to natural isomorphisms) exact functor $\hat{F}\colon \mathbb{C}_{\text{ex}} \longrightarrow \mathbb{A}$ making commutative the following diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[rr, "\Gamma"] \arrow[dr, "F"'] & & \mathbb{C}_{\text{ex}} \arrow[dl, "\hat{F}"] \\
& \mathbb{A} &
\end{tikzcd}$$

Let us start with the (essential) uniqueness of $\hat{F}$ under the hypothesis that it is exact and makes the previous diagram commutative.

By proposition 1.3.2, we can see an object $r_1, r_2\colon R \rightrightarrows X$ of $\mathbb{C}_{\text{ex}}$ as the object part of a coequalizer

$$\begin{tikzcd}
\Gamma R \arrow[r, shift left, "\Gamma r_1"] \arrow[r, shift right, "\Gamma r_2"'] \arrow[dr, "p"'] & \Gamma X \arrow[r, "q"] & (R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X) \\
& N(q) \arrow[u, shift left, "i_1"] \arrow[u, shift right, "i_2"'] & &
\end{tikzcd}$$

where the triangle is the regular epi-jointly monic factorization of the pair Γr_1, Γr_2.

As p is an epimorphism, q is also the coequalizer of i_1, i_2; but Γ is a left covering functor, so that, by 1.4.9, (i_1, i_2) is an equivalence relation in the exact category $\mathbb{C}_{\text{ex}}$ and then it is the kernel pair of its coequalizer q.

Now $\hat{F}$ sends regular epis into regular epis, so that $\hat{F}q$ must be a coequalizer; in particular, $\hat{F}q$ is the coequalizer of its kernel pair which is, by left exactness of $\hat{F}$, $(\hat{F}i_1, \hat{F}i_2)$. Once again, p is regular epi and so is $\hat{F}p$; but then $\hat{F}q$, which is the coequalizer of $(\hat{F}i_1, \hat{F}i_2)$, is also the coequalizer of $(\hat{F}p \cdot \hat{F}i_1, \hat{F}p \cdot \hat{F}i_2)$, that is of $(\hat{F}(\Gamma r_1), \hat{F}(\Gamma r_2))$. Now, taking into account that $\Gamma \cdot \hat{F} \simeq F$, one has that the following is a coequalizer in $\mathbb{A}$

$$\begin{tikzcd} FR \arrow[r, shift left, "Fr_1"] \arrow[r, shift right, "Fr_2"'] & FX \arrow[r, "\hat{F}q"] & \hat{F}(R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X) \end{tikzcd}$$

This proves that $\hat{F}$ is completely determined on the object of $\mathbb{C}_{\text{ex}}$; the uniqueness on the arrows follows in the same way using again proposition 1.3.2.

Consider now an arrow in $\mathbb{C}_{\text{ex}}$

$$\begin{tikzcd} R \arrow[r, "\overline{f}"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

The previous discussion says that $\hat{F}$ must be defined by the following diagram in $\mathbb{A}$

$$\begin{tikzcd} FR \arrow[r, shift left, "Fr_1"] \arrow[r, shift right, "Fr_2"'] \arrow[d, "F\overline{f}"'] & FX \arrow[r, "\rho"] \arrow[d, "Ff"] & \hat{F}(R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] \arrow[d, "\hat{F}[f]"] & X) \\ FS \arrow[r, shift left, "Fs_1"] \arrow[r, shift right, "Fs_2"'] & FY \arrow[r, "\sigma"'] & \hat{F}(S \arrow[r, shift left, "s_1"] \arrow[r, shift right, "s_2"'] & Y) \end{tikzcd}$$

where the two horizontal lines are coequalizers and the last vertical arrow is the unique extension to the quotients.

Once again, it is theorem 1.4.9 which makes possible this definition; in fact it shows that the jointly monic part of the factorizations of (Fr_1, Fr_2) and (Fs_1, Fs_2) are equivalence relations in $\mathbb{A}$ exact, so that the coequalizers involved in the previous diagram exist.

It is obvious to verify the functoriality of $\hat{F}$ and the commutativity $\Gamma \cdot \hat{F} \simeq F$. It remains to show that $\hat{F}$ is exact. $\hat{F}$ preserves regular epis: with no loss of generality (see step 2 in the proof of theorem 1.2.3) we can consider a regular epi of the form

$$\begin{tikzcd} R \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & I \arrow[d, shift right] \arrow[d, shift left] \\ X \arrow[r, "1_X"'] & X \end{tikzcd}$$

Its image by $\hat{F}$ makes commutative the following square

$$\begin{tikzcd} FX \arrow[r, "\rho"] \arrow[d, "1_{FX}"'] & \hat{F}(R \rightrightarrows X) \arrow[d] \\ FX \arrow[r, "\sigma"'] & \hat{F}(I \rightrightarrows X) \end{tikzcd}$$

and so it is a regular epi.

We divide the proof of the left exactness of $\hat{F}$ into three steps.

Step 1: $\hat{F}$ is left covering with respect to the terminal object.

Recall that, if T is a weak terminal object in $\mathbb{C}$, the terminal object of $\mathbb{C}_{\text{ex}}$ is $\pi_1, \pi_2 \colon T \times T \rightrightarrows T$ (cf. step 1 of 1.2.3). Its image by $\hat{F}$ is then the coequalizer of $F\pi_1, F\pi_2 \colon F(T \times T) \rightrightarrows FT$; F is left covering (with respect to the terminal object), so that the unique arrow from FT to the terminal object of $\mathbb{A}$ is a regular epi and so also the unique arrow from

$$\hat{F}(T \times T \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} T)$$

to the terminal object of $\mathbb{A}$ must be a regular epi.

Step 2: $\hat{F}$ is left covering with respect to binary products.

Consider two objects $r_1, r_2 \colon R \rightrightarrows X$ and $t_1, t_2 \colon T \rightrightarrows Z$ in $\mathbb{C}_{\text{ex}}$ and their product with the corresponding projections

$$\begin{tikzcd} R \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & E \arrow[l] \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & T \arrow[d, shift right, "t_1"'] \arrow[d, shift left, "t_2"] \\ X & P \arrow[l, "x"] \arrow[r, "z"'] & Z \end{tikzcd}$$

To construct this product one could follow the construction given in step 1 of 1.2.3 (replacing, as usual, $s_1, s_2 \colon S \rightrightarrows Y$ by the terminal object of $\mathbb{C}_{\text{ex}}$); but it suffices to perform the first weak limit to realize that the resulting diagram $X \xleftarrow{x} P \xrightarrow{z} Z$ is a weak product in $\mathbb{C}$. We can apply the functor $\hat{F}$ and we obtain the following regular epis in $\mathbb{A}$

$$q_1 \colon FX \longrightarrow \hat{F}(R \rightrightarrows X)$$

$$q_2 \colon FZ \longrightarrow \hat{F}(T \rightrightarrows Z)$$

$$q_3 \colon FP \longrightarrow \hat{F}((R \rightrightarrows X) \times (T \rightrightarrows Z))$$

Consider now the following commutative diagram, where λ and μ are the canonical factorizations

$$\begin{tikzcd}
FP \arrow[r, "\lambda"] \arrow[d, "q_3"'] & FX \times FZ \arrow[d, "q_1 \times q_2"] \\
\hat{F}((R \rightrightarrows X) \times (T \rightrightarrows Z)) \arrow[r, "\mu"'] & \hat{F}(R \rightrightarrows X) \times \hat{F}(T \rightrightarrows Z))
\end{tikzcd}$$

By lemma 1.4.11, $q_1 \times q_2$ is a regular epi; λ is a regular epi because F is left covering (with respect to binary products). Then also μ is a regular epi, as required.

Step 3: $\hat{F}$ is left covering with respect to equalizers.

Consider a pair of parallel arrows in $\mathbb{C}_{\text{ex}}$ and their equalizer as in step 1 of 1.2.3

$$\begin{tikzcd}
\underline{R} \arrow[r, "\overline{e}"] \arrow[d, shift right, "e_1"'] \arrow[d, shift left, "e_2"] & R \arrow[r, shift left, "\overline{f}"] \arrow[r, shift right, "\overline{g}"'] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\
E \arrow[r, "e"'] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd}$$

Applying $\hat{F}$ we obtain

$$\begin{tikzcd}
F\underline{R} \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & FR \arrow[r, shift left, "F\overline{f}"] \arrow[r, shift right, "F\overline{g}"'] \arrow[d, shift right, "Fr_1"'] \arrow[d, shift left, "Fr_2"] & FS \arrow[d, shift right, "Fs_1"'] \arrow[d, shift left, "Fs_2"] \arrow[dr, "p_1"] & \\
FE \arrow[r, "Fe"] \arrow[d, "q_1"'] & FX \arrow[r, shift left, "Ff"] \arrow[r, shift right, "Fg"'] \arrow[d, "q_2"] & FY \arrow[d, "q_3"] & N(q_3) \arrow[l, shift right, "n_1"'] \arrow[l, shift left, "n_2"] \\
\hat{F}(R \rightrightarrows E) \arrow[r, "\hat{F}[e]"] \arrow[d, "t"'] & \hat{F}(R \rightrightarrows X) \arrow[r, shift left, "\hat{F}[f]"] \arrow[r, shift right, "\hat{F}[g]"'] & \hat{F}(S \rightrightarrows Y) & \\
L \arrow[ur, "h"'] & & &
\end{tikzcd}$$

where the triangle on the right is the regular epi-jointly monic factorization of (Fs_1, Fs_2) and the triangle at the bottom is the equalizer in $\mathbb{A}$ of $\hat{F}[f]$ and $\hat{F}[g]$ with the corresponding factorization $t\colon \hat{F}(R \rightrightarrows E) \longrightarrow L$; we need to prove that t is a regular epi.

Consider the following pullback

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, "i"'] & N(q_3) \arrow[d, "\langle n_1, n_2 \rangle"] \\ FX \arrow[r, "\langle Ff, Fg \rangle"'] & FY \times FY \end{tikzcd}$$

As, in $\mathbb{C}_{\mathrm{ex}}$, $[e]$ equalizes $[f]$ and $[g]$, there exists a homotopy $\Sigma\colon E \longrightarrow S$ such that $\Sigma \cdot s_1 = e \cdot f$ and $\Sigma \cdot s_2 = e \cdot g$; this implies that $F\Sigma \cdot p_1 \cdot n_1 = Fe \cdot Ff$ and $F\Sigma \cdot p_1 \cdot n_2 = Fe \cdot Fg$, so that there exists an arrow $\sigma\colon FE \longrightarrow A$ such that $\sigma \cdot i = Fe$ and $\sigma \cdot k = F\Sigma \cdot p_1$. Let us assume that σ is a regular epi (it will be proved later).

Consider again the following diagram

$$\begin{tikzcd} FE \arrow[r, "\sigma"] \arrow[d, "q_1"'] & A \arrow[d, "i"] \\ \hat{F}(R \rightrightarrows E) \arrow[d, "t"'] & FX \arrow[d, "q_2"] \\ L \arrow[r, "h"'] & \hat{F}(R \rightrightarrows X) \end{tikzcd}$$

It is commutative, so there exists $\tau\colon A \longrightarrow L$ making commutative the two resulting triangles.

We want now to show that τ is a regular epi (this immediately implies that also t is a regular epi, as required). To this end, let us show that the following diagram is a pullback.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\tau"'] & FX \arrow[d, "q_2"] \\ L \arrow[r, "h"'] & \hat{F}(R \rightrightarrows X) \end{tikzcd}$$

Suppose there exist two arrows $L \xleftarrow{x} M \xrightarrow{y} FX$ such that $x \cdot h = y \cdot q_2$; this implies $y \cdot Ff \cdot q_3 = y \cdot Fg \cdot q_3$ (in fact $y \cdot Ff \cdot q_3 = y \cdot q_2 \cdot \hat{F}[f] = x \cdot h \cdot \hat{F}[f] = x \cdot h \cdot \hat{F}[g] = y \cdot q_2 \cdot \hat{F}[g] = y \cdot Fg \cdot q_3$). Now we use for the first time that $\mathbb{A}$ is exact (and not only regular): this means that (n_1, n_2) is the kernel pair of q_3, so that the previous equation implies that there exists $\varphi\colon M \longrightarrow N(q_3)$ such that $\varphi \cdot n_1 = y \cdot Ff$ and $\varphi \cdot n_2 = y \cdot Fg$. Now we have two arrows $FX \xleftarrow{y} M \xrightarrow{\varphi} N(q_3)$ and the last two equations imply that there exists $z\colon M \longrightarrow A$ such that $z \cdot k = \varphi$ and $s \cdot i = y$. To prove that z is the required factorization, it remains to observe that $z \cdot \tau \cdot h = z \cdot i \cdot q_2 = y \cdot q_2 = x \cdot h$ and then $z \cdot \tau = x$ because h is a mono. The factorization z is obviously unique because i is mono.

It remains to prove that $\sigma\colon FE \longrightarrow A$ is a regular epi. First of all, consider the following pullback

$$
\begin{tikzcd}
\underline{A} \arrow[r, "\underline{k}"] \arrow[d, "\underline{i}"'] & FS \arrow[d, "\langle Fs_1, Fs_2 \rangle"] \\
FX \arrow[r, "\langle Ff, Fg \rangle"'] & FY \times FY
\end{tikzcd}
$$

so that the unique arrow $m\colon \underline{A} \longrightarrow A$ such that $m \cdot i = \underline{i}$ and $m \cdot k = \underline{k} \cdot p_1$ is a regular epi.

Now observe that the previous pullback is the limit as in the following diagram

$$
\begin{tikzcd}
& \underline{A} \arrow[dl, "\underline{i}"'] \arrow[dr, "\underline{k}"] & \\
FX \arrow[d, "Ff"'] \arrow[drr, "Fg" near start] & & FS \arrow[d, "Fs_2"] \arrow[dll, "Fs_1"' near start] \\
FY & & FY
\end{tikzcd}
$$

(the verification of this fact is an easy argument of diagram chasing which holds in every left exact category). Coming back to the construction of equalizers in $\mathbb{C}_{\mathrm{ex}}$, we get a factorization $\alpha\colon FE \longrightarrow \underline{A}$ such that $\alpha \cdot \underline{i} = Fe$ and $\alpha \cdot \underline{k} = F\varphi$. Since F is left covering, this factorization is a regular epimorphism.

It remains to prove that the regular epimorphism

$$FE \xrightarrow{\alpha} \underline{A} \xrightarrow{m} A$$

coincides with σ. Since i is a monomorphism, it suffices to show that $\sigma \cdot i = \alpha \cdot m \cdot i$. This is the case because $\sigma \cdot i = Fe$ and $\alpha \cdot m \cdot i = \alpha \cdot \underline{i} = Fe$.

The proof of step 3 is now complete; putting together the three steps, we can deduce by proposition 1.4.10 that the functor $\hat{F}\colon \mathbb{C}_{\mathrm{ex}} \longrightarrow \mathbb{A}$ is left covering. Since $\mathbb{C}_{\mathrm{ex}}$ is, in particular, left exact, proposition 1.4.3 allows us to conclude that $\hat{F}$ is left exact.

To end the proof of the theorem, we need to show that natural transformations between two left covering functors F and G are in bijection with natural transformations between $\hat{F}$ and $\hat{G}$. But this is a corollary of the next proposition. ■

Proposition 1.5.4 *Let $\Gamma\colon \mathbb{C} \longrightarrow \mathbb{C}_{ex}$ be as in 1.3.1 and consider a left covering functor $F\colon \mathbb{C} \longrightarrow \mathbb{A}$ with $\mathbb{A}$ exact; then the unique exact extension $\hat{F}\colon \mathbb{C}_{ex} \longrightarrow \mathbb{A}$ described in theorem 1.5.2 is the left Kan-extension of F along Γ.*

Proof: We show that the set of natural transformations $\mathrm{Nat}(F, \Gamma \cdot H)$ is in bijection with $\mathrm{Nat}(\hat{F}, H)$, where H is an arbitrary functor $\mathbb{C}_{\mathrm{ex}} \longrightarrow \mathbb{A}$. Consider a natural transformation $\beta\colon F \longrightarrow \Gamma \cdot H$ and an object $r_1, r_2\colon R \rightrightarrows X$ in $\mathbb{C}_{\mathrm{ex}}$; by definition of $\hat{F}$, we have that in the following diagram the upper horizontal line is a coequalizer

$$\begin{tikzcd}
FR \arrow[r, shift left, "Fr_1"] \arrow[r, shift right, "Fr_2"'] \arrow[d, "\beta_R"'] & FX \arrow[r, "\hat{F}(p)"] \arrow[d, "\beta_X"] & \hat{F}(R \rightrightarrows X) \\
H(\Gamma R) \arrow[r, shift left, "H(\Gamma r_1)"] \arrow[r, shift right, "H(\Gamma r_2)"'] & H(\Gamma X) \arrow[r, "H(p)"'] & H(R \rightrightarrows X)
\end{tikzcd}$$

(where $p\colon \Gamma X \longrightarrow (R \rightrightarrows X)$ is as in proposition 1.3.2).

By naturality of $\beta\colon F \longrightarrow \Gamma \cdot H$, the left-hand square is two-time commutative, so that there exists exactly one arrow $\hat{F}(R \rightrightarrows X) \longrightarrow H(R \rightrightarrows X)$ making commutative the right-hand part.

We take this arrow as component at the point $r_1, r_2\colon R \rightrightarrows X$ of a natural transformation $\hat{F} \longrightarrow H$. The rest of the proof is straightforward. ■

Corollary 1.5.5 *Let $\Gamma\colon \mathbb{C} \longrightarrow \mathbb{C}_{ex}$ be as in 1.3.1; for each exact category $\mathbb{A}$, composition with Γ gives us an equivalence between the category of exact functors $Ex(\mathbb{C}_{ex}, \mathbb{A})$ and the category of left covering functors $Lco(\mathbb{C}, \mathbb{A})$.*

Proof: The functor induced by Γ is essentially surjective on the objects by theorem 1.5.2; as far as natural trasformations are concerned, we need to prove that, given two left covering functors F,G from $\mathbb{C}$ to $\mathbb{A}$, there is a bijection $\mathrm{Nat}(F, G) \simeq \mathrm{Nat}(\hat{F}, \hat{G})$. For this, put $H = \hat{G}$ (the unique exact extension of G) in the previous proposition. ■

1.6 Characterization of free exact categories

In this section we complete our plan, giving a characterization of free exact categories over weakly lex ones.

We know from section 1.3 that the exact completion $\mathbb{C}_{\mathrm{ex}}$ has enough projectives; conversely, if an exact category has enough projectives, it is the exact completion of the full subcategory of projective objects.

More exactly, we have the following theorem

Theorem 1.6.1 *Let $\mathbb{A}$ be an exact category and $\mathbb{P}$ a projective cover of $\mathbb{A}$; the unique exact extension $\hat{F}\colon \mathbb{P}_{\mathrm{ex}} \longrightarrow \mathbb{A}$ of the inclusion $F\colon \mathbb{P} \longrightarrow \mathbb{A}$ is an equivalence.*

Proof: Observe that the statement makes sense because by 1.1.6 $\mathbb{P}$ is weakly lex and, obviously, the inclusion $\mathbb{P} \longrightarrow \mathbb{A}$ is left covering.

Let us recall that, given an arrow in $\mathbb{P}_{\mathrm{ex}}$

$$\begin{tikzcd} R \arrow[r, "\overline{f}"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

its image by $\hat{F}$ is given by the unique extension in the following diagram (where the two horizontal lines are coequalizers which exist by 1.1.9)

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] \arrow[d, "\overline{f}"'] & X \arrow[r, "a"] \arrow[d, "f"] & A \arrow[d, "\varphi"] \\ S \arrow[r, shift left, "s_1"] \arrow[r, shift right, "s_2"'] & Y \arrow[r, "b"'] & B \end{tikzcd}$$

First, we show that $\hat{F}$ is essentially surjective on the objects: let A be an object of $\mathbb{A}$; we can find an object $X \in \mathbb{P}$ and a regular epi $a\colon X \longrightarrow A$; consider now its kernel pair in $\mathbb{A}$

$$\begin{tikzcd} N \arrow[r, "a_2"] \arrow[d, "a_1"'] & X \arrow[d, "a"] \\ X \arrow[r, "a"'] & A \end{tikzcd}$$

and take again an object $R \in \mathbb{P}$ and a regular epi $r\colon R \longrightarrow N$; call $r \cdot a_1 = r_1$ and $r \cdot a_2 = r_2$. Obviously

$$R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X \xrightarrow{a} A$$

is a coequalizer diagram in $\mathbb{A}$ (because a is a regular epi, so that it is the coequalizer of its kernel pair) and we omit the straightforward verification that $r_1, r_2\colon R \rightrightarrows X$ is a pseudo equivalence-relation in $\mathbb{P}$ (not in $\mathbb{A}$).

Second, $\hat{F}$ is faithful: consider an arrow

$$[\overline{f}, f]\colon (r_1, r_2\colon R \rightrightarrows X) \longrightarrow (s_1, s_2\colon S \rightrightarrows Y)$$

in $\mathbb{P}_{\text{ex}}$ and its image by $\hat{F}$ as at the beginning of the proof; consider also a second arrow in $\mathbb{P}_{\text{ex}}$ and its image by $\hat{F}$

$$\begin{tikzcd}
R \arrow[r, "\overline{g}"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\
X \arrow[r, "g"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] \arrow[d, "\overline{g}"'] & X \arrow[r, "a"] \arrow[d, "g"] & A \arrow[d, "\psi"] \\
S \arrow[r, shift left, "s_1"] \arrow[r, shift right, "s_2"'] & Y \arrow[r, "b"'] & B
\end{tikzcd}$$

If $\varphi = \psi$, then $a \cdot \varphi = a \cdot \psi$ and so $f \cdot b = g \cdot b$; if we consider the regular epi-jointly monic factorization

$$\begin{tikzcd}
S \arrow[r, shift left, "s_1"] \arrow[r, shift right, "s_2"'] \arrow[dr, "s"'] & Y \\
& M \arrow[u, shift left, "b_1"] \arrow[u, shift right, "b_2"']
\end{tikzcd}$$

we know that $b_1, b_2\colon M \rightrightarrows Y$ is an equivalence relation in $\mathbb{A}$ exact, so that it is the kernel pair of its coequalizer b. The last equation implies then that there exists $\sigma\colon X \longrightarrow M$ such that $\sigma \cdot b_1 = f$ and $\sigma \cdot b_2 = g$; but X is projective and s is a regular epi, so that there exists $\Sigma\colon X \longrightarrow S$ such that $\Sigma \cdot s = \sigma$; this implies that $\Sigma \cdot s_1 = f$ and $\Sigma \cdot s_2 = g$, that is Σ is a homotopy establishing the equality of $[f]$ and $[g]$ in $\mathbb{P}_{\text{ex}}$.

Third, $\hat{F}$ is full: suppose there exists an arrow $\eta\colon A \longrightarrow B$ in $\mathbb{A}$; X is projective and b is a regular epi, so that there exists $f\colon X \longrightarrow Y$ such that $a \cdot \eta = f \cdot b$; this implies that $r_1 \cdot f \cdot b = r_2 \cdot f \cdot b$ and then there exists $\tilde{f}\colon R \longrightarrow M$ such that $r_1 \cdot f = \tilde{f} \cdot b_1$ and $r_2 \cdot f = \tilde{f} \cdot b_2$; but also R is projective, so that there exists $\overline{f}\colon R \longrightarrow S$ such that $\overline{f} \cdot s = \tilde{f}$. It remains to obverse that $\overline{f} \cdot s_1 = r_1 \cdot f$ and $\overline{f} \cdot s_2 = r_2 \cdot f$ and we conclude that

$$\begin{tikzcd} R \arrow[r, "\overline{f}"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is an arrow in $\mathbb{P}_{\text{ex}}$ whose image by $\hat{F}$ is the given arrow η. ■

As a left covering functor defined on a left exact category is exactly a left exact functor (proposition 1.4.3), we have the following particular case of theorem 1.5.2:

Given a left exact category $\mathbb{C}$, there exists an exact category $\mathbb{C}_{\text{ex}}$ and a left exact functor $\Gamma\colon \mathbb{C} \longrightarrow \mathbb{C}_{\text{ex}}$ such that, for each exact category $\mathbb{A}$, composing with Γ induces an equivalence of categories

$$\text{Ex}(\mathbb{C}_{\text{ex}}, \mathbb{A}) \xrightarrow{\simeq} \text{Lex}(\mathbb{C}, \mathbb{A}).$$

This is the main theorem contained in [15]; as it is shown there, this universal property becomes part of the left biadjoint to the obvious forgetful functor

$$\text{Ex} \hookrightarrow \text{Lex}$$

where Ex is the 2-category of exact categories and exact functors and Lex is the 2-category of left exact categories and left exact functors.

The question naturally arising is then whether, with a good choice of morphisms between weakly lex categories, the universal property stated in theorem 1.5.2 becomes part of the analogous adjunction between exact categories and weakly lex ones.

The answer is negative. Suppose that we have organized the weakly lex categories in a 2-category, say WLex, and that the exact completion of a weakly lex category $\Gamma\colon \mathbb{C} \longrightarrow \mathbb{C}_{\text{ex}}$ is the unit of a biadjunction

$$\begin{tikzcd} \text{Ex} \arrow[r, shift right] & \text{WLex} \arrow[l, shift right] \end{tikzcd}$$

Then, if $\mathbb{A} \in \text{Ex}$ and $\mathbb{C} \in \text{WLex}$, the morphisms of $\text{WLex}(\mathbb{C}, \mathbb{A})$ must correspond, via the composition with $\Gamma\colon \mathbb{C} \longrightarrow \mathbb{C}_{\text{ex}}$, to the exact functors from $\mathbb{C}_{\text{ex}}$ to $\mathbb{A}$, that is they are exactly the left covering functors from $\mathbb{C}$ to $\mathbb{A}$.

Of course, the unit $\Gamma\colon \mathbb{C} \longrightarrow \mathbb{C}_{\text{ex}}$ is in $\text{WLex}(\mathbb{C}, \mathbb{C}_{\text{ex}})$; but now, if we perform again the exact completion, we get a functor

$$\mathbb{C} \xrightarrow{\Gamma} \mathbb{C}_{\text{ex}} \xrightarrow{\Gamma} (\mathbb{C}_{\text{ex}})_{\text{ex}}$$

which must be in $\text{WLex}(\mathbb{C}, (\mathbb{C}_{\text{ex}})_{\text{ex}})$ (because it is the composition of two morphisms). Since $(\mathbb{C}_{\text{ex}})_{\text{ex}}$ is exact, this implies that the functor $\mathbb{C} \xrightarrow{\Gamma} \mathbb{C}_{\text{ex}} \xrightarrow{\Gamma} (\mathbb{C}_{\text{ex}})_{\text{ex}}$ must be a left covering functor. But, in general, this is not the case, as the following example shows.

Example 1.6.2

Using theorem 1.6.1, as composition $\mathbb{C} \xrightarrow{\Gamma} \mathbb{C}_{\text{ex}} \xrightarrow{\Gamma} (\mathbb{C}_{\text{ex}})_{\text{ex}}$ we can choose $\mathbb{P} \xrightarrow{i} \mathbb{A} \xrightarrow{\Gamma} (\mathbb{A})_{\text{ex}}$, where $\mathbb{A}$ is an exact category, $\mathbb{P}$ a projective cover of $\mathbb{A}$ and $i \colon \mathbb{P} \longrightarrow \mathbb{A}$ the full inclusion.

Now cover the terminal object τ of $\mathbb{A}$ with an object T of $\mathbb{P}$ and a regular epi $t \colon T \longrightarrow \tau$, so that T is a weak terminal in $\mathbb{P}$. $\Gamma \colon \mathbb{A} \longrightarrow \mathbb{A}_{\text{ex}}$ is left exact, so that the terminal object of $\mathbb{A}_{\text{ex}}$ is exactly $\Gamma\tau$.

If the composition $i \cdot \Gamma \colon \mathbb{P} \longrightarrow \mathbb{A}_{\text{ex}}$ is left covering, then the unique arrow from $\Gamma(iT)$ to $\Gamma\tau$, that is $\Gamma t \colon \Gamma T \longrightarrow \Gamma\tau$, is a regular epi. But $\Gamma\tau$ is projective in $\mathbb{A}_{\text{ex}}$ so that the regular epi Γt has a section, say $s \colon \Gamma\tau \longrightarrow \Gamma T$, in $\mathbb{A}_{\text{ex}}$. Since $\Gamma \colon \mathbb{A} \longrightarrow \mathbb{A}_{\text{ex}}$ is full and faithful, this implies that $t \colon T \longrightarrow \tau$ has a section in $\mathbb{A}$.

To show that, in general, this is not true, choose $\mathbb{A}$ as the category of rings with unit and $\mathbb{P}$ as the full subcategory of projective rings; as $t \colon T \longrightarrow \tau$, one can choose the unique morphism $\mathbb{Z} \longrightarrow (0 = 1)$ which, obviously, has no section in $\mathbb{A}$. The inclusion of the projective rings into the category of rings gives us also an example of functor $\Gamma \colon \mathbb{C} \longrightarrow \mathbb{C}_{\text{ex}}$ which does not preserve the finite weak limits, as announced in the first section.

1.7 More on the exact completion

In this last section of the first chapter, we list some elementary facts on the exact completion which will be useful for applications.

Proposition 1.7.1 *(Functoriality of the $(-)_{ex}$ construction)*
Let $F \colon \mathbb{C} \longrightarrow \mathbb{D}$ be a weakly lex functor; there exists a unique (up to natural isomorphisms) exact functor $F_{ex} \colon \mathbb{C}_{ex} \longrightarrow \mathbb{D}_{ex}$ such that the following diagram is commutative

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\Gamma"] \arrow[d, "F"'] & \mathbb{C}_{ex} \arrow[d, "F_{ex}"] \\ \mathbb{D} \arrow[r, "\Gamma"'] & \mathbb{D}_{ex} \end{tikzcd}$$

Proof: By 1.4.7 and 1.5.2, putting $F_{\text{ex}} = \widehat{F \cdot \Gamma}$. ■

In the following proposition, $\mathbb{C}_{\text{cc}}$ is the splitting of idempotents of a category $\mathbb{C}$ (cc stands for "Cauchy-completion") and $P(\mathbb{A})$ is the full subcategory of projective objects of $\mathbb{A}$.

Proposition 1.7.2 *Let $\mathbb{C}$ and $\mathbb{D}$ be two weakly lex categories and let $\mathbb{A}$ and $\mathbb{B}$ be two exact categories with enough projectives;*

1) $P(\mathbb{C}_{ex}) \simeq \mathbb{C}_{cc}$

2) *the corestriction of* $\Gamma\colon \mathbb{C} \longrightarrow \mathbb{C}_{ex}$ *to* $P(\mathbb{C}_{ex})$ *induces an equivalence* $\mathbb{C}_{ex} \simeq (P(\mathbb{C}_{ex}))_{ex}$

3) $\mathbb{C}_{ex} \simeq \mathbb{D}_{ex}$ *if and only if* $\mathbb{C}_{cc} \simeq \mathbb{D}_{cc}$

4) $\mathbb{A} \simeq \mathbb{B}$ *if and only if* $P(\mathbb{A}) \simeq P(\mathbb{B})$ *if and only if there exist a projective cover* $P_{\mathbb{A}}$ *of* $\mathbb{A}$*, a projective cover* $P_{\mathbb{B}}$ *of* $\mathbb{B}$ *and an equivalence* $P_{\mathbb{A}} \simeq P_{\mathbb{B}}$

Proof: Taking into account that $\mathbb{C}$ is (equivalent to) a projective cover of $\mathbb{C}_{\text{ex}}$, the first point follows from proposition 1.1.5; the other points easily follow from theorem 1.6.1 and proposition 1.7.1. In particular: if $P_{\mathbb{A}} \simeq P_{\mathbb{B}}$, then $(P_{\mathbb{A}})_{\text{ex}} \simeq (P_{\mathbb{B}})_{\text{ex}}$ (by 1.7.1); moreover, by 1.6.1, $(P_{\mathbb{A}})_{\text{ex}} \simeq \mathbb{A}$ and $(P_{\mathbb{B}})_{\text{ex}} \simeq \mathbb{B}$ so that $\mathbb{A} \simeq \mathbb{B}$. ∎

It is the last point of the previous proposition which will be crucial to characterize our major examples of free exact categories (see sections 2.1 and 2.2).

From theorem 1.6.1, we are able to recognize free exact categories; the analogous result for functors is now quite obvious.

Proposition 1.7.3 *Let* $\mathbb{A}$ *and* $\mathbb{B}$ *be two exact categories with enough projectives; an exact functor* $H\colon \mathbb{A} \longrightarrow \mathbb{B}$ *is the exact extension (in the sense of 1.7.1) of a weakly lex functor* $F\colon \mathbb{P}(\mathbb{A}) \longrightarrow \mathbb{P}(\mathbb{B})$ *if and only if* H *sends projectives into projectives.*

Proof: the (only if) is obvious; for the (if), observe that if H is exact and sends projectives into projectives, then its restriction $H'\colon \mathbb{P}(\mathbb{A}) \longrightarrow \mathbb{P}(\mathbb{B})$ preserves weak finite limits, so that H is $(H')_{\text{ex}}$ (cf. 1.7.1). ∎

Corollary 1.7.4 *The exact completion induces a biequivalence between the 2-category of weakly lex categories in which idempotents split and weakly lex functors and the 2-category of exact categories with enough projectives and exact functors which preserve projective objects.* ∎

Chapter 2

Examples and applications

2.1 Monadic categories over $\mathcal{SET}$

In this section we show that the characterization of monadic categories over $\mathcal{SET}$ (due, I think, to Duskin, cf. chapter 9 of [3]) easily follows from the characterization of free exact categories established in section 1.6. Moreover, we use the theory developed in the first chapter to characterize reflections, epireflections and localizations of monadic categories over $\mathcal{SET}$. All results can be easily generalized to monads over powers of $\mathcal{SET}$.

Let us recall the theorem (cf. [8] and [42]).

Theorem 2.1.1 *Let $\mathbb{A}$ be a category; the following conditions are equivalent:*

1) *$\mathbb{A}$ is equivalent to the category of algebras EM($\mathbb{T}$) for a monad $\mathbb{T}$ over $\mathcal{SET}$ (EM stands for Eilenberg-Moore)*

2) *$\mathbb{A}$ is a locally small, exact category and there exists an object $G \in \mathbb{A}$ such that*

   i) G is projective

   ii) $\forall\ I \in \mathcal{SET}\ \ \exists\ I \bullet G$ (the I-indexed copower of G)

   iii) $\forall\ A \in \mathbb{A}\ \ \exists\ I \bullet G \longrightarrow A$ regular epi.

The implication 1) $\Rightarrow$ 2) follows from a more general discussion on the exactness of monadic categories which can be found in section 2.5.

Let us concentrate on the implication 2) $\Rightarrow$ 1).

As the coproduct of projective objects is still a projective, a category $\mathbb{A}$ satisfying condition 2) is an exact category with enough projectives. We know, from theorem 1.6.1, that such a category is completely determined by a projective cover. Recall now that the category KL($\mathbb{T}$) of free algebras (KL stands for Kleisli) is a projective cover of EM($\mathbb{T}$). This means that what really need to characterize EM($\mathbb{T}$) is to characterize KL($\mathbb{T}$).

But this is an easy problem.

Proposition 2.1.2 *Let $\mathbb{C}$ be a category; the following conditions are equivalent:*

1) *$\mathbb{C}$ is equivalent to the category KL($\mathbb{T}$) for a monad $\mathbb{T}$ over $\mathcal{SET}$*

2) *$\mathbb{C}$ is locally small and there exists an object $G \in \mathbb{C}$ such that*

   i) $\forall\, I \in \mathcal{SET} \quad \exists\, I \bullet G$
   ii) $\forall\, X \in \mathbb{C} \quad \exists\, I \in \mathcal{SET}$ *such that* $X \simeq I \bullet G$

Proof: 1) $\Rightarrow$ 2) : take as G the free $\mathbb{T}$-algebra over the singleton.
2) $\Rightarrow$ 1) : consider the pair of functors

$$\begin{tikzcd} \mathcal{SET} \arrow[r, shift right, "- \bullet G"'] & \mathbb{C} \arrow[l, shift right, "\mathbb{C}(G,-)"'] \end{tikzcd}$$

The first condition says that $-\bullet G \colon \mathcal{SET} \longrightarrow \mathbb{C}$ exists, and so it is automatically the left adjoint to $\mathbb{C}(G,-)$. Call now $\mathbb{T}$ the monad induced by $-\bullet G \dashv \mathbb{C}(G,-)$. The second condition says that the comparison functor $\mathrm{KL}(\mathbb{T}) \longrightarrow \mathbb{C}$ is essentially surjective on the objects, so that it is an equivalence (recall that it is always full and faithful). ■

Proof: Proof of theorem 2.1.1: 2) $\Rightarrow$ 1) : let $\mathbb{C}$ be the full subcategory of $\mathbb{A}$ spanned by $I \bullet G$ for $I \in$ *Set*. By proposition 2.1.2, $\mathbb{C}$ is equivalent to KL($\mathbb{T}$) for a monad $\mathbb{T}$ over $\mathcal{SET}$. But KL($\mathbb{T}$) is a projective cover of EM($\mathbb{T}$) and, by assumption, $\mathbb{C}$ is a projective cover of $\mathbb{A}$. So, by proposition 1.7.2, $\mathbb{A}$ is equivalent to EM($\mathbb{T}$). ■

Now we can study localizations of EM($\mathbb{T}$). We use the fact that EM($\mathbb{T}$) is the exact completion of KL($\mathbb{T}$), so that we can work with its formal description as a free exact category.

Proposition 2.1.3 *Consider a category $\mathbb{B}$; the following conditions are equivalent:*

1) *$\mathbb{B}$ is equivalent to a localization of EM($\mathbb{T}$) for a monad $\mathbb{T}$ over $\mathcal{SET}$ (that is a reflective subcategory such that the reflector is lex)*

2) *$\mathbb{B}$ is locally small and exact and has a regular generator which admits all copowers.*

Proof: the implication 1⇒2 is quite obvious, so let us look at the implication 2⇒1. Let G be a regular generator as in condition 2 and let us fix some notations: if S is a set, $S \bullet G$ is the S-indexed copower of G and $i_s \colon G \longrightarrow S \bullet G$ is the s'th canonical injection ($s \in S$); if $\alpha \colon S \longrightarrow T$ is in $\mathcal{SET}$, $\alpha' \colon S \bullet G \longrightarrow T \bullet G$ is the arrow in $\mathbb{B}$ defined by $i_s \cdot \alpha' = i_{\alpha(s)}$

$$\begin{tikzcd} S \bullet G \arrow[rr, "\alpha'"] & & T \bullet G \\ & G \arrow[ul, "i_s"] \arrow[ur, "i_{\alpha(s)}"'] & \end{tikzcd}$$

Given an object A in $\mathbb{B}$, the canonical cover of A by G is the unique arrow $a: \mathbb{B}(G,A) \bullet G \longrightarrow A$ such that for each $f: G \longrightarrow A$ the following diagram commutes

$$\begin{tikzcd} \mathbb{B}(G,A) \bullet G \arrow[rr, "a"] & & A \\ & G \arrow[ul, "i_f"] \arrow[ur, "f"'] & \end{tikzcd}$$

The fact that G is a regular generator means exactly that, for each object A of $\mathbb{B}$, such cover is a regular epimorphism. Now consider the full subcategory $\mathbb{C}$ of $\mathbb{B}$ spanned by copowers of G.

First step: the full inclusion $F: \mathbb{C} \longrightarrow \mathbb{B}$ is a left covering functor.
For the sake of brevity, we prove that F is left covering with respect to binary products, but the argument can be easily adapted to any finite limit. Consider two objects in $\mathbb{C}$ together with their product in $\mathbb{B}$

$$I \bullet G \xleftarrow{\pi_1} I \bullet G \times J \bullet G \xrightarrow{\pi_2} J \bullet G$$

We obtain a weak product in $\mathbb{C}$ precomposing with the canonical cover by G

$$\gamma: \mathbb{B}(G, I \bullet G \times J \bullet G) \bullet G \longrightarrow I \bullet G \times J \bullet G$$

which, by assumption, is a regular epimorphism. In fact, given an object and two arrows in $\mathbb{C}$

$$I \bullet G \xleftarrow{f} S \bullet G \xrightarrow{g} J \bullet G$$

a possible factorization is given by

$$\alpha': S \bullet G \longrightarrow \mathbb{B}(G, I \bullet G \times J \bullet G) \bullet G$$

where

$$\alpha: S \longrightarrow \mathbb{B}(G, I \bullet G \times J \bullet G)$$

sends $s \in S$ into

$$(i_s \cdot f, i_s \cdot g): G \longrightarrow I \bullet G \times J \cdot G$$

with $i_s: G \longrightarrow S \bullet G$. In fact, for each $s \in S$,

$$i_s \cdot \alpha' \cdot \gamma \cdot \pi_1 = i_{(i_s \cdot f, i_s \cdot g)} \cdot \gamma \cdot \pi_1 = (i_s \cdot f, i_s \cdot g) \cdot \pi_1 = i_s \cdot f$$

so that $\alpha' \cdot \gamma \cdot \pi_1 = f$; analogously $\alpha' \cdot \gamma \cdot \pi_2 = g$. In the rest of the proof, we omit the verification of equations, which can always be done precomposing with canonical injections in some copowers. The fact that γ is a regular epimorphism means exactly that F is left covering (with respect to binary products). By the universal property of the exact completion $\Gamma: \mathbb{C} \longrightarrow \mathbb{C}_{\mathrm{ex}}$, this implies that there exists an exact functor $\hat{F}: \mathbb{C}_{\mathrm{ex}} \longrightarrow \mathbb{B}$ such that F and $\Gamma \cdot \hat{F}$ are naturally isomorphic. Let us recall that, if

$$\begin{tikzcd}
R \bullet G \arrow[r, "\overline{f}"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \bullet G \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\
X \bullet G \arrow[r, "f"'] & Y \bullet G
\end{tikzcd}$$

is an arrow in $\mathbb{C}_{\text{ex}}$, $\hat{F}[\overline{f}, f]$ is the unique extension to the quotient as in the following diagram

$$\begin{tikzcd}
R \bullet G \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] \arrow[d, "\overline{f}"'] & X \bullet G \arrow[r, "q_1"] \arrow[d, "f"] & A \arrow[d, "{\hat{F}[\overline{f}, f]}"] \\
S \bullet G \arrow[r, shift left, "s_1"] \arrow[r, shift right, "s_2"'] & Y \bullet G \arrow[r, "q_2"] & B
\end{tikzcd}$$

Second step: embedding of $\mathbb{B}$ in $\mathbb{C}_{\text{ex}}$.
Given an object A in $\mathbb{B}$, consider its canonical cover by G

$$a \colon \mathbb{B}(G, A) \bullet G \longrightarrow A,$$

the kernel pair of a

$$a_1, a_2 \colon N(a) \rightrightarrows \mathbb{B}(G, A) \bullet G$$

and again the canonical cover by G

$$n \colon \mathbb{B}(G, N(a)) \bullet G \longrightarrow N(a)$$

The pair of arrows

$$n \cdot a_1, n \cdot a_2 \colon \mathbb{B}(G, N(a)) \bullet G \rightrightarrows \mathbb{B}(G, A) \bullet G$$

is an object of $\mathbb{C}_{\text{ex}}$. Let us verify its transitivity (reflexivity and symmetry can be verified in an analogous way). Consider the following diagram

$$\begin{tikzcd}
P \arrow[rr, "p_1"] \arrow[dd, "p_2"'] & & \mathbb{B}(G, N(a)) \bullet G \arrow[d, "n"] \\
& N(a) \star N(a) \arrow[r, "d_1"] \arrow[d, "d_2"'] & N(a) \arrow[d, "a_2"] \\
\mathbb{B}(G, N(a)) \bullet G \arrow[r, "n"'] & N(a) \arrow[r, "a_1"'] & \mathbb{B}(G, A) \bullet G
\end{tikzcd}$$

where both squares are pullbacks, so that there exists a unique factorization $d\colon P \longrightarrow N(a) \star N(a)$. We obtain a weak pullback of $n \cdot a_1$ and $n \cdot a_2$ in $\mathbb{C}$ precomposing with the cover

$$\gamma\colon \mathbb{B}(G,P) \bullet G \longrightarrow P$$

Consider again the transitivity of

$$a_1, a_2\colon N(a) \rightrightarrows \mathbb{B}(G,A) \bullet G$$

that is the unique arrow $t\colon N(a) \star N(a) \longrightarrow N(a)$ such that $t \cdot a_1 = d_1 \cdot a_1$ and $t \cdot a_2 = d_2 \cdot a_2$. The transitivity of

$$n \cdot a_1, n \cdot a_2\colon \mathbb{B}(G,N(a)) \bullet G \rightrightarrows \mathbb{B}(G,A) \bullet G$$

is given by

$$\alpha'\colon \mathbb{B}(G,P) \bullet G \longrightarrow \mathbb{B}(G,N(a)) \bullet G$$

where

$$\alpha\colon \mathbb{B}(G,P) \longrightarrow \mathbb{B}(G,N(a))$$

sends $h\colon G \longrightarrow P$ into $h \cdot d \cdot t\colon G \longrightarrow P \longrightarrow N(a) \star N(a) \longrightarrow N(a)$.

Consider now an arrow $\varphi\colon A \longrightarrow B$ in $\mathbb{B}$; we can build up the following diagram, commutative in each part

$$\begin{tikzcd}
\mathbb{B}(G,N(a)) \bullet G \arrow[r, "n"] \arrow[d, "\overline{\alpha}'"'] & N(a) \arrow[r, shift left, "a_1"] \arrow[r, shift right, "a_2"'] \arrow[d, "t"'] & \mathbb{B}(G,A) \bullet G \arrow[r, "a"] \arrow[d, "\alpha'"] & A \arrow[d, "\varphi"] \\
\mathbb{B}(G,N(b)) \bullet G \arrow[r, "m"] & N(b) \arrow[r, shift left, "b_1"] \arrow[r, shift right, "b_2"'] & \mathbb{B}(G,B) \bullet G \arrow[r, "b"] & B
\end{tikzcd}$$

The construction of the horizontal lines has just been explained; as far as the columns are concerned, α' is induced by

$$\alpha\colon \mathbb{B}(G,A) \longrightarrow \mathbb{B}(G,B)$$

which sends $h\colon G \longrightarrow A$ into $h \cdot \varphi\colon G \longrightarrow A \longrightarrow B$; the existence of a unique t such that $t \cdot b_1 = a_1 \cdot \alpha'$ and $t \cdot b_2 = a_2 \cdot \alpha'$ follows from $a \cdot \varphi = \alpha' \cdot b$ and the universal property of $N(b)$; $\overline{\alpha}'$ is induced by

$$\overline{\alpha}\colon \mathbb{B}(G,N(a)) \longrightarrow \mathbb{B}(G,N(b))$$

which sends $h\colon G \longrightarrow N(a)$ into $h \cdot t\colon G \longrightarrow N(a) \longrightarrow N(b)$.

In particular

$$\begin{tikzcd}
\mathbb{B}(G,N(a))\bullet G \arrow[r, "\overline{\alpha}'"] \arrow[d, shift right, "n\cdot a_1"'] \arrow[d, shift left, "n\cdot a_2"] & \mathbb{B}(G,N(b))\bullet G \arrow[d, shift right, "m\cdot b_1"'] \arrow[d, shift left, "m\cdot b_2"] \\
\mathbb{B}(G,A)\bullet G \arrow[r, "\alpha'"'] & \mathbb{B}(G,B)\bullet G
\end{tikzcd}$$

gives us an arrow in $\mathbb{C}_{\text{ex}}$ which we take as value of a functor

$$r\colon \mathbb{B} \longrightarrow \mathbb{C}_{\text{ex}}$$

The functoriality of r is quite obvious. As far as its faithfulness is concerned, consider a second arrow $\psi\colon A \longrightarrow B$ in $\mathbb{B}$ and build up an arrow $[\overline{\beta}', \beta']$ in $\mathbb{C}_{\text{ex}}$ in the same way as $[\overline{\alpha}', \alpha']$ has been built up from $\varphi\colon A \longrightarrow B$; if $[\overline{\beta}', \beta'] = [\overline{\alpha}', \alpha']$, there exists $\Sigma\colon \mathbb{B}(G,A)\bullet G \longrightarrow \mathbb{B}(G,N(b))\bullet G$ such that $\Sigma\cdot m\cdot b_1 = \alpha'$ and $\Sigma\cdot m\cdot b_2 = \beta'$. Now we have

$$a\cdot\varphi = \alpha'\cdot b = \Sigma\cdot m\cdot b_1\cdot b = \Sigma\cdot m\cdot b_2\cdot b = \beta'\cdot b = a\cdot\psi$$

and then $\varphi = \psi$ because a is a (regular) epimorphism.

It remains to show that r is full. For this, consider an arrow in $\mathbb{C}_{\text{ex}}$ as in the following diagram

$$\begin{tikzcd}
\mathbb{B}(G,N(a))\bullet G \arrow[r, "\overline{f}"] \arrow[d, shift right, "n\cdot a_1"'] \arrow[d, shift left, "n\cdot a_2"] & \mathbb{B}(G,N(b))\bullet G \arrow[d, shift right, "m\cdot b_1"'] \arrow[d, shift left, "m\cdot b_2"] \\
\mathbb{B}(G,A)\bullet G \arrow[r, "f"'] & \mathbb{B}(G,B)\bullet G
\end{tikzcd}$$

Since

$$a\colon \mathbb{B}(G,A)\bullet G \longrightarrow A$$

is a regular epimorphism, it is the coequalizer of a_1 and a_2 and then of $n\cdot a_1$ and $n\cdot a_2$ because also n is a (regular) epimorphism. The condition on $[\overline{f}, f]$ to be an arrow in $\mathbb{C}_{\text{ex}}$ implies then that there exists a unique arrow $\varphi\colon A \longrightarrow B$ such that $a\cdot\varphi = f\cdot b$. Now we can build up an arrow $r(\varphi) = [\overline{\alpha}', \alpha']$ in $\mathbb{C}_{\text{ex}}$ as explained before and we need to show that $[\overline{f}, f] = [\overline{\alpha}', \alpha']$, that is we need an arrow

$$\Sigma\colon \mathbb{B}(G,A)\bullet G \longrightarrow \mathbb{B}(G,N(b))\bullet G$$

such that $\Sigma\cdot m\cdot b_1 = f$ and $\Sigma\cdot m\cdot b_2 = \alpha'$. Since $a\cdot\varphi = f\cdot b$ and $a\cdot\varphi = \alpha'\cdot b$, there exists a unique arrow

$$\tau\colon \mathbb{B}(G,A)\bullet G \longrightarrow N(b)$$

such that $\tau\cdot b_1 = f$ and $\tau\cdot b_2 = \alpha'$. Now we can take as Σ the arrow induced by

$$\sigma\colon \mathbb{B}(G,A) \longrightarrow \mathbb{B}(G,N(b))$$

which sends $f: G \longrightarrow A$ into $i_f \cdot \tau: G \longrightarrow \mathbb{B}(G,A) \bullet G \longrightarrow N(b)$.

Third step: adjunction $\hat{F} \dashv r$.

Let $r_1, r_2: R \bullet G \rightrightarrows X \bullet G$ be an object in $\mathbb{C}_{\mathrm{ex}}$, consider its coequalizer $q: X \bullet G \longrightarrow A$ in $\mathbb{B}$ (that is $A = \hat{F}(r_1, r_2)$) and build up

$$r(A) = (n \cdot a_1, n \cdot a_2: \mathbb{B}(G, N(a)) \bullet G \rightrightarrows \mathbb{B}(G,A) \bullet G$$

The unit of the adjunction $\hat{F} \dashv r$ must be an arrow in $\mathbb{C}_{\mathrm{ex}}$ of the following kind

$$\begin{tikzcd}
R \bullet G \arrow[r, "\overline{\eta}'"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & \mathbb{B}(G, N(a)) \bullet G \arrow[d, shift right, "n \cdot a_1"'] \arrow[d, shift left, "n \cdot a_2"] \\
X \bullet G \arrow[r, "\eta'"'] & \mathbb{B}(G,A) \bullet G
\end{tikzcd}$$

As η' we take the arrow induced by $\eta: X \longrightarrow \mathbb{B}(G,A)$ which sends $x \in X$ into $i_x \cdot q: G \longrightarrow X \bullet G \longrightarrow A$. Now observe that with this definition $\eta' \cdot a = q$ and then $r_1 \cdot \eta' \cdot a = r_2 \cdot \eta' \cdot a$. This implies that there exists a unique arrow $\tau: R \bullet G \longrightarrow N(a)$ such that $\tau \cdot a_1 = r_1 \cdot \eta'$ and $\tau \cdot a_2 = r_2 \cdot \eta'$. Now as $\overline{\eta}'$ we can take the arrow induced by $\overline{\eta}: R \longrightarrow \mathbb{B}(G, N(a))$ which sends $r \in R$ into $i_r \cdot \tau: G \longrightarrow R \bullet G \longrightarrow N(a)$.

Consider now an object B of $\mathbb{B}$ and the following arrow in $\mathbb{C}_{\mathrm{ex}}$

$$\begin{tikzcd}
R \bullet G \arrow[r, "\overline{H}"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & \mathbb{B}(G, N(b)) \bullet G \arrow[d, shift right, "m \cdot b_1"'] \arrow[d, shift left, "m \cdot b_2"] \\
X \bullet G \arrow[r, "H"'] & \mathbb{B}(G,B) \bullet G
\end{tikzcd}$$

Since $r_1 \cdot H \cdot b = r_2 \cdot H \cdot b$, there exists a unique arrow $\varphi: A \longrightarrow B$ such that $q \cdot \varphi = H \cdot b$ and we can build up $r(\varphi) = [\overline{\alpha}', \alpha']$. We need to prove that $[\overline{\eta}', \eta'] \cdot [\overline{\alpha}', \alpha'] = [\overline{H}, H]$ and for this we need an arrow

$$\Sigma: X \bullet G \longrightarrow \mathbb{B}(G, N(b)) \bullet G$$

such that $\Sigma \cdot m \cdot b_1 = H$ and $\Sigma \cdot m \cdot b_2 = \eta' \cdot \alpha'$. Since $H \cdot b = \eta' \cdot \alpha' \cdot b$, there exists a unique arrow $\tau: X \bullet G \longrightarrow N(b)$ such that $\tau \cdot b_1 = H$ and $\tau \cdot b_2 = \eta' \cdot \alpha'$. Now we can take as Σ the arrow induced by $\sigma: X \longrightarrow \mathbb{B}(G, N(b))$ which sends $x \in X$ into $i_x \cdot \tau: G \longrightarrow X \bullet G \longrightarrow N(b)$.

As far as the uniqueness of the factorization is concerned, consider a second arrow $\psi: A \longrightarrow B$ and suppose that $r(\psi) = [\overline{\beta}', \beta']$ is such that $[\overline{\eta}', \eta'] \cdot [\overline{\alpha}', \alpha'] =$

$[\overline{\eta}', \eta'] \cdot [\overline{\beta}', \beta']$. This means that there exists an arrow $\Sigma \colon X \bullet G \longrightarrow \mathbb{B}(G, N(b)) \bullet G$ such that $\Sigma \cdot m \cdot b_1 = \eta' \cdot \alpha'$ and $\Sigma \cdot m \cdot b_2 = \eta' \cdot \beta'$. Now

$$q \cdot \varphi = \eta' \cdot a \cdot \varphi = \eta' \cdot \alpha' \cdot b = \Sigma \cdot m \cdot b_1 \cdot b = \Sigma \cdot m \cdot b_2 \cdot b = \eta' \cdot \beta' \cdot b = \eta' \cdot a \cdot \psi = q \cdot \psi$$

and then $\varphi = \psi$ because q is a (regular) epimorphism.

Conclusion: we have just proved that $\mathbb{B}$ is (equivalent to) a localization of $\mathbb{C}_{\text{ex}}$. But, from proposition 2.1.2, we know that the full subcategory $\mathbb{C}$ of $\mathbb{B}$ is equivalent to KL($\mathbb{T}$) for a monad $\mathbb{T}$ over $\mathcal{SET}$ and then its exact completion $\mathbb{C}_{\text{ex}}$ is equivalent to EM($\mathbb{T}$). The proof of proposition 2.1.3 is now complete. ■

Let us look more carefully at the proof of proposition 2.1.3. If, instead of exact, $\mathbb{B}$ is assumed to be only left exact but with coequalizers, we can again define $\hat{F} \colon \mathbb{C}_{\text{ex}} \longrightarrow \mathbb{B}$ as at the end of the first step. Since in the second and the third steps we do not use the exactness of $\mathbb{B}$, we have the following

Proposition 2.1.4 *Consider a category $\mathbb{B}$; the following conditions are equivalent:*

1) *$\mathbb{B}$ is equivalent to a reflective subcategory of EM($\mathbb{T}$) for a monad $\mathbb{T}$ over $\mathcal{SET}$*

2) *$\mathbb{B}$ is a locally small and left exact category with coequalizers and has a regular generator which admits all copowers.* ■

A warning: in the previous proposition it does not suffice to assume the existence of coequalizers of pseudo equivalence relations. This is because a pseudo equivalence relation in $\mathbb{C}$ is not necessarily a pseudo equivalence relation in the whole category $\mathbb{B}$.

Working essentially in the same way (that is working with the formal description of $\mathbb{C}_{\text{ex}}$ and forgetting that it is equivalent to EM($\mathbb{T}$)) we can also prove the following proposition.

Proposition 2.1.5 *Consider a category $\mathbb{B}$; the following conditions are equivalent:*

1) *$\mathbb{B}$ is equivalent to an epireflective subcategory of EM($\mathbb{T}$) for a monad $\mathbb{T}$ over $\mathcal{SET}$ (epireflective = units are regular epimorphisms)*

2) *$\mathbb{B}$ is locally small and regular with coequalizers of equivalence relations and has a regular projective regular generator which admits all copowers.*

Proof: first of all, observe that to define $\hat{F} \colon \mathbb{C} \longrightarrow \mathbb{B}$ as in the first step of proof of 2.1.3, it suffices to have coequalizers of equivalence relations. This is because the jointly monic part of the (regular epi, mono) factorization of a pseudo equivalence relation in $\mathbb{C}$ is an equivalence relation in $\mathbb{B}$. It remains only to prove that units are regular epimorphisms. For this, with the same notations used in the third step of the proof of 2.1.3, consider a unit $[\overline{\eta}', \eta']$ and the monic part of its (regular epi, mono) factorization. It is given by the following diagram (cf. step 2 of theorem 1.2.3)

$$\begin{tikzcd}
\mathbb{B}(G,L)\bullet G \arrow[r, "l\cdot\lambda"] \arrow[d, shift right, "l\cdot l_1"'] \arrow[d, shift left, "l\cdot l_2"] & \mathbb{B}(G,N(a))\bullet G \arrow[d, shift right, "n\cdot a_1"'] \arrow[d, shift left, "n\cdot a_2"] \\
X\bullet G \arrow[r, "\eta'"'] & \mathbb{B}(G,A)\bullet G
\end{tikzcd}$$

where

$$\begin{tikzcd}
X\bullet G \arrow[d, "\eta'"'] & L \arrow[l, "l_1"'] \arrow[r, "l_2"] \arrow[d, "\lambda"'] & X\bullet G \arrow[d, "\eta'"] \\
\mathbb{B}(G,A)\bullet G & \mathbb{B}(G,N(a))\bullet G \arrow[l, "n\cdot a_1"'] \arrow[r, "n\cdot a_2"] & \mathbb{B}(G,A)\bullet G
\end{tikzcd}$$

is a limit in $\mathbb{B}$ and $l: \mathbb{B}(G,L)\bullet G \longrightarrow L$ is the canonical cover of L by G. To prove that $[\overline{\eta}', \eta']$ is a regular epimorphism, it suffices to prove that $[l\cdot\lambda, \eta']$ has a left inverse $[\overline{\sigma}', \sigma']$ (so that it is an isomorphism). Since G is regular projective, for each arrow $f: G \longrightarrow A$ we can choose an arrow $f': G \longrightarrow X\bullet G$ such that $f'\cdot q = f$ (where $q: X\bullet G \longrightarrow A$ is the coequalizer of $r_1, r_2: R\bullet G \rightrightarrows X\bullet G$ in $\mathbb{B}$). The following diagram gives us the definition of σ'

$$\begin{tikzcd}
\mathbb{B}(G,A)\bullet G \arrow[rr, "\sigma'"] & & X\bullet G \\
& G \arrow[ul, "i_f"] \arrow[ur, "f'"'] &
\end{tikzcd}$$

Observe that $\sigma'\cdot q = a$ (which is the canonical cover of A by G). This implies that $n\cdot a_1\cdot\sigma'\cdot\eta'\cdot a = n\cdot a_2\cdot\sigma'\cdot\eta'\cdot a$, so that there exists a unique arrow $t: \mathbb{B}(G,N(a))\bullet G \longrightarrow N(a)$ such that $t\cdot a_1 = n\cdot a_1\cdot\sigma'\cdot\eta'$ and $t\cdot a_2 = n\cdot a_2\cdot\sigma'\cdot\eta'$. Now we can consider the arrow

$$\tau': \mathbb{B}(G,N(a))\bullet G \longrightarrow \mathbb{B}(G,N(a))\bullet G$$

induced by

$$\tau: \mathbb{B}(G,N(a)) \longrightarrow \mathbb{B}(G,N(a))$$

which sends $g: G \longrightarrow N(a)$ into $i_g\cdot t: G \longrightarrow \mathbb{B}(G,N(a))\bullet G \longrightarrow N(a)$. Observe now that $n\cdot a_1\cdot\sigma'\cdot\eta' = \tau'\cdot n\cdot a_1$ and $n\cdot a_2\cdot\sigma'\cdot\eta' = \tau'\cdot n\cdot a_2$, so that there exists a unique arrow $x: \mathbb{B}(G,N(a))\bullet G \longrightarrow L$ such that $x\cdot\lambda = \tau'$, $x\cdot l_1 = n\cdot a_1\cdot\sigma'$ and $x\cdot l_2 = n\cdot a_2\cdot\sigma'$. We take as

$$\overline{\sigma}': \mathbb{B}(G,N(a))\bullet G \longrightarrow \mathbb{B}(G,L)\bullet G$$

the arrow induced by

$$\overline{\sigma}: \mathbb{B}(G,N(a)) \longrightarrow \mathbb{B}(G,L)$$

which sends $g: G \longrightarrow N(a)$ into $i_g \cdot x: G \longrightarrow \mathbb{B}(G, N(a)) \bullet G \longrightarrow L$. It remains to verify that $[\overline{\sigma}', \sigma']$ is a left inverse for $[l \cdot \lambda, \eta']$. For this we need an arrow

$$\Sigma' : \mathbb{B}(G, A) \bullet G \longrightarrow \mathbb{B}(G, N(a)) \bullet G$$

such that $\Sigma' \cdot n \cdot a_1 = \sigma' \cdot \eta'$ and $\Sigma' \cdot n \cdot a_2 = 1$. Since $\sigma' \cdot \eta' \cdot a = a$, there exists a unique arrow $s: \mathbb{B}(G, A) \bullet G \longrightarrow N(a)$ such that $s \cdot a_1 = \sigma' \cdot \eta'$ and $s \cdot a_2 = 1$. We can take as Σ' the arrow induced by

$$\Sigma : \mathbb{B}(G, A) \longrightarrow \mathbb{B}(G, N(a))$$

which sends $f: G \longrightarrow A$ into $i_f \cdot s: G \longrightarrow \mathbb{B}(G, A) \bullet G \longrightarrow N(a)$. ∎

Remarks: i) In the characterization of EM($\mathbb{T}$) as well as in proposition 2.1.5, the regularity of the category is a little bit redundant; in fact the stability of regular epimorphisms under pullbacks follows from the other assumptions (cf. lesson 2 in [31]). This is not true in proposition 2.1.3, because there the regular generator in general is not regular projective.

ii) Observe that, in the case of epireflections, $\mathbb{B}$ contains not only $\mathbb{C}$, but also $\mathbb{C}_{\text{reg}}$ (the regular completion of $\mathbb{C}$). In fact, $F: \mathbb{C} \longrightarrow \mathbb{B}$ is full and faithful and factors through the regular projective objects of $\mathbb{B}$, so that its extension $\mathbb{C}_{\text{reg}} \longrightarrow \mathbb{B}$ is full and faithful (cf. proposition 3.4.1). This agrees with the fact that $\mathbb{C}_{\text{reg}}$ is the epireflective hull of $\mathbb{C}$ in $\mathbb{C}_{\text{ex}}$ (cf. section 4.3).

iii) Proposition 2.1.5 is well-known (see [36], where this kind of characterizations are used to study Malcev conditions in varietal and quasi-varietal categories, [40] and [30]). I have quoted it here because I think it is remarkable that the theory of the exact completion gives us a general framework to prove (in a quite straightforward way) all characterization theorems contained in this section.

iv) All results can be obviously generalized to monads over a power $\mathcal{SET}^X$ of $\mathcal{SET}$. To characterize KL($\mathbb{T}$), we need an X-indexed family of objects G_x in $\mathbb{C}$ such that: i) for each $f: S \longrightarrow X$ in $\mathcal{SET}$ there exists $\coprod_{s \in S} G_{f(s)}$; ii) for each object C in $\mathbb{C}$ there exists $f: S \longrightarrow X$ in $\mathcal{SET}$ such that $C \simeq \coprod_{s \in S} G_{f(s)}$. Now the other results hold replacing the single generator with an X-indexed family of generators which admit all sums.

2.2 Presheaf categories

Working in the same way as in the previous section, we give here an easy proof of a well-known characterization of presheaf categories (cf. [12] and [42]).

We need some preliminary facts.

Let $\mathcal{D}$ be a small category; the sum-completion Fam$\mathcal{D}$ of $\mathcal{D}$ is defined as follows:

objects: functors $f: I \longrightarrow \mathcal{D}$ with I a discrete category (a set)

arrows: an arrow $(f: I \longrightarrow \mathcal{D}) \longrightarrow (g: J \longrightarrow \mathcal{D})$ is a pair (a, α) with a an application $I \longrightarrow J$ and α a natural transformation $f \Rightarrow a \cdot g$.

It is an easy fact to check that Fam$\mathcal{D}$ has sums. In particular, the initial object is the unique functor $\emptyset \longrightarrow \mathcal{D}$ with $\emptyset$ the empty set.

There exists an obvious functor

$$\mathcal{I}\colon \mathcal{D} \longrightarrow \mathrm{Fam}\mathcal{D}$$

defined by

$$X \rightsquigarrow (X\colon \{*\} \longrightarrow \mathcal{D})$$

where $\{*\}$ is the singleton.

The name "sum-completion" comes from the following universal property which characterizes Fam$\mathcal{D}$ uniquely up to equivalences:
for each functor $F\colon \mathcal{D} \longrightarrow \mathbb{B}$, with $\mathbb{B}$ a category with sums, there exists a unique sum-preserving functor $F'\colon \mathrm{Fam}\mathcal{D} \longrightarrow \mathbb{B}$ such that the following diagram is commutative

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "\mathcal{I}"] \arrow[dr, "F"'] & & \mathrm{Fam}\mathcal{D} \arrow[dl, "F'"] \\
& \mathbb{B} &
\end{tikzcd}$$

In fact, it suffices to define

$$F'(f\colon I \longrightarrow \mathcal{D}) = \coprod_{i \in I} F(f(i))$$

The uniqueness of F' on the objects follows from the fact that in Fam$\mathcal{D}$ each object is a sum of objects like $(\{*\} \longrightarrow \mathcal{D})$. To define F' on the arrow, it suffices to consider an arrow as $(i, \alpha)\colon (X\colon \{*\} \longrightarrow \mathcal{D}) \longrightarrow (f\colon I \longrightarrow \mathcal{D})$ with $i \in I$, that is an arrow $\alpha\colon X \longrightarrow f(i)$ in $\mathcal{D}$; obviously, one puts $F'(i, \alpha) = F(\alpha)$.

This functor $F'\colon \mathrm{Fam}\mathcal{D} \longrightarrow \mathbb{B}$ is also the left Kan extension of F along $\mathcal{I}$. In fact, given a functor $T\colon \mathrm{Fam}\mathcal{D} \longrightarrow \mathbb{B}$ and a natural transformation $\alpha\colon F \longrightarrow \mathcal{I} \cdot T$, we can build up a natural transformation $\beta\colon F' \longrightarrow T$. For this, consider an object $f\colon I \longrightarrow \mathcal{D}$ in Fam$\mathcal{D}$; as $F'(f\colon I \longrightarrow \mathcal{D}) = \coprod_{i \in I} F(f(i))$, to have an arrow $\beta_f\colon F'(f\colon I \longrightarrow \mathcal{D}) \longrightarrow T(f\colon I \longrightarrow \mathcal{D})$, it suffices to have an I-indexed family of arrows $F(f(i)) \longrightarrow T(f\colon I \longrightarrow \mathcal{D})$. But we can choose $\alpha_{f(i)}\colon F(f(i)) \longrightarrow T(\mathcal{I}(f(i)))$ followed by the image by T of the canonical inclusion of $f(i)\colon \{*\} \longrightarrow \mathcal{D}$ in the coproducts $\coprod_{i \in I}(f(i) : \{*\} \longrightarrow \mathcal{D}) = (f\colon I \longrightarrow \mathcal{D})$.

The next proposition gives an alternative description of Fam$\mathcal{D}$.

Proposition 2.2.1 *Let $\mathcal{D}$ be a small category and $\mathbb{B}$ the full subcategory of $\mathcal{SET}^{\mathcal{D}^{op}}$ spanned by sums of representable functors; $\mathbb{B}$ is equivalent to Fam$\mathcal{D}$.*

Proof: Consider the unique extension $Y'\colon \mathrm{Fam}\mathcal{D} \longrightarrow \mathbb{B}$ of the Yoneda embedding $Y\colon \mathcal{D} \longrightarrow \mathbb{B}$; obviously, Y' is surjective on the objects. To show that Y'

is full and faithful, it suffices to consider the arrow-part of Y' between hom-sets of the following kind

$$\mathrm{Fam}\mathcal{D}(\{*\} \xrightarrow{X} \mathcal{D}, I \xrightarrow{f} \mathcal{D})$$

and

$$\mathbb{B}(Y'(\{*\} \xrightarrow{X} \mathcal{D}), Y'(I \xrightarrow{f} \mathcal{D}))$$

But the first home set is exactly $\coprod_{i \in I} \mathcal{D}(X, f(i))$ and the second one is $\mathrm{Nat}(\mathcal{D}(-, X), \coprod_{i \in I} \mathcal{D}(-, f(i)))$. If $\alpha: X \longrightarrow f(i)$ is in $\coprod_{i \in I} \mathcal{D}(X, f(i))$, then $Y'(\alpha)$ is the composition

$$\mathcal{D}(-, X) \xrightarrow{- \cdot \alpha} \mathcal{D}(-, f(i)) \longrightarrow \coprod_{i \in I} \mathcal{D}(-, f(i))$$

that is the (inverse of the) bijection of the Yoneda lemma. ■

Corollary 2.2.2 *Sums in Fam$\mathcal{D}$ are disjoint and the initial object is strict.* ■

From the previous proposition, one can deduce more: Fam$\mathcal{D}$ is an extensive category, so that sums in Fam$\mathcal{D}$ are also universal. (Recall that a category $\mathbb{B}$ is extensive when it has finite sums and, for each pair of objects B_1, B_2, the canonical functor

$$\mathbb{B}/B_1 \times \mathbb{B}/B_2 \longrightarrow \mathbb{B}/B_1 \coprod B_2$$

is an equivalence ($\mathbb{B}/B$ is the comma category). In an extensive category sums are disjoint and the initial object is strict. A left exact category with finite sums is extensive if and only if sums are disjoint and universal. A category $\mathbb{B}$ with finite sums and finite products is distributive if, for each triple of objects A, B and C, the canonical arrow

$$(A \times B) \coprod (A \times C) \longrightarrow A \times (B \coprod C)$$

is an isomorphism. A left exact and extensive category is distributive. All this can be found in [16]. In this work with "extensive" we always mean the obvious extension from "finite sums" to "arbitrary (small) sums" of the facts just quoted.)

Since Fam$\mathcal{D}$ is equivalent to a projective cover of $\mathcal{SET}^{\mathcal{D}^{\mathrm{op}}}$ (proposition 2.2.1), the next step to characterize presheaf categories is to characterize categories which are the sum-completion of a small category.

Proposition 2.2.3 *Consider a category $\mathbb{B}$; the following conditions are equivalent:*

1) *$\mathbb{B}$ is equivalent to the category Fam$\mathcal{D}$ for a small category $\mathcal{D}$*

2) *$\mathbb{C}$ is locally small with disjoint sums and strict initial object and there exists a small subcategory $\mathcal{D}$ of $\mathbb{B}$ such that*

   i) $\forall\ B \in \mathbb{B}\ \exists\ \{X_i\}_I$ with $X_i \in \mathcal{D}$ such that $B \simeq \coprod_{i \in I} X_i$

ii) $\forall\, f\colon X \longrightarrow \coprod_{i\in I} X_i$ *with* $X, X_i \in \mathcal{D}$ *there exists* $i_0 \in I$ *such that* f *can be factored through the injection* $X_{i_0} \longrightarrow \coprod_{i\in I} X_i$

iii) *the initial object* 0 *is not in* $\mathcal{D}$

Proof: 1) $\Rightarrow$ 2) : take the full subcategory of Fam$\mathcal{D}$ constituted by objects $(X\colon \{*\} \longrightarrow \mathcal{D})$ with X in $\mathcal{D}$.

2) $\Rightarrow$ 1) : consider the unique extension $F'\colon \mathrm{Fam}\mathcal{D} \longrightarrow \mathbb{B}$ of the full inclusion $F\colon \mathcal{D} \longrightarrow \mathbb{B}$. The first condition on $\mathcal{D}$ means exactly that F' is essentially surjective on the objects.

Consider an arrow $(i,\alpha)\colon (X\colon \{*\} \longrightarrow \mathcal{D}) \longrightarrow (f\colon I \longrightarrow \mathcal{D})$ in Fam$\mathcal{D}$, that is an object $i \in I$ and an arrow $\alpha\colon X \longrightarrow f(i)$ in $\mathcal{D}$; F' sends it on the composition $X \xrightarrow{\alpha} f(i) \longrightarrow \coprod_{i\in I} f(i)$. Clearly the second condition means that F' is full.

Now suppose that $F'(i,\alpha) = F'(j,\beta)$, with $i,j \in I$ and $\alpha\colon X \longrightarrow f(i)$, $\beta\colon X \longrightarrow f(j)$; this means exactly that the following diagram is commutative

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\beta"'] & f(i) \arrow[d] \\ f(j) \arrow[r] & \coprod_{i\in I} f(i) \end{tikzcd}$$

If $i = j$, then $\alpha = \beta$ because $f(i) \longrightarrow \coprod_{i\in I} f(i)$ is a mono. If $i \neq j$, then there exists a unique arrow $t\colon X \longrightarrow 0$ such that

$$\begin{tikzcd} X \arrow[rr, "t"] \arrow[dr, "\alpha"'] & & 0 \arrow[dl] \\ & f(i) & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "t"] \arrow[dr, "\beta"'] & & 0 \arrow[dl] \\ & f(j) & \end{tikzcd}$$

are commutative; but 0 is strict so that t is an isomorphism. This implies that $0 \in \mathcal{D}$, in contradiction with the last condition. ∎

We are ready now to give the announced characterization of presheaf categories.

Proposition 2.2.4 *Consider a category* $\mathbb{A}$*; the following conditions are equivalent:*

1) $\mathbb{A}$ *is equivalent to the category of presheaves on a small category*

2) $\mathbb{A}$ *is exact with disjoint sums and strict initial object and has a set* $\{G_j\}_J$ *of generators (that is regular generators) such that*

   i) $\forall\, j \in J$, G_j *is projective*

   ii) $\forall\, f\colon G \longrightarrow \coprod_{i\in I} G_i$ *with* $G, G_i \in \{G_j\}_J$, $\exists\, i_0 \in I$ *such that* f *can be factored through the injection* $G_{i_0} \longrightarrow \coprod_{i\in I} G_i$

3) $\mathbb{A}$ *has a family of absolutely presentable generators.*

Proof: 1) $\Rightarrow$ 3) and 3) $\Rightarrow$ 2) are obvious (recall that an object $G \in \mathbb{A}$ is absolutely presentable if $\mathbb{A}(G,-)\colon \mathbb{A} \longrightarrow \mathcal{SET}$ preserves colimits). 2) $\Rightarrow$ 1): two cases.

First, if the initial object $0 \in \{G_j\}_J$, but $\{G_j\}_J \setminus 0$ is not a family of generators, then $\{G_j\}_J = \{0\}$ and so $\mathbb{A} \simeq \mathcal{SET}^{\emptyset}$.

Second, if $0 \notin \{G_j\}_J$, let $\mathcal{D}$ be the full subcategory of generators and $\mathbb{B}$ the full subcategory spanned by sums of generators; by proposition 2.2.3, $\mathbb{B}$ is equivalent to Fam$\mathcal{D}$ which is, by proposition 2.2.1, a projective cover of $\mathcal{SET}^{\mathcal{D}^{\mathrm{op}}}$; but, by assumption, $\mathbb{B}$ is a projective cover of $\mathbb{A}$. So, by proposition 1.7.2, $\mathbb{A}$ is equivalent to $\mathcal{SET}^{\mathcal{D}^{\mathrm{op}}}$. ■

2.3 The Giraud theorem

In this section we want to revisit another celebrated characterization theorem, that is Giraud theorem characterizing Grothendieck topo.

Let us start with a brief and informal discussion. Giraud theorem states that the following three conditions are equivalent for a given category $\mathbb{A}$:

1) $\mathbb{A}$ satifies Giraud axioms for a topos

2) $\mathbb{A}$ is a localization of a presheaf category

3) $\mathbb{A}$ is equivalent to the category of sheaves on a site.

Usually (cf. [8], [28], [32], [34]) the proof of this theorem runs as follows.

First, one proves that the associated sheaf functor exhibits a category of sheaves as a localization of the corresponding presheaf category (that is 3) $\Rightarrow$ 2)).

Second, one observes that Giraud axioms are verified by a presheaf category and that they are stable under localizations (that is 2) $\Rightarrow$ 1)).

Third, starting from the family of generators involved in Giraud axioms, one has to construct a site and an equivalence between the given category and the resulting category of sheaves (that is 1) $\Rightarrow$ 3)); clearly, this is the most difficult part.

In my opinion, one can look at condition 1) as the definition of a Grothendieck topos; at condition 2) as the characterization of Grothendieck topo and at condition 3) as a representation theorem.

For this reason, I think it is of some interest to have a direct proof that 1) implies 2).

Moreover, as a presheaf category is a free exact one, it must be possible to express the conditions making the reflector a left exact functor in terms of a left covering functor.

For this, we look more carefully at the sum-completion Fam$\mathcal{D}$ of a small category $\mathcal{D}$.

We know (proposition 2.2.1) that Fam$\mathcal{D}$ is (equivalent to) a projective cover of $\mathcal{SET}^{\mathcal{D}^{\mathrm{op}}}$, so that it is a weakly lex category.

Let us describe explicitely some weak finite limits in Fam$\mathcal{D}$, identified with the full subcategory of $\mathcal{SET}^{\mathcal{D}^{\mathrm{op}}}$ spanned by sums of representable functors. Each time, one can consider the corresponding strong limit in $\mathcal{SET}^{\mathcal{D}^{\mathrm{op}}}$ and use the canonical presentation of a presheaf as colimit (that is quotient of a sum) of representable presheaves. We give directly the resulting formula.

A weak terminal object in Fam$\mathcal{D}$ is the coproduct $\coprod_{X \in \mathcal{D}_0} \mathcal{D}(-, X)$ of all the representable presheaves.

A weak product of two objects $\mathcal{D}(-, A)$ and $\mathcal{D}(-, B)$ in Fam$\mathcal{D}$ is the coproduct $\coprod \mathcal{D}(-, X)$ indexed over all the pairs of arrows $A \xleftarrow{u} X \xrightarrow{v} B$ in $\mathcal{D}$ with X varying in $\mathcal{D}_0$.

A weak equalizer of two parallel arrows $u, v \colon \mathcal{D}(-, A) \rightrightarrows \mathcal{D}(-, B)$ in Fam$\mathcal{D}$ is the coproduct $\coprod \mathcal{D}(-, X)$ indexed over all the arrows $x \colon X \longrightarrow A$ in $\mathcal{D}$ such that $x \cdot u = x \cdot v$ with X varying in $\mathcal{D}_0$.

In each case the projections are the obvious ones; for example, the projection from the weak product $\coprod \mathcal{D}(-, X) \longrightarrow \mathcal{D}(-, B)$ is the unique arrow induced by the family of arrows $\{v \colon X \longrightarrow B\}$ indexed by all the pairs $A \xleftarrow{u} X \xrightarrow{v} B$ in $\mathcal{D}$.

So, we have described binary products and equalizers of objects and arrows of Fam$\mathcal{D}$ coming from $\mathcal{D}$ via the (Yoneda) embedding $\mathcal{I} \colon \mathcal{D} \longrightarrow$ Fam$\mathcal{D}$.

Lemma 2.3.1 *Let $\mathcal{D}$ be a small category and $\mathbb{A}$ a left exact and extensive category. Consider a sum-preserving functor $F \colon$ Fam$\mathcal{D} \longrightarrow \mathbb{A}$ left covering with respect to binary products and equalizers of objects and arrows of Fam$\mathcal{D}$ coming from $\mathcal{D}$; F is left covering with respect to binary products and equalizers.*

Proof: Taking into account that in Fam$\mathcal{D}$ each object is a sum of representable ones, the proof proceeds by induction.

For the sake of clarity, we consider only binary sums, but all the arguments are valid for arbitrary sums.
Products: consider three objects $X, A, B \in$ Fam$\mathcal{D}$ and assume, by induction, that the canonical factorizations $\varphi \colon F(X \times A) \longrightarrow FX \times FA$ and $\psi \colon F(X \times B) \longrightarrow FX \times FB$ are strong epi's. In Fam$\mathcal{D}$ there exists a factorization

$$\gamma \colon (X \times A) \coprod (X \times B) \longrightarrow X \times (A \coprod B)$$

(commuting with the appropriate projections); on the other hand, the canonical factorization in $\mathbb{A}$

$$(FX \times FA) \coprod (FX \times FB) \longrightarrow FX \times (FA \coprod FB)$$

is an iso, because $\mathbb{A}$ is distributive.

Now we can build up a commutative diagram in $\mathbb{A}$

$$\begin{tikzcd}
F(X\times(A\coprod B)) \arrow[r, "\alpha"] & FX\times F(A\coprod B) \arrow[d, "\simeq"] \\
F((X\times A)\coprod(X\times B)) \arrow[u, "F\gamma"] \arrow[d, "\simeq"'] & FX\times(FA\coprod FB) \arrow[d, "\simeq"] \\
F(X\times A)\coprod F(X\times B) \arrow[r, "\varphi\coprod\psi"'] & (FX\times FA)\coprod(FX\times FB)
\end{tikzcd}$$

but $\varphi\coprod\psi$ is a strong epi, so that also the canonical factorization α is a strong epi, as required.

(Observe that the coproduct of a family of strong epimorphisms is again a strong epimorphism.)

Equalizers: first case. Consider in Fam$\mathcal{D}$ a coproduct $X \xrightarrow{i_X} X\coprod Y \xleftarrow{i_Y} Y$ and two arrows $f,g\colon X\coprod Y \rightrightarrows A$; consider two weak equalizers

$$E_X \xrightarrow{e_X} X \overset{i_X\cdot f}{\underset{i_X\cdot g}{\rightrightarrows}} A \qquad E_Y \xrightarrow{e_Y} Y \overset{i_Y\cdot f}{\underset{i_Y\cdot g}{\rightrightarrows}} A$$

and then apply F

$$\begin{tikzcd}
FE_X \arrow[d, "p_X"'] \arrow[dr, "Fe_X"] & & \\
L_X \arrow[r, "l_X"'] & FX \arrow[r, shift left, "Fi_X\cdot Ff"] \arrow[r, shift right, "Fi_X\cdot Fg"'] & FA
\end{tikzcd}
\qquad
\begin{tikzcd}
FE_Y \arrow[d, "p_Y"'] \arrow[dr, "Fe_Y"] & & \\
L_Y \arrow[r, "l_Y"'] & FY \arrow[r, shift left, "Fi_Y\cdot Ff"] \arrow[r, shift right, "Fi_Y\cdot Fg"'] & FA
\end{tikzcd}$$

(where l_X and l_Y are the equalizers in $\mathbb{A}$).

Consider again a weak equalizer

$$E \xrightarrow{e} X\coprod Y \overset{f}{\underset{g}{\rightrightarrows}} A$$

and the corresponding equalizer in $\mathbb{A}$

$$\begin{tikzcd}
FE \arrow[d, "p"'] \arrow[dr, "Fe"] & & \\
L \arrow[r, "l"'] & FX\coprod FY \arrow[r, shift left, "Ff"] \arrow[r, shift right, "Fg"'] & FA
\end{tikzcd}$$

(where we have identified for brevity $F(X \coprod Y)$ with $FX \coprod FY$); we want to show that the canonical factorization p is a strong epi. By definition of E, E_X and E_Y, there exists a factorization $t: E_X \coprod E_Y \longrightarrow E$ commuting with the appropriate projections. On the other hand, the analogous factorization in $\mathbb{A}$ $L_X \coprod L_Y \longrightarrow L$ is an iso (this easily follows from the universality of sums and keeping in mind the construction of equalizers via pullbacks and products). Now we have the following commutative diagram

$$\begin{tikzcd} FE_X \coprod FE_Y \arrow[r, "Ft"] \arrow[d, "p_X \coprod p_Y"'] & FE \arrow[d, "p"] \\ L_X \coprod L_Y \arrow[r, "\simeq"] & L \end{tikzcd}$$

but by induction p_X and p_Y are strong epimorphisms, so that also p is a strong epimorphism.

Equalizers: second case. Consider in Fam$\mathcal{D}$ two arrows $f, g: X \rightrightarrows A \coprod B$ with $A, B \in \mathcal{D}$. As usual, consider a weak equalizer $e: E \longrightarrow X$ in Fam$\mathcal{D}$, the strong equalizer in $\mathbb{A}$ and the corresponding factorization

$$\begin{tikzcd} FE \arrow[d, "p"'] \arrow[dr, "Fe"] & & \\ L \arrow[r, "l"'] & FX \arrow[r, shift left, "Ff"] \arrow[r, shift right, "Fg"'] & FA \coprod FB \end{tikzcd}$$

(once again we identify $F(A \coprod B)$ with $FA \coprod FB$).

Taking into account the first case just discussed, we can suppose also X in $\mathcal{D}$, so that f and g have to factor through one of the canonical inclusions $A \xrightarrow{i_A} A \coprod B \xleftarrow{i_B} B$.

Suppose they factor as $f = \overline{f} \cdot i_A$, $g = \overline{g} \cdot i_B$. Then L factors through the pullback of Ff and Fg, which factors through the pullback of Fi_A and Fi_B which is the initial object (sums are disjoint); but the initial object is strict, so that all the involved factorizations (included $p: FE \longrightarrow L$) are isomorphisms.

Suppose now they factor as $f = \overline{f} \cdot i_A$, $g = \overline{g} \cdot i_A$. Taking into account that $i_A: A \longrightarrow A \coprod B$ is a mono (sums in Fam$\mathcal{D}$ are performed in $\mathcal{SET}^{\mathcal{D}^{\mathrm{op}}}$), we have that $e: E \longrightarrow X$ is also a weak equalizer of $\overline{f}, \overline{g}: X \rightrightarrows A$. By assumption, the factorization on the corresponding equalizer in $\mathbb{A}$

$$\begin{tikzcd} FE \arrow[d, "\underline{p}"'] \arrow[dr, "Fe"] & & \\ \underline{L} \arrow[r, "\underline{l}"'] & FX \arrow[r, shift left, "F\overline{f}"] \arrow[r, shift right, "F\overline{g}"'] & FA \end{tikzcd}$$

is a strong epimorphism.

But $\underline{l}\colon \underline{L} \longrightarrow FX$ is also the equalizer of

$$FX \underset{F\overline{g}\cdot i_{FA}}{\overset{F\overline{f}\cdot i_{FA}}{\rightrightarrows}} FA \coprod FB$$

(because i_{FA} is a mono), that is of $Ff, Fg\colon FX \rightrightarrows FA \coprod FB$. Therefore $p = \underline{p}$ is a strong epimorphism, as required. ■

Corollary 2.3.2 *Under the hypothesis of the previous lemma, if moreover $\mathbb{A}$ is regular and F is left covering with respect to the terminal object, then F is left covering.*

Proof: By lemma 2.3.1 and proposition 1.4.10. ■

We are ready to see how left covering functors can be used in the proof of Giraud theorem. To make this more evident, we have chosen a list of axioms which is by no way minimal (for example, "cocomplete" can be replaced by "with sums" and "dense" can be omitted) and we only sketch the classical arguments involved in the proof.

Proposition 2.3.3 *Let $\mathbb{A}$ be an exact, extensive and cocomplete category. If $\mathbb{A}$ admits a dense family $\{G_i\}_I$ of generators, then $\mathbb{A}$ is a localization of a presheaf category.*

Proof: Consider the full subcategory $\mathcal{D}$ of $\mathbb{A}$ whose objects are the generators G_i and call $F\colon \mathcal{D} \longrightarrow \mathbb{A}$ the inclusion. Since the family $\{G_i\}_I$ is dense, the functor $\mathbb{A}(F-,-)\colon \mathbb{A} \longrightarrow \mathcal{SET}^{\mathcal{D}^{\mathrm{op}}}$ is full and faithful and, since $\mathbb{A}$ is cocomplete, it is the right adjoint of the left Kan extension $\hat{F}\colon \mathcal{SET}^{\mathcal{D}^{\mathrm{op}}} \longrightarrow \mathbb{A}$ of F along the Yoneda embedding $Y\colon \mathcal{D} \longrightarrow \mathcal{SET}^{\mathcal{D}^{\mathrm{op}}}$. Taking into account the results of section 2.2, we can factor Y as

$$\mathcal{D} \xrightarrow{\mathcal{I}} \mathrm{Fam}\mathcal{D} \xrightarrow{\Gamma} (\mathrm{Fam}\mathcal{D})_{\mathrm{ex}} \simeq \mathcal{SET}^{\mathcal{D}^{\mathrm{op}}}$$

(where $\mathcal{I}$ is described at the beginning of section 2.2 and Γ is described in 1.3.1).

Also $\hat{F}$ can be built up in two steps: first, consider the left Kan extension $F'\colon \mathrm{Fam}\mathcal{D} \longrightarrow \mathbb{A}$ of F along $\mathcal{I}$ (that is, the sum-preserving extension of F) and then take the left Kan extension $\hat{F}$ of F' along Γ. By proposition 1.5.4 and theorem 1.5.2, to prove that $\hat{F}$ is left exact, it suffices to prove that F' is left covering. Keeping in mind the description of weak limits in Fam$\mathcal{D}$, we have to look at the three following canonical factorizations

$$\coprod G_i \longrightarrow 1 \ , \ \coprod G_i \longrightarrow C \times D \ , \ \coprod G_i \longrightarrow E_{u,v}$$

where: C, D, u, v are objects and arrows in $\mathcal{D}$; the first coproduct is indexed by all the objects G_i in $\mathcal{D}$; the second coproduct is indexed by all the pairs of arrows $C \longleftarrow G_i \longrightarrow D$ in $\mathcal{D}$ with G_i varying in $\mathcal{D}$; the third coproduct is indexed by all the arrows $w: G_i \longrightarrow C$ such that $w \cdot u = w \cdot v$ with G_i varying in $\mathcal{D}$; 1 is the terminal in $\mathbb{A}$; $C \times D$ is the product in $\mathbb{A}$; $E_{u,v} \longrightarrow C \overset{u}{\underset{v}{\rightrightarrows}} D$ is the equalizer in $\mathbb{A}$.

But the hypothesis that the family $\{G_i\}_I$ generates $1, C \times D$ and $E_{u,v}$ means exactly that these three arrows are regular epis. By corollary 2.3.2, we have that F' is left covering and the proof is complete. ∎

The previous proof is not, in some sense, satisfactory. In fact, we have used only the universal property of $\mathcal{SET}^{\mathcal{D}^{\mathrm{op}}}$ as free exact category, but not its formal description. As a consequence, we are forced to use some non trivial facts on dense generators and pointwise Kan extensions. We give now a different proof, which is less elegant but completely straightforward. What is the "best" proof of a theorem is, of course, a matter of personal taste. Nevertheless, I think it is interesting to point out that, once some basic facts on exact categories are achieved, the characterization of localizations of presheaf categories becomes an easy exercise.

We work as in the proof of proposition 2.1.3. The major difference is that $\mathbb{C}$ must be the sum-completion of the full subcategory $\mathcal{D}$ of generators, so that $\mathbb{C}_{\mathrm{ex}}$ is equivalent to $\mathcal{SET}^{\mathcal{D}^{\mathrm{op}}}$. For this, we take as arrows from a generator to a sum of generators only the arrows which factor through a canonical injection (generators are indecomposable in $\mathcal{SET}^{\mathcal{D}^{\mathrm{op}}}$). Now the inclusion of $\mathbb{C}$ in $\mathbb{B}$ is not full, but it remains left covering in virtue of the extensivity assumption on $\mathbb{B}$.

Proposition 2.3.4 *Consider a category $\mathbb{B}$; the following conditions are equivalent:*

1) *$\mathbb{B}$ is equivalent to a localization of a presheaf category*

2) *$\mathbb{B}$ is exact, extensive and has a set of regular generators which admit all sums.*

Proof: let $\{G_x\}_{x \in X}$ be the set of generators. Consider the following category $\mathbb{C}$: objects are sums of generators, so that a generic object of $\mathbb{C}$ is of the form $\coprod_{s \in S} G_{f(s)}$ for a map $f: S \longrightarrow X$; an arrow

$$\coprod_{s \in S} G_{f(s)} \longrightarrow \coprod_{r \in R} G_{g(r)}$$

in $\mathbb{C}$ is an arrow in $\mathbb{B}$ such that, for each $s \in S$, there exist $r_s \in R$ and $G_{f(s)} \longrightarrow G_{g(r_s)}$ in $\mathbb{B}$ making commutative the following diagram

$$\begin{tikzcd}
\coprod_{s\in S} G_{f(s)} \arrow[r] & \coprod_{r\in R} G_{g(r)} \\
G_{f(s)} \arrow[u, "i_s"] \arrow[r] & G_{g(r)} \arrow[u, "i_{r_s}"']
\end{tikzcd}$$

We write

$$a\colon \coprod_{x,\mathbb{B}(G_x,A)} G_x \longrightarrow A$$

for the canonical cover of an object A of $\mathbb{B}$ by generators. Now observe that $\mathbb{C}$ is equivalent to the sum-completion of the full subcategory of $\mathbb{B}$ spanned by generators. For this, it suffices to observe that, if

$$\coprod_{s\in S} G_{f(s)} \longrightarrow \coprod_{t\in T} G_{h(t)} \longleftarrow \coprod_{r\in R} G_{g(r)}$$

are arrows in $\mathbb{C}$, then the unique factorization in $\mathbb{B}$

$$(\coprod_{s\in S} G_{f(s)}) \coprod (\coprod_{r\in R} G_{g(r)}) \longrightarrow \coprod_{t\in T} G_{h(t)}$$

is again an arrow in $\mathbb{C}$.

First step: the inclusion $F\colon \mathbb{C} \longrightarrow \mathbb{B}$ is a left covering functor. Consider a finite diagram $\mathcal{L}\colon \mathcal{D} \longrightarrow \mathbb{C}$ and its limit in $\mathbb{B}$

$$\lim \mathcal{L} \cdot F = (\pi_D\colon L \longrightarrow \mathcal{L}(D))_{D\in\mathcal{D}_0}$$

If we precompose each π_D with the canonical cover

$$l\colon \coprod_{x,\mathbb{B}(G_x,L)} G_x \longrightarrow L$$

we do not know if the resulting projections $l \cdot \pi_D$ are arrows in $\mathbb{C}$. What we can do is to take a "subcover"

$$l'\colon \coprod_{x,\mathbb{C}(G_x,L)} G_x \longrightarrow L$$

where $\mathbb{C}(G_x, L) = \{f\colon G_x \longrightarrow L \mid \ \forall\, D \in \mathcal{D}_0 \quad f \cdot \pi_D \in \mathbb{C}\}$. One can prove that

$$(l' \cdot \pi_D\colon \coprod_{x,\mathbb{C}(G_x,L)} G_x \longrightarrow \mathcal{L}(D))_{D\in\mathcal{D}_0}$$

is a weak limit of $\mathcal{L}$ in $\mathbb{C}$ working as in 2.1.3. But now we have a new problem: by assumption, the canonical cover l is a regular epimorphism; how can we prove that also the subcover l' is a regular epimorphism? If, for each $D \in \mathcal{D}_0$, $\mathcal{L}(D)$ is reduced to a single generator, the difference between $\mathbb{B}(G_x, L)$ and $\mathbb{C}(G_x, L)$ vanishes, so that $l' = l$ is a regular epimorphism. The general case follows from this particular case using lemma 2.3.1.

Second and third steps: here the only difference with respect to the algebraic case (proposition 2.1.3) is in the definition of $r\colon \mathbb{B} \longrightarrow \mathbb{C}_{\mathrm{ex}}$ on the objects of $\mathbb{B}$. Given an object A in $\mathbb{B}$, we start with the canonical cover

$$a\colon \coprod_{x,\mathbb{B}(G_x,A)} G_x \longrightarrow A$$

and we take its kernel pair in $\mathbb{B}$

$$a_1, a_2 \colon N(a) \rightrightarrows \coprod_{x,\mathbb{B}(G_x,A)} G_x$$

But now, to construct a pseudo equivalence relation in $\mathbb{C}$, we take the subcover

$$n' \colon \coprod_{x,\mathbb{C}(G_x,N(a))} G_x \longrightarrow N(a)$$

where $\mathbb{C}(G_x, N(a)) = \{f \colon G_x \longrightarrow N(a) \mid \quad f \cdot a_1, f \cdot a_2 \in \mathbb{C}\}$. It remains only to prove that n' is a (regular) epimorphism, which is essential to show that $r \colon \mathbb{B} \longrightarrow \mathbb{C}_{\text{ex}}$ is full (we can not use here lemm 2.3.1 as in the first step, because A is not necessarily in $\mathbb{C}$). Since the rest of the proof runs exactly as in the algebraic case, we can conclude our proof showing that n' is a regular epimorphism in a separated lemma.

<u>Lemma:</u> Consider a pullback in $\mathbb{B}$

$$\begin{tikzcd} P \arrow[r, "k'"] \arrow[d, "h'"'] & \coprod_{s \in S} G_{f(s)} \arrow[d, "h"] \\ \coprod_{r \in R} G_{g(r)} \arrow[r, "k"] & A \end{tikzcd}$$

and the subcover

$$p' \colon \coprod_{x,\mathbb{C}(G_x,P)} G_x \longrightarrow P$$

where $\mathbb{C}(G_x, P) = \{f \colon G_x \longrightarrow P \mid \quad f \cdot h', f \cdot k' \in \mathbb{C}\}$; the arrow p' is a regular epimorphism.

Proof: such as the one of lemma 2.3.1, the proof is by induction. If $R = S = \{\star\}$, the difference between $\mathbb{C}(G_x, P)$ and $\mathbb{B}(G_x, P)$ vanishes, so that $p' = p$ which is a regular epimorphism by assumption on generators. If $R = \{\star\}$ but S is arbitrary, consider the pullback

$$\begin{tikzcd} P_s \arrow[r, "k_s"] \arrow[d, "j_s"'] & G_{f(s)} \arrow[d, "i_s"] \\ P \arrow[r, "k'"'] & \coprod_{s \in S} G_{f(s)} \end{tikzcd}$$

By associativity of pullbacks, we can apply the induction, so that the subcover

$$p'_s \colon \coprod_{x,\mathbb{C}(G_x,P_s)} G_x \longrightarrow P_s$$

is a regular epimorphism, where $\mathbb{C}(G_x, P_s) = \{f \colon G_x \longrightarrow P_s \mid f \cdot j_s \cdot h', f \cdot k_s \in \mathbb{C}\}$. Let us write A_s for $\coprod_{x,\mathbb{C}(G_x,P_s)} G_x$; we are in the following situation

$$\begin{tikzcd} A_s \arrow[r, "a_s"] \arrow[d, "p'_s"'] & \coprod_{s\in S} A_s \\ P_s \arrow[r, "j_s"'] & P \end{tikzcd}$$

so that there exists a unique arrow $\sigma\colon \coprod_{s\in S} A_s \longrightarrow P$ such that $a_s \cdot \sigma = p'_s \cdot j_s$ for each s in S. By universality of sums, $(j_s\colon P_s \longrightarrow P) = \coprod_{s\in S} P_s$. Since, for each s, p'_s is a regular epimorphism, by commutativity of colimits also σ is a regular epimorphism. Consider now

$$\alpha'_s\colon A_s \longrightarrow \coprod_{x,\mathbb{C}(G_x,P)} G_x$$

induced by

$$\alpha_s\colon \coprod_{x\in X} \mathbb{C}(G_x, P_s) \longrightarrow \coprod_{x\in X} \mathbb{C}(G_x, P)$$

which sends $f\colon G_x \longrightarrow P_s$ into $f \cdot j_s\colon G_x \longrightarrow P_s \longrightarrow P$. We obtain a unique arrow

$$\alpha\colon \coprod_{s\in S} A_s \longrightarrow \coprod_{x,\mathbb{C}(G_x,P)} G_x$$

such that $a_s \cdot \alpha = \alpha'_s$ for each s. We have built up the diagram

$$\begin{tikzcd} \coprod_{s\in S} A_s \arrow[rr, "\sigma"] & & P \\ & \coprod_{x,\mathbb{C}(G_x,P)} G_x \arrow[ul, "\alpha"] \arrow[ur, "p'"'] & \end{tikzcd}$$

and its commutativity shows that p' is a regular epimorphism. In the same way the case R and S arbitrary follows inductively from the case $R = \{\star\}$ and S arbitrary. ■

Remarks: i) Recall that, in the previous proposition, the assumption of regularity on generators is redundant. In fact, in an exact and extensive category each monomorphism is regular and then each epimorphism is regular.

ii) One can be tempted to adapt also propositions 2.1.4 and 2.1.5 to presheaf categories. Unfortunately, if the reflector is not lex, the disjointness of sums is not preserved, so that $\mathbb{C}$ is no more equivalent to the sum-completion of the full subcategory of generators. Moreover, if one tries to transpose the argument used to prove the regularity of units from the algebraic case to the presheaves case, one realizes immediately that, in the second case, it runs if and only if generators are indecomposable in $\mathbb{B}$. This is true in some particular cases (for example when the right adjoint preserves regular projective objects), but not for any epireflection.

2.4 Geometric morphisms

We want to develop here some facts arising from the two previous sections. More exactly, we want to revisit the study of geometric morphisms from an (elementary) topos $\mathcal{E}$ to a topos of presheaves $\mathcal{SET}^{\mathcal{D}^{\mathrm{op}}}$. We follow the terminology of [32].

Definition 2.4.1 *Consider a small category $\mathcal{D}$, the covariant Yoneda embedding $Y\colon \mathcal{D} \longrightarrow \mathcal{SET}^{\mathcal{D}^{\mathrm{op}}}$ and a cocomplete left exact category $\mathbb{A}$; a functor $F\colon \mathcal{D} \longrightarrow \mathbb{A}$ is flat if its left Kan extension $\tilde{F}\colon \mathcal{SET}^{\mathcal{D}^{\mathrm{op}}} \longrightarrow \mathbb{A}$ along Y is left exact.*

Observe that in the previous definition one can equivalently require that $\tilde{F}$ is exact; in fact $\tilde{F}$ is computed pointwise, so that it has always right adjoint given by $\mathbb{A}(F-,-)\colon \mathbb{A} \longrightarrow \mathcal{SET}^{\mathcal{D}^{\mathrm{op}}}$.

Proposition 2.4.2 *With the notations of the previous definition and supposing $\mathbb{A}$ exact, we have that F is flat if and only if its sum-preserving extension $F'\colon \mathrm{Fam}\mathcal{D} \longrightarrow \mathbb{A}$ is left covering.*

Proof: Consider the following diagram

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "\mathcal{I}"] \arrow[dr, "F"'] & \mathrm{Fam}\mathcal{D} \arrow[d, "F'"] \arrow[dr, "\Gamma"] & \\
& \mathbb{A} & \mathcal{SET}^{\mathcal{D}^{\mathrm{op}}} \arrow[l, shift right, "\tilde{F}"'] \arrow[l, shift left, "\hat{F}"]
\end{tikzcd}$$

where $\mathcal{I} \cdot \Gamma =$ Yoneda embedding, $\mathcal{I}$ is as in section 2.2 and Γ is (up to the equivalence of proposition 2.2.1) as in proposition 1.3.1.
(if): if F' is left covering, then (by 1.5.4 and 1.5.2) its left Kan extension $\hat{F}$ along Γ is exact. But F' is the left Kan extension of F along $\mathcal{I}$, so that $\hat{F}$ is the left Kan extension of F along Y and then $\hat{F}$ coincides with $\tilde{F}$.
(only if): if F is flat, then by definition $\tilde{F}$ is exact and then (by 1.4.8) $\Gamma \cdot \tilde{F}$ is left covering. But $\Gamma \cdot \tilde{F}$ coincides with F' because $\mathcal{I} \cdot \Gamma \cdot \tilde{F} = F$ and $\Gamma \cdot \tilde{F}$ is sum-preserving. ∎

Definition 2.4.3 *Let $\mathcal{D}$ be a small category and $\mathbb{A}$ a left exact category. A functor $F\colon \mathcal{D} \longrightarrow \mathbb{A}$ is filtering if the following three coditions hold:*

1) *the family of all maps $F(X) \longrightarrow 1$ (with X varying in $\mathcal{D}_0$) is epimorphic*

2) *for each pair of objects A, B in $\mathcal{D}$, the family of all maps*

$$\langle F(u), F(v)\rangle\colon FX \longrightarrow FA \times FB$$

(with $A \xleftarrow{u} X \xrightarrow{v} B$ in $\mathcal{D}$ with X varying in $\mathcal{D}_0$) is epimorphic

3) *for each pair of arrows* $u, v\colon A \rightrightarrows B$ *in* $\mathcal{D}$, *the family of all maps* $F(X) \longrightarrow E_{u,v}$ *(induced via the equalizer* $E_{u,v} \longrightarrow FA \overset{Fu}{\underset{Fv}{\rightrightarrows}} FB$ *by maps* $w\colon X \longrightarrow A$ *in* $\mathcal{D}$ *such that* $w \cdot u = w \cdot v$ *with* X *varying in* $\mathcal{D}_0$*) is epimorphic.*

(Recall that if $\mathbb{A} = \mathcal{SET}$, this definition means exactly that the category of elements of F is filtering (cf. [32]) or, equivalently, that F is a filtered colimit of representable functors (cf. [8])).

Proposition 2.4.4 *Consider a functor* $F\colon \mathcal{D} \longrightarrow \mathbb{A}$ *with* $\mathcal{D}$ *small and* $\mathbb{A}$ *an exact and extensive category. Let* $F'\colon \mathrm{Fam}\mathcal{D} \longrightarrow \mathbb{A}$ *be the sum-preserving extension of* F*;* F *is filtering if and only if* F' *is left covering.*

Proof: Recall that in such a category $\mathbb{A}$ every epimorphism is regular (cf. [17]). Keeping in mind the description of weak limits in Fam$\mathcal{D}$ given at the beginning of section 2.3, the three conditions of the previous definition are equivalent respectively to the fact that F' is left covering with respect to the terminal object, products of objects coming from $\mathcal{D}$ and equalizers of arrows coming from $\mathcal{D}$. One immediately concludes by lemma 2.3.1 and corollary 2.3.2. ∎

Putting together propositions 2.4.2 and 2.4.4, we obtain an easy proof of the characterization of geometric morphisms $\mathbb{A} \longrightarrow \mathcal{SET}^{\mathcal{D}^{\mathrm{op}}}$ in terms of filtering functors $\mathcal{D} \longrightarrow \mathbb{A}$; our proof holds if $\mathbb{A}$ is a cocomplete pretopos.

Moreover, proposition 2.4.4 explains in some sense the (a little bit mysterious) definition of filtering functor.

2.5 Sup-lattices as free exact categories

In section 2.1, we have characterized the category of algebras for a monad over $\mathcal{SET}$ as an exact category with enough projectives such that the full subcategory of projectives is equivalent to the category of free algebras. The fact that free algebras are projective depends on the axiom of choice in $\mathcal{SET}$.

In this section we give an example of a monadic category which is exact and whose free algebras are projective without using the axiom of choice. For this, we need some elementary facts about the regularity and exactness of monadic categories.

In the following, $\mathbb{T} = (T, \mu\colon T^2 \longrightarrow T, \epsilon\colon 1 \longrightarrow T)$ is a monad over a category $\mathbb{A}$, EM($\mathbb{T}$) and KL($\mathbb{T}$) are respectively the Eilenberg-Moore and the Kleisli category of $\mathbb{T}$.

Proposition 2.5.1 *Let* $\mathbb{A}$ *be regular;*

1) T *preserves regular epis if and only if the forgetful functor*

$$U\colon EM(\mathbb{T}) \longrightarrow \mathbb{A}$$

preserves regular epis

2) if T preserves regular epis, then EM($\mathbb{T}$) is regular and U preserves and reflects the regular epi-mono factorization

Proof: 1) (if): obvious, because $T = F \cdot U$ with F the left adjoint of U. (only if): it follows from the construction of the regular epi-mono factorization in EM($\mathbb{T}$) contained in the proof of part 2).
2): EM($\mathbb{T}$) is left exact because $\mathbb{A}$ is left exact and U creates limits; let us consider now a morphism $f\colon (C,c) \longrightarrow (D,d)$ in EM($\mathbb{T}$) that is, C is an object of $\mathbb{A}$ and $c\colon TC \longrightarrow C$ is the structural map, $f\colon C \longrightarrow D$ is a morphism in $\mathbb{A}$ such that the following diagram commutes

$$\begin{tikzcd} TC \arrow[r, "Tf"] \arrow[d, "c"'] & TD \arrow[d, "d"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

Since $\mathbb{A}$ is regular, we can factor $f\colon C \longrightarrow D$ as a regular epi followed by a mono

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "e"'] & & D \\ & I \arrow[ur, "m"'] & \end{tikzcd}$$

We have to prove that this is also the regular epi-mono factorization in EM($\mathbb{T}$). By assumption, Te is a regular epi, so that there exists $i\colon TI \longrightarrow I$ making commutative the following diagram in each part

$$\begin{tikzcd} TC \arrow[r, "Te"] \arrow[d, "c"'] & TI \arrow[r, "Tm"] \arrow[d, "i"] & TM \arrow[d, "d"] \\ C \arrow[r, "e"'] & I \arrow[r, "m"'] & D \end{tikzcd}$$

Since $m\colon I \longrightarrow D$ is a mono in $\mathbb{A}$, (I,i) is a $\mathbb{T}$-algebra (cf. section 3.2 of [3]) and clearly $e\colon (C,c) \longrightarrow (I,i)$ and $m\colon (I,i) \longrightarrow (D,d)$ are morphisms of $\mathbb{T}$-algebras. m is a mono in EM($\mathbb{T}$) because it is a mono in $\mathbb{T}$ and $U\colon \mathrm{EM}(\mathbb{T}) \longrightarrow \mathbb{A}$ reflects monomorphisms.

Consider the kernel pair $e_1, e_2\colon (N,n) \rightrightarrows (C,c)$ of $e\colon (C,c) \longrightarrow (I,i)$ in EM($\mathbb{T}$), so that $e_1, e_2\colon N \rightrightarrows C$ is the kernel pair of $e\colon C \longrightarrow I$ in $\mathbb{A}$; but e is a regular epi, so that it is the coequalizer of e_1 and e_2 in $\mathbb{A}$. It is the coequalizer of e_1 and e_2 also in EM($\mathbb{T}$): in fact, if $h\colon (C,c) \longrightarrow (A,a)$ is a $\mathbb{T}$-algebra morphism such that $e_1 \cdot h = e_2 \cdot h$, then there exists a unique $g\colon I \longrightarrow A$ in $\mathbb{A}$ such that $e \cdot g = h$. But g is in EM($\mathbb{T}$); in fact $Te \cdot Tg \cdot a = T(e \cdot g) \cdot a = Th \cdot a = c \cdot h = c \cdot e \cdot g = Te \cdot i \cdot g$ and then $Tg \cdot a = i \cdot g$ because Te is an epimorphism.

From the above discussion, it follows that $U\colon \mathrm{EM}(\mathbb{T}) \longrightarrow A$ preserves and reflects regular epis (and then the regular epi-mono factorization) so that regular epis are pullback stable in EM($\mathbb{T}$) because they are pullback stable in $\mathbb{T}$. ■

Proposition 2.5.2 *Let $\mathbb{A}$ be regular;*

1) *T sends regular epis into split epis (that is, epis with a section) if and only if U sends regular epis into split epis*

2) *if T sends regular epis into split epis, then the free algebras are projective.*

Proof: 1) (if): obvious. (only if): let $f\colon (X,x) \longrightarrow (Y,y)$ be a regular epi in EM($\mathbb{T}$); by proposition 2.5.1, $f\colon X \longrightarrow Y$ is a regular epi in $\mathbb{A}$ and then $Tf : TX \longrightarrow TY$ has a section $s\colon TY \longrightarrow TX$. A section of f in $\mathbb{A}$ is given by the following composition $Y \xrightarrow{\epsilon_Y} TY \xrightarrow{s} TX \xrightarrow{x} X$; in fact,

$$\epsilon_Y \cdot s \cdot x \cdot f = \epsilon_Y \cdot s \cdot Tf \cdot y = \epsilon_Y \cdot y = 1_Y.$$

2): by 2.5.1, EM($\mathbb{T}$) is regular, so that to prove that a free algebra is projective it suffices to show that in EM($\mathbb{T}$) a regular epi $f\colon (D,d) \longrightarrow (TC,\mu_C)$ splits. Once again $Tf\colon TD \longrightarrow T(TC)$ has a section $s\colon T(TC) \longrightarrow TD$ in $\mathbb{A}$. Consider now $g\colon C \longrightarrow D$ in $\mathbb{A}$ given by the following composition

$$C \xrightarrow{\epsilon_C} TC \xrightarrow{\epsilon_{TC}} T(TC) \xrightarrow{s} TD \xrightarrow{d} D;$$

$TC \xrightarrow{Tg} TD \xrightarrow{d} D$ is a morphism $(TC,\mu_C) \longrightarrow (D,d)$ in EM($\mathbb{T}$) and the verification that it gives the required section of $f\colon (D,d) \longrightarrow (TC,\mu_C)$ is straightforward. ■

Lemma 2.5.3 *Let $\mathbb{A}$ be an exact category; consider an equivalence relation $e_1, e_2\colon (E,e) \rightrightarrows (X,x)$ in EM($\mathbb{T}$) and its coequalizer $q\colon X \longrightarrow Q$ in $\mathbb{A}$; if*

$$\begin{tikzcd} TE \arrow[r, shift left, "Te_1"] \arrow[r, shift right, "Te_2"'] & TX \arrow[r, "Tq"] & TQ \end{tikzcd}$$

is a coequalizer diagram in $\mathbb{A}$, then $e_1, e_2\colon (E,e) \rightrightarrows (X,x)$ is effective.

Proof: Since U is left exact, $e_1, e_2\colon E \rightrightarrows X$ is an equivalence relation in $\mathbb{A}$ and then we can complete it in an exact sequence

$$\begin{tikzcd} E \arrow[r, shift left, "e_1"] \arrow[r, shift right, "e_2"'] & X \arrow[r, "q"] & Q \end{tikzcd}$$

By assumption, there exists a unique arrow $\lambda\colon TQ \longrightarrow Q$ making commutative the right-hand square in the following diagram

$$\begin{tikzcd} TE \arrow[r, shift left, "Te_1"] \arrow[r, shift right, "Te_2"'] \arrow[d, "c"'] & TX \arrow[r, "Tq"] \arrow[d, "x"] & TQ \arrow[d, "\lambda"] \\ E \arrow[r, shift left, "e_1"] \arrow[r, shift right, "e_2"'] & X \arrow[r, "q"'] & Q \end{tikzcd}$$

Using the fact that q and Tq are epimorphisms, one can easily show that (Q, λ) is a $\mathbb{T}$-algebra. Moreover (e_1, e_2) is the kernel pair of q in $\mathbb{A}$ and then it is the kernel pair of q also in EM($\mathbb{T}$) because U reflects limits. ■

Proposition 2.5.4 *Let $\mathbb{A}$ be exact; if*

- *T preserves regular epis and*
- *T preserves the coequalizers in $\mathbb{A}$ of the equivalence relation in EM($\mathbb{T}$)*

then EM($\mathbb{T}$) is exact.

Proof: By proposition 2.5.1, the first assumption implies that EM($\mathbb{T}$) is regular; by lemma 2.5.3, the second assumption implies that in EM($\mathbb{T}$) each equivalence relation is effective. ■

Corollary 2.5.5 *Let $\mathbb{A}$ be exact,*

1) *if T is left exact and preserves regular epis, then EM($\mathbb{T}$) is exact;*
2) *if the coequalizer in $\mathbb{A}$ of an equivalence relation in EM($\mathbb{T}$) is a split epi in $\mathbb{A}$, then EM($\mathbb{T}$) is exact and the free algebras are projective;*
3) *the axiom of choice holds in $\mathbb{A}$ if and only if, for every monad $\mathbb{T}$ over $\mathbb{A}$, EM($\mathbb{T}$) is exact and the free algebras are projective.*

Proof: With the notation of lemma 2.5.3, 1) Tq is a regular epi, so that

$$TQ = \mathrm{coeq}(\ker(Tq)) = \mathrm{coeq}(T(\ker q)) = \mathrm{coeq}(Te_1, Te_2)$$

and we can apply proposition 2.5.4;

2)

$$\begin{tikzcd} E \arrow[r, shift left, "e_1"] \arrow[r, shift right, "e_2"'] & X \arrow[r, "q"] & Q \end{tikzcd}$$

is a split exact sequence, then it is an absolute coequalizer and by lemma 2.5.3 in EM($\mathbb{T}$) equivalence relations are effective. Let us prove now that U sends regular epis into split epis (so that by proposition 2.5.1 EM($\mathbb{T}$) is regular and by proposition 2.5.2 free algebras are projective). Consider a regular epi $f \colon (X, x) \longrightarrow (Y, y)$ in EM($\mathbb{T}$) and the corresponding exact sequence

$$\begin{tikzcd} (N, n) \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & (X, x) \arrow[r, "f"] & (Y, y). \end{tikzcd}$$

Since U is left exact, we can complete $f_1, f_2 \colon N \rightrightarrows X$ in an exact sequence

$$N \underset{f_2}{\overset{f_1}{\rightrightarrows}} X \xrightarrow{q} Q$$

in $\mathbb{A}$; by assumption, this is a splitting exact sequence and then an absolute coequalizer. Now working as in lemma 2.5.3, one can give to Q a $\mathbb{T}$-algebra structure making $q\colon X \longrightarrow Q$ a morphism of $\mathbb{T}$-algebras. Moreover, using the fact that $Tq\colon TX \longrightarrow TQ$ is an epimorphism, one has immediately that $q\colon X \longrightarrow Q$ is the coequalizer of $f_1, f_2\colon (N,n) \rightrightarrows (X,x)$ in EM($\mathbb{T}$), so that it is isomorphic to $f\colon (X,x) \longrightarrow (Y,y)$ and then f splits in $\mathbb{A}$;

3) here the axiom of choice means that each regular epi splits; using the previous point, one has one implication. For the opposite implication, just consider the identity functor of $\mathbb{A}$. ∎

As each algebra is a quotient of a free algebra, if free algebras are projective, then EM($\mathbb{T}$) has enough projectives; if, moreover, EM($\mathbb{T}$) is exact, one has that EM($\mathbb{T}$) is the free exact category over its full subcategory KL($\mathbb{T}$) of free algebras.

In section 2.1 we have discussed the case $\mathbb{A} = \mathcal{SET}$ using the axiom of choice; it can be easily generalized to a power of $\mathcal{SET}$. Now we can give a topos-theoretic example of a free exact category.

Proposition 2.5.6 *Let $\mathcal{E}$ be an elementary topos; the category of sup-lattices in $\mathcal{E}$ is the free exact category over the category of relations in $\mathcal{E}$.*

Proof: Let us consider the covariant monad "power-set" $\mathcal{P}\colon \mathcal{E} \longrightarrow \mathcal{E}$. It is well-known that EM($\mathcal{P}$) $\simeq$ SL($\mathcal{E}$), the category of suplattices in $\mathcal{E}$. Moreover, KL($\mathcal{P}$) $\simeq$ Rel($\mathcal{E}$), the category of relations in $\mathcal{E}$: in fact, sup-preserving arrows $\mathcal{P}X \longrightarrow \mathcal{P}Y$ are in bijection with arrows $X \longrightarrow \mathcal{P}Y$ which are in bijection with relations $R \hookrightarrow X \times Y$. Let us prove now that the second point of the previous corollary holds. We sketch the proof using the internal language of $\mathcal{E}$.

Consider an equivalence relation $e_1, e_2\colon E \rightrightarrows X$ in SL($\mathcal{E}$) and its coequalizer $q\colon X \longrightarrow Q$ in $\mathcal{E}$; we obtain a section $s\colon Q \longrightarrow X$ of q defining

$$\forall\, y \in Q \quad s(y) = \mathrm{Sup}\{x \in X \mid \; q(x) = y\}.$$

Let us show that $q \cdot s \cdot q = q$ (then $s \cdot q = 1_Q$ because q is an epimorphism). Consider $\overline{x} \in X$, then $q(s(q(\overline{x}))) = q(\alpha)$ where $\alpha = \mathrm{Sup}\{x \in X \mid \; q(x) = q(\overline{x})\}$, but $q(x) = q(\overline{x})$ if and only if $xE\overline{x}$, so that $\alpha = \mathrm{Sup}\{x \in X \mid \; xE\overline{x}\}$. Since E is a congruence, this implies that $\alpha E \overline{x}$ and then $q(\alpha) = q(\overline{x})$ as required. ∎

To complete our analysis, let us observe that the conditions stated in 2.5.2 and in 2.5.5.2 are also necessary. More precisely, we have the following lemma.

Lemma 2.5.7 *Let $\mathbb{T}$ be a monad over a category $\mathbb{A}$;*

1) *if EM($\mathbb{T}$) is regular and free algebras are projective, then U sends regular epis into split epis*

2) *if U sends regular epis into (split) regular epis, then the coequalizer in $\mathbb{A}$ of an exact sequence in EM($\mathbb{T}$) is a (split) regular epi in $\mathbb{A}$.*

Proof: 1) Let us consider the following situation in EM($\mathbb{T}$)

$$\begin{tikzcd} (TA, \mu_A) \arrow[r, "a"] & (A, a) \arrow[d, "f"] \\ (TB, \mu_B) \arrow[r, "b"'] & (B, b) \end{tikzcd}$$

a is a regular epi in EM($\mathbb{T}$) and b has a section $\epsilon_B : B \longrightarrow TB$ in $\mathbb{A}$; if (TB, μ_B) is projective and f is a regular epi in EM($\mathbb{T}$), then there exists $y : (TB, \mu_B) \longrightarrow (TA, \mu_A)$ such that $y \cdot a \cdot f = b$; clearly $\epsilon_B \cdot y \cdot a$ is a section of f in $\mathbb{A}$.
2) consider an exact sequence in EM($\mathbb{T}$)

$$(E, e) \underset{e_2}{\overset{e_1}{\rightrightarrows}} (X, x) \xrightarrow{q} (Q, \lambda).$$

Obviously

$$E \underset{e_2}{\overset{e_1}{\rightrightarrows}} X \xrightarrow{q} Q$$

is a kernel pair in $\mathbb{A}$; but, by assumption, q is a regular epi in $\mathbb{A}$, so that it is an exact sequence also in $\mathbb{A}$. ■

Now we can summarize the previous discussion as follows:

Proposition 2.5.8 *Let $\mathbb{A}$ be an exact category and $\mathbb{T}$ a monad over $\mathbb{A}$; the following conditions are equivalent:*

1) *EM($\mathbb{T}$) is exact and the free algebras are projective*

2) *the coequalizer in $\mathbb{A}$ of an equivalence relation in EM($\mathbb{T}$) is a split epi in $\mathbb{A}$.* ■

Example 2.5.9

Let us recall that a further example of an exact category with enough projective objects is given by the dual $\mathcal{E}^{\mathrm{op}}$ of a topos $\mathcal{E}$. One can prove this fact using our previous proposition. For this, consider the contravariant "power-set" functor $\mathcal{P} : \mathcal{E}^{\mathrm{op}} \longrightarrow \mathcal{E}$. The proof that it is monadic runs as follows (cf. [8] or [32]): consider a reflexive pair in $\mathcal{E}^{\mathrm{op}}$, that is a pair of arrows $f, g : X \rightrightarrows Y$ in $\mathcal{E}$ with a common retraction $r : Y \longrightarrow X$ (that is $f \cdot r = 1_X = g \cdot r$).

Consider now the equalizer in $\mathcal{E}$

$$E \xrightarrow{e} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

and apply the functor $\mathcal{P} : \mathcal{E}^{\mathrm{op}} \longrightarrow \mathcal{E}$.

Using Beck condition, one can easily prove that

$$\mathcal{P}X \underset{\mathcal{P}g}{\overset{\mathcal{P}f}{\rightrightarrows}} \mathcal{P}Y \xrightarrow{\mathcal{P}e} \mathcal{P}E$$

is a split coequalizer.

If the given pair of arrows in $\mathcal{E}^{\mathrm{op}}$ is, in particular, an equivalence relation, the previous argument shows that condition 2) of proposition 2.5.8 is satisfied.

Chapter 3

The regular completion

3.1 A glance at the left exact case

In this chapter, we want to build up the regular completion of a weakly lex category and to establish its universal property. In the first chapter, we have seen that the exact completion of a weakly lex category is defined formally in the same way as the exact completion of a left exact category. On the contrary, the regular completion of a weakly lex category needs some modifications with respect to the left exact case. For this reason, we prefer to start by briefly recalling the construction of the regular completion of a left exact category, which has been introduced in [19] (see also [14]).

Definition 3.1.1 *Let $\mathbb{C}$ be a left exact category; we define a new category $\mathbb{C}_{reg}$ as follows:*

- *objects: an object of $\mathbb{C}_{reg}$ is an arrow $f\colon X \longrightarrow X'$ in $\mathbb{C}$*
- *arrows: an arrow between two objects $f\colon X \longrightarrow X'$ and $g\colon Y \longrightarrow Y'$ of $\mathbb{C}_{reg}$ is an equivalence class of arrows $\alpha\colon X \longrightarrow Y$ of $\mathbb{C}$ such that $f_0 \cdot \alpha \cdot g = f_1 \cdot \alpha \cdot g$*

$$\begin{tikzcd} N(f) \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "f_1"] & \\ X \arrow[r, "\alpha"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X' & Y' \end{tikzcd}$$

(where $f_0, f_1\colon N(f) \rightrightarrows X$ is the kernel pair of f); two arrows of this kind $\alpha\colon X \longrightarrow Y$ and $\beta\colon X \longrightarrow Y$ are said to be equivalent if $\alpha \cdot g = \beta \cdot g$.

- *composition and identities are the obvious ones.*

Once again, the previous definition needs some comments: think of $\mathbb{C}$ as a left exact category with coequalizer of kernel pairs and in which each object is projective; an arrow $[\alpha]: (f: X \longrightarrow X') \longrightarrow (g: Y \longrightarrow Y')$ as in definition 3.1.1 is then exactly an arrow between the image of f (that is, the coequalizer of $f_0, f_1: N(f) \rightrightarrows X$)and the image of g making commutative the following diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d] & Y \arrow[d] \\ \mathrm{Im} f \arrow[r] & \mathrm{Im} g \end{tikzcd}$$

Now let us make an obvious remark which will be helpful in next sections. Consider two arrows in $\mathbb{C}$ $f_1: X \longrightarrow X_1$ and $f_2: X \longrightarrow X_2$; since $\mathbb{C}$ is left exact, this is equivalent to giving a unique arrow $x: X \longrightarrow X_1 \times X_2$, that is an object of $\mathbb{C}_{\text{reg}}$. Consider its kernel pair $x_0, x_1: N(x) \rightrightarrows X$; it is universal with the property that $x_0 \cdot f_1 = x_1 \cdot f_1$ and $x_0 \cdot f_2 = x_1 \cdot f_2$.

We are ready now to discuss the regular completion of a weakly lex category following the same steps as in the first chapter.

3.2 The regular completion

In this section we construct a regular category from a weakly lex one.

Definition 3.2.1 *Let $\mathbb{C}$ be a weakly lex category; we define a new category $\mathbb{C}_{\text{reg}}$ as follows:*

- *objects: an object of $\mathbb{C}_{\text{reg}}$ is a finite (possibly empty) family of arrows $(f_i: X \longrightarrow X_i)_I$ in $\mathbb{C}$ (all the arrows f_i of the family have the same domain)*

- *arrows: an arrow between two objects*

$$(f_i: X \longrightarrow X_i)_I \quad \text{and} \quad (g_j: Y \longrightarrow Y_j)_J$$

of $\mathbb{C}_{\text{reg}}$ is an equivalence class of arrows $\alpha: X \longrightarrow Y$ of $\mathbb{C}$ such that $\forall\, j \in J$, $x_0 \cdot \alpha \cdot g_j = x_1 \cdot \alpha \cdot g_j$

$$
\begin{tikzcd}
\overline{X} \arrow[d, shift right, "x_0"'] \arrow[d, shift left, "x_1"] & \\
X \arrow[r, "\alpha"] \arrow[d, "f_i"'] & Y \arrow[d, "g_j"] \\
X_i & Y_j
\end{tikzcd}
$$

(where $x_0, x_1 : \overline{X} \rightrightarrows X$ is a pair which is weak universal with respect to the property that $\forall\ i \in I$, $x_0 \cdot f_i = x_1 \cdot f_i$). Two arrows of this kind $\alpha : X \longrightarrow Y$ and $\beta : X \longrightarrow Y$ are said to be equivalent if $\forall j \in J$, $\alpha \cdot g_j = \beta \cdot g_j$

- *compositions and identities are the obvious ones.*

When it will not be ambiguous, we will use the notation $[\alpha] : (f_i) \longrightarrow (g_j)$ for the equivalence class of

$$\alpha : (f_i : X \longrightarrow X_i)_I \longrightarrow (g_j : Y \longrightarrow Y_j)_J .$$

Observe that a pair of arrows $x_0, x_1 : \overline{X} \rightrightarrows X$ with the required weak universal property certainly exists in $\mathbb{C}$. In fact this property means exactly that $(\overline{X}; x_0, x_1)$ is a weak limit in the following diagram

$$
\begin{tikzcd}
& \overline{X} \arrow[dl, "x_0"'] \arrow[dr, "x_1"] & \\
X \arrow[d, "f_i"'] \arrow[drr, "f_j" near start] & & X \arrow[d, "f_j"] \arrow[dll, "f_i"' near start] \\
X_i & & X_j
\end{tikzcd}
$$

(think of two distinct copies of f_i for each i in I). If the family (f_i) is indexed over the empty set, then the pair $x_0, x_1 : \overline{X} \rightrightarrows X$ in the previous description is nothing but the two projections from a weak product $X \times X \rightrightarrows X$.

Observe also that the conditions on $\alpha : X \longrightarrow Y$ for being an arrow do not depend on the choice of $\overline{X} \rightrightarrows X$ (which is, by no way, uniquely determined). In fact, if $t_0, t_1 : T \rightrightarrows X$ is another pair with the same weak universal property, then there exists $m : T \longrightarrow \overline{X}$ such that $m \cdot x_0 = t_0$ and $m \cdot x_1 = t_1$ and there exists $n : \overline{X} \longrightarrow T$ such that $n \cdot t_0 = x_0$ and $n \cdot t_1 = x_1$; now $x_0 \cdot \alpha \cdot g_j = x_1 \cdot \alpha \cdot g_j$ if and only if $t_0 \cdot \alpha \cdot g_j = t_1 \cdot \alpha \cdot g_j$.

Now observe that if $\alpha: X \longrightarrow Y$ is such that $\forall\, j \in J$, $x_0 \cdot \alpha \cdot g_j = x_1 \cdot \alpha \cdot g_j$, then there exists $\overline{\alpha}: \overline{X} \longrightarrow \overline{Y}$ such that $\overline{\alpha} \cdot y_0 = x_0 \cdot \alpha$ and $\overline{\alpha} \cdot y_1 = x_1 \cdot \alpha$; moreover, if $\alpha': X \longrightarrow Y$ is such that $\forall\, j \in J$, $\alpha \cdot g_j = \alpha' \cdot g_j$, then there exists $\Sigma: X \longrightarrow \overline{Y}$ such that $\Sigma \cdot y_0 = \alpha$ and $\Sigma \cdot y_1 = \alpha'$; the situation is illustrated in the following two diagrams

$$\begin{tikzcd}
\overline{X} \arrow[r, "\overline{\alpha}"] \arrow[d, shift right, "x_0"'] \arrow[d, shift left, "x_1"] & \overline{Y} \arrow[d, shift right, "y_0"'] \arrow[d, shift left, "y_1"] \\
X \arrow[r, "\alpha"'] \arrow[d, "f_i"'] & Y \arrow[d, "g_j"] \\
X_i & Y_j
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{X} \arrow[d, shift right, "x_0"'] \arrow[d, shift left, "x_1"] & \overline{Y} \arrow[d, shift right, "y_0"'] \arrow[d, shift left, "y_1"] \\
X \arrow[ur, "\Sigma"] \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\alpha'"'] \arrow[d, "f_i"'] & Y \arrow[d, "g_j"] \\
X_i & Y_j
\end{tikzcd}$$

From these two last observations, one can immediately deduce that the relation between arrows is really an equivalence relation stable under composition and the composite of two arrows in $\mathbb{C}_{\mathrm{reg}}$ is again an arrow. We can conclude that the previous definition is well posed.

Now an easy observation; it will be crucial in section 3.5 to compare the regular and the exact completion (cf. also the last remark at the end of section 1.4).

Proposition 3.2.2 *Let $(f_i: X \longrightarrow X_i)_I$ be a finite family of arrows in a weakly lex category $\mathbb{C}$ and consider a pair $x_0, x_1: \overline{X} \rightrightarrows X$ weakly universal with respect to the condition $x_0 \cdot f_i = x_1 \cdot f_i$ for each i in I. Such a pair x_0, x_1 is a pseudo equivalence-relation.*

Proof: (Think of the pair $x_0, x_1: \overline{X} \rightrightarrows X$ as the kernel pair of the arrow $\langle f_i \rangle: X \longrightarrow \coprod_I X_i$.) Transitivity: consider a weak pullback

$$\begin{tikzcd}
\overline{X} * \overline{X} \arrow[r, "l_0"] \arrow[d, "l_1"'] & \overline{X} \arrow[d, "x_1"] \\
\overline{X} \arrow[r, "x_0"'] & X
\end{tikzcd}$$

As $l_0 \cdot x_0 \cdot f_i = l_0 \cdot x_1 \cdot f_i = l_1 \cdot x_0 \cdot f_i = l_1 \cdot x_1 \cdot f_i$ for each i, there exists $t_{\overline{X}}: \overline{X} * \overline{X} \longrightarrow \overline{X}$ such that $t_{\overline{X}} \cdot x_0 = l_0 \cdot x_0$ and $t_{\overline{X}} \cdot x_1 = l_1 \cdot x_1$. This means exactly that $t_{\overline{X}}$ is the transitivity of $x_0, x_1: \overline{X} \rightrightarrows X$. Analogously for the reflexivity and symmetry. ■

Theorem 3.2.3 *Let $\mathbb{C}$ be a weakly lex category and let $\mathbb{C}_{\mathrm{reg}}$ be as in definition 3.2.1; $\mathbb{C}_{\mathrm{reg}}$ is a regular category.*

Step 1: $\mathbb{C}_{\mathrm{reg}}$ is a left exact category. Consider two objects (f_i) and (g_j) in $\mathbb{C}_{\mathrm{reg}}$; their product is given in the following diagram

$$\begin{tikzcd}
X \arrow[d, "f_i"'] & & X \times Y \arrow[ll, "\pi_X"'] \arrow[rr, "\pi_Y"] \arrow[dl, "\pi_X \cdot f_i"'] \arrow[dr, "\pi_Y \cdot g_j"] & & Y \arrow[d, "g_j"] \\
X_i & X_i & & Y_j & Y_j
\end{tikzcd}$$

where $X \xleftarrow{\pi_X} X \times Y \xrightarrow{\pi_Y} Y$ is a weak product in $\mathbb{C}$.

If T is a weak terminal object in $\mathbb{C}$, then the empty family $(T \longrightarrow)_\emptyset$ of arrows with domain T is the terminal object of $\mathbb{C}_{\mathrm{reg}}$.

Consider now two parallel arrows in $\mathbb{C}_{\mathrm{reg}}$ $[\alpha], [\beta] \colon (f_i) \rightrightarrows (g_j)$; their equalizer is given in the following diagram

$$\begin{tikzcd}
E \arrow[r, "e"] \arrow[d, "e \cdot f_i"'] & X \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] \arrow[d, "f_i"] & Y \arrow[d, "g_j"] \\
X_i & X_i & Y_j
\end{tikzcd}$$

where $e \colon E \longrightarrow X$ is a weak limit over the diagram

$$\begin{tikzcd}
X \arrow[r, shift left, "\alpha \cdot g_j"] \arrow[r, shift right, "\beta \cdot g_j"'] & Y_j
\end{tikzcd}$$

(think of a pair of parallel arrows for each j in J).

Let us verify that $[e] \colon (e \cdot f_i) \longrightarrow (f_i)$ is the equalizer of $[\alpha]$ and $[\beta]$ in $\mathbb{C}_{\mathrm{reg}}$. Observe that $[e]$ is a mono: given two arrows $[a]$ and $[b]$, the equations $[a] \cdot [e] = [b] \cdot [e]$ and $[a] = [b]$ both means $a \cdot e \cdot f_i = b \cdot e \cdot f_i$ for each i.

Now if $[\gamma] \colon (h_K \colon Z \longrightarrow Z_K)_K \longrightarrow (f_i \colon X \longrightarrow X_i)_I$ is such that $[\gamma] \cdot [\alpha] = [\gamma] \cdot [\beta]$, one has $\gamma \cdot \alpha \cdot g_j = \gamma \cdot \beta \cdot g_j$ for each$j$ and then there exists $\tilde{\gamma} \colon Z \longrightarrow E$ such that $\tilde{\gamma} \cdot e = \gamma$.

It remains only to prove that $[\tilde{\gamma}] \colon (h_K) \longrightarrow (e \cdot f_i)$ is an arrow in $\mathbb{C}_{\mathrm{reg}}$. By assumption $z_0 \cdot \gamma \cdot f_i = z_j \cdot \gamma \cdot f_i$, so that $z_0 \cdot \tilde{\gamma} \cdot e \cdot f_i = z_0 \cdot \gamma \cdot f_i = z_i \cdot \gamma \cdot f_i = z_j \cdot \tilde{\gamma} \cdot e \cdot f_i$.

As far as the terminal object and products are concerned, the verifications run in a similar way and we omit the details.

Step 2: $\mathbb{C}_{\mathrm{reg}}$ has regular epi-mono factorization and regular epis are stable under pullbacks.

Consider an arrow $[\alpha] \colon (f_i) \longrightarrow (g_j)$ in $\mathbb{C}_{\mathrm{reg}}$; its factorization is given in the following diagram

$$\begin{tikzcd}
X \arrow[rr, "\alpha"] \arrow[d, "f_i"'] \arrow[dr, "1_X"] & & Y \arrow[d, "g_j"] \\
X_i & X \arrow[ur, "\alpha"'] \arrow[d, "\alpha \cdot g_j"] & Y_j \\
& Y_j &
\end{tikzcd}$$

Obviously the second part is a monomorphism in $\mathbb{C}_{\text{reg}}$. Let us show that in $\mathbb{C}_{\text{reg}}$ an arrow like

$$[1_X]: (f_i: X \longrightarrow X_i)_I \longrightarrow (g_j: X \longrightarrow Y_j)_J$$

is always the coequalizer of its kernel pair: the kernel pair is given in the following diagram

$$\begin{tikzcd}
& E \arrow[r, shift left, "e \cdot \pi_1"] \arrow[r, shift right, "e \cdot \pi_2"'] \arrow[dl, "e \cdot \pi_1 \cdot f_i"'] \arrow[d, "e \cdot \pi_2 \cdot f_i"] & X \arrow[r, "1_X"] \arrow[d, "f_i"] & X \arrow[d, "g_j"] \\
X_i & X_i & X_i & Y_j
\end{tikzcd}$$

where $X \xleftarrow{\pi_1} X \times X \xrightarrow{\pi_2} X$ and $E \xrightarrow{e} X \times X \underset{\pi_2 \cdot g_j}{\overset{\pi_1 \cdot g_j}{\rightrightarrows}} Y_j$ are weak limits computed in $\mathbb{C}$ to obtain a pullback in $\mathbb{C}_{\text{reg}}$ via the description of products and equalizers given in step 1.

Obviously the arrow $[1_X]: (f_i) \longrightarrow (g_j)$ coequalizes the previous diagram in $\mathbb{C}_{\text{reg}}$. Suppose now that $[\alpha]: (f_i) \longrightarrow (h_k)$ is a second arrow in $\mathbb{C}_{\text{reg}}$ such that $e \cdot \pi_1 \cdot \alpha$ is equivalent to $e \cdot \pi_2 \cdot \alpha$. We can take $[\alpha]: (g_j) \longrightarrow (h_k)$ as factorization; the only problem is to show that it is really an arrow in $\mathbb{C}_{\text{reg}}$.

For this, consider a pair of arrows $y_0, y_1: \overline{Y} \rightrightarrows X$ such that $\forall\, j \in J$, $y_0 \cdot g_j = y_1 \cdot g_j$; those relations imply the existence of an arrow $\overline{y}: \overline{Y} \longrightarrow E$ such that $\overline{y} \cdot e \cdot \pi_1 = y_0$ and $\overline{y} \cdot e \cdot \pi_2 = y_1$; but now $y_0 \cdot \alpha \cdot h_k = \overline{y} \cdot e \cdot \pi_1 \cdot \alpha \cdot h_k = \overline{y} \cdot e \cdot \pi_2 \cdot \alpha \cdot h_k = y_1 \cdot \alpha \cdot h_k\ \forall\, k \in K$ because α coequalizes $e \cdot \pi_1$ and $e \cdot \pi_2$ in $\mathbb{C}_{\text{reg}}$. The factorization is unique because $[1_X]: (f_i) \longrightarrow (g_j)$ is obviously an epimorphism in $\mathbb{C}_{\text{reg}}$.

The previous argument shows that in $\mathbb{C}_{\text{reg}}$ a regular epi is always, up to isomorphisms, of the form

$$[1_X]: (f_i: X \longrightarrow X_i)_I \longrightarrow (g_j: X \longrightarrow Y_j)_J.$$

Using this fact, the verification of the stability conditions becomes easy. For this, consider the pullback of $[1_X]\colon (f_i\colon X \longrightarrow X_i)_I \longrightarrow (g_j\colon X \longrightarrow Y_j)_J$ along an arrow $[\beta]\colon (h_k\colon Z \longrightarrow Z_k)_K \longrightarrow (g_j\colon X \longrightarrow Y_j)_J$.

It is given by

$$\begin{tikzcd}
 & E \arrow[r, "e \cdot \pi_z"] \arrow[dl, "e \cdot \pi_X \cdot f_i"'] \arrow[d, "e \cdot \pi_z \cdot h_K"] & Z \arrow[d, "h_K"] \\
X_i & Z_K & Z_K
\end{tikzcd}$$

where $X \xleftarrow{\pi_X} X \times Z \xrightarrow{\pi_Z} Z$ and $E \xrightarrow{e} X \times Z \underset{\pi_Z \cdot \beta \cdot g_j}{\overset{\pi_X \cdot g_j}{\rightrightarrows}} Y_j$ are weak limits computed in $\mathbb{C}$ to obtain a pullback in $\mathbb{C}_{\mathrm{reg}}$ following the descriptions given in step 1.

Now we can consider the regular epi-mono factorization of $[e \cdot \pi_Z]$. It is given by the following diagram

$$\begin{tikzcd}
 & E \arrow[r, "1_E"] \arrow[dl, "e \cdot \pi_X \cdot f_i"'] \arrow[d, "e \cdot \pi_z \cdot h_K"] & E \arrow[r, "e \cdot \pi_z"] \arrow[d, "e \cdot \pi_Z \cdot h_K"] & Z \arrow[d, "h_K"] \\
X_i & Z_K & Z_K & Z_K
\end{tikzcd}$$

We look for an inverse to the monic part. Consider $b\colon Z \longrightarrow X \times Z$ such that $b \cdot \pi_X = \beta$ and $b \cdot \pi_Z = 1_Z$.

Since $b \cdot \pi_X \cdot g_j = \beta \cdot g_j = b \cdot \pi_Z \cdot \beta \cdot g_j$ for each j, there exists $a\colon Z \longrightarrow E$ such that $a \cdot e = b$.

Now observe that $z_0 \cdot a \cdot e \cdot \pi_Z \cdot h_K = z_0 \cdot h_K = z_1 \cdot h_K = z_1 \cdot a \cdot e \cdot \pi_Z \cdot h_K$, so that $[a]\colon (h_k) \longrightarrow (e \cdot \pi_Z \cdot h_k)$ is an arrow in $\mathbb{C}_{\mathrm{reg}}$. Since $a \cdot e \cdot \pi_Z = 1_Z$, $[e \cdot \pi_Z]$ is a split epi in $\mathbb{C}_{\mathrm{reg}}$ and then it is an isomorphism. ∎

The rest of this section is devoted to point out two properties of $\mathbb{C}_{\mathrm{reg}}$ which will be the characterizing properties for free regular categories over weakly lex ones. More precisely, we will show that $\mathbb{C}_{\mathrm{reg}}$ has enough projectives and that each object of $\mathbb{C}_{\mathrm{reg}}$ can be embedded in a product of projective objects.

Proposition 3.2.4 *Let $\mathbb{C}$ be a weakly lex category and $\mathbb{C}_{\mathrm{reg}}$ its regular completion as in definition 3.2.1; there exists a functor*

$$\Gamma\colon \mathbb{C} \longrightarrow \mathbb{C}_{\mathrm{reg}}$$

defined by

$$f\colon X \longrightarrow Y \quad \rightsquigarrow \quad [f]\colon (1_X\colon X \longrightarrow X) \longrightarrow (1_Y\colon Y \longrightarrow Y)$$

which is full and faithful and preserves monomorphic families. ∎

Proposition 3.2.5 *Let*

$$\Gamma\colon \mathbb{C} \longrightarrow \mathbb{C}_{reg}$$

be as in the previous proposition and consider an object $Y \in \mathbb{C}$; ΓY is a projective object in $\mathbb{C}_{reg}$.

Proof: Let $[\alpha]\colon (f_i\colon X \longrightarrow X_i)_I \longrightarrow (1_Y\colon Y \longrightarrow Y)$ be an arrow in $\mathbb{C}_{\mathrm{reg}}$ and its factorization

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "f_i"'] & X \arrow[r, "\alpha"] \arrow[d, "\alpha"] & Y \arrow[d, "1_Y"] \\ X_i & Y & Y \end{tikzcd}$$

Suppose that the monic part is an iso; there exists $\beta\colon Y \longrightarrow X$ such that $\beta \cdot \alpha = 1_Y$; the arrow $[\beta]\colon (1_Y) \longrightarrow (f_i)$ gives us the required section. ■

Proposition 3.2.6 *Let*

$$\Gamma\colon \mathbb{C} \longrightarrow \mathbb{C}_{reg}$$

be as in proposition 3.2.4 and consider an arrow in $\mathbb{C}_{reg}$

$$[\alpha]\colon (f_i) \longrightarrow (g_j);$$

we have the following commutative diagram in $\mathbb{C}_{reg}$

$$\begin{tikzcd} & \Gamma(X_i) & \\ \Gamma X \arrow[ur, "\Gamma f_i"] \arrow[r, "regepi"] \arrow[d, "\Gamma\alpha"'] & (f_i) \arrow[r, "mono"] \arrow[d, "{[\alpha]}"] & \prod_I \Gamma(X_i) \arrow[ul, "\pi_i"'] \\ \Gamma Y \arrow[r, "regepi"] \arrow[dr, "\Gamma g_j"'] & (g_j) \arrow[r, "mono"] & \prod_J \Gamma(Y_j) \arrow[dl, "\pi_j"] \\ & \Gamma(Y_i) & \end{tikzcd}$$

Before starting with the proof, let us read some consequences from the above diagram; keep in mind proposition 3.2.5.

Corollary 3.2.7

1) each object of $\mathbb{C}_{reg}$ is a quotient of a projective object coming from $\mathbb{C}$ and can be embedded in a product of projective objects coming from $\mathbb{C}$

2) each arrow of $\mathbb{C}_{reg}$ is the unique extension to the images of an arrow coming from $\mathbb{C}$

3) $\mathbb{C}_{reg}$ has enough projectives and $\Gamma(\mathbb{C})$ is a projective cover of $\mathbb{C}_{reg}$.

■

Proof: Proof of 3.2.6: from the diagram in step 2 of theorem 3.2.3, we know that

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "1_X"'] & X \arrow[d, "f_i"] \\ X & X_i \end{tikzcd}$$

is a regular epi in $\mathbb{C}_{reg}$; let us show that the family of arrows

$$\begin{tikzcd} X \arrow[r, "f_i"] \arrow[d, "f_i"'] & X_i \arrow[d, "1_{X_i}"] \\ X_i & X_i \end{tikzcd}$$

is monomorphic: given $[a], [b] \colon (h_k) \longrightarrow (f_i)$, $a \cdot f_i = b \cdot f_i \quad \forall\, i \in I$ in $\mathbb{C}_{reg}$ if and only if $a \cdot f_i = b \cdot f_i$ in $\mathbb{C}$ which means exactly $a = b$ in $\mathbb{C}_{reg}$. The commutativity of

$$\begin{tikzcd} \Gamma X \arrow[r] \arrow[d, "\Gamma\alpha"'] & (f_i) \arrow[d, "{[\alpha]}"] \\ \Gamma Y \arrow[r] & (g_j) \end{tikzcd}$$

is obvious. ■

3.3 The universality of the regular completion

In this section we show that the embedding $\Gamma\colon \mathbb{C} \longrightarrow \mathbb{C}_{\mathrm{reg}}$ is universal.

Theorem 3.3.1 *let $\mathbb{C}$ be a weakly lex category and $\mathbb{A}$ a regular one; consider the regular completion $\Gamma\colon \mathbb{C} \longrightarrow \mathbb{C}_{reg}$ described in 3.2.3 and in 3.2.4; composing with Γ induces an equivalence*

$$\Gamma_{reg}\colon Ex(\mathbb{C}_{reg}, \mathbb{A}) \longrightarrow Lco(\mathbb{C}, \mathbb{A})$$

between the category of exact functors from $\mathbb{C}_{reg}$ to $\mathbb{A}$ and the category of left covering functors from $\mathbb{C}$ to $\mathbb{A}$.

Corollary 3.3.2 *With the notation of 3.3.1, the left covering functor $\Gamma\colon \mathbb{C} \longrightarrow \mathbb{C}_{reg}$ is uniquely determined (up to equivalences) by the previous universal property*

■

Proof: Proof of 3.3.1:The more difficult part consists in proving that, given a left covering functor $F\colon \mathbb{C} \longrightarrow \mathbb{A}$, there exists a unique (up to natural isomorphisms) exact functor $\hat{F}\colon \mathbb{C}_{\mathrm{reg}} \longrightarrow \mathbb{A}$ making commutative the following diagram

$$\begin{tikzcd} \mathbb{C} \arrow[rr, "\Gamma"] \arrow[dr, "F"'] & & \mathbb{C}_{\mathrm{reg}} \arrow[dl, "\hat{F}"] \\ & \mathbb{A} & \end{tikzcd}$$

Keeping in mind the diagram in proposition 3.2.6 (and using the same notations), if $\hat{F}\colon \mathbb{C}_{\mathrm{reg}} \longrightarrow \mathbb{A}$ preserves regular epis, mono's and finite products and if $\Gamma \cdot \hat{F} \simeq F$, then $\hat{F}(f_i)$ must be the image in $\mathbb{A}$ of

$$\langle F f_i \rangle\colon FX \longrightarrow \prod_I F(X_i)$$

and $\hat{F}[\alpha]$ must be the unique extension to the images as in the following diagram

$$\begin{tikzcd}
& F(X_i) & \\
FX \arrow[ur, "Ff_i"] \arrow[r, "p"'] \arrow[d, "F\alpha"'] & \hat{F}(f_i) \arrow[r, "m"'] \arrow[d, "\hat{F}[\alpha]"] & \prod_I F(X_i) \arrow[ul, "\pi_i"'] \\
FY \arrow[r, "q"] \arrow[dr, "Fg_j"'] & \hat{F}(g_j) \arrow[r, "n"] & \prod_J F(Y_j) \arrow[dl, "\pi_j"] \\
& F(Y_j) &
\end{tikzcd}$$

This gives us the uniqueness of $\hat{F}$.

The existence of $\hat{F}$ on the objects depends only on the regularity of $\mathbb{A}$; if the extension $\hat{F}[\alpha]\colon \hat{F}(f_i) \longrightarrow \hat{F}(g_j)$ exists, then the functoriality of $\hat{F}$ and the fact that $\Gamma \cdot \hat{F} \simeq F$ are obvious.

Step 1: existence of $\hat{F}$ on the arrows. Consider an arrow in $\mathbb{C}_{\text{reg}}$

$$[\alpha]\colon (f_i) \longrightarrow (g_j)$$

By definition of an arrow in $\mathbb{C}_{\text{reg}}$, we have that $x_0 \cdot \alpha \cdot g_j = x_1 \cdot \alpha \cdot g_j \; \forall\, j \in J$ (cf. 3.2.1). Now take the kernel pair $p_0, p_1\colon N(p) \rightrightarrows FX$ of $\langle Ff_i \rangle\colon FX \longrightarrow \prod_I F(X_i)$ (that is of p).

Observe that

$$FX \xleftarrow{p_0} N(p) \xrightarrow{p_1} FX$$

is nothing but the limit on the diagram

$$\begin{tikzcd}
FX \arrow[d, "Ff_i"'] \arrow[drr, "Ff_j"] & & FX \arrow[d, "Ff_j"] \arrow[dll, "Ff_i"'] \\
FX_i & & FX_j
\end{tikzcd}$$

This implies that there exists a factorization $x\colon F\overline{X} \longrightarrow N(p)$ such that $x \cdot p_0 = Fx_0$ and $x \cdot p_1 = Fx_1$. Moreover, since F is left covering, this factorization is a regular epimorphism.

Analogously, $Fy_0, Fy_1 : F\overline{Y} \rightrightarrows FY$ can be factored through the kernel pair $q_0, q_1 : N(q) \rightrightarrows FY$ of the arrow $\langle Fg_j \rangle : FY \longrightarrow \prod_J F(Y_j)$. Recall also that there exists $\overline{\alpha} : \overline{X} \longrightarrow \overline{Y}$ such that $\overline{\alpha} \cdot y_0 = x_0 \cdot \alpha$ and $\overline{\alpha} \cdot y_1 = x_1 \cdot \alpha$.

We are now in the following situation

$$\begin{tikzcd}
F\overline{X} \arrow[r, "x"] \arrow[d, "F\overline{\alpha}"'] & N(p) \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & FX \arrow[r, "p"] \arrow[d, "F\alpha"] & \hat{F}(f_i) \\
F\overline{Y} \arrow[r, "y"'] & N(q) \arrow[r, shift left, "q_0"] \arrow[r, shift right, "q_1"'] & FY \arrow[r, "q"'] & \hat{F}(g_j)
\end{tikzcd}$$

Since

$$\begin{tikzcd}
N(p) \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & FX \arrow[r, "p"] & \hat{F}(f_i)
\end{tikzcd}$$

is a coequalizer diagram, to obtain the arrow $\hat{F}[\alpha] : \hat{F}(f_i) \longrightarrow \hat{F}(g_j)$ it suffices to show that $p_0 \cdot F\alpha \cdot q = p_1 \cdot F\alpha \cdot q$. It suffices to show that this is the case when we compose with x on the left and with n on the right (because x is an epimorphism and n is a mono). This is equivalent to show that

$$Fx_0 \cdot F\alpha \cdot Fg_j = Fx_1 \cdot F\alpha \cdot Fg_j \quad \forall\, j \in J$$

, but this comes from the condition on α to be an arrow in $\mathbb{C}_{\text{reg}}$.

(By the way, observe that also

$$\begin{tikzcd}
F\overline{X} \arrow[r, shift left, "Fx_0"] \arrow[r, shift right, "Fx_1"'] & FX \arrow[r, "p"] & \hat{F}(f_i)
\end{tikzcd}$$

is a coequalizer diagram because x is an epimorphism.)

Step 2: $\hat{F} : \mathbb{C}_{\text{reg}} \longrightarrow \mathbb{A}$ is exact. To show that $\hat{F}$ preserves regular epis, we use the fact that in $\mathbb{C}_{\text{reg}}$ a regular epi is, up to isomorphisms, of the form

$$\begin{tikzcd}
X \arrow[r, "1_X"] \arrow[d, "f_i"'] & X \arrow[d, "g_j"] \\
X_i & Y_j
\end{tikzcd}$$

Its image by $\hat{F}$ is given in the following diagram

$$\begin{tikzcd}
FX \arrow[r, "p"] \arrow[d, "F1_X"'] & \hat{F}(f_i) \arrow[d] \\
FX \arrow[r, "q"'] & \hat{F}(g_j)
\end{tikzcd}$$

and then it is a regular epi.

To prove that $\hat{F}$ is left exact, we show that it is left covering with respect to binary products, equalizers and terminal objects (cf. proposition 1.4.3 and proposition 1.4.10).

Consider two objects (f_i) and (g_j) in $\mathbb{C}_{\text{reg}}$ and their product

$$\begin{tikzcd}
X \arrow[d, "f_i"'] & & X \times Y \arrow[ll, "\pi_X"'] \arrow[rr, "\pi_Y"] \arrow[dl, "\pi_X \cdot f_i"'] \arrow[dr, "\pi_Y \cdot g_j"] & & Y \arrow[d, "g_j"] \\
X_i & X_i & & Y_j & Y_j
\end{tikzcd}$$

By definition of $\hat{F}$, we obtain the following commutative diagram in $\mathbb{A}$

$$\begin{tikzcd}
FX \arrow[d, "p"'] & F(X \times Y) \arrow[l, "F(\pi_X)"'] \arrow[r, "F(\pi_Y)"] \arrow[d, "r"] & FY \arrow[d, "q"] \\
\hat{F}(f_i) & \hat{F}((f_i) \times (g_j)) \arrow[l, "\hat{F}[\pi_X]"] \arrow[r, "\hat{F}[\pi_Y]"'] & \hat{F}(g_j)
\end{tikzcd}$$

Consider again the canonical factorization $s\colon F(X \times Y) \longrightarrow FX \times FY$ which is a regular epi (because F is left covering). Now the canonical factorization $t\colon \hat{F}((f_i) \times (g_j)) \longrightarrow \hat{F}(f_i) \times \hat{F}(g_j)$ makes commutative the following diagram

$$\begin{tikzcd}
F(X \times Y) \arrow[r, "r"] \arrow[d, "s"'] & \hat{F}((f_i) \times (g_j)) \arrow[d, "t"] \\
FX \times FY \arrow[r, "p \times q"'] & \hat{F}(f_i) \times \hat{F}(g_j)
\end{tikzcd}$$

and then it is a regular epi because s, p and q are regular epis.

Consider now two parallel arrows $[\alpha], [\beta]\colon (f_i) \rightrightarrows (g_j)$ in $\mathbb{C}_{\text{reg}}$ and take its equalizer

$$\begin{tikzcd}
E \arrow[r, "e"] \arrow[d, "e \cdot f_i"'] & X \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] \arrow[d, "f_i"] & Y \arrow[d, "g_j"] \\
X_i & X_i & Y_j
\end{tikzcd}$$

(with the notations of theorem 3.2.3, step 1). By definition of $\hat{F}$, we obtain the following commutative diagram in $\mathbb{A}$

$$\begin{tikzcd}
FE \arrow[r, "Fe"] \arrow[d, "r"'] & FX \arrow[r, shift left, "F\alpha"] \arrow[r, shift right, "F\beta"'] \arrow[d, "p"'] & FY \arrow[d, "q"] \\
\hat{F}(e \cdot f_i) \arrow[r, "\hat{F}[e]"'] & \hat{F}(f_i) \arrow[r, shift left, "\hat{F}[\alpha]"] \arrow[r, shift right, "\hat{F}[\beta]"'] & \hat{F}(g_j)
\end{tikzcd}$$

Consider the following limit in $\mathbb{A}$

$$\begin{tikzcd}
H \arrow[r, "h"] & FX \arrow[r, shift left, "F(\alpha \cdot g_j)"] \arrow[r, shift right, "F(\beta \cdot g_j)"'] & FY_j
\end{tikzcd}$$

(think of a pair of parallel arrows for each j in J).

Since F is left covering, the unique arrow $t\colon FE \longrightarrow H$ such that $t \cdot h = Fe$ is a regular epimorphism. Observe that $t \cdot h$ is the regular epi-mono factorization of Fe.

Since

$$FY \xleftarrow{q_0} N(q) \xrightarrow{q_1} FY$$

is the limit over the diagram

$$\begin{tikzcd}
FY \arrow[d, "Fg_j"'] \arrow[drr, "Fg_{j'}" near start] & & FY \arrow[d, "Fg_{j'}"] \arrow[dll, "Fg_j"' near start] \\
FY_j & & FY_{j'}
\end{tikzcd}$$

there exists $u\colon H \longrightarrow N(q)$ such that $u \cdot q_0 = h \cdot F\alpha$ and $u \cdot q_1 = h \cdot F\beta$.

Moreover, we can factor $\hat{F}[e]$ through the equalizer of $\hat{F}[\alpha]$ and $\hat{F}[\beta]$ as

$$\begin{tikzcd}
\hat{F}(e \cdot f_i) \arrow[rr, "\hat{F}[e]"] \arrow[dr, "v"'] & & \hat{F}(f_i) \\
& L \arrow[ur, "l"'] &
\end{tikzcd}$$

So we have the following commutative diagram

$$\begin{tikzcd} FE \arrow[r, "t"] \arrow[d, "r"'] & H \arrow[r, "h"] & FX \arrow[d, "p"] \\ \hat{F}(e \cdot f_i) \arrow[r, "v"'] & L \arrow[r, "l"'] & \hat{F}(f_i) \end{tikzcd}$$

and then there exists $\tau\colon H \longrightarrow L$ such that the two resulting squares are commutative.

Recall that we want to prove that $v\colon \hat{F}(e \cdot f_i) \longrightarrow L$ is a regular epi; clearly for this it suffices to show that $\tau\colon H \longrightarrow L$ is a regular epi. But this is true because the following diagram is a pullback

$$\begin{tikzcd} H \arrow[r, "h"] \arrow[d, "\tau"'] & FX \arrow[d, "p"] \\ L \arrow[r, "l"'] & \hat{F}(f_i) \end{tikzcd}$$

This easily follows from the fact that

$$\begin{tikzcd} H \arrow[r, "h"] \arrow[d, "u"'] & FX \arrow[d, "(F\alpha, F\beta)"] \\ N(q) \arrow[r, "(q_0, q_1)"'] & FY \times FY \end{tikzcd}$$

is a pullback, which can be proved with a straightforward argument of diagram chasing.
In fact, it is commutative by construction of u. Now if $N(q) \xleftarrow{y} \mathbb{A} \xrightarrow{x} FX$ are such that $x \cdot (F\alpha, F\beta) = y \cdot (q_0, q_1)$, one has $x \cdot F\alpha \cdot q = y \cdot q_0 \cdot q = y \cdot q_1 \cdot q = x \cdot F\beta \cdot q$ and then, for each j, $x \cdot F\alpha \cdot q \cdot q \cdot u \cdot \pi_j = x \cdot F\beta \cdot q \cdot u \cdot \pi_j$, that is $x \cdot F\alpha \cdot Fg_j = x \cdot F\beta \cdot Fg - J$. This implies that there exists $a\colon A \longrightarrow H$ such that $a \cdot h = x$. This is the required factorization; in fact $a \cdot u \cdot (q_0, q_1) = a \cdot h \cdot (F\alpha, F\beta) = x \cdot (F\alpha, F\beta) = y \cdot (q_0, q_1)$ and then $a \cdot u = y$ because (q_0, q_1) is a mono.

The factorization is unique because h is a monomorphism.

Finally we can prove that τ is the pullback of p along l. Suppose $L \xleftarrow{b} B \xrightarrow{c} FX$ are such that $b \cdot l = c \cdot p$. Since $c \cdot F\alpha \cdot q = c \cdot p \cdot \hat{F}[\alpha] = b \cdot l \cdot \hat{F}[\alpha] = b \cdot l \cdot \hat{F}[\beta] =$

$c \cdot p \cdot \hat{F}[\beta] = c \cdot F\beta \cdot q$, there exists $d\colon B \longrightarrow N(q)$ such that $d \cdot q_0 = c \cdot F\alpha$ and $d \cdot q_1 = c \cdot F\beta$.

This implies that there exists $a\colon B \longrightarrow H$ such that $a \cdot u = d$ and $a \cdot h = c$. It remains to prove that $a \cdot \tau = b$. Since l is a monomorphism, it suffices to observe that $a \cdot \tau \cdot l = a \cdot h \cdot p = c \cdot p = b \cdot l$. Once again, this factorization $a\colon B \longrightarrow H$ is unique because h is a monomorphism.

It remains to prove that $\hat{F}$ is left covering with respect to the terminal object $(T \longrightarrow)_\emptyset$ of $\mathbb{C}_{\mathrm{reg}}$. It is clear, because $\hat{F}(T \longrightarrow)_\emptyset$ is the image of the unique arrow $t\colon FT \longrightarrow \tau$, where τ is the terminal object of $\mathbb{A}$

$$\begin{tikzcd} FT \arrow[rr, "t"] \arrow[dr, "p"'] & & \tau \\ & \hat{F}(T \to)_\emptyset \arrow[ur, "m"'] & \end{tikzcd}$$

But, by assumption on F, t is a regular epi and so also m is a regular epi.

To end the proof, we need to show that the natural transformations between two left covering functors F and G are in bijection with the natural transformations between $\hat{F}$ and $\hat{G}$. But this is a corollary of the next proposition. ■

Proposition 3.3.3 *Let $\Gamma\colon \mathbb{C} \longrightarrow \mathbb{C}_{reg}$ be as in 3.2.4 and consider a left covering functor $F\colon \mathbb{C} \longrightarrow \mathbb{A}$ with $\mathbb{A}$ regular; the unique exact extension $\hat{F}\colon \mathbb{C}_{reg} \longrightarrow \mathbb{A}$ described in theorem 3.3.1 is the left Kan-extension of F along Γ.*

Proof: given a functor $H\colon \mathbb{C}_{\mathrm{reg}} \longrightarrow \mathbb{A}$ and a natural transformation $\beta\colon F \longrightarrow \Gamma \cdot H$, consider the following diagram

$$\begin{tikzcd} F\overline{X} \arrow[r, shift left, "Fx_0"] \arrow[r, shift right, "Fx_1"'] \arrow[d, "\beta_{\overline{X}}"'] & FX \arrow[r, "p"] \arrow[d, "\beta_X"] & \hat{F}(f_i) \\ H(\Gamma\overline{X}) \arrow[r, shift left, "H(\Gamma x_0)"] \arrow[r, shift right, "H(\Gamma x_1)"'] & H(\Gamma X) \arrow[r, "H(t)"'] & H(f_i) \end{tikzcd}$$

(where $t\colon \Gamma X \longrightarrow (f_i\colon X \longrightarrow X_i)_I$ is given by $1_X\colon X \longrightarrow X$ as in 3.2.6, so that $\Gamma x_0 \cdot t = \Gamma x_1 \cdot t$).

In the proof of theorem 3.3.1, first step, we have shown that the upper line is a coequalizer diagram; moreover, by naturality of β, the left-hand square is two-time commutative. This implies that there exists exactly one arrow $\hat{F}(f_i) \longrightarrow H(f_i)$ making commutative the right-hand part. We take this arrow as a component at the point (f_i) of a natural transformation $\hat{F} \longrightarrow H$. The rest of the proof is straightforward. ■

Corollary 3.3.4 *Let $\Gamma : \mathbb{C} \longrightarrow \mathbb{C}_{reg}$ be as in 3.2.4; for each regular category $\mathbb{A}$, composition with Γ gives us an equivalence between the category of exact functors $Ex(\mathbb{C}_{reg}, \mathbb{A})$ and the category of left covering functors $Lco(\mathbb{C}, \mathbb{A})$.*

Proof: From theorem 3.3.1 and the previous proposition. ∎

Proposition 3.3.5 *(Functoriality of the $(-)_{reg}$ construction)*
Let $F : \mathbb{C} \longrightarrow \mathbb{D}$ be a weakly lex functor; there exists a unique (up to natural isomorphisms) exact functor $F_{reg} : \mathbb{C}_{reg} \longrightarrow \mathbb{D}_{reg}$ making commutative the following diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\Gamma"] \arrow[d, "F"'] & \mathbb{C}_{reg} \arrow[d, "F_{reg}"] \\
\mathbb{D} \arrow[r, "\Gamma"'] & \mathbb{D}_{reg}
\end{tikzcd}$$

Proof: By 1.4.7 and 3.3.1, putting $F_{\text{reg}} = \widehat{F \cdot \Gamma}$. ∎

3.4 Characterization of free regular categories

As announced in section 3.2, we characterize a free regular category over a weakly lex one in terms of the properties of $\mathbb{C}_{\text{reg}}$ pointed out in corollary 3.2.7.

Let us start with a general remark on the regular completion; an analogous one can be done for the exact completion.

Proposition 3.4.1 *Consider a left covering functor $F : \mathbb{C} \longrightarrow \mathbb{A}$ with $\mathbb{A}$ regular and its exact extension $\hat{F} : \mathbb{C}_{reg} \longrightarrow \mathbb{A}$ as in theorem 3.3.1;*

1) if F is faithful, then $\hat{F}$ is faithful

2) if F if full, faithful and factors in the full subcategory of projective objects of $\mathbb{A}$, then $\hat{F}$ is full.

Proof: Recall that, given an arrow $[\alpha] : (f_i) \longrightarrow (g_j)$ in $\mathbb{C}_{\text{reg}}$, $\hat{F}(\alpha)$ is defined as the unique arrow such that $p \cdot \hat{F}[\alpha] = F\alpha \cdot q$, where p and q are regular epimorphisms and m and n are monomorphisms

$$\begin{tikzcd}
 & F(X_i) & \\
FX \arrow[ur, "Ff_i"] \arrow[r, "p"'] \arrow[d, "F\alpha"'] & \hat{F}(f_i) \arrow[r, "m"'] \arrow[d, "\hat{F}[\alpha]"] & \prod_I F(X_i) \arrow[ul, "\pi_i"'] \\
FY \arrow[r, "q"] \arrow[dr, "Fg_j"'] & \hat{F}(g_j) \arrow[r, "n"] & \prod_J F(Y_j) \arrow[dl, "\pi_j"] \\
 & F(Y_j) &
\end{tikzcd}$$

1) given $[\beta]: (f_i) \longrightarrow (g_j)$ in $\mathbb{C}_{\text{reg}}$ such that $\hat{F}[\alpha] = \hat{F}[\beta]$, we have $F\alpha \cdot Fg_j = F\alpha \cdot q \cdot n \cdot \pi_j = p \cdot \hat{F}[\alpha] \cdot n \cdot \pi_j = p \cdot \hat{F}[\beta] \cdot n \cdot \pi_j = F\beta \cdot q \cdot n \cdot \pi_j = F\beta \cdot Fg_j$ for each j. Since F is faithful, this implies $\alpha \cdot g_j = \beta \cdot g_j$ for each j and then $[\alpha] = [\beta]$ in $\mathbb{C}_{\text{reg}}$;

2) recall, from the proof of theorem 3.3.1, that

$$F\overline{X} \underset{Fx_1}{\overset{Fx_0}{\rightrightarrows}} FX \xrightarrow{p} \hat{F}(f_i)$$

is a coequalizer diagram. Consider an arrow $a: \hat{F}(f_i) \longrightarrow \hat{F}(g_j)$; since FX is projective and q is a regular epi, there exists $b: FX \longrightarrow FY$ such that $b \cdot q = p \cdot a$. Since F is full, there exists $\alpha: X \longrightarrow Y$ such that $F\alpha = b$. It remains to show that α induces an arrow $[\alpha]: (f_i) \longrightarrow (g_j)$ in $\mathbb{C}_{\text{reg}}$, that is $x_0 \cdot \alpha \cdot g_j = x_1 \cdot \alpha \cdot g_j$ for each j. Since F is faithful, it suffices to show that $Fx_0 \cdot F\alpha \cdot Fg_j = Fx_1 \cdot F\alpha \cdot Fg_j$. This is true because $Fx_0 \cdot F\alpha \cdot Fg_j = Fx_0 \cdot F\alpha \cdot q \cdot n \cdot \pi_j = Fx_0 \cdot p \cdot a \cdot n \cdot \pi_j = Fx_1 \cdot p \cdot a \cdot n \cdot \pi_j = Fx_1 \cdot F\alpha \cdot q \cdot n \cdot \pi_j = Fx_1 \cdot F\alpha \cdot Fg_j$. ■

Theorem 3.4.2 *Let $\mathbb{A}$ be a regular category and $\mathbb{P}$ a projective cover of $\mathbb{A}$. Suppose that, for each object A of $\mathbb{A}$, there exists a finite family $(X_i)_I$ of objects of $\mathbb{P}$ and a monomorphism $A \longrightarrow \prod_I X_i$; the unique exact extension $\hat{F}: \mathbb{P}_{\text{reg}} \longrightarrow \mathbb{A}$ of the full inclusion $F: \mathbb{P} \longrightarrow \mathbb{A}$ is an equivalence.*

Proof: Consider an object $(f_i: X \longrightarrow X_i)_I$ in $\mathbb{P}_{\text{reg}}$ and the factorization

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "f_i"'] & & \prod_I X_i \arrow[dl, "\pi_i"] \\
& X_i &
\end{tikzcd}$$

$\hat{F}(f_i)$ is the image of f in its regular epi-mono factorization

$$X \longrightarrow \hat{F}(f_i) \longrightarrow \prod_I X_i$$

$\hat{F}$ is essentially surjective on the object: given an object A in $\mathbb{A}$, by assumption, there exist objects $X, (X_i)_I$ in $\mathbb{P}$ together with a regular epi $p: X \longrightarrow A$ and a mono $m: A \longrightarrow \prod_I X_i$. Consider the object $(p \cdot m \cdot \pi_i : X \longrightarrow X_i)_I$ in $\mathbb{P}_{\mathrm{reg}}$; clearly its image under $\hat{F}$ is (isomorphic to) A.

Fullness and faithfulness of $\hat{F}$ immediately follow from the previous proposition. ■

3.5 Comparing regular and exact completion

The aim of this section is to show that, under some additional hypotheses, the regular completion of a weakly lex category is an epireflective subcategory of the corresponding exact completion (epireflective means regular epireflective).

In this section, to avoid confusion, the regular completion of a weakly lex category $\mathbb{C}$ will be indicated with $\Gamma_{\mathrm{reg}}: \mathbb{C} \longrightarrow \mathbb{C}_{\mathrm{reg}}$ and its exact completion with $\Gamma_{\mathrm{ex}}: \mathbb{C} \longrightarrow \mathbb{C}_{\mathrm{ex}}$.

Proposition 3.5.1 *Given a weakly lex category $\mathbb{C}$, the unique exact extension $\mathbb{C}_{reg} \longrightarrow \mathbb{C}_{ex}$ as in the following commutative diagram*

$$\begin{tikzcd}
\mathbb{C} \arrow[rr, "\Gamma_{reg}"] \arrow[dr, "\Gamma_{ex}"'] & & \mathbb{C}_{reg} \arrow[dl] \\
& \mathbb{C}_{ex} &
\end{tikzcd}$$

is full and faithful, i.e. $\mathbb{C}_{reg}$ is equivalent to a full subcategory of $\mathbb{C}_{ex}$.

Proof: By 1.3.1, 1.3.3 and 3.4.1. ■

Let us call

$$\mathrm{Ker}: \mathbb{C}_{\mathrm{reg}} \longrightarrow \mathbb{C}_{\mathrm{ex}}$$

this full and faithful functor; the reason for this name becomes evident considering an explicit description of the functor.

Given an object (f_i) in $\mathbb{C}_{\text{reg}}$, $\text{Ker}(f_i)$ is "the" weak universal pair $x_0, x_1 \colon \overline{X} \rightrightarrows X$ involved in definition 3.2.1 (i.e., the kernel pair of the arrow $\langle f_i \rangle \colon X \longrightarrow \prod_I X_i$ if $\mathbb{C}$ was left exact); in this case we can speak of "the" weak universal pair because two pairs of this kind are isomorphic in $\mathbb{C}_{\text{ex}}$. If $[\alpha] \colon (f_i) \longrightarrow (g_j)$ is an arrow in $\mathbb{C}_{\text{reg}}$, then $\text{Ker}[\alpha]$ is given by

$$\begin{tikzcd} \overline{X} \arrow[r, "\overline{\alpha}"] \arrow[d, shift right, "x_0"'] \arrow[d, shift left, "x_1"] & \overline{Y} \arrow[d, shift right, "y_0"'] \arrow[d, shift left, "y_1"] \\ X \arrow[r, "\alpha"'] & Y \end{tikzcd}$$

where $\overline{\alpha} \colon \overline{X} \longrightarrow \overline{Y}$ exists because $x_0 \cdot \alpha \cdot g_j = x_1 \cdot \alpha \cdot g_j \; \forall \; j \in J$ (and the class of α in $\mathbb{C}_{\text{ex}}$ does not depend on the component $\overline{\alpha}$).

In order to construct a left adjoint to the functor Ker, let us point out what remains true of the universal property of the exact completion when the codomain is only regular.

Proposition 3.5.2 *Let $\mathbb{C}$ be a weakly lex category and $\Gamma_{\text{ex}} \colon \mathbb{C} \longrightarrow \mathbb{C}_{\text{ex}}$ its exact completion. Consider a regular category $\mathbb{A}$ with coequalizers of equivalence relations.*

1) *if $G, H \colon \mathbb{C}_{\text{ex}} \longrightarrow \mathbb{A}$ are two exact functors such that $\Gamma_{\text{ex}} \cdot G \simeq \Gamma_{\text{ex}} \cdot H$, then $G \simeq H$*

2) *if $F \colon \mathbb{C} \longrightarrow \mathbb{A}$ is a left covering functor, then there exists a functor $\hat{F} \colon \mathbb{C}_{\text{ex}} \longrightarrow \mathbb{A}$ making commutative the following diagram*

$$\begin{tikzcd} \mathbb{C} \arrow[rr, "\Gamma_{ex}"] \arrow[dr, "F"'] & & \mathbb{C}_{ex} \arrow[dl, "\hat{F}"] \\ & \mathbb{A} & \end{tikzcd}$$

$\hat{F}$ is the left Kan extension of F along Γ_{ex} and preserves regular epis; moreover, $\hat{F}$ is left covering with respect to the terminal object and finite products.

Proof: If we look at the proofs of theorem 1.5.2 and proposition 1.5.4, we can observe that we have used the exactness of $\mathbb{A}$ (and not only its regularity) in two points.

The first point is the definition of $\hat{F}$: for this, let us consider a pseudo equivalence-relation $r_1, r_2 \colon R \rightrightarrows Y$ in $\mathbb{C}$ and the regular epi-jointly monic factorization

$$\begin{tikzcd} FR \arrow[r, shift left, "Fr_1"] \arrow[r, shift right, "Fr_2"'] \arrow[dr, "p"'] & FX \\ & \underline{R} \arrow[u, shift left, "i_1"] \arrow[u, shift right, "i_2"'] \end{tikzcd}$$

By 1.4.9, $i_1, i_2 : \underline{R} \rightrightarrows FX$ is an equivalence relation. Since we have assumed that in $\mathbb{A}$ there exists the coequalizer of an equivalence relation, we can define once again $\hat{F}(R \rightrightarrows X)$ as the coequalizer of $\underline{R} \rightrightarrows FX$ and then of $FR \rightrightarrows FX$.

The second point is the verification that $\hat{F}$ is left covering with respect to equalizers. But we do not require this fact in the present statement. ■

Using propositions 1.4.3 and 1.4.10, one has that, under the condition of the previous proposition, the functor $\hat{F}: \mathbb{C}_{\text{ex}} \longrightarrow \mathbb{A}$ is left exact exactly when it is left covering with respect to equalizers.

Proposition 3.5.3 *Let $\mathbb{C}$ be a weakly lex category; its regular completion $\mathbb{C}_{\text{reg}}$ is an epireflective subcategory of the exact completion $\mathbb{C}_{\text{ex}}$ if and only if in $\mathbb{C}_{\text{reg}}$ there exist coequalizers of equivalence relations.*

Proof: the (only if) part is an obvious fact which is true for a reflective subcategory of a left exact category with coequalizers of equivalence relations.

Now the (if) part: the previous proposition gives us a functor

$$\text{Coker}: \mathbb{C}_{\text{ex}} \longrightarrow \mathbb{C}_{\text{reg}}$$

making commutative the following diagram

$$\begin{tikzcd} \mathbb{C} \arrow[rr, "\Gamma_{\text{ex}}"] \arrow[dr, "\Gamma_{\text{reg}}"'] & & \mathbb{C}_{\text{ex}} \arrow[dl, "\text{Coker}"] \\ & \mathbb{C}_{\text{reg}} & \end{tikzcd}$$

To show that Coker is left adjoint to Ker, let us explicit the definition of Coker.

Given an object $r_0, r_1 : R \rightrightarrows X$ in $\mathbb{C}_{\text{ex}}$, we know that the monic part of the regular epi-jointly monic factorization of $\Gamma_{\text{reg}}(r_0, r_1 : R \rightrightarrows X)$ is an equivalence relation in $\mathbb{C}_{\text{reg}}$ (because Γ_{reg} is left covering and by theorem 1.4.9). Then the extra assumption on $\mathbb{C}_{\text{reg}}$ tells us that there exists its coequalizer which is, up to isomorphisms, of the following kind

$$\begin{tikzcd} R \arrow[r, shift left, "r_0"] \arrow[r, shift right, "r_1"'] \arrow[d, "1_R"'] & X \arrow[r, "1_X"] \arrow[d, "1_X"] & X \arrow[d, "q_i"] \\ R & X & Q_i \end{tikzcd}$$

and one has

$$\mathrm{Coker}(R \overset{r_0}{\underset{r_1}{\rightrightarrows}} X) = (q_i \colon X \longrightarrow Q_i)_I.$$

Given an arrow

$$\begin{tikzcd} R \arrow[r, "\overline{f}"] \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & S \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

it induces an arrow

$$[f] \colon \Gamma_{\mathrm{reg}}(X) \longrightarrow \mathrm{Coker}(S \overset{s_0}{\underset{s_1}{\rightrightarrows}} Y)$$

such that $\Gamma_{\mathrm{reg}}(r_0) \cdot [f] = \Gamma_{\mathrm{reg}}(r_1) \cdot [f]$ so that there exists a canonical extension of $[f]$ to an arrow

$$\mathrm{Coker}[f] \colon \mathrm{Coker}(R \overset{r_0}{\underset{r_1}{\rightrightarrows}} X) \longrightarrow \mathrm{Coker}(S \overset{s_0}{\underset{s_1}{\rightrightarrows}} Y).$$

Clearly the composition

$$\mathbb{C}_{\mathrm{reg}} \xrightarrow{\mathrm{Ker}} \mathbb{C}_{\mathrm{ex}} \xrightarrow{\mathrm{Coker}} \mathbb{C}_{\mathrm{reg}}$$

is isomorphic to the identity functor on $\mathbb{C}_{\mathrm{reg}}$ (as it must be, because the "right-adjoint" Ker is full and faithful) simply because

$$\begin{tikzcd} \overline{X} \arrow[r, shift left, "x_0"] \arrow[r, shift right, "x_1"'] \arrow[d, "1_{\overline{X}}"'] & X \arrow[r, "1_X"] \arrow[d, "1_X"] & X \arrow[d, "f_i"] \\ \overline{X} & X & X_i \end{tikzcd}$$

is a coequalizer diagram in $\mathbb{C}_{\mathrm{reg}}$.

Let us look at the unit of the adjunction: given $r_0, r_1 \colon R \rightrightarrows X$ in $\mathbb{C}_{\mathrm{ex}}$, consider the coequalizer in $\mathbb{C}_{\mathrm{reg}}$

$$\begin{tikzcd} R \arrow[r, shift left, "r_0"] \arrow[r, shift right, "r_1"'] \arrow[d, "1_R"'] & X \arrow[r, "1_X"] \arrow[d, "1_X"] & X \arrow[d, "q_i"] \\ R & X & Q_i \end{tikzcd}$$

and a pair $q_0, q_1 \colon \overline{X} \rightrightarrows X$ weak universal with respect to the property that $q_0 \cdot q_i = q_1 \cdot q_i \ \forall\ i \in I$. If 1_X coequalizes r_0 and r_1 in $\mathbb{C}_{\mathrm{reg}}$, then, $\forall\ i \in I$, $r_0 \cdot q_i = r_1 \cdot q_i$ and then there exists $\eta \colon R \longrightarrow \overline{X}$ such that $\eta \cdot q_0 = r_0$ and $\eta \cdot q_1 = r_1$.

The arrow in $\mathbb{C}_{\mathrm{ex}}$

$$\begin{tikzcd} R \arrow[r, "\eta"] \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & \overline{X} \arrow[d, shift right, "q_0"'] \arrow[d, shift left, "q_1"] \\ X \arrow[r, "1_X"'] & X \end{tikzcd}$$

gives us the component at the point $(r_0, r_1 \colon R \rightrightarrows X)$ of the unit in the adjunction Coker $\dashv$ Ker; since such an arrow is a regular epi in $\mathbb{C}_{\mathrm{ex}}$, the adjunction is in fact an epireflection. (In other words, we have a coequalizer in $\mathbb{C}_{\mathrm{reg}}$

$$\Gamma_{\mathrm{reg}}(R \overset{r_0}{\underset{r_1}{\rightrightarrows}} X) \longrightarrow \mathrm{Coker}(R \overset{r_0}{\underset{r_1}{\rightrightarrows}} X)$$

and we apply the exact functor Ker

$$\Gamma_{\mathrm{ex}}(R \overset{r_0}{\underset{r_1}{\rightrightarrows}} X) \longrightarrow \mathrm{Ker}(\mathrm{Coker}(R \overset{r_0}{\underset{r_1}{\rightrightarrows}} X))$$

but in $\mathbb{C}_{\mathrm{ex}}$ the coequalizer is given by

$$\Gamma_{\mathrm{ex}}(R \overset{r_0}{\underset{r_1}{\rightrightarrows}} X) \longrightarrow (R \overset{r_0}{\underset{r_1}{\rightrightarrows}} X)$$

so that there exists a canonical factorization

$$(R \overset{r_0}{\underset{r_1}{\rightrightarrows}} X) \longrightarrow \mathrm{Ker}(\mathrm{Coker}(R \overset{r_0}{\underset{r_1}{\rightrightarrows}} X))$$

which is a regular epi because Ker preserves regular epis.).

As far as the universal property is concerned, let $(g_j \colon Y \longrightarrow Y_j)_J$ be an object in $\mathbb{C}_{\mathrm{reg}}$ and consider an arrow

$$\begin{tikzcd} R \arrow[r, "\overline{a}"] \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & \overline{Y} \arrow[d, shift right, "y_0"'] \arrow[d, shift left, "y_1"] \\ X \arrow[r, "a"'] & Y \end{tikzcd}$$

in $\mathbb{C}_{\mathrm{ex}}$. As $[a] \colon \Gamma_{\mathrm{reg}}(X) \longrightarrow (g_j)$ coequalizes $\Gamma_{\mathrm{reg}}(r_0)$ and $\Gamma_{\mathrm{reg}}(r_1)$, there exists $[\alpha] \colon (q_i) \longrightarrow (g_j)$ such that

$$\begin{tikzcd}
\Gamma_{\mathrm{reg}}(X) \arrow[rr, "{[a]}"] \arrow[dr, "{[1_X]}"'] & & (g_j) \\
& (q_i) \arrow[ur, "{[\alpha]}"'] &
\end{tikzcd}$$

commutes in $\mathbb{C}_{\mathrm{reg}}$. But then $\forall\ j \in J,\ a \cdot g_j = \alpha \cdot g_j$ so that there exists $\Sigma\colon X \longrightarrow \overline{Y}$ such that $\Sigma \cdot y_0 = \alpha$ and $\Sigma \cdot y_1 = a$. This Σ is the homotopy allowing the composition in $\mathbb{C}_{\mathrm{ex}}$

$$\begin{tikzcd}
R \arrow[r, "\eta"] \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & \overline{X} \arrow[r, "\overline{\alpha}"] \arrow[d, shift right, "x_0"'] \arrow[d, shift left, "x_1"] & \overline{Y} \arrow[d, shift right, "y_0"'] \arrow[d, shift left, "y_1"] \\
X \arrow[r, "1_X"'] & X \arrow[r, "\alpha"'] & Y
\end{tikzcd}$$

to be equal to

$$\begin{tikzcd}
R \arrow[r, "\overline{a}"] \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & \overline{Y} \arrow[d, shift right, "y_0"'] \arrow[d, shift left, "y_1"] \\
X \arrow[r, "a"'] & Y
\end{tikzcd}$$

Since the second part of the previous composition is exactly Ker$[\alpha]$, we have obtained the required factorization. Its uniqueness follows from the fact that the first part is a regular epi and the functor Ker is faithful. ■

Let us conclude this section translating in terms of $\mathbb{C}$ the condition that $\mathbb{C}_{\mathrm{reg}}$ has coequalizers of equivalence relations.

Proposition 3.5.4 *Let $\mathbb{C}$ be a weakly lex category; its regular completion $\mathbb{C}_{\mathrm{reg}}$ has coequalizers of equivalence relations if and only if the following condition holds: given a pseudo equivalence relation $r_0, r_1\colon R \rightrightarrows X$ in $\mathbb{C}$, there exists a finite family $(q_i\colon X \longrightarrow Q_i)_I$ of arrows in $\mathbb{C}$ such that*

1) $r_0 \cdot q_i = r_i \cdot q_1 \quad \forall\ i \in I$

2) if $f\colon X \longrightarrow Y$ is such that $r_0 \cdot f = r_1 \cdot f$, then $q_0 \cdot f = q_1 \cdot f$ (where $q_0, q_1\colon \overline{X} \rightrightarrows X$ is weak universal such that $q_0 \cdot q_i = q_1 \cdot q_i \quad \forall\ i \in I$).

Proof: (only if): recall that the coequalizer of $\Gamma_{\mathrm{reg}}(r_0, r_1\colon R \rightrightarrows X)$ exists because $\Gamma_{\mathrm{reg}}\colon \mathbb{C} \longrightarrow \mathbb{C}_{\mathrm{reg}}$ is left covering (cf. 1.4.9); moreover, it must be of the form

$$\begin{tikzcd}
X \arrow[r, "1_X"] \arrow[d, "1_X"'] & X \arrow[d, "q_i"] \\
X & Q_i
\end{tikzcd}$$

because in $\mathbb{C}_{\text{reg}}$ all the regular epis have, up to isomorphisms, this form. Now observe that the conditions mean exactly that

$$\begin{tikzcd}
R \arrow[r, shift left, "r_0"] \arrow[r, shift right, "r_1"'] \arrow[d, "1_R"'] & X \arrow[r, "1_X"] \arrow[d, "1_X"] & X \arrow[d, "q_i"] \\
R & X & Q_i
\end{tikzcd}$$

is a coequalizer diagram in $\mathbb{C}_{\text{reg}}$ with respect to the arrows of $\mathbb{C}_{\text{reg}}$ coming from $\mathbb{C}$.

For the (if) part, suppose that $[f]\colon \Gamma_{\text{reg}}(X) \longrightarrow (g_j)$ is an arrow in $\mathbb{C}_{\text{reg}}$ such that $[r_0] \cdot [f] = [r_1] \cdot [f]$, that is $\forall\, j \in J$, $r_0 \cdot f \cdot g_j = r_1 \cdot f \cdot g_j$. Then for each j in J, we can apply our hypothesis so that $q_0 \cdot f \cdot g_j = q_1 \cdot f \cdot g_j$. This implies that $[f]\colon (q_i) \longrightarrow (g_j)$ is the unique required factorization and then the previous diagram is a coequalizer in all $\mathbb{C}_{\text{reg}}$.

To conclude, observe that, in the previous proposition, to give the explicit description of $\text{Coker}\colon \mathbb{C}_{\text{ex}} \longrightarrow \mathbb{C}_{\text{reg}}$, we only need the existence of this kind of coequalizers. ∎

3.6 Two examples

This section is devoted to two examples of the situation studied in the previous section.

Example 3.6.1

The Effective topos. There exists a large literature on this topos (originally introduced in [27]) and on the way it can be described via some free constructions. We follow here the description given in [14], which is essentially the same given in [38].

The category $\mathcal{S}^*_R$ is defined as follows: an object is a surjective map $p\colon X \longrightarrow I$ with $X, I \in \mathcal{SET}$ and $I \subseteq \mathbb{N}$; an arrow is a commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ I \arrow[r, "\varphi"'] & J \end{tikzcd}$$

where $\varphi\colon I \longrightarrow J$ is induced by a partial recursive function $\Phi\colon \mathbb{N} \longrightarrow \mathbb{N}$ (that is, $\Phi(n)$ is defined and is in J for all n in I). It is not difficult to show that $\mathcal{S}^*_R$ is a left exact category. Then, using the axiom of choice in $\mathcal{SET}$, one can prove that $(\mathcal{S}^*_R)_{\mathrm{ex}}$ is a topos; in fact, it is (equivalent to) the effective topos.

Consider now the regular completion $(\mathcal{S}^*_R)_{\mathrm{reg}}$; to show that it is reflective in $(\mathcal{S}^*_R)_{\mathrm{ex}}$, we can use proposition 3.5.4. Given a pseudo equivalence-relation in $\mathcal{S}^*_R$

$$\begin{tikzcd} R \arrow[r, shift left, "r_0"] \arrow[r, shift right, "r_1"'] \arrow[d] & X \arrow[d] \\ I \arrow[r, shift left, "\rho_0"] \arrow[r, shift right, "\rho_1"'] & J \end{tikzcd}$$

its coequalizer in $(\mathcal{S}^*_R)_{\mathrm{reg}}$ is given by the arrow of $\mathcal{S}^*_R$

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[d] & Q \arrow[d] \\ J \arrow[r] & * \end{tikzcd}$$

where $q\colon X \longrightarrow Q$ is the coequalizer of r_0 and r_1 in $\mathcal{SET}$.

One can also prove that the epireflector Coker: $(\mathcal{S}^*_R)_{\mathrm{ex}} \longrightarrow (\mathcal{S}^*_R)_{\mathrm{reg}}$ preserves finite products and monomorphisms, so that it exhibits $(\mathcal{S}^*_R)_{\mathrm{reg}}$ as the category of separated objects for a topology on $(\mathcal{S}^*_R)_{\mathrm{ex}}$ (cf. [17]). This topology turns out to be the topology of double negation. In other words, $(\mathcal{S}^*_R)_{\mathrm{reg}}$ is a quasi-topos (cf. [9]).

Example 3.6.2

Stone spaces. This second example needs a quite obvious modification of the regular completion of a weakly lex category. This generalization consists of taking as object of $\mathbb{C}_{\mathrm{reg}}$ a family of arrows $(f_i\colon X \longrightarrow X_i)_I$ of $\mathbb{C}$ where I is a small but not necessarily finite set. It is clear that one has to start with a category $\mathbb{C}$ with all small weak limits. This generalization will be discussed in section 4.1; now the example.

We know, from section 2.1, that a monadic category over $\mathcal{SET}$ is the free exact category over the category of free algebras. By a theorem by Manes, this is the case for the category $\mathcal{CH}$ of compact Hausdorff spaces (cf. chapter 4, vol. II of [8]).

The problem is: what is the regular completion for the category of free compact Hausdorff spaces?

Equivalently, what is the regular completion of the full subcategory of projective objects in $\mathcal{CH}$? (Such a full subcategory has clearly all the small weak limits, because $\mathcal{CH}$ is complete).

A theorem by Gleason (see [29]) shows that projectives in $\mathcal{CH}$ are exactly the extremally disconnected spaces. So they are contained in the category of Stone spaces, which is a regular and epireflective subcategory of $\mathcal{CH}$ (see [29] and [10] chapter II, 4, n.4, prop. 7).

Moreover, a standard argument shows that each Stone space can be embedded in a product of projective objects (the product of as many copies of the two-point discrete space as the points of the Stone space).

This means exactly that the category of Stone spaces is the regular completion (in the infinitary sense) of the category of extremally disconnected spaces.

<u>Remark:</u> in both examples, the reflector $\mathbb{C}_{\text{ex}} \longrightarrow \mathbb{C}_{\text{reg}}$ preserves products (cf. [14] and [18]).

It is not clear to me how general is this fact: for example, one can prove that if $\mathbb{C}$ is left exact and $\mathbb{C}_{\text{reg}}$ is cartesian closed, then the reflector preserves products. These conditions are verified in the first example but not in the second one.

<u>Remark:</u> let me point out another open problem. We already know two examples of free exact categories which are topo, a presheaf topos $\mathcal{SET}^{\mathcal{D}^{\text{op}}}$ and the Effective topos. Two other examples of this kind are studies in [41]: they are two "...toposes generalizing notions of extensional realizability in the same way as Hyland's Effective topos generalizes Kleene's realizability."; both of them have enough projectives, so that they are free exact categories.

The open problem is to find conditions on a weakly lex category $\mathbb{C}$ such that $\mathbb{C}_{\text{ex}}$ is a topos.

When $\mathbb{C}$ is left exact, some partial answers to this problem are given in [14], but much remains to be done in this direction.

Chapter 4

Appendix

4.1 Completely regular categories

In order to put example 3.6.2 in our theory, we need a minor modification that we discuss briefly in this section.

Definition 4.1.1 *A completely regular category $\mathbb{C}$ is a category which is complete and regular and such that the following condition holds: if $(p_i \colon X_i \longrightarrow Y_i)_I$ is a family of regular epis, then the unique factorization*

$$\prod_I p_i \colon \prod_I X_i \longrightarrow \prod_I Y_i$$

is again a regular epi.

If the family is finite, the condition is redundant, as shown in lemma 1.4.11; but if the family is infinite, the condition can not be deduced from the completeness and the regularity of the category. For example, one can show that the condition does not hold in a topos of sheaves on a topological space.

Let us consider now a category $\mathbb{C}$ with all small weak limits; we can construct $\mathbb{C}^{\infty}_{\text{reg}}$ as in definition 3.2.1, but an object is now a small (but not necessarily finite) family of arrows $(f_i \colon X \longrightarrow X_i)_I$.

Clearly, $\mathbb{C}^{\infty}_{\text{reg}}$ is a complete and regular category. Moreover, using once again the fact that a regular epi is, up to isomorphisms, of the form

$$[1_X] \colon (f_i \colon X \longrightarrow X_i)_I \longrightarrow (g_j \colon X \longrightarrow Y_j)_J,$$

it is quite obvious to prove that $\mathbb{C}^{\infty}_{\text{reg}}$is a completely regular category. As in proposition 3.2.6, each object of $\mathbb{C}^{\infty}_{\text{reg}}$ can be embedded in a product of projective objects, but this product can be now infinite.

Also the universal property of the embedding

$$\Gamma \colon \mathbb{C} \longrightarrow \mathbb{C}^{\infty}_{\text{reg}}$$

needs a modification; it runs as follows:

for each completely regular category $\mathbb{A}$ and for each functor $F: \mathbb{C} \longrightarrow \mathbb{A}$ which covers each small limit, there exists a unique (up to natural isomorphisms) exact and continuous functor $\hat{F}: \mathbb{C}^{\infty}_{\mathrm{reg}} \longrightarrow \mathbb{A}$ making commutative the following diagram

$$\begin{tikzcd} \mathbb{C} \arrow[rr, "\Gamma"] \arrow[dr, "F"'] & & \mathbb{C}^{\infty}_{\mathrm{reg}} \arrow[dl, "\hat{F}"] \\ & \mathbb{A} & \end{tikzcd}$$

In this statement, we need the stability of regular epis under arbitrary products in $\mathbb{A}$ only to show that $\hat{F}$ is left covering with respect to arbitrary products, but not to define $\hat{F}$.

As a consequence, the characterization of free regular categories remains unchanged. This means that $\mathbb{A}$ is equivalent to $\mathbb{C}^{\infty}_{\mathrm{reg}}$ if and only if $\mathbb{A}$ is complete, regular, has enough projectives and each object can be embedded in a product of projectives. In fact in the proof of proposition 3.4.1 and theorem 3.4.2, we only use the definition of $\hat{F}$ and not its exactness.

The previous discussion allows us to say that the category of Stone spaces is $\mathbb{C}^{\infty}_{\mathrm{reg}}$, where $\mathbb{C}$ is the category of extremally disconnected spaces, as stated in section 3.6.

We want now to point out some elementary properties of the notion of completely regular category.

First of all, an example: clearly, the category $\mathcal{SET}$ is completely regular, because in $\mathcal{SET}$ regular epi means surjective. Working at each point, also a presheaf category is completely regular.

Now a first stability of the notion of completeley regular.

Proposition 4.1.2 *Let $\mathbb{T}$ be a monad on a completely regular category $\mathbb{A}$; if T sends regular epis into regular epis, then EM$(\mathbb{T})$ is completely regular.*

Proof: Clearly EM$(\mathbb{T})$ is complete because so is $\mathbb{A}$; moreover, from 2.5.1.2, we know that EM$(\mathbb{T})$ is regular.

Consider a family $\{f_i: (A_i, a_i) \longrightarrow (B_i, b_i)\}_I$ of regular epis in EM$(\mathbb{T})$; the limit

$$\prod_I f_i: \prod_I (A_i, a_i) \longrightarrow \prod_I (B_i, b_i)$$

is built up in $\mathbb{A}$ as $\prod_I f_i: \prod_I A_i \longrightarrow \prod_I B_i$. By 2.5.1.1, the f_i's are regular epis in $\mathbb{A}$ and then $\prod_I f_i$ is a regular epi in $\mathbb{A}$ because $\mathbb{A}$ is completely regular; once again by 2.5.1.2 we have that $\prod_I f_i$ is a regular epi in EM$(\mathbb{T})$. ∎

Corollary 4.1.3 *Let $\mathbb{T}$ be a monad over $\mathcal{SET}$; EM$(\mathbb{T})$ is completely regular.* ∎

The second stability of the notion of completely regular follows from the first point of next proposition.

Proposition 4.1.4 *Let $i: \mathbb{A} \hookrightarrow \mathbb{B}$ be a reflective subcategory, with reflector $r: \mathbb{B} \longrightarrow \mathbb{A}$:*

1) if $\mathbb{B}$ is regular and r is an epireflector, that is $\forall\, B \in \mathbb{B}$ the unit $\eta_B: B \longrightarrow i(rB)$ is a regular epi, then $i: \mathbb{A} \longrightarrow \mathbb{B}$ preserves regular epis and $\mathbb{A}$ is regular;

2) if $i: \mathbb{A} \hookrightarrow \mathbb{B}$ preserves regular epis, then $r: \mathbb{B} \longrightarrow \mathbb{A}$ preserves projective objects;

3) if $r: \mathbb{B} \longrightarrow \mathbb{A}$ preserves projective objects and $\mathbb{B}$ has enough projectives, then $\mathbb{A}$ has enough projectives.

Proof: 1): consider a regular epi $f: X \longrightarrow Y$ in $\mathbb{A}$ together with its kernel pair $f_0, f_1: N(f) \rightrightarrows X$; consider now the coequalizer in $\mathbb{B}$

$$\begin{tikzcd} i(N(f)) \arrow[r, shift left, "i(f_0)"] \arrow[r, shift right, "i(f_1)"'] & i(X) \arrow[r, "g"] & Z \end{tikzcd}$$

so that $X \xrightarrow{f} Y$ is isomorphic to $r(i(X) \xrightarrow{g} Z)$. From this and using the fact that counits are isomorphisms, one can easily show that the following diagram is commutative

$$\begin{tikzcd} i(X) \arrow[r, "g"] \arrow[d, "i(f)"'] & Z \arrow[d, "\eta_Z"] \\ i(Y) \arrow[r, "\simeq"] & i(rZ) \end{tikzcd}$$

so that $i(f)$ is a regular epi. The fact that $\mathbb{A}$ is regular is now obvious.

2): let X be a projective object in $\mathbb{B}$ and consider the two following diagrams

$$\begin{tikzcd} & rX \arrow[d, "f"] \\ Z \arrow[r, "p"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta_X"] & i(rX) \arrow[d, "i(f)"] \\ i(Z) \arrow[r, "i(p)"'] & i(Y) \end{tikzcd}$$

where p is a regular epi in $\mathbb{A}$ and then, by assumption, $i(p)$ is a regular epi in $\mathbb{B}$. Since X is projective, there exists $g: X \longrightarrow i(Z)$ making commutative the right-hand diagram. By adjunction, there exists $g': r(X) \longrightarrow Z$ making commutative the left-hand diagram (here we do not need the fullness and faithfulness of i).

3): consider an object X in $\mathbb{A}$ and a projective cover in $\mathbb{B}$ $p: P \longrightarrow i(X)$; $r(p): r(P) \longrightarrow r(i(X)) \simeq X$ is a projective cover in $\mathbb{A}$. ■

Corollary 4.1.5 *Let $i: \mathbb{A} \hookrightarrow \mathbb{B}$ be an epireflective subcategory of a completely regular category $\mathbb{B}$ (with enough projectives); $\mathbb{A}$ is completely regular (with enough projectives).* ∎

In particular, this proves directly that the category of Stone spaces is completeley regular with enough projectives: in fact, it is an epireflective subcategory of the category of compact Hausdorff spaces, which is monadic over $\mathcal{SET}$.

4.2 Colimits in the exact completion

In our two major examples of free exact categories, that is the category of algebras for a monad over $\mathcal{SET}$ and the category of presheaves on a small category, we have the following situation: the weakly lex base $\mathbb{C}$ has small sums which are computed in $\mathbb{C}_{\mathrm{ex}}$ and $\mathbb{C}_{\mathrm{ex}}$ is cocomplete. This section is devoted to the study of this situation.

Lemma 4.2.1 *Let $\mathbb{A}$ be a category with weak kernel pairs and $\mathbb{P}$ a projective cover of $\mathbb{A}$; the full inclusion $\mathbb{P} \longrightarrow \mathbb{A}$ preserves the sums which turn out to exist in $\mathbb{P}$.*

Proof: We write the proof for a binary sum, but the argument is general.
Consider a sum in $\mathbb{P}$

$$P_1 \xrightarrow{s_1} P \xleftarrow{s_2} P_2$$

and two arrows in $\mathbb{A}$

$$P_1 \xrightarrow{x_1} X \xleftarrow{x_2} P_2$$

with $\mathbb{P}$-cover $q: Q \longrightarrow X$; we obtain two extensions $y_1: P_1 \longrightarrow Q$ and $y_2 : P_2 \longrightarrow Q$ such that $y_1 \cdot q = x_1$ and $y_2 \cdot q = x_2$. Since Q is in $\mathbb{P}$, there exists $y: P \longrightarrow Q$ such that $s_1 \cdot y = y_1$ and $s_2 \cdot y = y_2$. Then $y \cdot q: P \longrightarrow X$ is the required factorization.

As far as uniqueness is concerned, suppose that $f, g: P \rightrightarrows X$ are two arrows such that $s_1 \cdot f = s_1 \cdot g$ and $s_2 \cdot f = s_2 \cdot g$.

Consider two extensions $\overline{f}: P \longrightarrow Q$ and $\overline{g}: P \longrightarrow Q$ such that $\overline{f} \cdot q = f$ and $\overline{g} \cdot q = g$. Now $s_1 \cdot \overline{f} \cdot q = s_1 \cdot f = s_1 \cdot g = s_1 \cdot \overline{g} \cdot q$, so that there exists $t_1: P_1 \longrightarrow N(q)$ such that $t_1 \cdot q_0 = s_1 \cdot \overline{f}$ and $t_1 \cdot q_1 = s_1 \cdot \overline{g}$, where $q_0, q_1: N(q) \rightrightarrows Q$ is a weak kernel pair of $q: Q \longrightarrow X$. Analogously, there exists $t_2: P_2 \longrightarrow N(q)$ such that $t_2 \cdot q_0 = s_2 \cdot \overline{f}$ and $t_2 \cdot q_1 = s_2 \cdot \overline{g}$.

Now, from the first part of the proof, we obtain $t: P \longrightarrow N(q)$ such that $s_1 \cdot t = t_1$ and $s_2 \cdot t = t_2$. Moreover, $s_1 \cdot \overline{f} = t_1 \cdot q_0 = s_1 \cdot t \cdot q_0$ and $s_2 \cdot \overline{f} = t_2 \cdot q_0 = s_2 \cdot t \cdot q_0$ so that $\overline{f} = t \cdot q_0$ because Q is in $\mathbb{P}$; analogously $\overline{g} = t \cdot q_1$, because $s_1 \cdot \overline{g} = t_1 \cdot q_1 = s_1 \cdot t \cdot q_1$ and $s_2 \cdot \overline{g} = t_2 \cdot q_1 = s_2 \cdot t \cdot q_1$.

Finally $f = \overline{f} \cdot q = t \cdot q_0 \cdot q = t \cdot q_1 \cdot q = \overline{g} \cdot q = g$. ∎

Corollary 4.2.2 *Let $\mathbb{C}$ be a weakly lex category; the functor $\Gamma: \mathbb{C} \longrightarrow \mathbb{C}_{ex}$ preserves the sums which turn out to exist in $\mathbb{C}$.* ∎

Let us fix some notations for the next lemma.

If $\mathbb{A}$ is a category, $\theta(\mathbb{A})$ is the ordered reflection of $\mathbb{A}$: if A and B are objects of $\mathbb{A}$, $A \leq B$ if and only if $\mathbb{A}(A, B)$ is not empty. Clearly $\leq$ is reflective and transitive in the class of objects $\mathbb{A}_0$ of $\mathbb{A}$. $\theta(\mathbb{A})$ is the quotient of $\mathbb{A}_0$ by the following equivalence relation: $A \sim B$ if and only if $A \leq B$ and $B \leq A$; clearly $\theta(\mathbb{A})$ is an ordered class.

If A is an object of $\mathbb{A}$, $\mathrm{Sub}(A)$ is the ordered class of subobjects of A and $\mathbb{A}/A$ is the usual comma category.

Lemma 4.2.3 *Let $\mathbb{A}$ be a category with strong epi-mono factorization and $\mathbb{P}$ a strong-projective cover of $\mathbb{A}$; for each object A of $\mathbb{A}$, $\mathrm{Sub}(A)$ and $\theta(\mathbb{P}/A)$ are isomorphic ordered classes.*

Proof: First, $\mathrm{Sub}(A) \longrightarrow \theta(\mathbb{P}/\mathbb{A})$: given a monomorphism $X \longrightarrow A$ we can consider a $\mathbb{P}$-cover $P \longrightarrow X$ of X and we obtain an element of $\theta(\mathbb{P}/A)$ taking the composit $P \longrightarrow X \longrightarrow A$; the order is preserved because the objects of $\mathbb{P}$ are strong projective.

Second, $\theta(\mathbb{P}/A) \longrightarrow \mathrm{Sub}(A)$: given an object $P \longrightarrow A$ of $\mathbb{P}/A$, we can take the monic part of its factorization; the order is preserved because, by definition, strong epimorphisms are orthogonal to monomorphisms.

Clearly, $\mathrm{Sub}(A) \longrightarrow \theta(\mathbb{P}/A)$ and $\theta(\mathbb{P}/A) \longrightarrow \mathrm{Sub}(A)$ are one the inverse of the other. ∎

Proposition 4.2.4 *Let $\mathbb{C}$ be a weakly lex category; if $\mathbb{C}$ has sums and $\mathbb{C}_{\mathrm{ex}}$ is well-powered, then $\mathbb{C}_{\mathrm{ex}}$ is cocomplete.*

Proof: First, the coequalizers: consider two arrows in $\mathbb{C}_{\mathrm{ex}}$ with their regular epi-jointly monic factorization

$$\begin{tikzcd} B \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] \arrow[dr, "r"'] & A \\ & R \arrow[u, shift left, "i_0"] \arrow[u, shift right, "i_1"'] \end{tikzcd}$$

Now we can consider the equivalence relation $j_0, j_1\colon E \rightrightarrows A$ generated by the relation $i_0, i_1\colon R \rightrightarrows A$, that is the intersection of all the equivalence relations in A which contain $i_0, i_1\colon R \rightrightarrows A$.

This intersection exists: by the previous lemma, $\mathrm{Sub}(A)$ is isomorphic to $\theta(\mathbb{P}/A)$ which is cocomplete because $\mathbb{P}$ has sums; so $\mathrm{Sub}(A)$ is also complete.

Since $\mathbb{C}_{\mathrm{ex}}$ is exact, $j_0, j_1\colon E \rightrightarrows A$ has a coequalizer which is clearly also the coequalizer of $i_0, i_1\colon R \rightrightarrows A$ and then of $a, b\colon B \rightrightarrows A$.

Second, the sums: once again we sketch the proof for a binary sum, but it is general.

The sum of two objects $r_1, r_2\colon R \rightrightarrows X$ and $s_1, s_2\colon S \rightrightarrows Y$ of $\mathbb{C}_{\mathrm{ex}}$ is built up in the following diagram

$$
\begin{tikzcd}
\Gamma R \arrow[r, shift left, "\Gamma r_1"] \arrow[r, shift right, "\Gamma r_2"'] \arrow[d] & \Gamma X \arrow[r] \arrow[d] & (R \rightrightarrows X) \arrow[d] \\
\Gamma(R \coprod S) \arrow[r, shift left, "\Gamma(r_1 \coprod s_1)"] \arrow[r, shift right, "\Gamma(r_2 \coprod s_2)"'] & \Gamma(X \coprod Y) \arrow[r] & Q \\
\Gamma S \arrow[r, shift left, "\Gamma s_1"] \arrow[r, shift right, "\Gamma s_2"'] \arrow[u] & \Gamma Y \arrow[r] \arrow[u] & (S \rightrightarrows Y) \arrow[u]
\end{tikzcd}
$$

By corollary 4.2.2, the first two columns are sums. The first and the third lines are coequalizers (cf. proposition 1.3.2). By the first part of the proof, we know that $\mathbb{C}_{\text{ex}}$ has coequalizers, so that we can complete the second line with a coequalizer. An interchange argument shows now that also the third column is a sum. ■

Clearly, the previous proposition applies to the case of a presheaf category as well as the case of the category of algebras for a monad over $\mathcal{SET}$.

Coming back to lemma 4.2.3, let us observe that, given an object A of $\mathbb{A}$ and a $\mathbb{P}$-cover $p: P \longrightarrow A$, the composition with p gives a surjection from the object of $\mathbb{P}/P$ to the object of $\mathbb{P}/A$. Using the axiom of choice, we can therefore inject the object of $\mathbb{P}/A$ in $\mathbb{P}/P$. Moreover, if two objects of $\mathbb{P}/A$ are identified in $\theta(\mathbb{P}/P)$, then they are identified also in $\theta(\mathbb{P}/A)$.

So, in virtue of lemma 4.2.3, we have proved the following lemma.

Lemma 4.2.5 *Let $\mathbb{A}$ be a category with strong epi-mono factorization and $\mathbb{P}$ a strong-projective cover of $\mathbb{A}$; $\mathbb{A}$ is well-powered if and only if for each P in $\mathbb{P}$, $\theta(\mathbb{P}/P)$ is a small set.* ■

Corollary 4.2.6 *Let $\mathbb{C}$ be a weakly lex category; $\mathbb{C}_{\text{ex}}$ is well-powered if and only if, for each X in $\mathbb{C}$, $\theta(\mathbb{C}/X)$ is a small set.* ■

4.3 The epireflective hull

In this section we come back to the problem, discussed in section 3.5, of the reflectivity of $\mathbb{C}_{\text{reg}}$ as subcategory of $\mathbb{C}_{\text{ex}}$. Our aim is to show that, under some assumptions on the size of $\mathbb{C}$, $\mathbb{C}_{\text{reg}}$ is the epireflective hull of $\mathbb{C}$ in $\mathbb{C}_{\text{ex}}$.

Let us start with a warning: we know that $\mathbb{C}_{\text{reg}}$ is equivalent, via the full and faithful functor $\mathrm{Ker}\colon \mathbb{C}_{\text{reg}} \longrightarrow \mathbb{C}_{\text{ex}}$, to a full subcategory of $\mathbb{C}_{\text{ex}}$; but in general this full subcategory is not replete. In order to use some general facts about reflective subcategories which work when the given subcategory is replete, in this section we suppose to add to $\mathbb{C}_{\text{reg}}$ all the objects of $\mathbb{C}_{\text{ex}}$ isomorphic to some objects of $\mathbb{C}_{\text{reg}}$. We adopt the same convention when we look at $\mathbb{C}$ as a full subcategory of $\mathbb{C}_{\text{ex}}$.

Lemma 4.3.1 *Let $\mathbb{C}$ be a weakly lex category; $\mathbb{C}_{\text{reg}}$ is closed in $\mathbb{C}_{\text{ex}}$ with respect to the formation of subobjects.*

Proof: Let $(f_i\colon X \longrightarrow X_i)_I$ be an object of $\mathbb{C}_{\text{reg}}$ and $x_0, x_1\colon \overline{X} \rightrightarrows X$ its embedding in $\mathbb{C}_{\text{ex}}$ (that is x_0, x_1 is a weak universal pair such that $x_0 \cdot f_i = x_1 \cdot f_i$ $\forall\ i \in I$); consider now a monomorphism in $\mathbb{C}_{\text{ex}}$

$$\begin{tikzcd} R \arrow[r, "\overline{f}"] \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & \overline{X} \arrow[d, shift right, "x_0"'] \arrow[d, shift left, "x_1"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

and the object $(f \cdot f_i\colon Y \longrightarrow X_i)_I$ of $\mathbb{C}_{\text{reg}}$ together with its embedding in $\mathbb{C}_{\text{ex}}$ $y_0, y_1\colon \overline{Y} \rightrightarrows Y$ (that is y_0, y_1 is a weak universal pair such that $y_0 \cdot f \cdot f_i = y_1 \cdot f \cdot f_i$ $\forall\ i \in I$).

Let us show that $r_0, r_1\colon R \rightrightarrows Y$ is isomorphic to $y_0, y_1\colon \overline{Y} \rightrightarrows Y$. As $r_0 \cdot f \cdot f_i = \overline{f} \cdot x_0 \cdot f_i = \overline{f} \cdot x_1 \cdot f_i = r_1 \cdot f \cdot f_i$, we have an arrow $r\colon R \longrightarrow \overline{Y}$ such that $r \cdot y_0 = r_0$ and $r \cdot y_1 = r_1$, that is an arrow in $\mathbb{C}_{\text{ex}}$

$$\begin{tikzcd} R \arrow[r, "r"] \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & \overline{Y} \arrow[d, shift right, "y_0"'] \arrow[d, shift left, "y_1"] \\ Y \arrow[r, "1_Y"'] & Y \end{tikzcd}$$

which is a regular epi. As $y_0 \cdot f \cdot f_i = y_1 \cdot f \cdot f_i$, we have an arrow $y\colon \overline{Y} \longrightarrow \overline{X}$ such that $y_0 \cdot f = y \cdot x_0$ and $y_1 \cdot f = y \cdot x_1$, that is an arrow in $\mathbb{C}_{\text{ex}}$

$$\begin{tikzcd} \overline{Y} \arrow[r, "y"] \arrow[d, shift right, "y_0"'] \arrow[d, shift left, "y_1"] & \overline{X} \arrow[d, shift right, "x_0"'] \arrow[d, shift left, "x_1"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

Clearly, the composition of the two arrows just constructed is the given monomorphism, so that also the first arrow is a monomorphism. But it is also a regular epi, so that it is an isomorphism. ■

Let us recall now a general fact whose proof can be found, for example, in section 37 of [25].

Proposition 4.3.2 *Let $\mathbb{A}$ be a complete, well-powered and regular category and $\mathbb{B}$ a full and replete subcategory of $\mathbb{A}$; $\mathbb{B}$ can be embedded in a smallest epireflective subcategory of $\mathbb{A}$ called the epireflective hull of $\mathbb{B}$ in $\mathbb{A}$. The objects of the hull are precisely the subobjects of products of $\mathbb{B}$-objects in $\mathbb{A}$.* ■

Proposition 4.3.3 *Let $\mathbb{C}$ be a weak complete category and suppose that $\mathbb{C}_{ex}$ is well-powered; $\mathbb{C}_{reg}^{\infty}$ is the epireflective hull of $\mathbb{C}$ in $\mathbb{C}_{ex}$.*

Proof: Since $\mathbb{C}$ is weak complete, $\mathbb{C}_{\mathrm{ex}}$ is complete and moreover it is completely regular (for this, use once again the characterization of regular epis in $\mathbb{C}_{\mathrm{ex}}$). $\mathbb{C}_{\mathrm{reg}}^{\infty}$ is then closed in $\mathbb{C}_{\mathrm{ex}}$ under the formation of products (because the embedding $\mathrm{Ker}\colon \mathbb{C}_{\mathrm{reg}}^{\infty} \longrightarrow \mathbb{C}_{\mathrm{ex}}$ is continuous).

An object of $\mathbb{C}_{\mathrm{reg}}^{\infty}$ is a subobject (in $\mathbb{C}_{\mathrm{reg}}^{\infty}$ and then in $\mathbb{C}_{\mathrm{ex}}$) of a product (in $\mathbb{C}_{\mathrm{reg}}^{\infty}$ and then in $\mathbb{C}_{\mathrm{ex}}$) of objects of $\mathbb{C}$ and then it is contained in the epireflective hull.

Conversely, an object of the hull is a subobject of a product of $\mathbb{C}$-objects in $\mathbb{C}_{\mathrm{ex}}$; then it is a subobject in $\mathbb{C}_{\mathrm{ex}}$ of an object of $\mathbb{C}_{\mathrm{reg}}^{\infty}$ (because the objects of $\mathbb{C}$ are in $\mathbb{C}_{\mathrm{reg}}^{\infty}$ which is closed in $\mathbb{C}_{\mathrm{ex}}$ under products). By lemma 4.3.1, it is then an object of $\mathbb{C}_{\mathrm{reg}}^{\infty}$. ■

Corollary 4.3.4

1) *let $\mathbb{T}$ be a monad over $\mathcal{SET}$; $KL(\mathbb{T})_{reg}^{\infty}$ is the epireflective hull of $KL(\mathbb{T})$ in $EM(\mathbb{T})$*

2) *let $\mathcal{D}$ be a small category; $Fam(\mathcal{D})_{reg}^{\infty}$ is the epireflective hull of $Fam(\mathcal{D})$ in $\mathcal{SET}^{\mathcal{D}^{op}}$.* ■

4.4 Further applications

The aim of this section is to point out that the theory developed in chapters 1 and 3 can be used to simplify a little bit the proof of two well-known theorems.

Milnor's theorem:

Let us fix some notations:

- $\underline{\Delta}$ is the simplicial category;
- $\underline{\mathrm{Ke}}$ is the category of Kelley spaces (also called "compactly generated spaces");

- $\Delta^?\colon \underline{\Delta} \longrightarrow \underline{\mathrm{Ke}}$ is the "standard simplex" functor;
- $|?\,|\colon \mathcal{SET}^{\underline{\Delta}^{\mathrm{op}}} \longrightarrow \underline{\mathrm{Ke}}$ is the "geometric realization" functor, that is the left Kan extension of $\Delta^?$ along the covariant Yoneda embedding $Y\colon \underline{\Delta} \longrightarrow \mathcal{SET}^{\underline{\Delta}^{\mathrm{op}}}$;
- $\underline{\mathrm{Haus}}$ is the category of Hausdorff spaces.

Theorem 4.4.1 *(Milnor) the geometric realization $|?\,|\colon \mathcal{SET}^{\underline{\Delta}^{\mathrm{op}}} \longrightarrow \underline{\mathrm{Ke}}$ is an exact functor.*

This theorem is discussed in detail for example in [23]. In order to apply our results, let us recall two basic facts about the category of Kelley spaces; for their proof we refer once again to [23].

Proposition 4.4.2 *$\underline{\mathrm{Ke}}$ is cartesian closed and coreflective in $\underline{\mathrm{Haus}}$; the coreflection $\mathbb{K}\colon \underline{\mathrm{Haus}} \longrightarrow \underline{\mathrm{Ke}}$ sends a Hausdorff space X into the space $\mathbb{K}X$ which has the same underlying set and whose closed subsets are the subsets C whose intersection with each compact subspace of X is closed in X; the counit $\epsilon_X\colon \mathbb{K}X \longrightarrow X$ is the identity on the underlying sets.* ■

Proposition 4.4.3 *Let $g\colon A \longrightarrow Y$ be in $\underline{\mathrm{Ke}}$ and $i\colon \underline{\mathrm{Ke}} \longrightarrow \underline{\mathrm{Top}}$ the full inclusion; consider a functor $\delta\colon \mathcal{T} \longrightarrow \underline{\mathrm{Ke}}/Y$ (where $\mathcal{T}$ is a small category and $\underline{\mathrm{Ke}}/Y$ the comma category) which associates to an object t of $\mathcal{T}$ the object $\underline{d}(t) \longrightarrow Y$ of $\underline{\mathrm{Ke}}/Y$ (so that $\underline{d}$ is a functor $\mathcal{T} \longrightarrow \underline{\mathrm{Ke}}$). Suppose that colim$\underline{d}\cdot i$ is a Hausdorff space. Then the canonical morphism between Kelley spaces*

$$colim_{t\in\mathcal{T}}\mathbb{K}(\underline{d}(t)\times_Y A) \longrightarrow \mathbb{K}((colim_{t\in\mathcal{T}}\underline{d}(t))\times_Y A)$$

is a homeomorphism.

(In the previous formula: the colimit on the left is taken in $\underline{\mathrm{Ke}}$, the colimit on the right is at the same time in $\underline{\mathrm{Top}}$ and in $\underline{\mathrm{Haus}}$, the pullbacks are in $\underline{\mathrm{Haus}}$ so that their Kelleyfications are the pullbacks in $\underline{\mathrm{Ke}}$). ■

Now we can prove that the category of Kelley spaces is regular.

Lemma 4.4.4 *Given a regular epi $f\colon X \longrightarrow Y$ in $\underline{\mathrm{Ke}}$, there exists a Kelley space K and two continuous maps $\alpha, \beta\colon K \rightrightarrows X$ such that*

$$K \underset{\beta}{\overset{\alpha}{\rightrightarrows}} X \xrightarrow{f} Y$$

is a coequalizer diagram in $\underline{\mathrm{Ke}}$ but also in $\underline{\mathrm{Top}}$.

Proof: Let us recall that in $\underline{\mathrm{Top}}$ each arrow can be factored as a regular epi followed by a monomorphism, so that an arrow is a regular epi if and only if it is an extremal epimorphism. Moreover, an extremal epimorphism in $\underline{\mathrm{Haus}}$ is also an extremal epimorphism in $\underline{\mathrm{Top}}$ (this follows from the fact that if $b\colon Z \longrightarrow Y$ is a monomorphism in $\underline{\mathrm{Top}}$ and Y is a Hausdorff space, then also Z is Hausdorff).

Now let us consider a regular epi $f\colon X \longrightarrow Y$ in $\underline{\mathrm{Ke}}$. Since $\underline{\mathrm{Ke}}$ is coreflective in $\underline{\mathrm{Haus}}$, f is a regular epi in $\underline{\mathrm{Haus}}$ and then, by the above observation, in $\underline{\mathrm{Top}}$.
Now consider its kernel pair $a, b\colon N \rightrightarrows X$ in $\underline{\mathrm{Top}}$, so that

$$N \overset{a}{\underset{b}{\rightrightarrows}} X \xrightarrow{f} Y$$

is a coequalizer diagram in $\underline{\mathrm{Top}}$ and also in $\underline{\mathrm{Haus}}$ (because N is Hausdorff). The two compositions

$$\mathbb{K}N \xrightarrow{1} N \overset{a}{\underset{b}{\rightrightarrows}} X$$

give us the required diagram $\alpha, \beta\colon K \rightrightarrows X$ in $\underline{\mathrm{Ke}}$. ■

Proposition 4.4.5 *The category of Kelley spaces is regular.*

Proof: Since $\underline{\mathrm{Ke}}$ is complete and cocomplete (because it is coreflective in $\underline{\mathrm{Haus}}$), it suffices to show that regular epis are pullback stable. Consider a regular epi $f\colon X \longrightarrow Y$ in $\underline{\mathrm{Ke}}$; by lemma 4.4.4, there exist two arrows $\alpha, \beta\colon K \rightrightarrows X$ between Kelley spaces such that

$$K \overset{\alpha}{\underset{\beta}{\rightrightarrows}} X \xrightarrow{f} Y$$

is a coequalizer in $\underline{\mathrm{Top}}$. Now we can apply proposition 4.4.3: consider the diagram $\delta\colon \mathcal{T} \longrightarrow \underline{\mathrm{Ke}}/Y$ given by

$$\begin{tikzcd}
K \arrow[rr, shift left, "\alpha"] \arrow[rr, shift right, "\beta"'] \arrow[dr, "\gamma"'] & & X \arrow[dl, "f"] \\
& Y &
\end{tikzcd}$$

(where $\gamma = \alpha \cdot f = \beta \cdot f$); pullback

$$\begin{tikzcd}
K \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] \arrow[dr, "\gamma"'] & X \arrow[r, "f"] \arrow[d, "f"] & Y \arrow[dl, "1"] \\
& Y &
\end{tikzcd}$$

along $g\colon A \longrightarrow Y$ to obtain

$$\begin{tikzcd}
K' \arrow[r, shift left, "\alpha'"] \arrow[r, shift right, "\beta'"'] \arrow[dr, "\gamma'"'] & X' \arrow[r, "f'"] \arrow[d, "f'"'] & A \arrow[dl, "1"] \\
& A &
\end{tikzcd}$$

As the colimit of $\alpha, \beta \colon K \rightrightarrows X$ in $\underline{\mathrm{Top}}$ is Y which is a Hausdorff space, we have that $f' \colon X' \longrightarrow A$ is isomorphic to the colimit in $\underline{\mathrm{Ke}}$ of $\alpha', \beta' \colon K' \rightrightarrows X'$, that is f' is a regular epi in $\underline{\mathrm{Ke}}$.[1] ■

Unfortunately, $\underline{\mathrm{Ke}}$ is not exact (cf. [13]). Any way, one can use proposition 3.5.2 to give an alternative proof of Milnor theorem. Showing that the sum-extension $\Delta' \colon \mathrm{Fam}(\underline{\Delta}) \longrightarrow \underline{\mathrm{Ke}}$ of $\Delta^? \colon \underline{\Delta} \longrightarrow \underline{\mathrm{Ke}}$ is left covering is not too much difficult, because the simplicial category $\underline{\Delta}$ has terminal object and equalizers and, moreover, $\underline{\mathrm{Ke}}$ is an extensive category.

Clearly, a more important simplification can be apported to the proof of the topos-theoretic analogous of Milnor theorem, which is a basic ingredient to show that the topos of simplicial sets classifies linear orders (see [32]).

<u>Herrlich - Strecker theorem:</u>

We know, from a well-known theorem by Manes, that the category $\mathcal{CH}$ of compact Hausdorff spaces is monadic over $\mathcal{SET}$. In [24] this theorem is, in some sense, inverted.

Theorem 4.4.6 *(Herrlich - Strecker) Let $\mathbb{A}$ be a not trivial epireflective subcategory of $\underline{\mathrm{Haus}}$; if $\mathbb{A}$ is monadic over $\mathcal{SET}$ then $\mathbb{A}$ is the category $\mathcal{CH}$.*

The inclusion of $\mathbb{A}$ in $\mathcal{CH}$ is proved using an argument on normal spaces and I am not able to simplify this part.

For the opposite inclusion, one can work as follows. Since $\mathbb{A}$ is epireflective in $\underline{\mathrm{Haus}}$, it is closed under the formation of products and subobjects (cf. 37.1 in [25]). But, since $\mathbb{A}$ is not trivial (that is it contains a space with at least two points), this implies that it contains all the powers of the two-point discrete spaces and then all the free compact Hausdorff spaces. But, since $\mathbb{A}$ is contained in $\mathcal{CH}$, this implies that the free $\mathcal{CH}$-space on a set I is also the free $\mathbb{A}$-space on I. By proposition 1.7.2, we have finished the proof.

4.5 Accessible categories

In this section we recall a quite different description of the exact completion $\Gamma \colon \mathbb{C} \longrightarrow \mathbb{C}_{\mathrm{ex}}$. It can be found in [33] if $\mathbb{C}$ is left exact and in [26] for the more general case.

Lemma 4.5.1 *Consider a functor $F \colon \mathbb{C} \longrightarrow \mathcal{SET}$ defined on a weakly lex category $\mathbb{C}$; the two following conditions are equivalent:*

1) F is left covering

2) F is a filtered colimit of representable functors

Proof: 1) $\Rightarrow$ 2): by 1.4.6, we know that, for each $A \in \mathcal{SET}$, the comma category (A, F) is filtering; this means that the category of elements of the

[1] F. Cagliari has simplified and generalized this argument showing that if $\mathbb{A}$ is a coreflective subcategory of $\underline{\mathrm{Haus}}$ which is closed for closed-subspaces and cartesian closed, then $\mathbb{A}$ is regular (cf. [13]).

functor $\mathcal{SET}(A, F-)$ is filtering. If we choose as A the singleton, we have that the category of elements of F is filtering and then F is a filtered colimit of representable functors.
2) $\Rightarrow$ 1): a representable functor is left covering simply because in $\mathcal{SET}$ strong epimorphism means surjective. If F is a filtered colimit of representable functors, say $F = \mathrm{colim}_X \mathbb{C}(X, -)$, and $\mathcal{L}: \mathcal{D} \longrightarrow \mathbb{C}$ is defined on a finite category $\mathcal{D}$, we have that $F(\mathrm{wlim}\mathcal{L}) = \mathrm{colim}_X \mathbb{C}(X, \mathrm{wlim}\mathcal{L})$ and also $\mathrm{lim}\mathcal{L} \cdot F = \mathrm{lim}(\mathcal{L} \cdot \mathrm{colim}_X \mathbb{C}(X, -)) = \mathrm{colim}_X \mathrm{lim}(\mathcal{L} \cdot \mathbb{C}(X, -)$ (because in $\mathcal{SET}$ filtered colimits commute with finite limits). This implies that the canonical factorization from $F(\mathrm{wlim}\mathcal{L})$ to $\mathrm{lim}\mathcal{L} \cdot F$ is a colimit of regular epis and then it is a regular epi by interchange of colimits. ■

Let $\mathbb{C}$ be a small weakly lex category and let us write $\mathrm{Lco}(\mathbb{C})$ for the category of left covering functors from $\mathbb{C}$ to $\mathcal{SET}$. The proof of the two following theorems can be found in [26].

Theorem 4.5.2 *A category is $\aleph_0$-accessible with products if and only if it is equivalent to Lco($\mathbb{C}$) for a small weakly lex category $\mathbb{C}$.* ■

Let us write $\prod \mathrm{filt}(\mathrm{Lco}(\mathbb{C}))$ for the category of functors from $\mathrm{Lco}(\mathbb{C})$ to $\mathcal{SET}$ which preserve products and filtered colimits.

Theorem 4.5.3 *Let $\mathbb{C}$ be a small weakly lex category;*

1) the evaluation functor $e: \mathbb{C} \longrightarrow \prod \mathit{filt}(\mathit{Lco}(\mathbb{C}))$ is full and faithful

2) $e(\mathbb{C})$ is a projective cover of the exact category $\prod \mathit{filt}(\mathit{Lco}(\mathbb{C}))$. ■

4.6 An unpleasant proof

In this section we give a detailed and direct proof of theorem 1.2.3. A shorter, but not self-contained, proof of this theorem and of theorem 1.5.2 will be given in the next section.

Theorem 4.6.1 *Let $\mathbb{C}$ be a weakly lex category and let $\mathbb{C}_{\mathrm{ex}}$ be as in definition 1.2.2; then $\mathbb{C}_{\mathrm{ex}}$ is an exact category.*

We divide the proof into three steps.

Step 1: $\mathbb{C}_{\mathrm{ex}}$ is a left exact category. Let T be a weak terminal object in $\mathbb{C}$ and consider a weak product $T \xleftarrow{\pi_1} T \times T \xrightarrow{\pi_2} T$. Then $\pi_1, \pi_2: T \times T \rightrightarrows T$ is the terminal object in $\mathbb{C}_{\mathrm{ex}}$. For this, consider an object $r_1, r_2: R \rightrightarrows X$ in $\mathbb{C}_{\mathrm{ex}}$. Since T is a weak terminal, there exists $x: X \longrightarrow T$; since $T \times T$ is a weak product, there exists $\overline{x}: R \longrightarrow T \times T$ such that $\overline{x} \cdot \pi_1 = r_1 \cdot x$ and $\overline{x} \cdot \pi_2 = r_2 \cdot x$. This means that

$$[\overline{x}, x]: (r_1, r_2: R \rightrightarrows X) \longrightarrow (\pi_1, \pi_2: T \times T \rightrightarrows T)$$

is an arrow in $\mathbb{C}_{\mathrm{ex}}$. The verification that $\pi_1, \pi_2: T \times T \rightrightarrows T$ is an object in $\mathbb{C}_{\mathrm{ex}}$ is straightforward and does not depend on the fact that T is weak terminal. As far as the uniqueness is concerned, observe that if $y\colon X \longrightarrow T$ is another arrow in $\mathbb{C}$, then there exists $\Sigma\colon X \longrightarrow T \times T$ such that $\Sigma \cdot \pi_1 = x$ and $\Sigma \cdot \pi_2 = y$.

Let us consider now the following pair of arrows in $\mathbb{C}_{\mathrm{ex}}$

$$\begin{tikzcd}
R \arrow[r, "\overline{f}"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] & T \arrow[l, "\overline{g}"'] \arrow[d, shift right, "t_1"'] \arrow[d, shift left, "t_2"] \\
X \arrow[r, "f"'] & Y & Z \arrow[l, "g"]
\end{tikzcd}$$

In order to build up its pullback, consider a weak limit $(P; \underline{f}, \varphi, \underline{g})$ as in the following diagram

$$\begin{tikzcd}
X \arrow[d, "f"'] & P \arrow[l, "\underline{f}"'] \arrow[r, "\underline{g}"] \arrow[d, "\varphi"] & Z \arrow[d, "g"] \\
Y & S \arrow[l, "s_1"] \arrow[r, "s_2"'] & Y
\end{tikzcd}$$

and a weak limit $(E; \rho, e_1, e_2, \tau)$ as in the following diagram

$$\begin{tikzcd}
& & & E \arrow[dlll, "\rho"'] \arrow[dll, "e_1"] \arrow[drr, "e_2"'] \arrow[drrr, "\tau"] & & & \\
R \arrow[d, "r_1"'] \arrow[dr, "r_2"] & P \arrow[dl, "\underline{f}"'] \arrow[drrrr, "\underline{g}"] & & & & P \arrow[dllll, "\underline{f}"'] \arrow[dr, "\underline{g}"] & T \arrow[dl, "t_1"'] \arrow[d, "t_2"] \\
X & X & & & & Z & Z
\end{tikzcd}$$

The required pullback in $\mathbb{C}_{\mathrm{ex}}$ is given by

$$\begin{tikzcd}
R \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & E \arrow[l, "\rho"'] \arrow[r, "\tau"] \arrow[d, shift right, "e_1"'] \arrow[d, shift left, "e_2"] & T \arrow[d, shift right, "t_1"'] \arrow[d, shift left, "t_2"] \\
X & P \arrow[l, "\underline{f}"] \arrow[r, "\underline{g}"'] & Z
\end{tikzcd}$$

Let us show that $e_1, e_2\colon E \rightrightarrows P$ is an object of $\mathbb{C}_{\mathrm{ex}}$.
Reflexivity: consider the reflexivity $r_R\colon X \longrightarrow R$ and $r_T\colon Z \longrightarrow T$ (that is $r_R \cdot r_1 = 1_X = r_R \cdot r_2$ and $r_T \cdot t_1 = 1_Z = r_T \cdot t_2$). We obtain a cone

$$\begin{tikzcd}
 & & P \arrow[dll, "\underline{f} \cdot r_R"'] \arrow[dl, "1_P"] \arrow[d, "1_P"] \arrow[dr, "\underline{g} \cdot r_T"] & \\
R & P & P & T
\end{tikzcd}$$

on the diagram defining E, so that there exists $r_E \colon P \longrightarrow E$ such that, in particular, $r_E \cdot e_1 = 1_P = r_E \cdot e_2$.
Symmetry: consider the symmetries $s_R \colon R \longrightarrow R$ and $s_T \colon T \longrightarrow T$ (that is $s_R \cdot r_1 = r_2$, $s_R \cdot r_2 = r_1$, $s_T \cdot t_1 = t_2$ and $s_T \cdot t_2 = t_1$).

We obtain a cone

$$\begin{tikzcd}
 & & E \arrow[dll, "\rho \cdot s_R"'] \arrow[dl, "e_2"] \arrow[dr, "e_1"] \arrow[drr, "\tau \cdot s_T"] & & \\
R & P & & P & T
\end{tikzcd}$$

on the diagram defining E, so that there exists $s_E \colon E \longrightarrow E$ such that, in particular, $s_E \cdot e_1 = e_2$ and $s_E \cdot e_2 = e_1$.
Transitivity: consider the following weak pullbacks

$$\begin{tikzcd}
R * R \arrow[r, "a_2"] \arrow[d, "a_1"'] & R \arrow[d, "r_1"] \\
R \arrow[r, "r_2"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
T * T \arrow[r, "b_2"] \arrow[d, "b_1"'] & T \arrow[d, "t_1"] \\
T \arrow[r, "t_2"'] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
E * E \arrow[r, "c_2"] \arrow[d, "c_1"'] & E \arrow[d, "e_1"] \\
E \arrow[r, "e_2"'] & P
\end{tikzcd}$$

Since $\rho \cdot r_1 = e_1 \cdot \underline{f}$ and $\rho \cdot r_2 = e_2 \cdot \underline{f}$, there exists $\rho * \rho \colon E * E \longrightarrow P * P$ such that $\rho * \rho \cdot a_1 = c_1 \cdot \rho$ and $\rho * \rho \cdot a_2 = c_2 \cdot \rho$. Analogously there exists $\tau * \tau \colon E * E \longrightarrow T * T$ such that $\tau * \tau \cdot b_1 = c_1 \cdot \tau$ and $\tau * \tau \cdot b_2 = c_2 \cdot \tau$.

Consider now the transitivities $t_R \colon R * R \longrightarrow R$ and $t_T \colon T * T \longrightarrow T$ (that is $t_R \cdot r_1 = a_1 \cdot r_1$, $t_R \cdot r_2 = a_2 \cdot r_R$, $t_T \cdot t_1 = b_1 \cdot t_1$ and $t_T \cdot t_2 = b_2 \cdot t_2$).

We obtain a cone

$$\begin{tikzcd}
 & & E * E \arrow[dll, "\rho * \rho \cdot t_R"'] \arrow[dl, "c_1 \cdot e_1"] \arrow[dr, "c_2 \cdot e_2"'] \arrow[drr, "\tau * \tau \cdot t_T"] & & \\
R & P & & P & T
\end{tikzcd}$$

on the diagram defining E, so that there exists $t_E \colon E * E \longrightarrow E$ such that, in particular, $t_E \cdot e_1 = c_1 \cdot e_1$ and $t_E \cdot e_2 = c_2 \cdot e_2$.

Clearly, $[\underline{f}]$ and $[\underline{g}]$ are arrows in $\mathbb{C}_{\mathrm{ex}}$ (look at the definition of E) and $[\underline{f}] \cdot [f] = [\underline{g}] \cdot [g]$ (use $\varphi \colon P \longrightarrow S$ in the definition of P as homotopy).

Now suppose that

$$\begin{tikzcd} R \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & U \arrow[l, "\overline{x}"'] \arrow[r, "\overline{z}"] \arrow[d, shift right, "u_1"'] \arrow[d, shift left, "u_2"] & T \arrow[d, shift right, "t_1"'] \arrow[d, shift left, "t_2"] \\ X & W \arrow[l, "x"] \arrow[r, "z"'] & Z \end{tikzcd}$$

are two arrows in $\mathbb{C}_{\mathrm{ex}}$ such that $[x] \cdot [f] = [z] \cdot [g]$. This means that there exists a homotopy $\Sigma\colon W \longrightarrow S$ such that $\Sigma \cdot s_1 = x \cdot f$ and $\Sigma \cdot s_2 = z \cdot g$. But this means that

$$\begin{tikzcd} & W \arrow[dl, "x"'] \arrow[d, "\Sigma"] \arrow[dr, "z"] & \\ X & S & Z \end{tikzcd}$$

is a cone on the diagram defining P, so that there exists $p\colon W \longrightarrow P$ such that, in particular, $p \cdot \underline{f} = x$ and $p \cdot \underline{g} = z$.

These two equations say that

$$\begin{tikzcd} & & U \arrow[dll, "\overline{x}"'] \arrow[dl, "u_1 \cdot p"] \arrow[dr, "u_2 \cdot p"'] \arrow[drr, "\overline{z}"] & & \\ R & P & & P & T \end{tikzcd}$$

is a cone on the diagram defining E, so that there exists $\overline{p}\colon U \longrightarrow E$ such that, in particular, $\overline{p} \cdot e_1 = u_1 \cdot p$ and $\overline{p} \cdot e_2 = u_2 \cdot p$.

The last four equations mean that

$$\begin{tikzcd} U \arrow[r, "\overline{p}"] \arrow[d, shift right, "u_1"'] \arrow[d, shift left, "u_2"] & E \arrow[d, shift right, "e_1"'] \arrow[d, shift left, "e_2"] \\ W \arrow[r, "p"'] & P \end{tikzcd}$$

is the required factorization in $\mathbb{C}_{\mathrm{ex}}$.

It remains to show that $[\underline{f}]$ and $[\underline{g}]$ are a monomorphic pair.

Suppose that

$$\begin{tikzcd} U \arrow[r, "\overline{p}"] \arrow[d, shift right, "u_1"'] \arrow[d, shift left, "u_2"] & E \arrow[d, shift right, "e_1"'] \arrow[d, shift left, "e_2"] \\ W \arrow[r, "p"'] & P \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "\overline{q}"] \arrow[d, shift right, "u_1"'] \arrow[d, shift left, "u_2"] & E \arrow[d, shift right, "e_1"'] \arrow[d, shift left, "e_2"] \\ W \arrow[r, "q"'] & P \end{tikzcd}$$

are two arrows in $\mathbb{C}_{\text{ex}}$ such that $[p] \cdot [\underline{f}] = [q] \cdot [\underline{f}]$ and $[p] \cdot [\underline{g}] = [q] \cdot [\underline{g}]$.
This means that there exist two homotopies $\alpha \colon W \longrightarrow R$ and $\beta \colon W \longrightarrow T$ such that $\alpha \cdot r_1 = p \cdot \underline{f}$, $\alpha \cdot r_2 = q \cdot \underline{f}$, $\beta \cdot t_1 = p \cdot \underline{g}$ and $\beta \cdot t_2 = q \cdot \underline{g}$. But this means that

$$\begin{tikzcd}
& & & W \arrow[dlll, "\alpha"'] \arrow[dll, "p"] \arrow[drr, "q"'] \arrow[drrr, "\beta"] & & & \\
R & P & & & & P & T
\end{tikzcd}$$

is a cone on the diagram defining E, so that there exists $\Sigma \colon W \longrightarrow E$ such that, in particular, $\Sigma \cdot e_1 = p$ and $\Sigma \cdot e_2 = q$ and then $[p] = [q]$.

The argument about pullbacks in $\mathbb{C}_{\text{ex}}$ is now complete.

We also need an explicit description for equalizers in $\mathbb{C}_{\text{ex}}$; for this, consider two parallel arrows in $\mathbb{C}_{\text{ex}}$

$$\begin{tikzcd}
R \arrow[r, shift left, "\overline{f}"] \arrow[r, shift right, "\overline{g}"'] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd}$$

In order to build up their equalizer, consider a weak limit $(E; e, \varphi)$ as in the following diagram

$$\begin{tikzcd}
& E \arrow[dl, "e"'] \arrow[dr, "\varphi"] & \\
X \arrow[d, "f"'] \arrow[drr, "g"', near start] & & S \arrow[d, "s_2"] \arrow[dll, "s_1", near start] \\
Y & & Y
\end{tikzcd}$$

Consider again a weak limit $(\underline{R}; e_1, \overline{e}, e_2)$ as in the following diagram

$$\begin{tikzcd}
E \arrow[d, "e"'] & \underline{R} \arrow[l, "e_1"'] \arrow[r, "e_2"] \arrow[d, "\overline{e}"] & E \arrow[d, "e"] \\
X & R \arrow[l, "r_1"] \arrow[r, "r_2"'] & X
\end{tikzcd}$$

Then the equalizer in $\mathbb{C}_{\text{ex}}$ is given by

$$\begin{tikzcd}
\underline{R} \arrow[r, "\overline{e}"] \arrow[d, shift right, "e_1"'] \arrow[d, shift left, "e_2"] & R \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] \\
E \arrow[r, "e"'] & X
\end{tikzcd}$$

Here we omit details because they are similar (and easier) to those concerning the pullback.

<u>Step 2:</u> $\mathbb{C}_{\mathrm{ex}}$ has regular epi-mono factorization and regular epis are stable under pullbacks.

Given an arrow in $\mathbb{C}_{\mathrm{ex}}$

$$\begin{tikzcd}
R \arrow[r, "\overline{f}"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

consider a weak limit $(I; i_1, i, i_2)$ as in the following diagram

$$\begin{tikzcd}
X \arrow[d, "f"'] & I \arrow[l, "i_1"'] \arrow[r, "i_2"] \arrow[d, "i"] & X \arrow[d, "f"] \\
Y & S \arrow[l, "s_1"] \arrow[r, "s_2"'] & Y
\end{tikzcd}$$

Since

$$\begin{tikzcd}
 & R \arrow[dl, "r_1"'] \arrow[d, "\overline{f}"] \arrow[dr, "r_2"] & \\
X & S & X
\end{tikzcd}$$

is a cone on the diagram defining I, there exists $t \colon R \longrightarrow I$ such that, in particular, $t \cdot i_1 = r_1$ and $t \cdot i_2 = r_2$.

The required factorization is given by

$$\begin{tikzcd}
R \arrow[r, "t"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & I \arrow[r, "i"] \arrow[d, shift right, "i_1"'] \arrow[d, shift left, "i_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\
X \arrow[r, "1_X"'] & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Let us prove that $i_1, i_2 \colon I \rightrightarrows X$ is an object of $\mathbb{C}_{\text{ex}}$.
Reflexivity: consider the reflexivity $r_S \colon Y \longrightarrow S$ (that is $r_S \cdot s_1 = 1_Y = r_S \cdot s_2$). We obtain a cone

$$\begin{tikzcd} & X \arrow[dl, "1_X"'] \arrow[d, "f \cdot r_S"] \arrow[dr, "1_X"] & \\ X & S & X \end{tikzcd}$$

on the diagram defining I, so that there exists $r_I \colon X \longrightarrow I$ such that $r_I \cdot i_1 = 1_X = r_I \cdot i_2$.
Symmetry: consider the symmetry $s_S \colon S \longrightarrow S$ (that is $s_S \cdot s_1 = s_2$ and $s_S \cdot s_2 = s_1$). We obtain a cone

$$\begin{tikzcd} & I \arrow[dl, "i_2"'] \arrow[d, "i \cdot s_S"] \arrow[dr, "i_1"] & \\ X & S & X \end{tikzcd}$$

on the diagram defining I, so that there exists $s_I \colon I \longrightarrow I$ such that $s_I \cdot i_1 = i_2$ and $s_I \cdot i_2 = i_1$.
Transitivity: consider the following weak pullbacks

$$\begin{tikzcd} S * S \arrow[r, "c_2"] \arrow[d, "a_1"'] & S \arrow[d, "s_1"] \\ S \arrow[r, "s_2"'] & Y \end{tikzcd} \qquad \begin{tikzcd} I * I \arrow[r, "b_2"] \arrow[d, "b_1"'] & I \arrow[d, "i_1"] \\ I \arrow[r, "i_2"'] & X \end{tikzcd}$$

and the transitivity $t_S \colon S * S \longrightarrow S$ (that is $t_S \cdot s_1 = a_1 \cdot s_1$ and $t_S \cdot s_2 = a_2 \cdot s_2$).

Since $i \cdot s_1 = i_1 \cdot f$ and $i \cdot s_2 = i_2 \cdot f$, there exists $i * i \colon I * I \longrightarrow S * S$ such that $i * i \cdot a_2 = b_2 \cdot i$.

We obtain a cone

$$\begin{tikzcd} & I * I \arrow[dl, "b_1 \cdot i_1"'] \arrow[d, "i * i \cdot t_S"] \arrow[dr, "b_2 \cdot i_2"] & \\ X & S & X \end{tikzcd}$$

on the diagram defining I, so that there exists $t_I\colon I * I \longrightarrow I$ such that $t_I \cdot i_1 = b_1 \cdot i_1$ and $t_I \cdot i_2 = b_2 \cdot i_2$.

Coming back to the factorization of $[\overline{f}, f]$, let us show that the righy-hand square is a monomorphism in $\mathbb{C}_{\text{ex}}$.

Suppose

$$\begin{tikzcd}
T \arrow[r, shift left, "\overline{x}"] \arrow[r, shift right, "\overline{y}"'] \arrow[d, shift right, "t_1"'] \arrow[d, shift left, "t_2"] & I \arrow[d, shift right, "i_1"'] \arrow[d, shift left, "i_2"] \\
Z \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & X
\end{tikzcd}$$

are two arrows in $\mathbb{C}_{\text{ex}}$ such that $[x] \cdot [i, f] = [y] \cdot [i, f]$, so that there exists $\Sigma\colon Z \longrightarrow S$ such that $\Sigma \cdot s_1 = x \cdot f$ and $\Sigma \cdot s_2 = y \cdot f$. But this means that

$$\begin{tikzcd}
 & Z \arrow[dl, "x"'] \arrow[d, "z"] \arrow[dr, "y"] & \\
X & S & X
\end{tikzcd}$$

is a cone on the diagram defining I, so that there exists $\Sigma'\colon Z \longrightarrow I$ such that $\Sigma' \cdot i_1 = x$ and $\Sigma' \cdot i_2 = y$ and then $[x] = [y]$.

Now we have to prove that the right-hand square is a regular epi in $\mathbb{C}_{\text{ex}}$. For this, we build up its kernel pair and we show that it is the coequalizer of its kernel pair. Following the description of pullbacks given in the previous step, we have that the kernel pair of

$$\begin{tikzcd}
R \arrow[r, "t"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & I \arrow[d, shift right, "i_1"'] \arrow[d, shift left, "i_2"] \\
X \arrow[r, "1_X"'] & X
\end{tikzcd}$$

is given by

$$\begin{tikzcd}
E \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift right] \arrow[d, shift left] & R \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] \\
I \arrow[r, shift left, "i_1"] \arrow[r, shift right, "i_2"'] & X
\end{tikzcd}$$

Suppose that

$$\begin{tikzcd}
R \arrow[r, "\overline{z}"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & T \arrow[d, shift right, "t_1"'] \arrow[d, shift left, "t_2"] \\
X \arrow[r, "z"'] & Z
\end{tikzcd}$$

is an arrow in $\mathbb{C}_{\text{ex}}$ such that $[i_1] \cdot [z] = [i_2] \cdot [z]$.

This means that there exists $\Sigma\colon I \longrightarrow T$ such that $\Sigma \cdot t_1 = i_1 \cdot z$ and $\Sigma \cdot t_2 = i_2 \cdot Z$.

Then

$$\begin{tikzcd} I \arrow[r, "\Sigma"] \arrow[d, shift right, "i_1"'] \arrow[d, shift left, "i_2"] & T \arrow[d, shift right, "t_1"'] \arrow[d, shift left, "t_2"] \\ X \arrow[r, "z"'] & Z \end{tikzcd}$$

is the required factorization, that is $[t, 1_X] \cdot [\Sigma, z] = [z]$. The factorization is unique because $[t, 1_X]$is clearly an epimorphism in $\mathbb{C}_{\text{ex}}$.

Observe that to show that $[t, 1_X]$ is a regular epi, we do not use the explicit description of $i_1, i_2\colon I \rightrightarrows X$. So, we have proved that in $\mathbb{C}_{\text{ex}}$ regular epimorphisms are, up to isomorphisms, exactly the arrows of the form

$$\begin{tikzcd} R \arrow[r, "\overline{f}"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ X \arrow[r, "1_X"'] & X \end{tikzcd}$$

We use this fact to simplify a little bit the verification that in $\mathbb{C}_{\text{ex}}$ regular epis are pullback stable. For this, consider an arrow

$$[\overline{g}, g]\colon (t_1, t_2\colon T \rightrightarrows Z) \longrightarrow (s_1, s_2\colon S \rightrightarrows X)$$

in $\mathbb{C}_{\text{ex}}$ and construct the pullback of $[\overline{f}, 1_X]$ along $[\overline{g}, g]$ following the description given in the previous step.

Using the same notations, this pullback turn out to be

$$\begin{tikzcd} E \arrow[r, "\tau"] \arrow[d, shift right, "e_1"'] \arrow[d, shift left, "e_2"] & T \arrow[d, shift right, "t_1"'] \arrow[d, shift left, "t_2"] \\ S_2 \arrow[r, "s_2'"'] & Z \end{tikzcd}$$

where

$$\begin{tikzcd} S_2 \arrow[r, "s_2'"] \arrow[d, "g'"'] & Z \arrow[d, "g"] \\ S \arrow[r, "s_2"'] & X \end{tikzcd}$$

is a weak pullback.

Now we construct the regular epi-mono factorization of $[\tau, s_2']$. It is given by

$$\begin{tikzcd} E \arrow[r, "t"] \arrow[d, shift right, "e_1"'] \arrow[d, shift left, "e_2"] & I \arrow[r, "i"] \arrow[d, shift right, "i_1"'] \arrow[d, shift left, "i_2"] & T \arrow[d, shift right, "t_1"'] \arrow[d, shift left, "t_2"] \\ S_2 \arrow[r, "1_{S_2}"'] & S_2 \arrow[r, "s_2'"'] & Z \end{tikzcd}$$

where

$$\begin{tikzcd} S_2 \arrow[d, "s_2'"'] & I \arrow[l, "i_1"'] \arrow[r, "i_2"] \arrow[d, "i"] & S_2 \arrow[d, "s_2'"] \\ Z & T \arrow[l, "t_1"] \arrow[r, "t_2"'] & Z \end{tikzcd}$$

is a weak limit. We have to prove that in the previous factorization, the left-hand square is an isomorphism. Since it is a monomorphism, it suffices to build up a left inverse. Consider the reflexivity $r_T\colon Z \longrightarrow T$. Since $r_T \cdot \overline{g} \cdot s_2 = r_T \cdot t_2 \cdot g = g$, there exists $l\colon Z \longrightarrow S_2$ such that $l \cdot g' = r_T \cdot \overline{g}$ and $l \cdot s_2' = 1_Z$.

This implies that

$$\begin{tikzcd} & T \arrow[dl, "t_1 \cdot l"'] \arrow[d, "1_T"] \arrow[dr, "t_2 \cdot l"] & \\ S_2 & T & S_2 \end{tikzcd}$$

is a cone on the diagram defining I, so that there exists $\bar{l}\colon T \longrightarrow I$ such that $\bar{l} \cdot i_1 = t_1 \cdot l$ and $\bar{l} \cdot i_2 = t_2 \cdot l$.

Now we have constructed an arrow in $\mathbb{C}_{\mathrm{ex}}$

$$\begin{tikzcd} T \arrow[r, "\bar{l}"] \arrow[d, shift right, "t_1"'] \arrow[d, shift left, "t_2"] & I \arrow[d, shift right, "i_1"'] \arrow[d, shift left, "i_2"] \\ Z \arrow[r, "l"'] & S_2 \end{tikzcd}$$

which is left inverse to $[i, s_2']$ because $l \cdot s_2' = 1_Z$.

Step 3: in $\mathbb{C}_{\mathrm{ex}}$ an equivalence relation has a coequalizer and it is the kernel pair of its coequalizer.

Let

$$\begin{tikzcd} R \arrow[r, shift left, "\overline{h_1}"] \arrow[r, shift right, "\overline{h_2}"'] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ X \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & Y \end{tikzcd}$$

be an equivalence relation in $\mathbb{C}_{\mathrm{ex}}$; in order to construct its coequalizer, consider a weak limit $(V; p_1, \nu, p_2)$ as in the following diagram

$$\begin{tikzcd}
S \arrow[d, "s_2"'] & V \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "\nu"] & S \arrow[d, "s_2"] \\
Y & X \arrow[l, "h_1"] \arrow[r, "h_2"'] & Y
\end{tikzcd}$$

and call $v_1 = p_1 \cdot s_1$, $v_2 = p_2 \cdot s_1$.

Now we need an arrow $q\colon S \longrightarrow V$ making the diagram

$$\begin{tikzcd}
S \arrow[r, "q"] \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] & V \arrow[d, shift right, "v_1"'] \arrow[d, shift left, "v_2"] \\
Y \arrow[r, "1_Y"'] & Y
\end{tikzcd}$$

an arrow in $\mathbb{C}_{\mathrm{ex}}$.

For this, let us point out that the reflexivity of $\langle [h_1], [h_2] \rangle$ means that there exists an arrow $[p]\colon (s_1, s_2\colon S \rightrightarrows Y) \longrightarrow (r_1, r_2\colon R \rightrightarrows X)$ in $\mathbb{C}_{\mathrm{ex}}$ such that $[p] \cdot [h_1] = [1_Y]$ and $[p] \cdot [h_2] = [1_Y]$. This implies the existence of $\Sigma_1\colon Y \longrightarrow S$ such that $\Sigma_1 \cdot s_1 = p \cdot h_1$ and $\Sigma_1 \cdot s_2 = 1_Y$ and the existence of $\Sigma_2\colon Y \longrightarrow S$ such that $\Sigma_2 \cdot s_1 = p \cdot h_2$ and $\Sigma_2 \cdot s_2 = 1_Y$.

Consider now a weak pullback

$$\begin{tikzcd}
S * S \arrow[r, "d_1"] \arrow[d, "d_2"'] & S \arrow[d, "s_2"] \\
S \arrow[r, "s_1"'] & Y
\end{tikzcd}$$

and the transitivity $t_S\colon S * S \longrightarrow S$ of $s_1, s_2\colon S \rightrightarrows Y$.

Since $s_1 \cdot \Sigma_2 \cdot s_2 = s_1$, there exists $c\colon S \longrightarrow S * S$ such that $c \cdot d_1 = s_1 \cdot \Sigma_2$ and $c \cdot d_2 = 1_S$.

We obtain a cone

$$\begin{tikzcd}
 & S \arrow[dl, "s_1 \cdot \Sigma_1 \cdot s_S"'] \arrow[d, "s_1 \cdot p"] \arrow[dr, "c \cdot t_S \cdot s_S"] & \\
S & X & S
\end{tikzcd}$$

on the diagram defining V
(where $s_S\colon S \longrightarrow S$ is the symmetry of $s_1, s_2\colon S \rightrightarrows Y$), so that there exists a factorization $q\colon S \longrightarrow V$.

It remains to verify the double commutativity of

$$\begin{tikzcd} S \arrow[r, "q"] \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] & V \arrow[d, shift right, "v_1"'] \arrow[d, shift left, "v_2"] \\ Y \arrow[r, "1_Y"'] & Y \end{tikzcd}$$

$$q \cdot v_1 = q \cdot p_1 \cdot s_1 = s_1 \cdot \Sigma_1 \cdot s_S \cdot s_1 = s_1 \cdot \Sigma_1 \cdot s_2 = s_1$$

$$q \cdot v_2 = q \cdot p_2 \cdot s_1 = c \cdot t_S \cdot s_S \cdot s_1 = c \cdot t_S \cdot s_2 = c \cdot d_2 \cdot s_2 = s_2.$$

To check that the given equivalence relation is the kernel pair of this regular epi is more subtle, so as it is subtle to show that $v_1, v_2\colon V \rightrightarrows Y$ is indeed an object of $\mathbb{C}_{\text{ex}}$. Let us look at the symmetry of $v_1, v_2\colon V \rightrightarrows Y$.

First, the symmetry of the relation (in $\mathbb{C}_{\text{ex}}$) $\langle [h_1], [h_2] \rangle$ means that there exists an arrow

$$\begin{tikzcd} R \arrow[r, "\overline{\sigma}"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & R \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] \\ X \arrow[r, "\sigma"'] & X \end{tikzcd}$$

and two homotopies $\varphi_1\colon X \longrightarrow S$, $\varphi_2\colon X \longrightarrow S$ such that $\varphi_1 \cdot s_1 = \sigma \cdot h_1$ and $\varphi_1 \cdot s_2 = h_2$ (that is $[\sigma] \cdot [h_1] = [h_2]$) and such that $\varphi_2 \cdot s_1 = \sigma \cdot h_2$ and $\varphi_2 \cdot s_2 = h_1$ (that is $[\sigma] \cdot [h_2] = [h_1]$).

If $r_1, r_2\colon R \rightrightarrows X$ and $s_1, s_2\colon S \rightrightarrows Y$ are honest equivalence relations and if we build up V in the category $\mathcal{SET}$, we obtain

$$V = \{(x, y, \overline{y} \in X \times Y \times Y \quad | \quad (y, h(x)) \in S \text{ and } (\overline{y}, h_2(x)) \in S\}$$

But, informally, we can think

$$V = \{(y, \overline{y}) \in Y \times Y \mid \exists\, x \in X \text{ such that } (y, h, (x)) \in S \text{ and } (\overline{y}, h_2(x)) \in S\}$$

This is possible because if there exist $x, x' \in X$ such that $(y, h_1(x)) \in S$ and $(\overline{y}, h_2(x)) \in S$ but also $(y, h_1(x')) \in S$ and $(\overline{y}, h_2(x')) \in S$, then $(h_1(x), h_1(x')) \in S$ and $(h_2(x), h_2(x')) \in S$ (by symmetry and transitivity of S). This implies that $x = x'$ in the quotient of X by R because they are equal in the quotient of Y by S and h_1, h_2 are jointly monic.

The symmetry of $\langle [h_1], [h_2] \rangle$ means that $\forall\, x \in X (h_1(\sigma(x)), h_2(x)) \in S$ and $(h_2(\sigma(x)), h_1(x)) \in S$; now by symmetry and transitivity of S we have that if $(y, \overline{y}) \in V$ by means of a certain $x \in X$, then

$$(y, h_1(x)) \in S \text{ and } (h_2(\sigma(x)), h_1(x)) \in S \text{ so that } (y, h_2(\sigma(x)) \in S$$

$$(\overline{y}, h_2(x)) \in S \text{ and } (h_1(\sigma(x)), h_2(x)) \in S \text{ so that } (\overline{y}, h_1(\sigma(x))) \in S$$

and so $(\overline{y}, y) \in V$ by means of $\sigma(x) \in X$.

Now we translate the previous argument through the construction of $v_1, v_2 \colon V \rightrightarrows Y$: since $\nu \cdot \varphi_2 \cdot s_S \cdot s_1 = \nu \cdot \varphi_2 \cdot s_2 = \nu \cdot h_1 = p_1 \cdot s_2$, there exists $\alpha_1 \colon V \longrightarrow S * S$ such that $\alpha_1 \cdot d_1 = p_1$ and $\alpha_1 \cdot d_2 = \nu \cdot \varphi_2 \cdot s_S$. This implies that $\alpha_1 \cdot t_S \cdot s_1 = \alpha_1 \cdot d_1 \cdot s_1 = p_1 \cdot s_1$ and $d_1 \cdot t_S \cdot s_2 = \alpha_1 \cdot d_2 \cdot s_2 = \nu \cdot \varphi_2 \cdot s_S \cdot s_2 = \nu \cdot \varphi_2 \cdot s_1 = \nu \cdot \sigma \cdot h_2$.

Since $\nu \cdot \varphi_1 \cdot s_S \cdot s_1 = \nu \cdot \varphi_1 \cdot s_2 = \nu \cdot h_2 = p_2 \cdot s_2$, there exists $\alpha_2 \colon V \longrightarrow S * S$ such that $\alpha_2 \cdot d_1 = p_2$ and $\alpha_2 \cdot d_2 = \nu \cdot \varphi_1 \cdot s_S$. This implies that $\alpha_2 \cdot t_S \cdot s_1 = \alpha_2 \cdot d_1 \cdot s_1 = p_2 \cdot s_1$ and $\alpha_2 \cdot t_S \cdot s_2 = \alpha_2 \cdot d_2 \cdot s_2 = \nu \cdot \varphi_1 \cdot s_S \cdot s_2 = \nu \cdot \varphi_1 \cdot s_1 = \nu \cdot \sigma \cdot h_1$.

In this way, we obtain a cone

$$\begin{tikzcd}
 & V \arrow[dl, "\alpha_2 \cdot t_S"'] \arrow[d, "\nu \cdot \sigma"] \arrow[dr, "\alpha_1 \cdot t_S"] & \\
S & X & S
\end{tikzcd}$$

on the diagram defining V, so that there exists $s_V \colon V \longrightarrow V$ such that $s_V \cdot p_1 = \alpha_2 \cdot t_S$ and $s_V \cdot p_2 = \alpha_1 \cdot t_S$. This arrow s_V is the symmetry of $v_1, v_2 \colon V \rightrightarrows Y$; in fact $s_V \cdot v_1 = s_V \cdot p_1 \cdot s_1 = \alpha_2 \cdot t_S \cdot s_1 = p_2 \cdot s_1 = v_2$ and $s_V \cdot v_2 = s_V \cdot p_2 \cdot s_1 = \alpha_1 \cdot t_S \cdot s_1 = p_1 \cdot s_1 = v_1$.

Now we look at the transitivity of $v_1, v_2 \colon V \rightrightarrows Y$. For this, let us observe that the transitivity of $\langle [h_1], [h_2] \rangle$ means that there exists an arrow

$$\begin{tikzcd}
E \arrow[r, "\overline{\tau}"] \arrow[d, shift right, "e_1"'] \arrow[d, shift left, "e_2"] & R \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] \\
P \arrow[r, "\tau"'] & X
\end{tikzcd}$$

in $\mathbb{C}_{\text{ex}}$ and two homotopies $\psi_1 \colon P \longrightarrow S$ and $\psi_2 \colon P \longrightarrow S$ such that $\psi_1 \cdot s_1 = \tau \cdot h_1$, $\psi_1 \cdot s_2 = \underline{h}_2 \cdot h_1$, $\psi_2 \cdot s_1 = \tau \cdot h_2$ and $\psi_2 \cdot s_2 = \underline{h}_1 \cdot h_2$ (here $e_1, e_2 \colon E \rightrightarrows P$ is the pullback in $\mathbb{C}_{\text{ex}}$ of $[h_1]$ and $[h_2]$, so that the following is a weak limit

$$\begin{tikzcd}
X \arrow[d, "h_1"'] & P \arrow[l, "\underline{h}_1"'] \arrow[r, "\underline{h}_2"] \arrow[d, "y"] & X \arrow[d, "h_2"] \\
Y & S \arrow[l, "s_1"] \arrow[r, "s_2"'] & Y
\end{tikzcd}$$

Consider a weak pullback

$$\begin{tikzcd}
V * V \arrow[r, "u_1"] \arrow[d, "u_2"'] & V \arrow[d, "v_2"] \\
V \arrow[r, "v_1"'] & Y
\end{tikzcd}$$

We need an arrow $t_V\colon V*V \longrightarrow V$ such that $t_V \cdot v_1 = u_1 \cdot v_1$ and $t_V \cdot v_2 = u_2 \cdot v_2$.

Once again, let us start with an informal argument.

Transitivity of $\langle [h_1],[h_2]\rangle$ means that there exists $\tau\colon P \longrightarrow X$ such that $\forall\,(x,\overline{x}) \in P \quad (h_2(\tau(x,\overline{x})),h_2(x)) \in S$ and $(h_1(\tau(x,\overline{x})),h_1(\overline{x})) \in S$. If we suppose $(y,\overline{y}) \in V$ (that is there exists $\overline{x} \in X$ such that $(y,h_1(\overline{x})) \in S$) and $(\overline{y},\overline{\overline{y}}) \in V$ (that is there exists $x \in X$ such that $(\overline{y},h_1(x)) \in S$ and $(\overline{\overline{y}},h_2(x)) \in S$), by transitivity and symmetry of S we have $(h_1(x),h_2(\overline{x})) \in S$, which means exactly $(x,\overline{x}) \in P$. Using again transitivity and symmetry of S, we can deduce now $(y,h_1(\tau(x,\overline{x}))) \in S$ and $(\overline{\overline{y}},h_2(\tau(x,\overline{x}))) \in S$, that is $(y,\overline{\overline{y}}) \in V$ by means of $\tau(x,\overline{x})$.

Now we translate the previous argument through the construction of $v_1,v_2\colon V \rightrightarrows Y$.

Since $u_2 \cdot p_1 \cdot s_S \cdot s_2 = u_2 \cdot p_1 \cdot s_1 = u_2 \cdot v_1 = u_1 \cdot v_2 = u_1 \cdot p_2 \cdot s_1$, there exists $a\colon V*V \longrightarrow S*S$ such that $a \cdot d_1 = u_2 \cdot p_1 \cdot s_S$ and $a \cdot d_2 = u_1 \cdot p_2$.

We obtain a cone

$$\begin{tikzcd}
 & V*V \arrow[dl, "u_2 \cdot \nu"'] \arrow[d, "a \cdot t_S"] \arrow[dr, "u_1 \cdot \nu"] & \\
X & S & X
\end{tikzcd}$$

on the diagram defining P (in fact $u_2 \cdot \nu \cdot h_1 = u_2 \cdot p_1 \cdot s_2 = u_2 \cdot p_1 \cdot s_S \cdot s_1 = a \cdot d_1 \cdot s_1 = a \cdot t_S \cdot s_1$ and $u_1 \cdot \nu \cdot h_2 = u_1 \cdot p_2 \cdot s_2 = a \cdot d_2 \cdot s_2 = a \cdot t_S \cdot s_2$), so that there exists $\beta\colon V*V \longrightarrow P$ such that $\beta \cdot \underline{h}_1 = u_2 \cdot \nu$ and $\beta \cdot \underline{h}_2 = u_1 \cdot \nu$. Since $\beta \cdot \psi_1 \cdot s_S \cdot s_1 = \beta \cdot \psi_1 \cdot s_2 = \beta \cdot \underline{h}_2 \cdot h_1 = u_1 \cdot \nu \cdot h_1 = u_1 \cdot p_1 \cdot s_2$, there exists $b\colon V*V \longrightarrow S*S$ such that $b \cdot d_1 = u_1 \cdot p_1$ and $b \cdot d_2 = \beta \cdot \psi_1 \cdot s_S$.

Since $\beta \cdot \psi_2 \cdot s_S \cdot s_1 = \beta \cdot \psi_2 \cdot s_2 = \beta \cdot \underline{h}_1 \cdot h_2 = u_2 \cdot \nu \cdot h_2 = u_2 \cdot p_2 \cdot s_2$, there exists $d\colon V*V \longrightarrow S*S$ such that $d \cdot d_1 = u_2 \cdot p_2$ and $d \cdot d_2 = \beta \cdot \psi_2 \cdot s_S$.

We obtain a cone

$$\begin{tikzcd}
 & V*V \arrow[dl, "b \cdot t_S"'] \arrow[d, "\beta \cdot \tau"] \arrow[dr, "d \cdot t_S"] & \\
S & X & S
\end{tikzcd}$$

on the diagram defining V (in fact $b \cdot t_S \cdot s_2 = b \cdot d_2 \cdot s_2 = \beta \cdot \psi_1 \cdot s_S \cdot s_2 = \beta \cdot \psi_1 \cdot s_1 = \beta \cdot \tau \cdot h_1$ and $d \cdot t_S \cdot s_2 = d \cdot d_2 \cdot s_2 = \beta \cdot \psi_2 \cdot s_S \cdot s_2 = \beta \cdot \psi_2 \cdot s_1 = \beta \cdot \tau \cdot h_2$), so that there exists $t_V\colon V*V \longrightarrow V$ such that $t_V \cdot p_1 = b \cdot t_S$ and $t_V \cdot p_2 = b \cdot t_S$ and $t_V \cdot p_2 = d \cdot t_S$. Let us prove that t_V is the transitivity of $v_1,v_2\colon V \rightrightarrows Y$:

$$t_V \cdot v_1 = t_V \cdot p_1 \cdot s_1 = b \cdot t_S \cdot s_1 = b \cdot d_1 \cdot s_1 = u_1 \cdot p_1 \cdot s_1 = u_1 \cdot v_1$$

$$t_V \cdot v_2 = t_V \cdot p_2 \cdot s_1 = d \cdot t_S \cdot s_1 = d \cdot d_1 \cdot s_1 = u_2 \cdot p_2 \cdot s_1 = u_2 \cdot v_2$$

The reflexivity of $v_1, v_2 \colon V \rightrightarrows Y$ is easy to found: choose $r_V = r_S \cdot q$ (where $r_S \colon Y \longrightarrow S$ is the reflexivity of $s_1, s_2 \colon S \rightrightarrows Y$). One has $r_V \cdot v_1 = r_S \cdot q \cdot v_1 = r_S \cdot s_1 = 1_Y$ and analogously $r_V \cdot v_2 = 1_Y$.

It remains only to prove that

$$[h_1], [h_2] \colon (r_1, r_2 \colon R \rightrightarrows X) \rightrightarrows (s_1, s_2 \colon S \rightrightarrows Y)$$

is the kernel pair of $[q, 1_Y] \colon (s_1, s_2 \colon S \rightrightarrows Y) \longrightarrow (v_1, v_2 \colon V \rightrightarrows Y)$ (which implies that $[q, 1_Y]$ is the coequalizer of $[h_1]$ and $[h_2]$, because $[q, 1_Y]$ is a regular epi).

First, we need a homotopy $\epsilon \colon X \longrightarrow V$ to show that $[h_1] \cdot [q, 1_Y] = [h_2] \cdot [q, 1_Y]$.

It can be build up in the following way:

$$\begin{tikzcd} & X \arrow[dl, "h_1 \cdot r_S"'] \arrow[d, "1_X"] \arrow[dr, "h_2 \cdot r_S"] & \\ S & X & S \end{tikzcd}$$

is a cone on the diagram defining V, so that there exists a factorization $\epsilon \colon X \longrightarrow V$ which is the required homotopy. In fact $\epsilon \cdot v_1 = \epsilon \cdot p_1 \cdot s_1 = h_1 \cdot r_S \cdot s_1 = h_1$ and $\epsilon \cdot v_2 = \epsilon \cdot p_2 \cdot s_1 = h_2 \cdot r_S \cdot s_1 = h_2$.

Suppose now there exist in $\mathbb{C}_{\text{ex}}$ two arrows

$$\begin{tikzcd} S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] & T \arrow[l, "\overline{z}_1"'] \arrow[r, "\overline{z}_2"] \arrow[d, shift right, "t_1"'] \arrow[d, shift left, "t_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ Y & Z \arrow[l, "z_1"] \arrow[r, "z_2"'] & Y \end{tikzcd}$$

such that $[z_1] \cdot [q, 1_Y] = [z_2] \cdot [q, 1_Y]$.

We look for an arrow

$$\begin{tikzcd} T \arrow[r, "\overline{z}"] \arrow[d, shift right, "t_1"'] \arrow[d, shift left, "t_2"] & R \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] \\ Z \arrow[r, "z"'] & X \end{tikzcd}$$

such that $[z] \cdot [h_1] = [z_1]$ and $[z] \cdot [h_2] = [z_2]$ (if such a factorization exists, it is certainly unique because $[h_1]$ and $[h_2]$ are jointly monic).

By assumption, there exists $\chi \colon Z \longrightarrow V$ such that $\chi \cdot v_1 = z_1$ and $\chi \cdot v_2 = z_2$. As z we can choose $z = \chi \cdot \nu$. To show that $[z] \cdot [h_1] = [z_1]$ we can use $\chi \cdot p_1 \colon Z \cdot S$; in fact $\chi \cdot p_1 \cdot s_1 = \chi \cdot v_1 = z_1$ and $\chi \cdot p_1 \cdot s_2 = \chi \cdot \nu \cdot h_1 = z \cdot h_1$. Analogously, to show that $[z] \cdot [h_2] = [z_2]$ we can use $\chi \cdot p_2 \colon Z \longrightarrow S$.

It remains to find that $\overline{z} \colon T \longrightarrow R$ such that $\overline{z} \cdot r_1 = t_1 \cdot z$ and $\overline{z} \cdot r_2 = t_2 \cdot z$.

Once again, let us start with an informal argument. Let z, z' be in Z and suppose $(z_1(z), z_2(z)) \in V$ and $(z_1(z'), z_2(z')) \in V$; this means that there exist $x, x' \in X$ such that $(z_1(z), h_1(x)) \in S$, $(z_2(z), h_2(x)) \in S$ and $(z_1(z'), h_1(x')) \in S$, $(z_2(z'), h_2(x')) \in S$.

If moreover $(z, z') \in T$, since z_1 and z_2 are compatible with the relations T and S, we have that $(z_1(z), z_1(z')) \in S$ and $(z_2(z), z_2(z')) \in S$. Now using transitivity and symmetry of S, we obtain $(h_1(x), h_1(x')) \in S$ and $(h_2(x), h_2(x')) \in S$. As h_1 and h_2 are jointly monic, we can deduce now that $(x, x') \in R$, as required.

As a last effort to make readable this proof, let me translate the previous argument step by step by commutative diagrams.

$$\begin{tikzcd} T \arrow[r, "\overline{z}_1"] \arrow[d, shift right, "t_1"'] \arrow[d, shift left, "t_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ Z \arrow[r, "z_1"'] & Y \end{tikzcd}$$

means: $(z, z') \in T$ then $(z_1(z), z_1(z')) \in S$

$$\begin{tikzcd} T \arrow[r, "\overline{z}_2"] \arrow[d, shift right, "t_1"'] \arrow[d, shift left, "t_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ Z \arrow[r, "z_2"'] & Y \end{tikzcd}$$

means: $(z, z') \in T$ then $(z_2(z), z_2(z')) \in S$

$$\begin{tikzcd} Z \arrow[r, "x"] \arrow[d, shift right, "z_1"'] \arrow[d, shift left, "z_2"] & V \arrow[d, shift right, "v_1"'] \arrow[d, shift left, "v_2"] \\ Y \arrow[r, "1_Y"'] & Y \end{tikzcd}$$

means: $(z_1(z), z_2(z)) \in V$

$$\begin{tikzcd} & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ Z \arrow[ur, "\chi \cdot p_1"] \arrow[r, shift left, "z_1"] \arrow[r, shift right, "\chi \cdot \nu \cdot h_1"'] & Y \end{tikzcd} \qquad \begin{tikzcd} & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ Z \arrow[ur, "\chi \cdot p_2"] \arrow[r, shift left, "z_2"] \arrow[r, shift right, "\chi \cdot \nu \cdot h_2"'] & Y \end{tikzcd}$$

means: $\exists\, x \in X$ such that $(z_1(z), h_1(x)) \in S$ and $(z_2(z), h_2(x)) \in S$

$$\begin{tikzcd} & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ T \arrow[ur, "t_1 \cdot \chi \cdot p_1"] \arrow[r, shift left, "t_1 \cdot z_1"] \arrow[r, shift right, "t_1 \cdot \chi \cdot \nu \cdot h_1"'] & Y \end{tikzcd} \qquad \begin{tikzcd} & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ T \arrow[ur, "t_1 \cdot \chi \cdot p_2"] \arrow[r, shift left, "t_1 \cdot z_2"] \arrow[r, shift right, "t_1 \cdot \chi \cdot \nu \cdot h_2"'] & Y \end{tikzcd}$$

$$\begin{tikzcd} & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ T \arrow[ur, "t_2 \cdot \chi \cdot p_1"] \arrow[r, shift left, "t_2 \cdot z_1"] \arrow[r, shift right, "t_2 \cdot \chi \cdot \nu \cdot h_1"'] & Y \end{tikzcd} \qquad \begin{tikzcd} & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ T \arrow[ur, "t_2 \cdot \chi \cdot p_2"] \arrow[r, shift left, "t_2 \cdot z_2"] \arrow[r, shift right, "t_2 \cdot \chi \cdot \nu \cdot h_2"'] & Y \end{tikzcd}$$

mean respectively: $(z_1(z), h_1(x)) \in S$, $(z_2(z), h_2(x)) \in S$, $(z_1(z'), h_1(x')) \in S$ and $(z_2(z'), h_2(x')) \in S$

$$\begin{tikzcd} & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ T \arrow[ur, "t_1 \cdot \chi \cdot p_1 \cdot s_S"] \arrow[r, shift left, "t_1 \cdot \chi \cdot \nu \cdot h_1"] \arrow[r, shift right, "t_1 \cdot z_1"'] & Y \end{tikzcd}$$

means: $(h_1(x), z_1(z)) \in S$

Now, since $t_1 \cdot \chi \cdot p_1 \cdot s_S \cdot s_2 = \overline{z}_1 \cdot s_1$, there exists $a\colon T \longrightarrow S * S$ such that $a \cdot d_1 = t_1 \cdot \chi \cdot p_1 \cdot s_S$ and $a \cdot d_2 = \overline{z}_1$. This implies

$$\begin{tikzcd} & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ T \arrow[ur, "a \cdot t_S"] \arrow[r, shift left, "t_1 \cdot \chi \cdot \nu \cdot h_1"] \arrow[r, shift right, "t_2 \cdot z_1"'] & Y \end{tikzcd}$$

which means: $(h_1(x), z_1(z')) \in S$

Now, since $a \cdot t_S \cdot s_2 = t_2 \cdot \chi \cdot p_1 \cdot s_1$, there exists $b\colon T \longrightarrow S * S$ such that $b \cdot d_1 = a \cdot t_S$ and $b \cdot d_2 = t_2 \cdot \chi \cdot p_1$. This implies

$$\begin{tikzcd}
 & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\
T \arrow[ur, "b \cdot t_S"] \arrow[r, shift left, "t_1 \cdot \chi \cdot \nu \cdot h_1"] \arrow[r, shift right, "t_2 \cdot \chi \cdot \nu \cdot h_1"'] & Y
\end{tikzcd}$$

which means: $(h_1(x), h_1(x')) \in S$.

But the previous diagram means that

$$[t_1 \cdot \chi \cdot \nu] \cdot [h_1] = [t_2 \cdot \chi \cdot \nu] \cdot [h_1]$$

in $\mathbb{C}_{\text{ex}}$, where the arrows are

$$\begin{tikzcd}
T \arrow[r, shift left, "t_1 \cdot \chi \cdot \nu \cdot r_R"] \arrow[r, shift right, "t_2 \cdot \chi \cdot \nu \cdot r_R"'] \arrow[d, shift right, "1_T"'] \arrow[d, shift left, "1_T"] & R \arrow[r, "\overline{h}_1"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\
T \arrow[r, shift left, "t_1 \cdot \chi \cdot \nu"] \arrow[r, shift right, "t_2 \cdot \chi \cdot \nu"'] & X \arrow[r, "h_1"'] & Y
\end{tikzcd}$$

and the homotopy $T \longrightarrow S$ is given by $b \cdot t_S$.

Exactly in the same way, one can prove that $[t_1 \cdot \chi \cdot \nu] \cdot [h_2] = [t_2 \cdot \chi \cdot \nu] \cdot [h_2]$ in $\mathbb{C}_{\text{ex}}$. Since $[h_1], [h_2]$ is a jointly monic pair, the two previous equations in $\mathbb{C}_{\text{ex}}$ imply $[t_1 \cdot \chi \cdot \nu] = [t_2 \cdot \chi \cdot \nu]$.

But this means that there exists a homotopy $\overline{z}: T \longrightarrow R$ such that $\overline{z} \cdot r_1 = t_1 \cdot \chi \cdot \nu = t_1 \cdot z$ and $\overline{z} \cdot r_2 = t_2 \cdot \chi \cdot \nu = t_2 \cdot z$, as required.

The proof of the theorem is now complete. ■

4.7 An economical proof

In this section we obtain a different and more economical proof for the exactness of $\mathbb{C}_{\text{ex}}$ and for the universal property of $\Gamma: \mathbb{C} \longrightarrow \mathbb{C}_{\text{ex}}$. Nevertheless, I have chosen to maintain also the old proofs (theorem 4.6.1 and theorem 1.5.2) for two reasons. The first reason is affective: when I started to study the exact completion of a weakly lex category, I was not aware of the exact completion of a regular category, so I was forced to look for a direct proof of 1.2.3 and 1.5.2 which cost some efforts to me. The second reason is to have a self-contained work: the proof of the universal property of the exact completion of a regular category is largely based on the calculus of relations, a topic that I do not introduce in this work.

Let us start recalling some known facts. The existence of the free exact category on a regular one has been claimed by A. Joyal in a conference at Oberwolfach in 1972 and its description has been suggested by W. Lawvere in [31], where it is presented as a motivation for an axiomatic study of categories of relations.

Definition 4.7.1 *Let $\mathbb{B}$ be a regular category; we can define a new category $Rel(\mathbb{B})$ in the following way:*

- *objects of $Rel(\mathbb{B})$ are objects of $\mathbb{B}$*
- *an arrow $\alpha\colon X \mapsto Y$ in $Rel(\mathbb{B})$ is a relation in $\mathbb{B}$*

$$\begin{tikzcd} & \alpha \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2"] & \\ X & & Y \end{tikzcd}$$

- *composition: given a pair of arrows $\alpha\colon X \mapsto Y$ and $\beta\colon Y \mapsto Z$ in $\mathbb{B}$, its composite is the jointly monic part of the (regular epi, mono) factorization of*

$$\begin{tikzcd} & P \arrow[dl, "{\beta_1}' \cdot \alpha_1"'] \arrow[dr, "{\alpha_2}' \cdot \beta_2"] & \\ X & & Z \end{tikzcd}$$

where

$$\begin{tikzcd} P \arrow[r, "{\alpha_2}'"] \arrow[d, "{\beta_1}'"'] & \beta \arrow[d, "\beta_1"] \\ \alpha \arrow[r, "\alpha_2"'] & Y \end{tikzcd}$$

is a pullback

- *identities are diagonal relations*

Of course, to prove that the previous definition works requires some verifications, which has been accomplished in [31].

Definition 4.7.2 *Let $\mathbb{B}$ be a regular category; we can define a new category $\mathbb{B}_{EX}$ in the following way:*

- *objects: an object of $\mathbb{B}_{EX}$ is an equivalence relation $r_1, r_2\colon R \rightrightarrows X$ in $\mathbb{B}$*

- *arrows: an arrow* $\alpha\colon (R \rightrightarrows X) \longrightarrow (S \rightrightarrows Y)$ *in* $\mathbb{B}_{\mathrm{EX}}$ *is an arrow* $\alpha\colon X \mapsto Y$ *in Rel*($\mathbb{B}$) *such that*

  i) $R \cdot \alpha \cdot S = \alpha$

  ii) $\alpha^o \cdot \alpha < S$

  iii) $\alpha \cdot \alpha^o > R$

  (in the previous equations composition is that of Rel($\mathbb{B}$) *and* α^o *is the relation* $Y \mapsto X$ *defined by*

$$\begin{tikzcd} & \alpha \arrow[dl, "\alpha_2"'] \arrow[dr, "\alpha_1"] & \\ Y & & X \end{tikzcd}$$

- *composition is that of Rel*($\mathbb{B}$)

- *the identity on* $(R \rightrightarrows X)$ *is* $R\colon X \mapsto X$

We can define a functor $\gamma\colon \mathbb{B} \longrightarrow \mathbb{B}_{\mathrm{EX}}$ which sends an object X of $\mathbb{B}$ on the identity relation $1_X, 1_X\colon X \rightrightarrows X$ and an arrow $f\colon X \longrightarrow Y$ on its graph $f\colon X \mapsto Y$. This functor is full and faithful. As far as the fullness is concerned, observe that, if R is the identity relation on X and S is the identity relation on Y, then conditions ii) and iii) in definition 4.7.2 mean that α_1 is, respectively, a mono and a regular epi, so that $\alpha\colon X \mapsto Y$ is the graph of $\alpha_1^{-1} \cdot \alpha_2\colon X \longrightarrow Y$.

A complete proof of the exactness of $\mathbb{B}_{\mathrm{EX}}$ and $\gamma\colon \mathbb{B} \longrightarrow \mathbb{B}_{\mathrm{EX}}$ can be found in [38] and [22]. Let us only point out some useful facts to understand the universal property of $\gamma\colon \mathbb{B} \longrightarrow \mathbb{B}_{\mathrm{EX}}$. First, observe that, if $r_1, r_2\colon R \rightrightarrows X$ is an object in $\mathbb{B}_{\mathrm{EX}}$, the following is an exact sequence in $\mathbb{B}_{\mathrm{EX}}$

$$\begin{tikzcd} \gamma(R) \arrow[r, shift left, "\gamma(r_1)"] \arrow[r, shift right, "\gamma(r_2)"'] & \gamma(X) \arrow[r, "R"] & (R \rightrightarrows X) \end{tikzcd}$$

Moreover, if $\alpha\colon (R \rightrightarrows X) \longrightarrow (S \rightrightarrows Y)$ is an arrow in $\mathbb{B}_{\mathrm{EX}}$, we can consider the (regular epi, jointly monic) factorization of the pair $(\gamma(\alpha_1) \cdot R, \gamma(\alpha_2) \cdot S)$ as in the following diagram

$$\begin{tikzcd} \gamma(X) \arrow[d, "R"'] & \gamma(\alpha) \arrow[l, "\gamma(\alpha_1)"'] \arrow[r, "\gamma(\alpha_2)"] \arrow[d] & \gamma(Y) \arrow[d, "S"] \\ (R \rightrightarrows X) & \bullet \arrow[l, "m_1"] \arrow[r, "m_2"'] & (S \rightrightarrows Y) \end{tikzcd}$$

One can show that m_1 is an isomorphism and $\alpha = m_1^{-1} \cdot m_2$. In particular, this implies that $\gamma(\alpha_1) \cdot R \cdot \alpha = \gamma(\alpha_2) \cdot S$.

If $\mathbb{B}$ is an exact category, the functor $\gamma: \mathbb{B} \longrightarrow \mathbb{B}_{\mathrm{EX}}$ is an equivalence. In fact an object $r_1, r_2: R \rightrightarrows X$ is isomorphic to $\gamma(X/R)$, where $\rho: X \longrightarrow X/R$ is the coequalizer of r_1 and r_2 and the isomorphism is given by $\gamma(\rho)$ and $\gamma(\rho)^o$.

Proposition 4.7.3 *For each exact category $\mathbb{A}$ and for each exact functor $F: \mathbb{B} \longrightarrow \mathbb{A}$, there exists an essentially unique exact functor $\hat{F}: \mathbb{B}_{\mathrm{EX}} \longrightarrow \mathbb{A}$ making commutative the following diagram*

$$\begin{tikzcd} \mathbb{B} \arrow[rr, "\gamma"] \arrow[dr, "F"'] & & \mathbb{B}_{EX} \arrow[dl, "\hat{F}"] \\ & \mathbb{A} & \end{tikzcd}$$

Proof: let us only indicate how to define $\hat{F}$. If $r_1, r_2: R \rightrightarrows X$ is an object in $\mathbb{B}_{\mathrm{EX}}$, we put $\hat{F}(R \rightrightarrows X) = Q$, where

$$\begin{tikzcd} F(R) \arrow[r, shift left, "F(r_1)"] \arrow[r, shift right, "F(r_2)"'] & F(X) \arrow[r, "q"] & Q \end{tikzcd}$$

is a coequalizer. If $\alpha: (R \rightrightarrows X) \longrightarrow (S \rightrightarrows Y)$ is an arrow in $\mathbb{B}_{\mathrm{EX}}$, we can consider also the coequalizer $p: FY \longrightarrow P$ of Fs_1, Fs_2. Consider now the (regular epi, jointly monic) factorization of the pair $(F(\alpha_1) \cdot q, F(\alpha_2) \cdot p)$ as in the following diagram

$$\begin{tikzcd} F(X) \arrow[d, "q"'] & F(\alpha) \arrow[l, "F(\alpha_1)"'] \arrow[r, "F(\alpha_2)"] \arrow[d] & F(Y) \arrow[d, "p"] \\ Q & \bullet \arrow[l, "\mu_1"] \arrow[r, "\mu_2"'] & P \end{tikzcd}$$

But $\langle \mu_1, \mu_2 \rangle: Q \mapsto P$ is an arrow in $\mathbb{A}_{\mathrm{EX}}$ between $1_Q, 1_Q: Q \rightrightarrows Q$ and $1_P, 1_P: P \rightrightarrows P$, so that μ_1 is an isomorphism. It is now straightforward to define $\hat{F}(\alpha) = \mu_1^{-1} \cdot \mu_2: Q \longrightarrow P$. Once again, a complete proof based on the calculus of relations can be found in [38] or [22]. ■

Lemma 4.7.4 *With the notations of the previous proposition, $\hat{F}$ is the left Kan extension of F along γ.*

Proof: let $\lambda: F \longrightarrow \gamma \cdot H$ be a natural trasformation with $H: \mathbb{B}_{\mathrm{EX}} \longrightarrow \mathbb{A}$ an arbitrary functor. We can define a natural trasformation $\hat{\lambda}: \hat{F} \longrightarrow H$ as in the following diagram

$$\begin{tikzcd}
F(R) \arrow[r, shift left, "F(r_1)"] \arrow[r, shift right, "F(r_2)"'] \arrow[d, "\lambda_R"'] & F(X) \arrow[r, "\hat{F}(R)"] \arrow[d, "\lambda_X"] & \hat{F}(R \rightrightarrows X) \arrow[d, "\hat{\lambda}"] \\
H(\gamma(R)) \arrow[r, shift left, "H(\gamma(r_1))"] \arrow[r, shift right, "H(\gamma(r_2))"'] & H(\gamma(X)) \arrow[r, "H(R)"'] & H(R \rightrightarrows X)
\end{tikzcd}$$

The naturality of $\hat{\lambda}$ follows from that of λ and from the equation $\gamma(\alpha_1) \cdot R \cdot \alpha = \gamma(\alpha_2) \cdot S$. The rest of the proof is straightforward. ∎

Corollary 4.7.5 *Consider the functor $\gamma \colon \mathbb{B} \longrightarrow \mathbb{B}_{EX}$;*

1) *for each exact category $\mathbb{A}$, composing with γ induces an equivalence*

$$\gamma \cdot - \colon Ex(\mathbb{B}_{EX}, \mathbb{A}) \longrightarrow Ex(\mathbb{B}, \mathbb{A})$$

between the category of exact functors from $\mathbb{B}_{EX}$ to $\mathbb{A}$ and the category of exact functors from $\mathbb{B}$ to $\mathbb{A}$

2) *$\gamma \colon \mathbb{B} \longrightarrow \mathbb{B}_{EX}$ is the unit of the left biadjoint to the forgetful*

$$Ex \longrightarrow Reg$$

where Ex is the 2-category of exact categories and exact functors and Reg is the 2-category of regular categories and exact functors. ∎

Putting together theorem 3.3.1 and the first point of the previous corollary, we obtain the following theorem

Theorem 4.7.6 *Let $\mathbb{C}$ be a weakly lex category; consider $\Gamma_{reg} \colon \mathbb{C} \longrightarrow \mathbb{C}_{reg}$ as in proposition 3.2.4 and $\gamma \colon \mathbb{C}_{reg} \longrightarrow (\mathbb{C}_{reg})_{EX}$ as at the beginning of this section. For each exact category $\mathbb{A}$, composing with $\Gamma_{reg} \cdot \gamma$ induces an equivalence*

$$\Gamma_{reg} \cdot \gamma \cdot - \colon Ex((\mathbb{C}_{reg})_{EX}, \mathbb{A}) \longrightarrow Ex(\mathbb{C}_{reg}, \mathbb{A}) \longrightarrow Lco(\mathbb{C}, \mathbb{A})$$

between the category of exact functors from $(\mathbb{C}_{reg})_{EX}$ to $\mathbb{A}$ and the category of left covering functors from $\mathbb{C}$ to $\mathbb{A}$. ∎

Now, to obtain the exactness of $\mathbb{C}_{\mathrm{ex}}$ and the universal property of $\Gamma_{\mathrm{ex}} \colon \mathbb{C} \longrightarrow \mathbb{C}_{\mathrm{ex}}$ as a corollary of the previous theorem, it suffices to show that $\mathbb{C}_{\mathrm{ex}}$ is equivalent to $(\mathbb{C}_{\mathrm{reg}})_{\mathrm{EX}}$. For this, let us come back to the very beginning of the proof of theorem 1.5.2. Observe that, given a left covering functor $F \colon \mathbb{C} \longrightarrow \mathbb{A}$, the existence of an extension $\hat{F} \colon \mathbb{C}_{\mathrm{ex}} \longrightarrow \mathbb{A}$ of F along $\Gamma_{\mathrm{ex}} \colon \mathbb{C} \longrightarrow \mathbb{C}_{\mathrm{ex}}$ does not depend on the exactness of $\mathbb{C}_{\mathrm{ex}}$ but only on the exactness of $\mathbb{A}$. It is to this extension that we refer in the following lemma.

Lemma 4.7.7 *With the previous notations:*

1) *if F is full and faithful and, for each object X of $\mathbb{C}$, $F(X)$ is regular projective, then $\hat{F}$ is full and faithful*

2) *if, moreover, for each object A in $\mathbb{A}$ there exist an object X in $\mathbb{C}$ and a regular epimorphism $F(X) \longrightarrow A$, then $\hat{F}$ is an equivalence*

Proof: it suffices to observe that in the proof of theorem 1.6.1 we do not use the exactness of $\mathbb{C}_{\mathrm{ex}}$. ■

It remains only to prove that the functor

$$\Gamma_{\mathrm{reg}} \cdot \gamma \colon \mathbb{C} \longrightarrow \mathbb{C}_{\mathrm{reg}} \longrightarrow (\mathbb{C}_{\mathrm{reg}})_{\mathrm{EX}}$$

satisfies the hypothesis of the previous lemma. Clearly, it is full and faithful because both Γ_{reg} and γ are full and faithful. Moreover, each object of $\mathbb{C}_{\mathrm{reg}}$ can be covered by an object of $\mathbb{C}$ (proposition 3.2.6) and each object of $(\mathbb{C}_{\mathrm{reg}})_{\mathrm{EX}}$ can be covered by an object of $\mathbb{C}_{\mathrm{reg}}$. Since γ preserves regular epimorphisms, this implies that each object of $(\mathbb{C}_{\mathrm{reg}})_{\mathrm{EX}}$ can be covered by an object of $\mathbb{C}$. It remains to prove that each object of $\mathbb{C}$ is regular projective in $(\mathbb{C}_{\mathrm{reg}})_{\mathrm{EX}}$. We know, by proposition 3.2.5, that each object of $\mathbb{C}$ is regular projective in $\mathbb{C}_{\mathrm{reg}}$, so that it suffices to show that γ preserves regular projective objects. This follows as a particular case from the following lemma.

Lemma 4.7.8 *Let $G \colon \mathbb{B} \longrightarrow \mathbb{A}$ be a left exact functor;*

1) *if G is full and faithful, then it reflects regular epimorphism*

2) *if, moreover, $\mathbb{A}$ is regular and, for each object A in $\mathbb{A}$, there exist an object B in $\mathbb{B}$ and a regular epimorphism $G(B) \longrightarrow A$, then G preserves regular projective objects.*

Proof: 1): obviously a full and faithful functor reflects epimorphisms. Now consider an arrow $f \colon X \longrightarrow Y$ in $\mathbb{B}$ such that $G(f) \colon G(X) \longrightarrow G(Y)$ is a regular epimorphism. We can prove that f is the coequalizer of its kernel pair $f_1, f_2 \colon N(f) \rightrightarrows X$. In fact, if $g \colon X \longrightarrow Z$ is such that $f_1 \cdot g = f_2 \cdot g$, then $Gf_1 \cdot Gg = Gf_2 \cdot Gg$. But, by left exactness of G, the pair Gf_1,Gf_2 is the kernel pair of Gf, so that Gf is the coequalizer of Gf_1 and Gf_2. This implies that there exists a unique factorization of Gg along Gf and then, by fullness and faithfulness of G, a unique factorization of g along f.

2): suppose that P is regular projective in $\mathbb{B}$ and consider a regular epimorphism $p \colon A \longrightarrow G(P)$ in $\mathbb{A}$. We can cover A with an object B of $\mathbb{B}$ and a regular epimorphism $q \colon G(B) \longrightarrow A$. Now we have a regular epimorphism $q \cdot p \colon G(B) \longrightarrow G(P)$ so that there exists an arrow $f \colon B \longrightarrow P$ such that $G(f) = q \cdot p$. By the first part of the lemma, f is a regular epimorphism, so that there exists $g \colon P \longrightarrow B$ such that $g \cdot f = 1_P$. This implies that $G(g) \cdot q$ is a section of p. Since $\mathbb{A}$ is regular, this proves that $G(P)$ is regular projective. ■

Let us conclude this section observing that, as far as an explicit description of the exact extension $\hat{F}: \mathbb{C}_{\mathrm{ex}} \longrightarrow \mathbb{A}$ of a left covering functor $F: \mathbb{C} \longrightarrow \mathbb{A}$ is concerned, it is easier to use the argument concerning uniqueness at the beginning of the proof of 1.5.2 than going through the two steps of $\mathbb{C}_{\mathrm{reg}}$ and $(\mathbb{C}_{\mathrm{reg}})_{\mathrm{EX}}$. [2]

4.8 To be continued ...

I would like to leave the reader with three open problems which, in my opinion, can be of some interest.

Malcev condition:

The exact completion runs very well with respect to the additivity, as attested by the following proposition (cf. also [20] and [21]).

Proposition 4.8.1 *Let $\mathbb{C}$ be a weakly lex category; $\mathbb{C}_{\mathrm{ex}}$ is preadditive (that is abelian) if and only if $\mathbb{C}$ is preadditive.*

Proof: The (only if) is obvious because $\mathbb{C}$ is (equivalent to) a full subcategory of $\mathbb{C}_{\mathrm{ex}}$; for the converse, one has only to verify that the preadditive structure in $\mathbb{C}_{\mathrm{ex}}$ given by $[f] + [g] = [f + g]$ is well defined. ■

Unfortunately, the same problem is not so easy when we consider the not commutative analogous of abelian categories: here, a Malcev category is a weakly lex category in which every pseudo reflective-relation is also a pseudo equivalence-relation.

Proposition 4.8.2 *Let $\mathbb{C}$ be a weakly lex category; if $\mathbb{C}_{\mathrm{ex}}$ is Malcev, then $\mathbb{C}$ is Malcev.*

Proof: Let $\mathbb{A}$ be an exact category and $\mathbb{P}$ a projective cover of $\mathbb{A}$. Suppose $r_0, r_1: R \rightrightarrows X$ is a reflective pair in $\mathbb{P}$, so it is a reflective pair in $\mathbb{A}$ and then, by assumption, it is a pseudo equivalence-relation in $\mathbb{A}$. Clearly, this implies that it is symmetric in $\mathbb{P}$ only because $\mathbb{P}$ is full in $\mathbb{A}$.

As far as transitivity is concerned, we know that there exists an arrow $t: R * R \longrightarrow R$ such that $t \cdot r_0 = l_0 \cdot r_0$ and $t \cdot r_1 = l_1 \cdot r_1$, where

$$\begin{tikzcd} R * R \arrow[r, "l_0"] \arrow[d, "l_1"'] & R \arrow[d, "r_1"] \\ R \arrow[r, "r_0"'] & X \end{tikzcd}$$

is the pullback in $\mathbb{A}$.

Now if $p: P \longrightarrow R * R$ is a $\mathbb{P}$-cover of $R * R$, then

[2] This section has been written with the collaboration of M. Mosca. Further developements will be contained in [35].

$$\begin{tikzcd} P \arrow[r, "p \cdot l_0"] \arrow[d, "p \cdot l_1"'] & R \arrow[d, "r_1"] \\ R \arrow[r, "r_0"'] & X \end{tikzcd}$$

is a weak pullback in $\mathbb{P}$, so that $p \cdot t \colon P \longrightarrow R$ gives the transitivity of $r_0, r_1 \colon R \rightrightarrows X$ in $\mathbb{P}$. ∎

The open problem is clearly to find conditions on $\mathbb{C}$ such that $\mathbb{C}_{\mathrm{ex}}$ is a Malcev category.

A possible motivation to study Malcev condition in this context lies in the suggestion, due to M. Barr, that the right framework to study topology is given by exact categories with enough injectives in which unions are effective (cf. [4]). Both examples given by Barr in [4] (elementary topo and Grothendieck abelian categories) are categories such that the dual category is exact. In other words, they are exact categories such that the dual is a free exact and Malcev category (recall that a regular category is Malcev if and only if it is exact and the dual of the effective unions condition holds).

<u>The homotopy category:</u>

Let us recall the definition of the homotopy category $H_0\mathrm{Top}$:

objects: topological spaces

arrows: equivalence classes of continuous maps; two continuous maps $f, g : X \rightrightarrows Y$ are said to be equivalent if

- there exists a continuous map $H \colon I \times X \longrightarrow Y$ such that $i_0 \cdot H = f$ and $i_1 \cdot H = g$

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, shift right, "i_0"'] \arrow[d, shift left, "i_1"] & Y \\ I \times X \arrow[ur, "H"'] & \end{tikzcd}$$

  where $I = [0,1]$ euclideian and

$$i_0(x) = (0, x) \ , \ i_1(x) = (1, x) \quad \forall\, x \in X$$

or, equivalently, if

- there exists a continuous map $K \colon X \longrightarrow Y^I$ such that $K \cdot \mathrm{ev}_0 = f$ and $K \cdot \mathrm{ev}_1 = g$

$$\begin{tikzcd}
 & Y^I \arrow[d, shift right, "\mathrm{ev}_0"'] \arrow[d, shift left, "\mathrm{ev}_1"] \\
X \arrow[ur, "K"] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd}$$

where $\mathrm{ev}_0(\alpha) = \alpha(0)$ and $\mathrm{ev}_1(\alpha) = \alpha(1)$ $\forall\, \alpha \in Y^I$.

Proposition 4.8.3 *H_0Top is weakly lex.*

Proof: It is straightforward to prove that products in $\underline{\mathrm{Top}}$ are also (strong) products in H_0Top.

As far as equalizers are concerned, consider two parallel arrows in $\underline{\mathrm{Top}}$ $f, g\colon X \rightrightarrows Y$ and the evaluations $\mathrm{ev}_0, \mathrm{ev}_1\colon Y^I \rightrightarrows Y$; now take the pullbacks

$$\begin{tikzcd}
F \arrow[r, "f'"] \arrow[d, "f_0"'] & Y^I \arrow[d, "\mathrm{ev}_0"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "g'"] \arrow[d, "g_1"'] & Y^I \arrow[d, "\mathrm{ev}_1"] \\
X \arrow[r, "g"'] & Y
\end{tikzcd}$$

and then take the limit $F \xleftarrow{\pi_f} L \xrightarrow{\pi_g} G$ on the diagram

$$\begin{tikzcd}
F \arrow[d, "f_0"'] \arrow[dr, "f'"] & G \arrow[d, "g'"] \arrow[dl, "g_1"'] \\
X & Y^I
\end{tikzcd}$$

A weak equalizer in H_0Top is given by

$$L \xrightarrow{\pi_f \cdot f_0} X \overset{f}{\underset{g}{\rightrightarrows}} Y.$$

In fact, $\pi_f \cdot f' \cdot \mathrm{ev}_0 = \pi_g \cdot f_0 \cdot f$ and $\pi_f \cdot f' \cdot \mathrm{ev}_1 = \pi_g \cdot g' \cdot \mathrm{ev}_1 = \pi_g \cdot g_1 \cdot g = \pi_f \cdot f_0 \cdot g$ so that $\pi_f \cdot f_0 \cdot f$ is equivalent to $\pi_f \cdot f_0 \cdot g$. Now suppose that $h\colon Z \longrightarrow X$ is an arrow in $\underline{\mathrm{Top}}$ such that $h \cdot f$ is equivalent to $h \cdot g$, that is there exists $H\colon Z \longrightarrow Y^I$ such that $H \cdot \mathrm{ev}_0 = h \cdot f$ and $H \cdot \mathrm{ev}_1 = h \cdot g$. From the first equation, we have $\varphi\colon Z \longrightarrow F$ such that $\varphi \cdot f_0 = h$ and $\varphi \cdot f' = H$. From the second equation, we have $\psi\colon Z \longrightarrow G$ such that $\psi \cdot g_1 = h$ and $\psi \cdot g' = H$. But this implies the existence of $l\colon Z \longrightarrow L$ such that $l \cdot \pi_f = \varphi$ and $l \cdot \pi_g = \psi$. Finally $l \cdot \pi_f \cdot f_0 = \varphi \cdot f_0 = h$. ■

If one computes explicitly L, one finds $L = \{(x,\alpha) \in X \times Y^I \mid \alpha(0) = f(x) \text{ and } \alpha(1) = g(x)\}$. This is what is called homotopy-equalizer in [11]. In fact one can prove that the homotopy type of L depends only on the homotopy type of f and g. In the same way, if one builds up weak pullbacks in H_0Top using products in Top and weak equalizers as in the previous proposition, one obtains homotopy-pullbacks.

Two questions arise in a natural way from the previous proposition.

The first one, for the sake of unification, is

- what about weak limits in the homotopy category built up from a Quillen-model category (cf. [37]) or from a Baues-fibred category (cf. [5])?
- what about weak limits in categories of fractions? (cf. [5] and [23])

The second question, may be more interesting in view of further development, is

- what is the exact completion of H_0Top?

Exact embedding

As a last remark, let us point out a quite surprising property of categories which have a projective cover.

Lemma 4.8.4 *Let $f\colon X \longrightarrow Y$ be a regular epi in a weakly lex category and $f_0, f_1\colon N(f) \rightrightarrows X$ a weak kernel pair of f;*

$$N(f) \underset{f_1}{\overset{f_0}{\rightrightarrows}} X \xrightarrow{f} Y$$

is a coequalizer.

Proof: Work as in the "strong" case. ■

Proposition 4.8.5 *Let $\mathbb{A}$ be a small weakly lex category and $\mathbb{P}$ a projective cover of $\mathbb{A}$; the "Yoneda embedding"*

$$Y\colon \mathbb{A} \longrightarrow \mathcal{SET}^{\mathbb{P}^{\mathrm{op}}}$$

is full, faithful, left covering and preserves regular epis.

Proof: The fact that Y is left covering is obvious. For example, if T is a weak terminal object in $\mathbb{A}$, consider the unique natural transformation $\mathbb{A}(-,T) \Rightarrow \underline{1}$, where $\underline{1}$ is the terminal presheaf. For each object X (in $\mathbb{A}$), $\mathbb{A}(X,T)$ is not empty, so that the component at X $\mathbb{A}(X,T) \Rightarrow \underline{1}(X) = \{*\}$ is surjective.

Now consider a regular epi $p\colon A \longrightarrow B$ in $\mathbb{A}$ and the induced natural transformation

$$- \cdot p\colon \mathbb{A}(-,A) \Rightarrow \mathbb{A}(-,B)\colon \mathbb{P}^{\mathrm{op}} \longrightarrow \mathcal{SET}$$

Given an object P in $\mathbb{P}$ and an arrow $f\colon A \longrightarrow B$, the fact that P is projective and p is a regular epi means exactly that there exists $g\colon P \longrightarrow A$ such that $g \cdot p = f$. This means exactly that the component at P of $- \cdot p$ is surjective.

Also the faithfulness of Y is easy: consider two parallel arrows $a, b : A \rightrightarrows B$ in $\mathbb{A}$ and the natural transformations $- \cdot a, - \cdot b \colon \mathbb{A}(-, A) \Rightarrow \mathbb{A}(-, B)$. Consider now a $\mathbb{P}$-cover $p \colon P \longrightarrow A$ of A: if $- \cdot a = - \cdot b$ in particular $p \cdot a = p \cdot b$, but p is a regular epi, so that $a = b$.

It remains to prove that

$$Y \colon \mathbb{A} \longrightarrow \mathcal{SET}^{\mathbb{P}^{\mathrm{op}}}$$

is full. For this consider two objects A and B in $\mathbb{A}$ and a natural transformation $\tau \colon \mathbb{A}(-, A) \Rightarrow \mathbb{A}(-, B)$. The naturality of τ means that, for each $f \colon P \longrightarrow Q$ in $\mathbb{P}$, the following is a commutative diagram in $\mathcal{SET}$

$$\begin{tikzcd}
\mathbb{A}(P, A) \arrow[d, "\tau_P"'] & \mathbb{A}(Q, A) \arrow[l, "f \cdot -"'] \arrow[d, "\tau_Q"] \\
\mathbb{A}(P, B) & \mathbb{A}(Q, B) \arrow[l, "f \cdot -"]
\end{tikzcd}$$

that is, for each $x \colon Q \longrightarrow A$ in $\mathbb{A}$, the following is a commutative diagram in $\mathbb{A}$

$$\begin{tikzcd}
P \arrow[rr, "\tau_P(f \cdot x)"] \arrow[dr, "f"'] & & B \\
& Q \arrow[ur, "\tau_Q(x)"'] &
\end{tikzcd}$$

Now let us consider a $\mathbb{P}$-cover $a \colon \overline{A} \longrightarrow A$ of A, a weak kernel pair $a_0, a_1 \colon N(a) \rightrightarrows A$ of a and again a $\mathbb{P}$-cover $q \colon Q \longrightarrow N(a)$. Consider again $\tau_{\overline{A}}(a) \colon \overline{A} \longrightarrow B$; we have that $q \cdot a_0 \cdot \tau_{\overline{A}}(a) = \tau_Q(q \cdot a_0 \cdot a) = \tau_Q(q \cdot a_1 \cdot a) = q \cdot a_1 \cdot \tau_{\overline{A}}(a)$ and then $a_0 \cdot \tau_{\overline{A}}(a) = a_1 \cdot \tau_{\overline{A}}(a)$ because q is a regular epi. By the previous lemma, we have $t \colon A \longrightarrow B$ such that $a \cdot t = \tau_{\overline{A}}(a)$. Now we can prove that the two natural transformations τ and $- \cdot t$ are equal: consider $y \colon P \longrightarrow A$ with P in $\mathbb{P}$; since P is projective and a a regular epi, there exists $\overline{y} \colon P \longrightarrow \overline{A}$ such that $\overline{y} \cdot a = y$. Finally we have $y \cdot t = \overline{y} \cdot a \cdot t = \overline{y} \cdot \tau_{\overline{A}}(a) = \tau_P(\overline{y} \cdot a) = \tau_P(y)$ that is $\tau_P(y) = Y(t)_P(y)$.

$$\begin{tikzcd}
 & & & P \arrow[dl, "\overline{y}"'] \arrow[d, "y"] \\
Q \arrow[r, "q"] & N(a) \arrow[r, shift left, "a_0"] \arrow[r, shift right, "a_1"'] & \overline{A} \arrow[r, "a"] \arrow[dr, "\tau_{\overline{A}}(a)"'] & A \arrow[d, "t"] \\
 & & & B
\end{tikzcd}$$

■

The previous proposition gives us, as a particular case, the following corollary:

Corollary 4.8.6 *Let $\mathbb{A}$ be a small regular category and $\mathbb{P}$ a projective cover of $\mathbb{A}$; the Yoneda embedding*

$$Y : \mathbb{A} \longrightarrow \mathcal{SET}^{\mathbb{P}^{op}}$$

is a full, faithful and exact functor. ■

This means that the celebrated Barr theorem (which states that, given a small regular category $\mathbb{A}$, there exists a full and exact embedding of $\mathbb{A}$ in a topos of presheaves, cf. [2]) becomes obvious if we make the extra-assumption that $\mathbb{A}$ has enough projectives.

Keeping in mind the proof of Barr theorem given in [7], the question naturally arising is:

- can a regular category $\mathbb{A}$ be decomposed in a family of regular categories with enough projectives, so as to obtain Barr theorem by a suitable "glueing" of the easy case?

Enjoy yourself!

Bibliography

[1] **Adelman M.:** *Abelian categories over additive ones,* J. Pure Appl. Algebra 3 (1973), pp. 103-117.

[2] **Barr M., Grillet P. and van Osdol D.:** Exact Categories and Categories of Sheaves, Lecture Notes in Math. 236, Springer - Verlag, Berlin (1971).

[3] **Barr M. and Wells C.:** Toposes, Triples and Theories, Grundleheren der Mathematischen Wissenschaften 278, Springer - Verlag, Berlin (1984).

[4] **Barr M.:** *On categories with effective unions,* Categorical Algebra and its Applications, Louvain-la-Neuve 1987, Lecture Notes in Math. 1348, Springer-Verlag, Berlin (1988).

[5] **Baues H.J.:** Algebraic Homotopy, Cambridge University Press, Cambridge (1989).

[6] **Bénabou J.:** *Fibered categories,* draft of lectures in Louvain-la-Neuve, Belgium (1983).

[7] **Borceux F.:** *A propos d'un théorème de Barr,* Ann. Soc. Scientifique de Bruxelles, T.97, II (1983), pp. 41-58.

[8] **Borceux F.:** Handbook of Categorical Algebra, Cambridge University Press, Cambridge (1994).

[9] **Borceux F. and Pedicchio M.C.:** *A characterization of quasi-toposes,* J. of Algebra, vol. 139, n.2 (1991), pp. 505-526.

[10] **Bourbaki N.:** Topologie Générale, Hermann, Paris (1961).

[11] **Bousfield A.K. and Kan D.M.:** Homotopy Limits, Completions and Localizations, Lecture Notes in Math. 304, Springer-Verlag, Berlin (1972).

[12] **Bunge M.:** *Categories of set valued functors,* Dissertation, University of Pennsylvania (1966).

[13] **Cagliari F., Mantovani S. and Vitale E.M.:** *Regularity of the category of Kelley spaces* (to appear).

[14] **Carboni A.:** *Some free constructions in realizability and proof theory* (to appear in J. Pure Appl. Algebra).

[15] **Carboni A. and Celia Magno R.:** *The free exact category on a left exact one,* J. Austr. Math. Soc. (Series A) 33 (1982), pp. 295-301.

[16] **Carboni A., Lack S. and Walters R.F.C.:** *Introduction to extensive and distributive categories,* J. Pure Appl. Algebra 84 (1993), pp. 145-158.

[17] **Carboni A. and Mantovani S.:** *Elementary characterizations of categories of separated objects,* J. Pure Appl. Algebra 89 (1993), pp. 63-92.

[18] **Carboni A., Janelidze G., Kelly G.M. and Par R.:** *The monotone-light factorization in topology and the separable-purely inseparable factorization in algebra* (to appear).

[19] **Celia Magno R.:** Thesis, Universit della Calabria in Cosenza, Italy (1981).

[20] **Freyd P.:** *Representations in abelian categories,* Proceedings of the Conference on Categorical Algebra, La Jolla 1965, Springer - Verlag, Berlin (1966).

[21] **Freyd P.:** *Stable homotopy,* Proceedings of the Conference on Categorical Algebra, La Jolla 1965, Springer - Verlag, Berlin (1966).

[22] **Freyd P. and Scedrov A.:** Categories, Allegories, North - Holland, Amsterdam (1990).

[23] **Gabriel P. and Zisman M.:** Calculus of Fractions and Homotopy Theory, Springer - Verlag, Berlin (1987).

[24] **Herrlich H. and Strecker G.:** *Algebra* $\cap$ *Topology = Compactness,* General Topology Appl. 1 (1971), pp. 283-287.

[25] **Herrlich H. and Strecker G.:** Category Theory, Allyn and Bacon, Boston (1973).

[26] **Hu H.:** *Flat functors and free exact categories,* Report no. 94-05, University of York, Canada (1994).

[27] **Hyland J.M.E.:** *The effective topos,* The L.E.J. Brouwer Centenary Symposium, North-Holland, Amsterdam (1982).

[28] **Johnstone P.:** Topos Theory, London Math. Soc. Monographs, vol. 10, Academic Press, New York (1977).

[29] **Johnstone P.:** Stone Spaces, Cambridge University Press, Cambridge (1982).

[30] **Keane O.:** *Abstract Horn theories,* Lecture Notes in Math. 445, Springer-Verlag, Berlin (1975).

[31] **Lawvere F.W.:** *Category theory over a base topos,* mimeographed notes, Università di Perugia (1972-1973).

[32] **Mac Lane S. and Moerdijk I.:** Sheaves in Geometry and Logic, Springer - Verlag, Berlin (1992).

[33] **Makkai M.:** *A theorem on Barr-exact categories, with an infinitary generalization,* Ann. Pure Appl. Logic 47 (1990), pp. 225-268.

[34] **Makkai M. and Reyes G.E.:** First-Order Categorical Logic, Lecture Notes in Math. 611, Springer - Verlag, Berlin (1977).

[35] **Mosca M.:** Thesis, Università di Trieste (1995).

[36] **Pedicchio M.C.:** *On k-permutability for categories of T-algebras,* to appear.

[37] **Quillen D.:** Homotopical Algebra, Lecture Notes in Math. 43, Springer - Verlag, Berlin (1967).

[38] **Robinson E.P. and Rosolini G.:** *Colimit completions and the Effective topos,* The Jour. of Symbolic Logic (2) 55 (1990), pp. 678-699.

[39] **Succi Cruciani R.:** *La teoria delle relazioni nello studio di categorie regolari e di categorie esatte,* Riv. Mat. Univ. Parma (4) 1 (1975), pp. 143-158.

[40] **Tholen W.:** *Relative Bildzerlegungen und algebraische Kategorien,* Ph.D. thesis, Munster (1974).

[41] **van Oosten J.:** *Extensional realizability,* ILLC Prepublication Series ML-93-18, University of Amsterdam (1993).

[42] **Vitale E.M.:** *On the characterization of monadic categories over Set,* to appear in Cahiers de Topologie et G.D., vol. XXXVI (1995).