Representing and aggregating preferences using a stochastic interpretation

François Glineur

UCL/CORE & INMA, Tractebel Energy Networks chair
glineur@core.ucl.ac.be

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- ◇ Preference relations

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Markov chains

Definitions

$\mathbf{E}$ = set of n states = $\{E_i\}$
$\mathbf{X}$ = set of random variables $\{X_t\}$ such that $X_t \in \mathbf{E}$
(t is a discrete time index)

$\mathbf{X}$ is a finite discrete-time Markov chain iff

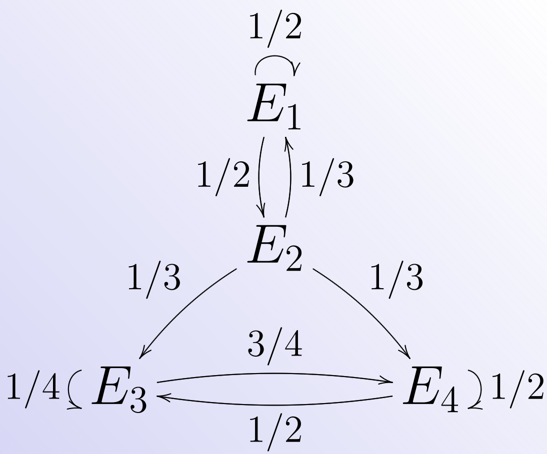
- X_t depends only on X_{t-1}
  (i.e. Markovian, memoryless process)
- This dependence is time-invariant
  (i.e. homogeneous process)

Matrix representation

$\mathbf{X}$ completely defined by matrix $\mathbf{M} = \{p_{i,j}\}$ where

$$p_{i,j} = \mathbb{P}(X_t = E_j \mid X_{t-1} = E_i)$$

Representation as a valued directed graph:
nodes $\equiv$ states, arcs $\equiv$ transitions



$$\mathbf{M} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \\ 0 & 0 & 1/4 & 3/4 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

Properties

- $\diamond$ $\mathbf{M}$ is a stochastic matrix (row sum equal to 1)
- $\diamond$ Given an initial probability distribution a_0 ($\rightarrow X_0$)

  a. Recursively computable a_i (distribution of X_i) $\forall i$

  b. Sequence $\{a_i\}$ tends to a well-defined limit a independently of initial distribution a_0 (under some regularity assumptions)

  c. Limit a, called stationary distribution, is efficiently computable (left eigenvector or linear system)

In the example, $a = \begin{pmatrix} 0 & 0 & 2/5 & 3/5 \end{pmatrix}$

Preference relations

Definitions

$\mathbf{A}$ = set of n alternatives = $\{A_i\}$

$\mathcal{R}$ = binary relation on $\mathbf{A}$ ($\subseteq \mathbf{A} \times \mathbf{A}$)

Let $a, b, c \in \mathbf{A}$. Relation $\mathcal{R}$ is

- reflexive if $a \;\mathcal{R}\; a$
- complete if $a \;\mathcal{R}\; b$ or $b \;\mathcal{R}\; a$
- symmetric if $a \;\mathcal{R}\; b \Rightarrow b \;\mathcal{R}\; a$
- asymmetric if $a \;\mathcal{R}\; b \Rightarrow b \;\neg\mathcal{R}\; a$
- transitive if $a \;\mathcal{R}\; b, b \;\mathcal{R}\; c \Rightarrow a \;\mathcal{R}\; c$

Interpretation

Relation $\mathcal{R}$ is understood as $\leq$, meaning

$$\boxed{a \; \mathcal{R} \; b \Leftrightarrow a \text{ is not strictly preferred to } b}$$

- $\mathcal{I} \equiv$ indifference part of $\mathcal{R}$, defined by
  $$\boxed{a \; \mathcal{I} \; b \Leftrightarrow a \; \mathcal{R} \; b \text{ and } b \; \mathcal{R} \; a}$$
- $\mathcal{S} \equiv$ strict preference part of $\mathcal{R}$, defined by
  $$\boxed{a \; \mathcal{S} \; b \Leftrightarrow a \; \mathcal{R} \; b \text{ and } b \; \neg\mathcal{R} \; a}$$
- $\mathcal{R} = \mathcal{I} \cup \mathcal{S}$
- $\mathcal{I}$ is symmetric and reflexive, $\mathcal{S}$ is asymmetric

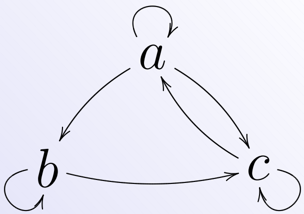
Representations and examples

- Matrix representation $\{r_{i,j}\}$ with $r_{i,j}$ equal to 1 if $A_i \mathcal{R} A_j$, $r_{i,j}$ equal to 0 otherwise
- Representation as a directed graph:

  nodes $\equiv$ alternatives, arcs $\equiv$ relation $\mathcal{R}$

A reflexive and complete example: $\mathbf{A} = \{a, b, c\}$ and $\mathcal{R} = \{(a, c), (c, a), (a, b), (b, c), (a, a), (b, b), (c, c)\}$

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$



Some specific preference relations

Let $\mathcal{R}$ be a complete and reflexive relation

- If $\mathcal{S}$ and $\mathcal{I}$ are both transitive,

  $\mathcal{R}$ is a complete preorder (weak order).

  It is possible to assign a number v_i to each alternative such that $A_i \ \mathcal{R} \ A_j \Leftrightarrow v_i \leq v_j$

- If $\mathcal{S}$ is transitive et $\mathcal{I} = \{(a, a)\} \ \forall a \in \mathcal{R}$,

  $\mathcal{R}$ is a total order.

  It is possible to assign a distinct number v_i to each alternative such that $A_i \ \mathcal{R} \ A_j \Leftrightarrow v_i \leq v_j$

Some specific preference relations (cont.)

- ◇ If it is possible to assign a real interval $U_i = [l_i \; r_i]$ to each alternative A_i such that $A_i \; \mathcal{I} \; A_j \Leftrightarrow U_i \cap U_j \neq \emptyset$ and $A_i \; \mathcal{S} \; A_j \Leftrightarrow r_i < l_j$, $\mathcal{R}$ is a (total) interval order.
- ◇ If it is possible to assign a real interval $U_i = [l_i \; r_i]$ to each alternative such that $A_i \; \mathcal{I} \; A_j \Leftrightarrow U_i \cap U_j \neq \emptyset$, $A_i \; \mathcal{S} \; A_j \Leftrightarrow r_i < l_j$ and no interval is *strictly included* in another one, $\mathcal{R}$ is a (total) semiorder.

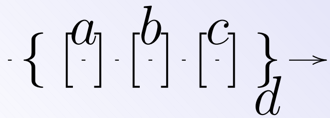
  In both cases, $\mathcal{S}$ is transitive but $\mathcal{I}$ isn't.

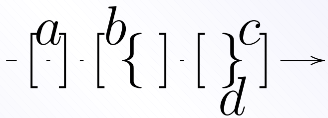
Some preference relations

Examples

- ◇ Total order $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ $\longrightarrow a \longrightarrow b \longrightarrow c \longrightarrow$
- ◇ Total preorder $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ $\longrightarrow a - b, c \longrightarrow$

Examples (cont.)

◇ Interval order $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ 

◇ Semiorder $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$ 

Properties of semiorders

Consider a preference relation $\mathcal{R}$ without any equivalent alternatives

(a and b are equivalent iff $\forall x \in \mathbf{A}$ we have $a \,\mathcal{R}\, x \Leftrightarrow b \,\mathcal{R}\, x$ and $x \,\mathcal{R}\, a \Leftrightarrow x \,\mathcal{R}\, b$)

- $\exists$ a unique total order underlying a semiorder $\mathcal{R}$.
- It is induced by the numbers $\{l_i\}$ (or $\{r_i\}$).
- Moreover, sorting rows and columns of the matrix representation of $\mathcal{R}$ in that order gives a step-type matrix

Properties of interval orders

◇ A step-type matrix: $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$

◇ $\exists$ two total orders underlying an interval order $\mathcal{R}$.

◇ They are induced by the numbers $\{l_i\}$ and $\{r_i\}$.

◇ Moreover, sorting rows (resp. columns) of the matrix representation of $\mathcal{R}$ in the first (resp. second) order also gives a step-type matrix.

Stochastic method

Single preference relation

Let $\mathcal{R}$ be a complete and reflexive binary preference relation. We proceed as follows :

a. Build a Markov chain:

   Associate to each alternative A_i a state E_i

b. Choose the transition probabilities,

   requiring the following conditions to hold

   - $p_{i,j} > 0$ when $A_i \ \mathcal{R} \ A_j$
   - $p_{i,j} = 0$ when $A_i \ \neg\mathcal{R} \ A_j$
   - Matrix $\{p_{i,j}\}$ is stochastic

Single preference relation (cont.)

a. Build a Markov chain.

b. Choose the transition probabilities,
requiring some intuitive conditions

c. Compute the stationary distribution and rank the alternatives according to the resulting probabilities

Justification

Intuitively, the process always moves from an alternative to a better or equivalent alternative. It is thus sensible to expect a high probability for the best alternatives in the stationary distribution.

Choice of $\{p_{i,j}\}$

We want the value of $p_{i,j}$ to be independent from other alternatives than A_i and A_j

$$\boxed{\text{If } A_i \mathcal{R} A_j \text{, let } p_{i,j} = \frac{1}{n}\text{, otherwise let } p_{i,j} = 0}$$

However $\mathbf{M}$ is not stochastic $\Rightarrow$ set $p_{i,i}$ according to

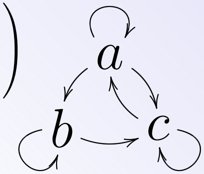
$$p_{i,i} = 1 - \sum_{j \neq i} p_{i,j}$$

With this choice, the Markov process can be described as

a. I am currently in state E_i
b. Choose randomly a state E_j (uniformly)
c. If $A_i \mathcal{R} A_j$, move to E_j, otherwise stay in E_i

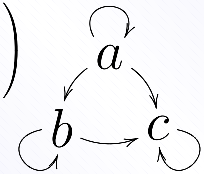
Examples

$\mathcal{R} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ gives $M = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}$



Stationary distribution is $\begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$: c is the best, a and b follow.

$\mathcal{R} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ gives $M = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}$



Stationary distribution is $\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$: c is the best but we have no information about the other alternatives (no way to "escape" from c)

Better choice of $\{p_{i,j}\}$

Let's add a neutral state E_0, representing no alternative, such that $p_{0,0} = 0$ and $p_{0,i} = \frac{1}{n}$.
Transitions from this state do not favor any alternative.

If $A_i \mathcal{R} A_j$, let $p_{i,j} = \frac{1}{n}$, otherwise let $p_{i,j} = 0$

and choose the $p_{i,0}$ to make the matrix stochastic, i.e.

$$p_{i,0} = 1 - \sum_{j \neq 0} p_{i,j}$$

a. I am currently in state E_i
b. Choose randomly a state E_j (uniformly)
c. If $A_i \mathcal{R} A_j$ move to E_j, otherwise move to neutral E_0

Examples

$\mathcal{R} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ gives $M = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$
Stationary distribution is $\begin{pmatrix} \frac{7}{27} & \frac{6}{27} & \frac{9}{27} & \frac{5}{27} \end{pmatrix}$: c is the best, b is the worst
$\mathcal{R} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ gives $M = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$
Stationary distribution is $\begin{pmatrix} \frac{4}{27} & \frac{6}{27} & \frac{9}{27} & \frac{8}{27} \end{pmatrix}$: c is the best, a is the worst

Property

If $\mathcal{R}$ is a complete preorder, the probabilities from the stationary distribution are ranked according to $\mathcal{R}$

Going multicriteria

Definitions

$\mathbf{R}$ = set of preference relations $\{\mathcal{R}_i\}$ where $\mathcal{R}_i$ is complete and reflexive.
These relations are weighted by $\{w_i\}$ (with $\sum w_i = 1$).
$\mathbf{M}_i$ = stochastic matrix for each $\mathcal{R}_i$ (computed as above).

Principle

Let
$$\mathbf{M} = \sum_i w_i \mathbf{M}_i$$
$\mathbf{M}$ is also stochastic and describes a Markov chain.

Principle (cont.)

We rank the alternatives according to the probabilities from the resulting stationary distribution (using $\mathbf{M}$).

Interpretation

a. I am currently in state E_i

b. Choose randomly a relation $\mathcal{R}_p$ (according to the $\{w_i\}$)

c. Choose randomly a state E_j (uniformly)

d. If $A_i \ \mathcal{R}_p \ A_j$, move to state E_j, otherwise move to neutral state E_0

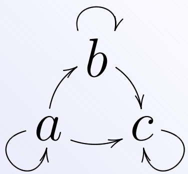
Examples

$r = 2, w_1 = \frac{1}{3}$ and $w_2 = \frac{2}{3}$

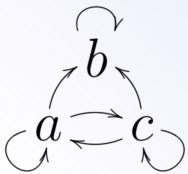
$$M_1 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix} \quad M_2 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

$$M = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{9} & \frac{5}{9} \\ \frac{2}{9} & \frac{2}{9} & \frac{1}{3} & \frac{2}{9} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

$\mathcal{R}_1$:



$\mathcal{R}_2$:



Stationary distribution is $\begin{pmatrix} \frac{16}{79} & \frac{24}{79} & \frac{21}{79} & \frac{18}{79} \end{pmatrix}$: b is the best, a is the worst

Expressing incomparability

Arrow's theorem

There is no good procedure to aggregate several complete preorders into a single complete preorder
(universality, monotonicity, independence, no dictators)

Method

In light of this result, we shouldn't expect too good properties for a preference analysis method producing a complete preorder.

$\Rightarrow$ a good method should output a weaker relation

Principle

Our previous method made no distinction between $A_i \; \mathcal{S} \; A_j$ and $A_i \; \mathcal{I} \; A_j$ (indifference and strict preference)

Idea: When A_i and A_j are indifferent, choose something less radical than $p_{i,j} = \frac{1}{n}$

- If $A_i \; \mathcal{S} \; A_j$, let $p_{i,j} = \frac{1}{n}$
- If $A_i \; \mathcal{I} \; A_j$, let $p_{i,j} = \frac{\alpha}{n}$
- Otherwise let $p_{i,j} = 0$

where α is a free parameter varying between 0 and 1.

Interpretation

a. I am currently in state E_i

b. Choose randomly a relation $\mathcal{R}_p$ (according to the $\{w_i\}$)

c. Choose randomly a state E_j

- $\diamond$ If $A_i \; \mathcal{S}_p \; A_j$, move to state E_j
- $\diamond$ If $A_i \; \mathcal{I}_p \; A_j$, move to
  - – state E_j with probability α
  - – neutral state E_0 with probability $(1 - \alpha)$
- $\diamond$ Otherwise, move to state E_0

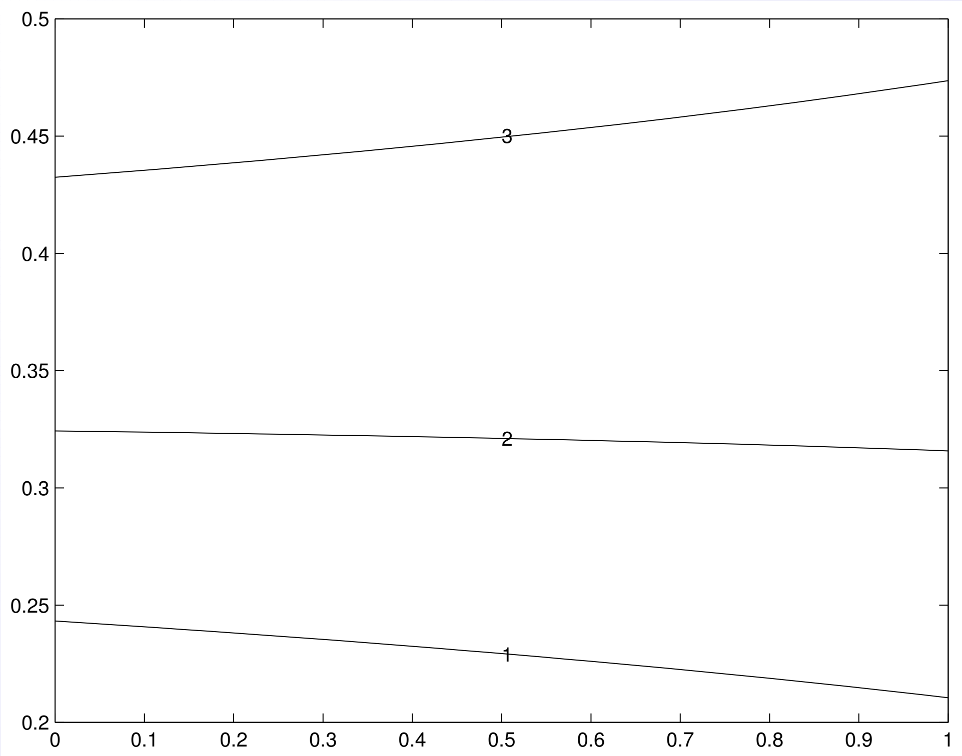
Interpretation (cont.)

How to exploit this family of complete preorders parameterized by α ?

Use it to deduce a partial preorder :

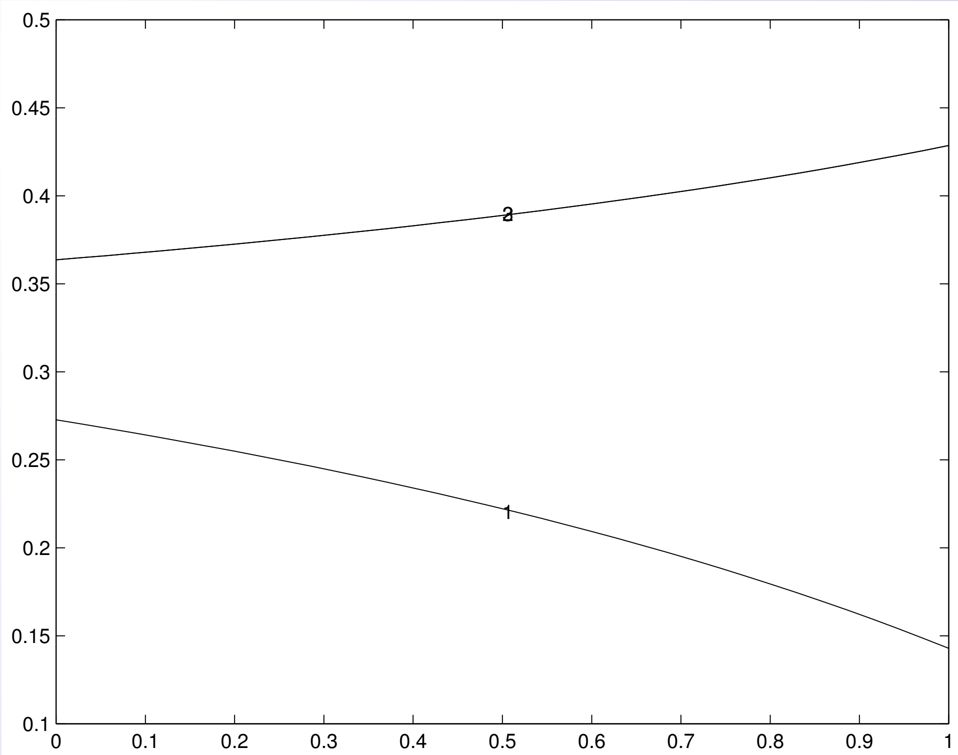
- ◇ If A_i is better than A_j for all values of α, declare $A_i > A_j$
- ◇ If A_i is worse than A_j for all values of α, declare $A_i < A_j$
- ◇ Otherwise, declare that A_i and A_j are not comparable

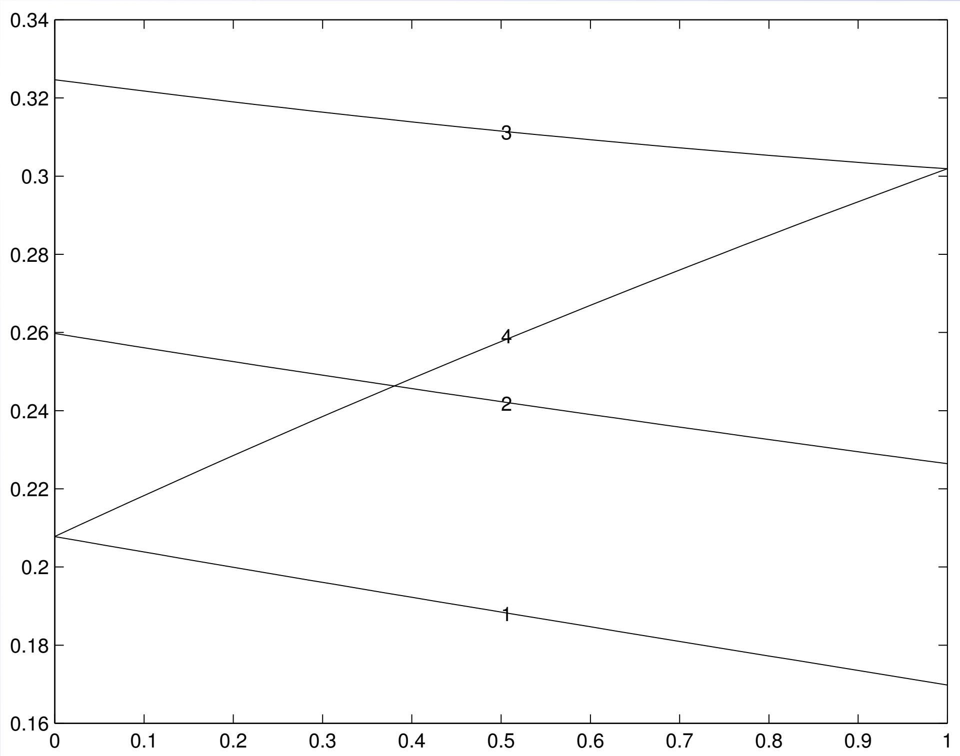
Examples



Total order

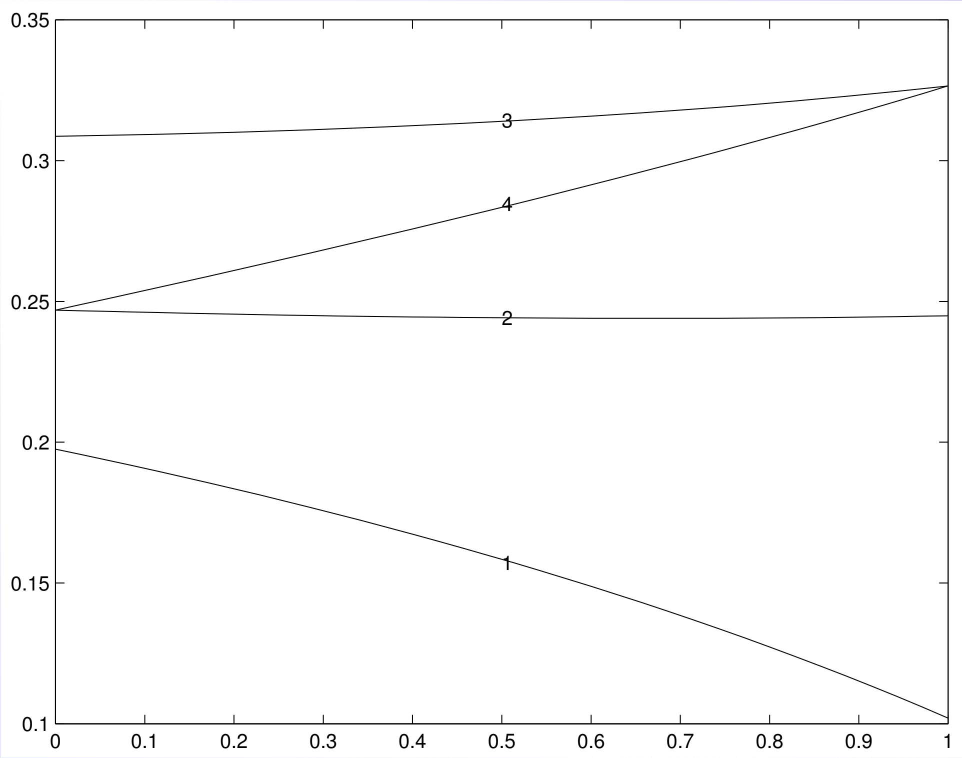
Total preorder





Interval order

Semiorder



Properties

- $\diamond$ If $\mathcal{R}$ is a total (pre)order, the rankings obtained $\forall \alpha \in [0\ 1]$ match $\mathcal{R}$
- $\diamond$ If $\mathcal{R}$ is a semiorder, the rankings obtained $\forall \alpha \in]0\ 1[$ match the unique total order underlying $\mathcal{R}$
- $\diamond$ If $\mathcal{R}$ is an interval order, the rankings obtained by letting α tend to 0 and 1 match the two total orders underlying $\mathcal{R}$

  $\Rightarrow$ there is at least one case of incomparability

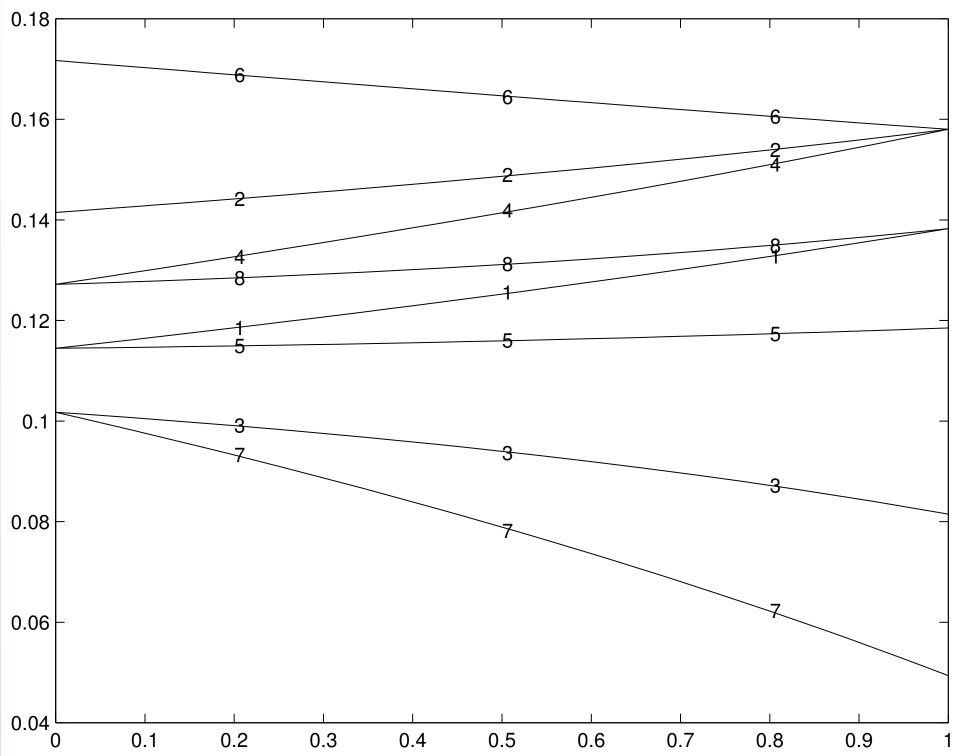
Usage

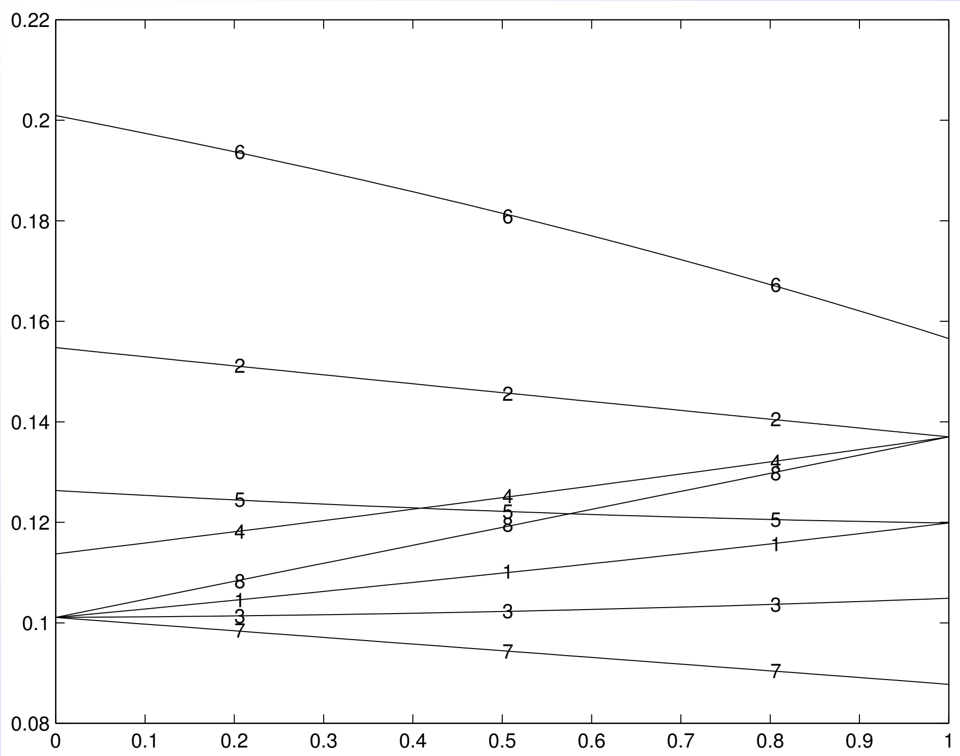
To select the best alternative, choose among those being ranked first for at least one value of α
$\Rightarrow$ these alternatives are incomparable to each other

Incomparability strength

Intuitively, one can suggest that the more a crossing is close to 0 or 1, the less the associated incomparability is strong

Illustrations: Semiorders and interval orders




0.22
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0.1
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4
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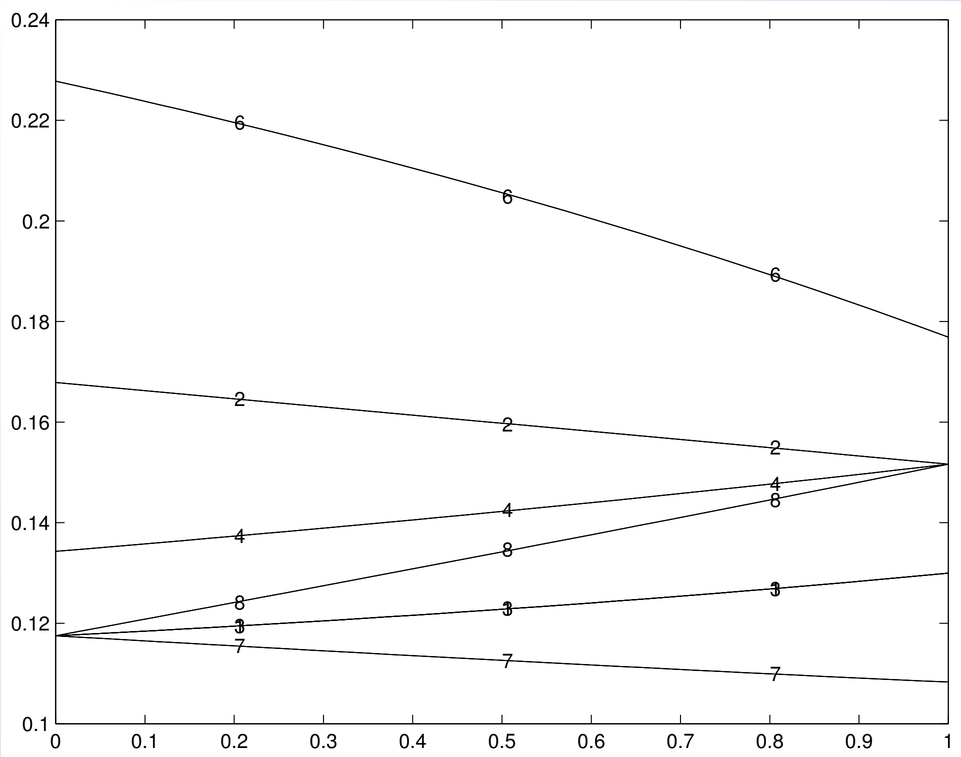
7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7

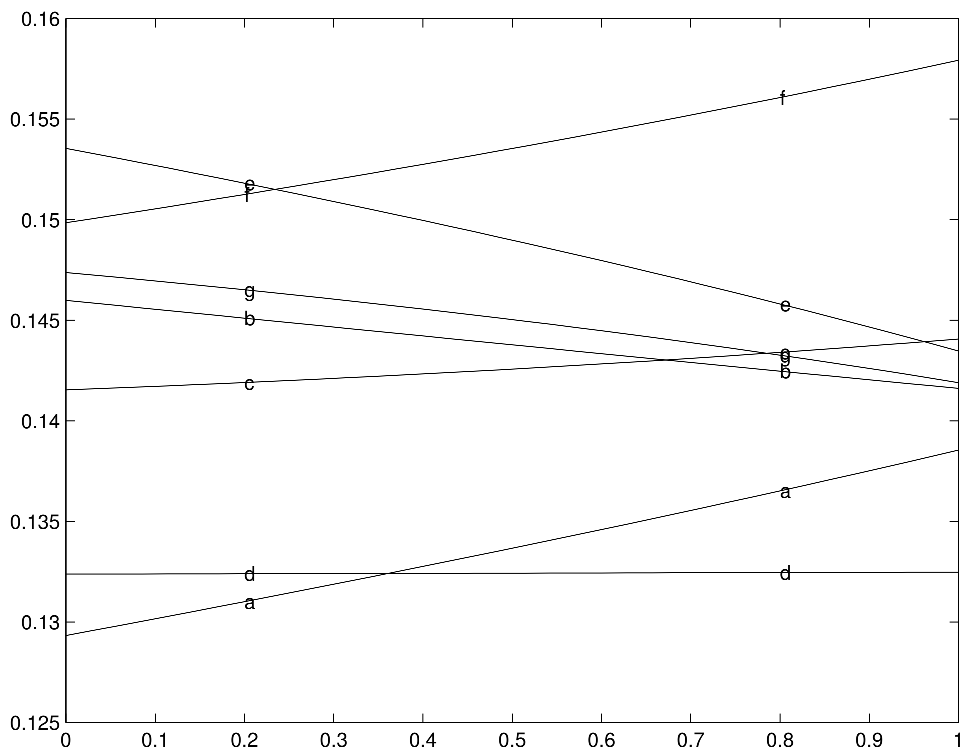
Factory location

Problem description and approach

- 5 numerical criteria : Price, Transport, Environment, Residential, Competition
- Could transform each criterion into a complete pre-order but introducing indifference thresholds adds more information

  $\Rightarrow$ each criterion produces a semiorder
- Use our stochastic procedure to aggregate these 5 semiorders

Results



0.16
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0.6
0.7
0.8
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1
f
e
f
g
b
c
e
g
b
a
d
d
a

Conclusions

Stochastic method

◇ Intuitive principle
◇ Graphical results
◇ No parameter value to choose
◇ Easy to implement

Multicriteria analysis

◇ Natural generalization
◇ Sensitivity analysis easy to perform
◇ Cardinal information allowed

Further research

◇ Valued relations
◇ Theoretical properties (independence)