Second-order cone optimization with a single second-order cone

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Outline

Introduction

- ◇ Convex, conic and second-order cone optimization
- ◇ Two easy subproblems

Second-order cone feasibility problem

- ◇ Main ideas: homogenization and minimum-norm solution
- ◇ Our algorithm: a three-case discussion

Second-order cone optimization problem

- ◇ Using the feasibility problem as a subproblem
- ◇ What about the dual problem?

Concluding remarks

- ◇ Summary, complexity and generalizations

Introduction

Convex optimization

Let $f_0 : \mathbb{R}^n \mapsto \mathbb{R}$ a convex function and $C \subseteq \mathbb{R}^n$ a convex set

$$\inf_{x \in \mathbb{R}^n} f_0(x) \quad \text{s.t.} \quad x \in C$$

Properties

- ◇ Local optima ⇒ global, form a convex optimal set
- ◇ Lagrange duality ⇒ related (asymmetric) dual problem
- ◇ Efficient *interior-point* methods (*self-concordant barriers*)

Conic optimization

Let $\mathcal{C} \subseteq \mathbb{R}^n$ a *solid, pointed, closed convex cone* :

$$\inf_{x \in \mathbb{R}^n} c^{\mathrm{T}} x \quad \text{s.t.} \quad Ax = b \text{ and } x \in \mathcal{C}$$ ⇒ **Equivalent** setting

Primal-dual pair

Dual cone is also a solid pointed closed convex cone

$$\mathcal{C}^* = \left\{ x^* \in \mathbb{R}^n \mid x^\mathrm{T} x^* \geq 0 \text{ for all } x \in \mathcal{C} \right\}$$

$\Rightarrow$ pair of primal-dual problems

$$\inf_{x \in \mathbb{R}^n} c^\mathrm{T} x \quad \text{s.t.} \quad Ax = b \text{ and } x \in \mathcal{C}$$

$$\sup_{(y,s) \in \mathbb{R}^{m+n}} b^\mathrm{T} y \quad \text{s.t.} \quad A^\mathrm{T} y + s = c \text{ and } s \in \mathcal{C}^*$$

Several cones: $x^1 \in \mathcal{C}^1, \ldots, x^r \in \mathcal{C}^r \Leftrightarrow (x^1, \ldots, x^r) \in \mathcal{C}^1 \times \cdots \times \mathcal{C}^r$

Advantages over classical formulation

- Remarkable primal-dual symmetry
- Special handling of (*easy*) linear equality constraints

Examples

$\mathcal{C} = \mathbb{R}^n_+ = \mathcal{C}^* \Rightarrow$ linear optimization

$\mathcal{C} = \mathbb{S}^n_+ = \mathcal{C}^* \Rightarrow$ semidefinite optimization

Both cones are *self-dual*.

A *single* $\mathbb{R}^n_+/\mathbb{S}^n_+$ cone can be considered w.l.o.g.

Second-order cone optimization

$$\mathbb{L}^n = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}_+ \times \mathbb{R}^n \mid \|(x_1, \ldots, x_n)\| \leq x_0\} \subset \mathbb{R}^{n+1}$$

Second-order or Lorentz cone $\mathbb{L}^n$ is self-dual

But a set of several constraints $x^i \in \mathbb{L}^{n_i}, i = 1, \ldots, r$ **cannot** be concatenated into a single second-order cone constraint

Goal of this talk: Study the problem with a single second-order cone

Single-constraint second-order cone problem

$$\inf_{x_0\in\mathbb{R},\ x\in\mathbb{R}^n} c_0 x_0 + c^{\mathrm{T}} x \quad \text{s.t.} \quad a_0 x_0 + Ax = b \text{ and } (x_0, x) \in \mathbb{L}^n$$

with $c_0 \in \mathbb{R}$, $c \in \mathbb{R}^n$; $a_0 \in \mathbb{R}^m$, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$

Known results

It is *well-known* that this problem can be solved *analytically* *[Alizadeh and Goldfarb 2003]*: solve primal-dual optimality conditions

$$\begin{aligned} a_0 x_0 + A_0 x = b &\quad \text{and} \quad && (x_0, x) \in \mathbb{L}^n \\ a_0^{\mathrm{T}} y + z_0 = c_0,\ A^{\mathrm{T}} y + z = c &\quad \text{and} \quad && (z_0, z) \in \mathbb{L}^n \\ (x_0, x) \circ (z_0, z) = 0 & && (\Leftrightarrow\ c_0 x_0 + c^{\mathrm{T}} x = b^{\mathrm{T}} y) \end{aligned}$$

where $(x_0, x) \circ (z_0, z) = (x_0 y_0 + x^{\mathrm{T}} y, x_0 y + y_0 x)$

But only works when *strong duality* holds !

Analytical solution of optimality conditions

Define $\bar{A} = (a_0 \ A)$, $\bar{c} = (c_0, c)$, $Q = P_{\mathrm{null}\,\bar{A}} = I - \bar{A}^\dagger \bar{A}$.

Assuming $x_0 > 0$ and $z_0 > 0$, one obtains after some linear algebra

$$\begin{aligned}
\gamma &= 1 - 2a_0^{\mathrm{T}}(\bar{A}\bar{A}^{\mathrm{T}})^{-1}a_0 \\
\alpha &= \frac{z_0}{x_0} = \sqrt{\frac{-\gamma\bar{c}^{\mathrm{T}}Q\bar{c} + 2(e^{\mathrm{T}}Q\bar{c})^2}{\gamma b^{\mathrm{T}}(\bar{A}\bar{A}^{\mathrm{T}})^{-1}b + 2(a_0^{\mathrm{T}}(\bar{A}\bar{A}^{\mathrm{T}})^{-1}b)^2}} \\
\delta &= \frac{\bar{c}^{\mathrm{T}}Q\bar{c} + \alpha^2 b^{\mathrm{T}}(\bar{A}\bar{A}^{\mathrm{T}})^{-1}b}{e^{\mathrm{T}}Q\bar{c} + \alpha a_0^{\mathrm{T}}(\bar{A}\bar{A}^{\mathrm{T}})^{-1}b} \\
y &= (\bar{A}\bar{A}^{\mathrm{T}})^{-1}(\bar{A}\bar{c} + \alpha b - \delta a_0) \\
(z_0, z) &= \bar{c} - \bar{A}^{\mathrm{T}}y = Q\bar{c} - \bar{A}^\dagger(\alpha b - \delta a_0) \\
(x_0, x) &= (z_0/\alpha, -z/\alpha)
\end{aligned}$$

Technical derivation but *why* is this possible? Special cases?

Classification of conic convex problems

Feasibility status of conic problem: define $\mathcal{L} = \{x \in \mathbb{R}^n \mid Ax = b\}$

$$\inf_{x \in \mathbb{R}^n} c^{\mathrm{T}} x \quad \text{s.t.} \quad Ax = b \text{ and } x \in \mathcal{C} \quad \Leftrightarrow \quad x \in \mathcal{L} \cap \mathcal{C}$$

- Feasible iff $\mathcal{L} \cap \mathcal{C} \neq \emptyset$. Moreover, in this case,
  - *Strictly* feasible (s.f.) if $\mathcal{L} \cap \operatorname{int} \mathcal{C} \neq \emptyset$
  - *Weakly* feasible (w.f.) otherwise
- Infeasible iff $\mathcal{L} \cap \mathcal{C} = \emptyset$. Moreover, in this case,
  - *Strictly* infeasible (s.i.) if $\operatorname{dist}(\mathcal{L}, \mathcal{C}) > 0$
  - *Weakly* infeasible (w.i.) otherwise, i.e. $\operatorname{dist}(\mathcal{L}, \mathcal{C}) = 0$
    $\Leftrightarrow \exists \{x^k\} \mid x^k \in \mathcal{L}$ and $\lim_{k \to \infty} \operatorname{dist}(x^k, \mathcal{C}) = 0$
    (which implies then that $\{x^k\}$ is unbounded)

All these cases already arise when $\mathcal{C} = \mathbb{L}^2$!

Two easy subproblems

Smallest-norm point on a line

Let $\alpha, \beta \in \mathbb{R}^n$ and $\beta \neq 0$

$$\min_{t \in \mathbb{R}} \phi(t) = \|\alpha - t\beta\|$$

$\phi(t)^2 = t^2 \|\beta\|^2 - 2t\alpha^T\beta + \|\alpha\|^2$ is easily minimized $\rightarrow t_* = \frac{\alpha^T\beta}{\|\beta\|^2}$

Minimum distance is $\phi(t_*)^2 = \|\alpha\|^2 - t_*^2 \|\beta\|^2 = \|\alpha\|^2 - \frac{(\alpha^T\beta)^2}{\|\beta\|^2}$

Intersecting a ball with an affine subspace

Decide the feasibility of the following convex problem

$$\text{Find } x \in \mathbb{R}^n \quad \text{s.t.} \quad Ax = b \text{ and } \|x\| \leq 1$$

with $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

Assume that A has full row rank ($\Rightarrow AA^T \succ 0$ and $Ax = b$ is feasible)

Idea: compute minimum-norm solution $\hat{x}$ of $Ax = b$

$$\hat{x} = A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}b$$

($A^\dagger = A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}$ is the Moore-Penrose generalized inverse of A)

Discuss according to $\|\hat{x}\|^2 = \left\|A^\dagger b\right\|^2 = b^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}b$

- $\|\hat{x}\|^2 > 1 \Rightarrow$ problem is strictly infeasible
- $\|\hat{x}\|^2 = 1 \Rightarrow$ problem is weakly feasible, $\hat{x}$ is unique solution
- $\|\hat{x}\|^2 < 1 \Rightarrow$ problem is strictly feasible, $\hat{x}$ is among the solutions

(obviously problem cannot be weakly infeasible)

Forming $AA^{\mathrm{T}} \in \mathbb{R}^{m \times m} \to \mathcal{O}\left(m^2 n\right)$ operations
Factorizing $AA^{\mathrm{T}} = LL^{\mathrm{T}}$, $L \in \mathbb{R}^{m \times m}$ triangular $\to \mathcal{O}\left(m^3\right)$ operations
$\Rightarrow$ Computing $x^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}y$ for any $x, y \in \mathbb{R}^m$ is $\mathcal{O}\left(m^2 n\right)$ operations.

Second-order cone feasibility problem

Our problem

Determine *feasibility* status of the following problem

$$\text{Find } (x_0, x) \in \mathbb{R} \times \mathbb{R}^n \quad \text{s.t.} \quad a_0 x_0 + Ax = b \text{ and } (x_0, x) \in \mathbb{L}^n$$

with $a_0 \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Assume w.l.o.g. that $a_0 x_0 + Ax = b$ is feasible and $(a_0 \ A)$ full row rank $(a_0 a_0^{\mathrm{T}} + AA^{\mathrm{T}} \succ 0)$

Main idea

Use the fact that $\mathbb{L}^n$ and $\mathbb{B}^n$ are strongly related:

$$(x_0, x) \in \mathbb{L}^n \quad \Leftrightarrow \quad (x/x_0) \in \mathbb{B}^n \text{ and } x_0 > 0 \quad \text{or} \quad (x_0, x) = (0, 0)$$

$\rightarrow$ use *homogenization* to get rid of x_0 variable

Our procedure: outline

Let $t > 0$ be a homogenizing variable. Problem

$$\text{Find } (x_0, x) \in \mathbb{R} \times \mathbb{R}^n \quad \text{s.t.} \quad a_0 x_0 + Ax = b \text{ and } (x_0, x) \in \mathbb{L}^n$$

becomes *equivalent* to the problem of finding

$$(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}_{++} \quad \text{s.t.} \quad a_0 x_0 + Ax = b\,t \text{ and } (x_0, x) \in \mathbb{L}^n$$

Each point (x_0, x) becomes a ray (tx_0, tx, t), $t > 0$, *except* for $(0, 0)$

Problem is completely *homogeneous* $\rightarrow$ arbitrarily fix $x_0 = 1$

$\Rightarrow$ constraint $(x_0, x) \in \mathbb{L}^n$ becomes equivalent to $x \in \mathbb{B}^n \quad \rightarrow$

$$\text{Find } (x, t) \in \mathbb{R}^n \times \mathbb{R}_{++} \quad \text{s.t.} \quad Ax = b\,t - a_0 \text{ and } x \in \mathbb{B}^n$$

which is the second easy problem with a parameter $t \in \mathbb{R}_{++}$

Soln (x, t) to this problem $\rightarrow$ soln $(1/t, x/t) \in \mathbb{L}^n$ to original problem

Special case

Do the linear constraints $a_0 x_0 + Ax = b$ imply that x_0 is constant?

$$\Leftrightarrow \quad \exists y \mid A^T y = 0 \quad (A \text{ is rank deficient})$$

$\Rightarrow x_0 = y^T b / a_0 = C$ for all feasible solutions

Not a *true* second-order cone problem

- $C < 0 \rightarrow$ problem strictly infeasible
- $C = 0 \rightarrow (0,0)$ unique potential solution
  - $b = 0 \rightarrow$ problem weakly feasible
  - $b \neq 0 \rightarrow$ problem strictly infeasible
- $C > 0$ problem becomes $A(x/x_0) = b/x_0 - a_0$ with $(x/x_0) \in \mathbb{B}^n$ which is *easy* (look at $\left\| A^\dagger (b/x_0 - a_0) \right\| \rightarrow$ s.i., w.f. or s.f.)

Main case

Assume A has full row rank. Problem is equivalent to

$$\text{Find } (x,t) \in \mathbb{R}^n \times \mathbb{R}_{++} \quad \text{s.t.} \quad Ax = b\,t - a_0 \text{ and } x \in \mathbb{B}^n$$

This suggests to look at $\left\|A^\dagger(b\,t - a_0)\right\|^2 = \|\alpha - t\beta\|^2$ with

$$\alpha = A^\dagger a_0 = A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}a_0 \text{ and } \beta = A^\dagger b = A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}b$$

$(\|\alpha\|^2 = a_0^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}a_0,\ \|\beta\|^2 = b^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}b,\ \alpha^{\mathrm{T}}\beta = a_0^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}b)$

However the possibility $x_0 = 0$ was left out!

In this case, we have $(x_0, x) = (0,0)$ which implies $b = 0$ and $\beta = 0$

We therefore have to distinguish two cases: $b = 0$ and $b \neq 0$

and add soln $(0,0)$ to homogenized solutions $(1/t, x/t)$ when $b = 0$

Find $(x, t) \in \mathbb{R}^n \times \mathbb{R}_{++}$ s.t. $Ax = b\,t - a_0$ and $x \in \mathbb{B}^n$

CASE A: $b = 0$

In this case, $\|\alpha - t\beta\|^2 = \|\alpha\|$ does not depend from t

- ◇ $\|\alpha\| > 1 \rightarrow$ no solution except $(0, 0) \rightarrow$ problem w.f.
- ◇ $\|\alpha\| = 1 \rightarrow$ ray $(t, t\alpha), t \in \mathbb{R}_+$ is solution $\rightarrow$ problem w.f.
- ◇ $\|\alpha\| < 1 \rightarrow$ interior solutions $\rightarrow$ problem s.f.

CASE B: $b \neq 0$

We can here safely ignore solutions with $x_0 = 0$

In theory, minimum value of $\|\alpha - t\beta\|^2$ is attained for $t_* = \frac{\alpha^T\beta}{\|\beta\|^2}$

But t is required to be positive

$\rightarrow$ distinguish whether t_* is positive or not $\Rightarrow$ discuss the *sign* of $\alpha^T\beta$

Geometrically: study intersection of open half-line with ball

$$\{\alpha - \mathbb{R}_{++}\beta\} \cap \mathbb{B}^n$$

CASE B$_1$: $\alpha^{\mathrm{T}}\beta > 0$

Minimum t_* is achieved. The smallest-norm solution is then

$$t_*\beta - \alpha \text{ with } \|t_*\beta - \alpha\|^2 = \|\alpha\|^2 - t_*^2\|\beta\|^2 = \|\alpha\|^2 - \frac{(\alpha^{\mathrm{T}}\beta)^2}{\|\beta\|^2} = \delta$$

- $\delta > 1 \longrightarrow$ no solution $\longrightarrow$ problem s.i.
- $\delta < 1 \longrightarrow$ there are interior solutions $\longrightarrow$ problem s.f.
- $\delta = 1 \longrightarrow$ only one solution $(1/t_*, \alpha/t_* - \beta) \longrightarrow$ problem w.f.

  Note this solution is easy to compute (in $\mathcal{O}\left(m^2 n\right)$ operations)

$$(x_0, x) = \left(1/t_*, A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}(b - a_0/t_*)\right) \text{ with } t_* = \frac{a_0^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}b}{b^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}b}$$

Case B$_2$: $\alpha^{\mathrm{T}}\beta \leq 0$

Minimum t_* cannot be reached

$\rightarrow$ actual minimum attained for $t \rightarrow 0^+$

This means we have to look at $\|\alpha\|^2$

- $\|\alpha\| > 1 \rightarrow$ no solution $\rightarrow$ problem s.i.
- $\|\alpha\| < 1 \rightarrow \exists t > 0$ such that $\|\alpha - t\beta\|^2 < 1 \rightarrow$ problem s.f.
- $\|\alpha\| = 1 \rightarrow$ no solution since $t = 0$ is forbidden

  But $\operatorname{dist}(\beta t - \alpha, \mathbb{B}^n) \rightarrow 0$. Let $x^t = (1/t, \beta - \alpha/t)$.

  What about $\operatorname{dist}(x^t, \mathbb{L}^n)$ as $t \rightarrow 0$?

  One has $\operatorname{dist}(x^t, \mathbb{L}^n) \approx (1/t) \operatorname{dist}(\beta t - \alpha, \mathbb{B}^n)$ and thus

  - $\alpha^{\mathrm{T}}\beta = 0 \rightarrow \operatorname{dist}(x^t, \mathbb{L}^n) \rightarrow 0 \rightarrow$ w.i.
  - $\alpha^{\mathrm{T}}\beta < 0 \rightarrow \operatorname{dist}(x^t, \mathbb{L}^n) > \epsilon \rightarrow$ s.i.

Summarizing table

◇ A not full row rank $\rightarrow x_0 = C$

- $C < 0 \rightarrow$ s.i.
- $C = 0$ and $b = 0 \rightarrow$ w.f. ; $C = 0$ and $b \neq 0 \rightarrow$ s.i.
- $C > 0 \rightarrow$ s.f., w.f. or s.i. when $\left\| A^\dagger (b/x_0 - a_0) \right\| <, =, > 1$

◇ A has full row rank, define α, β and $\delta = \|\alpha\|^2 - \frac{(\alpha^T \beta)^2}{\|\beta\|^2}$

	$\|\alpha\| < 1$	$\|\alpha\| = 1$	$\|\alpha\| > 1$	
$\alpha^T \beta < 0$	s.f.	s.i.	s.i.	
$b \neq 0$, $\alpha^T \beta = 0$	s.f.	w.i.	s.i.	
$b = 0$, $\alpha^T \beta = 0$	s.f.	w.f.	w.f.	
$\alpha^T \beta > 0$	s.f.	s.f.	*s.f., w.f., s.i.*	$(\delta <, =, > 1)$

Notes: "Loose" dependence from a_0 and b ; $\|\beta\|$ in single case

Simplifying notations

$$\text{Find } (x_0, x) \in \mathbb{R} \times \mathbb{R}^n \quad \text{s.t.} \quad a_0 x_0 + Ax = b \text{ and } (x_0, x) \in \mathbb{L}^n$$

$$\rightarrow \quad \text{Find } \bar{x} \in \mathbb{R} \times \mathbb{R}^n \quad \text{s.t.} \quad \bar{A}\bar{x} = b \text{ and } \bar{x} \in \mathbb{L}^n$$

Let

$$P = \bar{A}^{\mathrm{T}}(\bar{A}\bar{A}^{\mathrm{T}})^{-1}\bar{A}$$

(orthogonal projection on range $\bar{A}^{\mathrm{T}}$)

$$(d_0, d) = \bar{d} = \bar{A}^{\mathrm{T}}(\bar{A}\bar{A}^{\mathrm{T}})^{-1}b \in \operatorname{range} \bar{A}^{\mathrm{T}}$$

(minimum-norm solution of $\bar{A}\bar{x} = b$)

$$\rightarrow \quad \text{Find } \bar{x} \in \mathbb{R} \times \mathbb{R}^n \quad \text{s.t.} \quad P\bar{x} = \bar{d} \text{ and } \bar{x} \in \mathbb{L}^n$$

Summarizing table

Let $w = (1\ 0 \cdots 0) \in \mathbb{R}^{n+1}$ and $\lambda = \|Pw\|^2$

We have

$$\|\alpha\|^2 = \frac{\lambda}{1-\lambda}, \ \mathrm{sign}(\alpha^T\beta) = \mathrm{sign}\, d_0 \text{ and } \mathrm{sign}(\delta - 1) = \mathrm{sign}(\lambda \|\bar{d}\|^2 - d_0^2)$$

Main case (not considering rank-deficient A nor $b = 0$)

	$\lambda < 0$	$\lambda = 0$	$\lambda > 0$	
$d_0 < 0$	s.f.	s.i.	s.i.	
$d_0 = 0$	s.f.	w.i.	s.i.	
$d_0 > 0$	s.f.	s.f.	*s.f., w.f., s.i.*	$(\lambda \|\bar{d}\|^2 <, =, > d_0^2)$

Second-order cone optimization problem

Problem definition

$$\inf_{x_0 \in \mathbb{R},\ x \in \mathbb{R}^n} c_0 x_0 + c^{\mathrm{T}} x \quad \text{s.t.} \quad a_0 x_0 + Ax = b \text{ and } (x_0, x) \in \mathbb{L}^n$$

with $c_0 \in \mathbb{R}$, $c \in \mathbb{R}^n$; $a_0 \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Assume w.l.o.g. that $a_0 x_0 + Ax = b$ is feasible and $(a_0\ A)$ full row rank

Using feasibility problem as subproblem

Idea: add $c_0 x_0 + c^{\mathrm{T}} x = \gamma$ as a constraint with $\gamma \in \mathbb{R}$ as a parameter

$\rightarrow$ test whether γ is a *feasible* objective value

Find $(x_0, x) \in \mathbb{R} \times \mathbb{R}^n$ s.t. $\begin{pmatrix} c_0 \\ a_0 \end{pmatrix} x_0 + \begin{pmatrix} c^{\mathrm{T}} \\ A \end{pmatrix} x = \begin{pmatrix} \gamma \\ b \end{pmatrix}$ and $(x_0, x) \in \mathbb{L}^n$

which is a feasibility second-order cone problem with new data

Data of the subproblem

We have

$$\tilde{a}_0 = \begin{pmatrix} c_0 \\ a_0 \end{pmatrix}, \ \tilde{A} = \begin{pmatrix} c^{\mathrm{T}} \\ A \end{pmatrix}, \tilde{b} = \begin{pmatrix} \gamma \\ b \end{pmatrix}$$

What are the new quantities α and β?

We need to evaluate $(\tilde{A}\tilde{A}^{\mathrm{T}})^{-1} = \begin{pmatrix} c^{\mathrm{T}}c & c^{\mathrm{T}}A^{\mathrm{T}} \\ Ac & AA^{\mathrm{T}} \end{pmatrix}^{-1}$

Possible to compute this as a function of $(AA^{\mathrm{T}})^{-1}$ and c but tedious

Better approach

We can actually suppose *without loss of generality* that $\bar{c} \in \operatorname{null} \bar{A}$

New problem definition

$$\inf_{x_0 \in \mathbb{R},\ x \in \mathbb{R}^n} c_0 x_0 + c^{\mathrm{T}} x \quad \text{s.t.} \quad a_0 x_0 + Ax = b \text{ and } (x_0, x) \in \mathbb{L}^n$$

$$\rightarrow \qquad \inf_{\bar{x} \in \mathbb{R} \times \mathbb{R}^n} \bar{c}^{\mathrm{T}} \bar{x} \quad \text{s.t.} \quad P\bar{x} = \bar{d} \text{ and } \bar{x} \in \mathbb{L}^n$$

with $\bar{d} \in \operatorname{range} \bar{A}^{\mathrm{T}}$ and $\bar{c} \in \operatorname{null} \bar{A}$

Testing whether γ is a feasible objective value:

$$\text{Find } \bar{x} \in \mathbb{R} \times \mathbb{R}^n \quad \text{s.t.} \quad P\bar{x} = \bar{d},\ \bar{c}^{\mathrm{T}} \bar{x} = \gamma \text{ and } \bar{x} \in \mathbb{L}^n$$

Data of the original feasibility problem *becomes*

$$P \rightarrow P + \frac{\bar{c}\bar{c}^{\mathrm{T}}}{\|\bar{c}\|^2}, \quad \lambda \rightarrow \lambda + \frac{c_0^2}{\|\bar{c}\|^2}$$

$$d_0 \rightarrow d_0 + \gamma \frac{c_0}{\|\bar{c}\|^2}, \quad \|\bar{d}\|^2 \rightarrow \|\bar{d}\|^2 + \frac{\gamma^2}{\|\bar{c}\|^2}$$

Special case: $c_0 = 0$

In this case, one can simply read the values from the feasibility table

	$\lambda < 0$	$\lambda = 0$	$\lambda > 0$	
$d_0 < 0$	s.f.	s.i.	s.i.	
$d_0 = 0$	s.f.	w.i.	s.i.	
$d_0 > 0$	s.f.	s.f.	*s.f., w.f., s.i.*	$(\lambda(\left\|\bar{d}\right\|^2 + \frac{\gamma^2}{\|\bar{c}\|^2}) <, =, > d_0^2)$

Observations

s.i. $\rightarrow$ s.i, s.f. and w.f. $\rightarrow$ problem unbounded from above and below

w.i. $\rightarrow$ problem asymptotically unbounded from above and below

Exception: Case $(d_0 > 0, \lambda > 0)$: feasible γ satisfy

$$\left\|\bar{d}\right\|^2 + \frac{\gamma^2}{\|\bar{c}\|^2} \leq \frac{d_0^2}{\lambda}$$

$\Rightarrow \gamma^2 = \|\bar{c}\|^2 (d_0^2 - \lambda \left\|\bar{d}\right\|^2)/\lambda$ defines min and max

General case: $c_0 \neq 0$

Feasibility table becomes

	$\lambda + \frac{c_0^2}{\|\bar{c}\|^2} < 0$	$\lambda + \frac{c_0^2}{\|\bar{c}\|^2} = 0$	$\lambda + \frac{c_0^2}{\|\bar{c}\|^2} > 0$	
$d_0 + \gamma \frac{c_0}{\|\bar{c}\|^2} < 0$	s.f.	s.i.	s.i.	
$d_0 + \gamma \frac{c_0}{\|\bar{c}\|^2} = 0$	s.f.	w.i.	s.i.	
$d_0 + \gamma \frac{c_0}{\|\bar{c}\|^2} > 0$	s.f.	s.f.	*s.f., w.f., s.i.*	(*)

Condition (*) is

$$(\lambda + \frac{c_0^2}{\|\bar{c}\|^2})(\|\bar{d}\|^2 + \frac{\gamma^2}{\|\bar{c}\|^2}) <, =, > (d_0 + \gamma \frac{c_0}{\|\bar{c}\|^2})^2$$

Observations

First column: *unbounded* problems

Second column: problem is not *attained*

General case: $c_0 \neq 0$ (cont.)

	$\lambda + \frac{c_0^2}{\|\bar{c}\|^2} < 0$	$\lambda + \frac{c_0^2}{\|\bar{c}\|^2} = 0$	$\lambda + \frac{c_0^2}{\|\bar{c}\|^2} > 0$	
$d_0 + \gamma \frac{c_0}{\|\bar{c}\|^2} < 0$	s.f.	s.i.	s.i.	
$d_0 + \gamma \frac{c_0}{\|\bar{c}\|^2} = 0$	s.f.	w.i.	s.i.	
$d_0 + \gamma \frac{c_0}{\|\bar{c}\|^2} > 0$	s.f.	s.f.	*s.f., w.f., s.i.*	(*)

Observations (cont.)

Second column: *infimum* and *supremum* for problem are not *attained*

- $c_0 > 0$: infimum at $-\frac{d_0 \|\bar{c}\|^2}{c_0}$, unbounded from above
- $c_0 < 0$: supremum at $-\frac{d_0 \|\bar{c}\|^2}{c_0}$, unbounded from below

Third column: need to look at condition (*):

equality case $\Leftrightarrow$ w.f. $\Leftrightarrow$ *attained* min and max

General case: $c_0 \neq 0$ (cont.)

When $\lambda + \frac{c_0^2}{\|\bar{c}\|^2} > 0$, one has to solve

$$(\lambda + \frac{c_0^2}{\|\bar{c}\|^2})(\left\|\bar{d}\right\|^2 + \frac{\gamma^2}{\|\bar{c}\|^2}) = (d_0 + \gamma \frac{c_0}{\|\bar{c}\|^2})^2$$

Quadratic equation whose solutions are given by

$$\gamma = \frac{c_0 d_0 \pm \sqrt{(d_0^2 - \lambda \left\|\bar{d}\right\|^2)(c_0^2 + \lambda \left\|\bar{c}\right\|^2)}}{\lambda}$$

which define min and max for the problem.

Generalizes correctly previous special cases

Direction of vectors $\bar{c}$ and $\bar{d}$ is *irrelevant*!

Duality

Primal problem

$$p^* = \inf_{\bar{x} \in \mathbb{R} \times \mathbb{R}^n} \bar{c}^{\mathrm{T}} \bar{x} \quad \text{s.t.} \quad P\bar{x} = \bar{d} \text{ and } \bar{x} \in \mathbb{L}^n$$

with $\bar{d} \in \operatorname{range} \bar{A}^{\mathrm{T}}$ and $\bar{c} \in \operatorname{null} \bar{A}$

Dual problem

$$d^* = \inf_{\bar{z} \in \mathbb{R} \times \mathbb{R}^n} \bar{d}^{\mathrm{T}} \bar{z} \quad \text{s.t.} \quad Q\bar{z} = \bar{c} \text{ and } \bar{z} \in \mathbb{L}^n$$

with $Q = I - P$ (orthogonal projection on $\operatorname{null} \bar{A}$)

Properties

$p^* + d^* \geq 0$ (weak duality)

$p^* + d^* = 0$ (strong duality) under Slater condition

Primal and dual share the **same** format

Solving the dual problem

Reuse table for the primal while exchanging

$$\bar{c} \leftrightarrow \bar{d},\ P \leftrightarrow Q \text{ and } \lambda \leftrightarrow -\lambda$$

In particular, formula for min and max simply *changes its sign*

$$\gamma = \frac{c_0 d_0 \pm \sqrt{(d_0^2 - \lambda \left\|\bar{d}\right\|^2)(c_0^2 + \lambda \left\|\bar{c}\right\|^2)}}{\lambda}$$

$$\rightarrow \quad \gamma = \frac{d_0 c_0 \pm \sqrt{(c_0^2 + \lambda \left\|\bar{c}\right\|^2)(d_0^2 - \lambda \left\|\bar{d}\right\|^2)}}{-\lambda}$$

$\Rightarrow$ *duality gap* is *equal to* **zero**

Conclusions

Summary and perspectives

- Conic problems with a single second-order cone constraint can be solved analytically

  Bad cases (when strong duality fails) are detected and handled

- Interior-point method complexity for this problem is $\mathcal{O}\left(n^3 \log \frac{1}{\varepsilon}\right)$

  this procedure is $\mathcal{O}\left(m^2 n\right)$ $\rightarrow$ dependence on accuracy removed

- Application of the same technique to *dual* problem (study of *gap*)

- Possible *generalizations*

  – Allow additional *free* variables $\rightarrow$ still possible

  – Allow linear *inequality* constraints $\rightarrow$ quadratic programming

  – Use of this result for several second-order cones (*subproblem*?)