Topics in Convex Optimization: Interior-Point Methods, Conic Duality and Approximations

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Ph.D. dissertation
January 26, 2001

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Motivation

Operations research

Model real-life situations to help take the *best* decisions

$$\left.\begin{array}{rcl} \text{Decision} & \leftrightarrow & \text{vector of variables} \\ \text{Best} & \leftrightarrow & \text{objective function} \\ \text{Constraints} & \leftrightarrow & \text{feasible set} \end{array}\right\} \Rightarrow \text{Optimization}$$

General formulation

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad x \in \mathcal{D} \subseteq \mathbb{R}^n$$

Choice of design parameters, scheduling, planification

Two approaches

Solving all problems *efficiently* is impossible in practice!

Simple problem: $\min f(x_1, x_2, \ldots, x_{10})$
$\Rightarrow 10^{20}$ operations to be solved with 1% accuracy !

Reaction: two distinct orientations

- ◇ General nonlinear optimization
  Applicable to all problems but no efficiency guarantee
- ◇ Linear, quadratic, semidefinite, ... optimization
  Restrict set of problems to get efficiency guarantee

Tradeoff generality $\leftrightarrow$ efficiency (algorithmic complexity)

Restrict to which class of problems ?

Linear optimization : + specialized, very fast algorithms
− too restricted in practice

→ we focus on **Convex optimization**

- ◇ Convex objective and convex feasible set
- ◇ Many problems are convex or can be convexified
- ◇ Efficient algorithms and powerful duality theory
- ◇ Establishing convexity *a priori* is difficult

  → work with specific classes of convex constraints: *Structured* convex optimization (convexity by design)

Reward for a convex formulation is algorithmic efficiency

Overview of the thesis

Interior-point methods

- Linear optimization survey
- Self-concordant functions

Conic optimization

- Formulation and duality
- Geometric and l_p-norm optimization
- General framework: separable optimization

Approximations

- Geometric optimization with l_p-norm optimization
- Linearizing second-order cone optimization

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Applications

- *Classification* with ellipsoids and conic optimization
- *Isotopic dating* with geometric optimization

Self-concordant functions: the key to efficient algorithms for convex optimization
(chapter 2)

Interior-point methods

◇ Self-concordant functions

Conic optimization

◇ Formulation and duality

◇ Geometric optimization

◇ General framework: separable optimization

Applications

◇ Classification with ellipsoids and conic optimization

◇ Isotopic dating with geometric optimization

Convex optimization

Let $f_0 : \mathbb{R}^n \mapsto \mathbb{R}$ be a convex function, $C \subseteq \mathbb{R}^n$ be a convex set : optimize a vector $x \in \mathbb{R}^n$

$$\inf_{x \in \mathbb{R}^n} f_0(x) \quad \text{s.t.} \quad x \in C \tag{P}$$

Properties

- ◇ All local optima are global, optimal set is convex
- ◇ Lagrange duality $\rightarrow$ strongly related dual problem
- ◇ Objective can be taken linear w.l.o.g. $(f_0(x) = c^T x)$

Defining a problem

Two distinct approaches

a. **List of convex constraints.**

m convex functions $f_i : \mathbb{R}^n \mapsto \mathbb{R}$, $i = 1, 2, \ldots, m$

$$C = \{x \in \mathbb{R}^n \mid f_i(x) \leq 0 \text{ for all } i = 1, 2, \ldots, m\}$$

(intersection of convex level sets)

$$\inf_{x \in \mathbb{R}^n} f_0(x) \quad \text{s.t.} \quad f_i(x) \leq 0 \text{ for all } i = 1, 2, \ldots, m$$

b. **Use a barrier function.**

Feasible set $\equiv$ domain of a *barrier* function F s.t.

- $\diamond$ F is smooth
- $\diamond$ F is strongly convex $\operatorname{int} C$
- $\diamond$ $F(x) \to +\infty$ when $x \to \partial C$

$$\to \quad C = \operatorname{cl} \operatorname{dom} F = \operatorname{cl} \{x \in \mathbb{R}^n \mid F(x) < +\infty\}$$

Interior-point methods

Principle

Approximate a constrained problem by a *family* of unconstrained problems based on F

Let $\mu \in \mathbb{R}_{++}$ be a parameter and consider

$$\inf_{x \in \mathbb{R}^n} \frac{c^T x}{\mu} + F(x) \qquad (\mathrm{P}_\mu)$$

We have

$$x^*_\mu \rightarrow x^* \text{ when } \mu \searrow 0$$

where

- x^*_μ is the (unique) solution of (P_μ) ($\rightarrow$ central path)
- x^* is a solution of the original problem (P)

Ingredients

- ◇ A method for unconstrained optimization
- ◇ A barrier function

Interior-point methods rely on

- ◇ *Newton's method* to compute x^*_μ
- ◇ When C is defined with nonlinear functions f_i, one can introduce the *logarithmic* barrier function

$$F(x) = -\sum_{i=1}^n \ln(-f_i(x))$$

Question: What is a good barrier, i.e. a barrier for which Newton's method is efficient ?

Answer: A *self-concordant* barrier

Self-concordant barriers

Definition ***[Nesterov & Nemirovsky, 1988]***

$F : \operatorname{int} C \mapsto \mathbb{R}$ is called (κ, ν)-self-concordant on C iff

- ◇ F is convex
- ◇ F is three times differentiable
- ◇ $F(x) \to +\infty$ when $x \to \partial C$
- ◇ the following two conditions hold

$$\nabla^3 F(x)[h,h,h] \leq 2\kappa \left(\nabla^2 F(x)[h,h]\right)^{\frac{3}{2}}$$
$$\nabla F(x)^T (\nabla^2 F(x))^{-1} \nabla F(x) \leq \nu$$

for all $x \in \operatorname{int} C$ and $h \in \mathbb{R}^n$

Complexity result

Summary

Self-concordant barrier ⇒ polynomial number of iterations to solve (P) within a given accuracy

Principle of a short-step method

- ◇ Define a *proximity measure* $\delta(x, \mu)$ to central path
- ◇ Choose a starting iterate with a small $\delta(x_0, \mu_0)$
- ◇ While accuracy is not attained

  a. Decrease μ geometrically (δ increases)

  b. Take a Newton step to minimize barrier (δ decreases and is restored)

Geometric interpretation

Two self-concordancy conditions: each has its role

- ◇ Second condition bounds the size of the Newton step $\Rightarrow$ controls the increase of the proximity measure when μ is updated
- ◇ First condition bounds the variation of the Hessian $\Rightarrow$ guarantees that the Newton step restores the initial proximity to the central path

Complexity result

$$\mathcal{O}\left(\kappa\sqrt{\nu}\log\frac{1}{\epsilon}\right)$$

iterations lead a solution with ϵ accuracy on the objective

Optimal complexity result *[Glineur 00]*

Optimal values for two constants

- ◇ (maximum) proximity δ to the central path
- ◇ Constant of decrease of barrier parameter μ

lead to

$$\left\lceil (1.03 + 7.15\kappa\sqrt{\nu}) \log \frac{1.29\mu_0\kappa\sqrt{\nu}}{\epsilon} \right\rceil$$

iterations for a solution with ϵ accuracy

A useful lemma

Proving self-concordancy not always an easy task
$\Rightarrow$ improved version of lemma by *[Den Hertog et al.]*

Auxiliary functions

Let two increasing functions (see Figure 1)

$$r_1 : \mathbb{R} \mapsto \mathbb{R} : \gamma \mapsto \max\{1, \frac{\gamma}{\sqrt{3 - 2/\gamma}}\}$$

$$r_2 : \mathbb{R} \mapsto \mathbb{R} : \gamma \mapsto \max\{1, \frac{\gamma + 1 + 1/\gamma}{\sqrt{3 + 4/\gamma + 2/\gamma^2}}\}$$

We have $r_1(\gamma) \approx \frac{\gamma}{\sqrt{3}}$ and $r_2(\gamma) \approx \frac{\gamma+1}{\sqrt{3}}$ when $\gamma \to +\infty$.

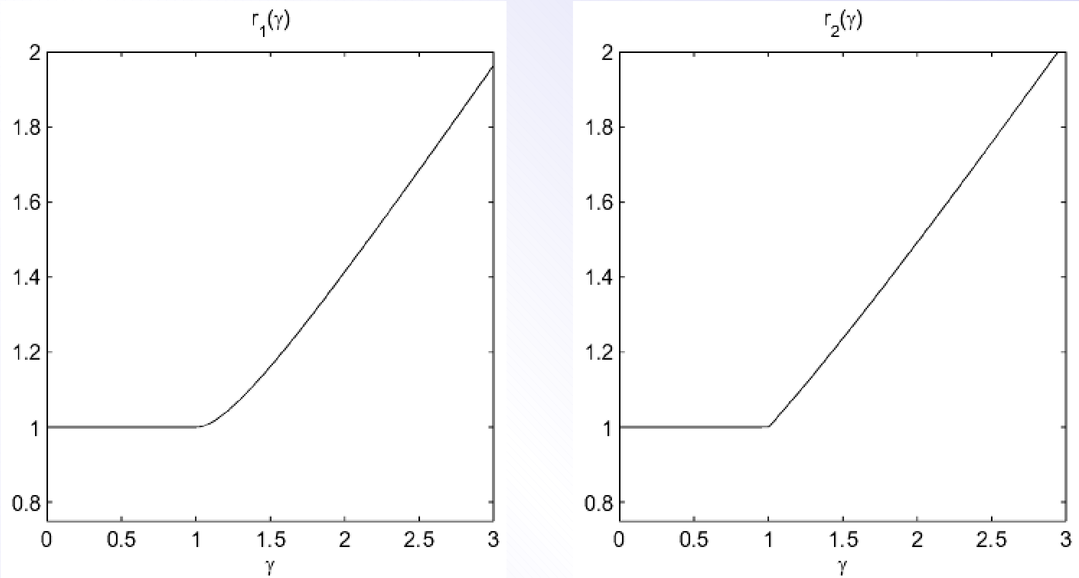

Figure 1: Graphs of functions r_1 and r_2

Lemma's statement *[Glineur 00]*

Let $F : \mathbb{R}^n \mapsto \mathbb{R}$ be a convex function on C.

If there is a constant $\gamma \in \mathbb{R}_+$ such that

$$\nabla^3 F(x)[h,h,h] \leq 3\gamma \nabla^2 F(x)[h,h] \sqrt{\sum_{i=1}^{n} \frac{h_i^2}{x_i^2}}$$

then the following barrier functions

$$F_1 : \mathbb{R}^n \mapsto \mathbb{R} : x \mapsto F(x) - \sum_{i=1}^{n} \ln x_i$$

$$F_2 : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R} : (x,u) \mapsto -\ln(u - F(x)) - \sum_{i=1}^{n} \ln x_i$$

satisfy the first self-concordancy condition with

$$\kappa_1 = r_1(\gamma) \quad \text{for } F_1 \text{ on } C$$

$$\kappa_2 = r_2(\gamma) \quad \text{for } F_2 \text{ on } \operatorname{epi} F = \{(x,u) \mid F(x) \leq u\}$$

Conic optimization: an elegant framework to formulate convex problems and study their duality properties

(chapter 3)

Interior-point methods

- ◇ Self-concordant functions

Conic optimization

- ◇ Formulation and duality
- ◇ Geometric optimization
- ◇ General framework: separable optimization

Applications

- ◇ Classification with ellipsoids and conic optimization
- ◇ Isotopic dating with geometric optimization

Conic formulation

Primal problem

Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a convex cone

$$\inf_{x \in \mathbb{R}^n} c^T x \quad \text{s.t.} \quad Ax = b \text{ and } x \in \mathcal{C}$$

Formulation is equivalent to convex optimization.

Dual problem

Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a *solid, pointed, closed* convex cone.
The dual cone $\mathcal{C}^* = \{x^* \in \mathbb{R}^n \mid x^T x^* \geq 0 \text{ for all } x \in \mathcal{C}\}$ is also convex, solid, pointed and closed $\rightarrow$ dual problem:

$$\sup_{(y,s) \in \mathbb{R}^{m+n}} b^T y \quad \text{s.t.} \quad A^T y + s = c \text{ and } s \in \mathcal{C}^*$$

Primal-dual pair

Symmetrical pair of primal-dual problems

$$p^* = \inf_{x \in \mathbb{R}^n} c^T x \text{ s.t. } Ax = b \text{ and } x \in \mathcal{C}$$

$$d^* = \sup_{(y,s) \in \mathbb{R}^{m+n}} b^T y \text{ s.t. } A^T y + s = c \text{ and } s \in \mathcal{C}^*$$

Optimum values p^* and d^* not necessarily attained !

Examples: $\mathcal{C} = \mathbb{R}^n_+ = \mathcal{C}^* \Rightarrow$ linear optimization,
$\mathcal{C} = \mathbb{S}^n_+ = \mathcal{C}^* \Rightarrow$ semidefinite optimization (self-duality)

Advantages over classical formulation

- ◇ Remarkable primal-dual symmetry
- ◇ Special handling of (*easy*) linear equality constraints

Weak duality

For every feasible x and y $\quad b^T y \leq c^T x$
with equality iff $x^T s = 0$ (*orthogonality* condition)

$\Delta = p^* - d^*$ is the *duality gap* $\Rightarrow$ always nonnegative

Definition: x *strictly feasible* $\Leftrightarrow$ x feasible and $x \in \operatorname{int} \mathcal{C}$

Strong duality (with Slater condition)

a. Strictly feasible dual point $\Rightarrow p^* = d^*$

b. If in addition primal is bounded
$\Rightarrow$ primal optimum is attained $\Leftrightarrow p^* = \min c^T x$

(dualized result obviously holds)

An application to classification
Pattern separation using ellipsoids and semidefinite optimization

(appendix)

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Pattern separation

Problem definition

Let us consider *objects* defined by *patterns*
Object $\equiv$ Pattern $\equiv$ Vector of n attributes

Assume it is possible to group these objects into c classes

Objective

Find a partition of $\mathbb{R}^n$ into c disjoint components such that each component corresponds to one class

Utility: classification

$\Rightarrow$ identify to which class an unknown pattern belongs

Classification

Consider

- ◇ Some well-known objects grouped into classes
- ◇ Some unknown objects

Two-step procedure

a. Separate the patterns of well-known objects
   $\equiv$ *learning* phase

b. Use that partition to classify the unknown objects
   $\equiv$ *generalization* phase

Examples

Medical diagnosis, species identification, credit approval

Our technique

W.l.o.g. consider two classes

Main idea

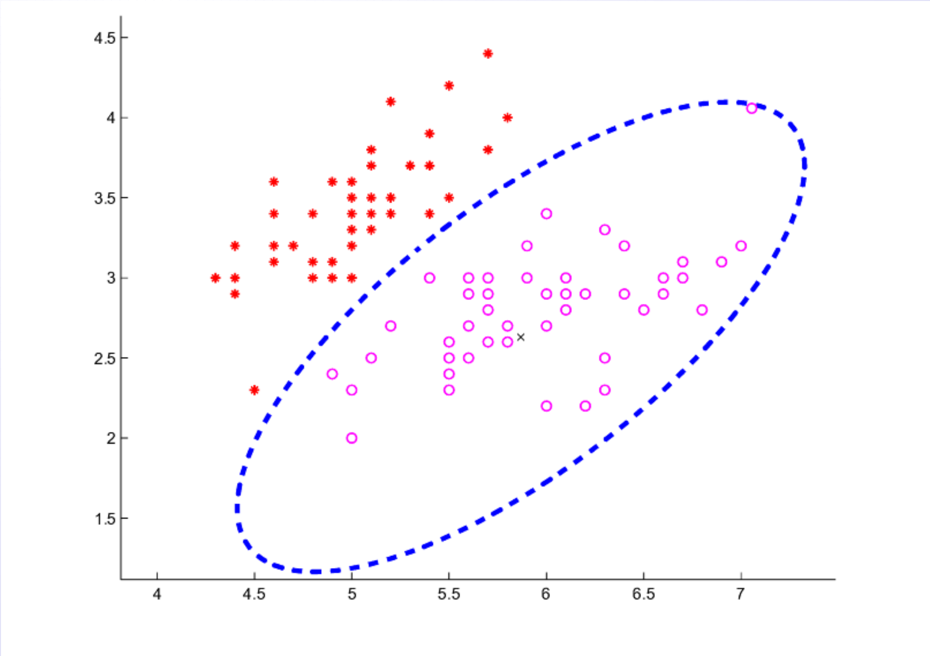
Use ellipsoids to separate the patterns

Ellipsoid

An ellipsoid $\mathcal{E} \subseteq \mathbb{R}^n \equiv$ a center $c \in \mathbb{R}^n$ and a positive semidefinite matrix $E \in \mathbb{S}^n_+$

$$\mathcal{E} = \{x \in \mathbb{R}^n \mid (x - c)^T E(x - c) \leq 1\}$$

But which ellipsoid performs the best separation ?



4.5
4
3.5
3
2.5
2
1.5
4
4.5
5
5.5
6
6.5
7

Separation ratio

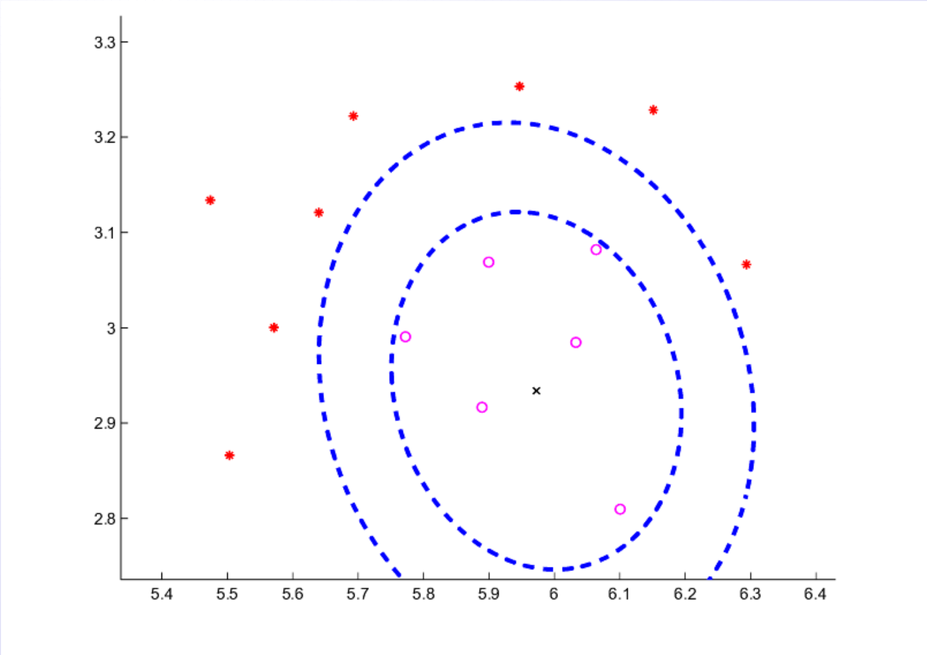
We want the best possible separation
$\Rightarrow$ define and maximize the *separation ratio*

Definition

Pair of homothetic ellipsoids sharing the same center
Separation ratio $\rho \equiv$ ratio of sizes

Mathematical formulation

$$\max \rho \quad \text{s.t.} \quad \begin{cases} (a_i - c)^T E (a_i - c) \leq 1 \ \forall i \\ (b_j - c)^T E (b_j - c) \geq \rho^2 \ \forall j \\ E \in \mathbb{S}^n_+ \end{cases}$$


3.3
3.2
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3
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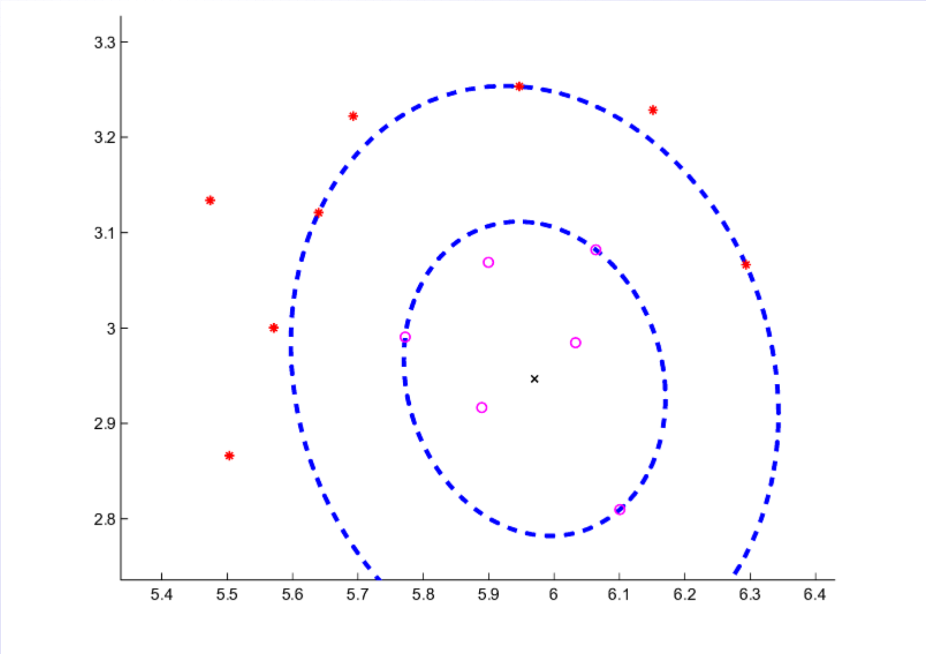

3.3
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6.2
6.3
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Analysis

This problem is not convex but can be convexified (*homogenizing* the description of the ellipsoid)

$\Rightarrow$ we obtain a semidefinite optimization problem
$\equiv$ conic optimization with $\mathcal{C} = \mathbb{S}^n_+$

A general semidefinite optimization problem

$$p^* = \inf_{X \in \mathbb{S}^n} C \bullet X \text{ s.t. } \mathcal{A}X = b \text{ and } X \in \mathbb{S}^n_+$$
$$d^* = \sup_{(y,S) \in \mathbb{R}^m \times \mathbb{S}^n} b^T y \text{ s.t. } \mathcal{A}^T y + S = C \text{ and } S \in \mathbb{S}^n_+$$

$\Rightarrow$ efficiently solvable in practice with interior-point method

Numerical experiments

◇ Implementation using MATLAB

◇ Test on sets from the *Repository of Machine Learning Databases and Domain Theories* maintained by the *University of California* at Irvine (widely used)

◇ Cross-validation
divide data set into *learning* and *validation* set

a. Compute best separating ellipsoid on learning set

b. Evaluate accuracy of separating ellipsoid on validation set (generalization capability)

Data sets

Three representative sets

a. **Wisconsin Breast Cancer**. Predict the benign or malignant nature of a breast tumor (683 patterns, 9 characteristics)

b. **Boston Housing**. Predict whether a housing value is above or below the median (596 patterns, 12 characteristics)

c. **Pima Indians Diabetes**. Predict whether a patient is showing signs of diabetes (768 patterns, 8 characteristics)

Comparison

	Best ellipsoid		LAD	Best other
Training	20 %	50 %	50 %	Variable (% tr.)
Cancer	5.1 %	4.2 %	**3.1** %	3.8 % (80 %)
Housing	15.8 %	**12.4** %	16.0 %	16.8 % (80 %)
Diabetes	28.5 %	28.9 %	28.1 %	**24.1** % (75 %)

- Competitive error rates
- Best results on the Housing problem (even 20 %)
- 50 % not always better than 20 % ($\Rightarrow$ overlearning)
- Results with small learning set already acceptable

A conic formulation for a well-known class of problems: geometric optimization

(chapter 5)

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Our approach

- ◇ Duality for general convex optimization weaker than for linear optimization (need Slater condition)
- ◇ But some classes of structured convex optimization problems feature better duality properties (i.e. zero duality gap even without Slater condition)

Our goal

- ◇ Prove these duality properties using general theorems for conic optimization
- ◇ $\Rightarrow$ Define new dedicated convex cones

Geometric optimization

Posynomials

Let $K = \{0, 1, 2, \ldots, r\}$, $I = \{1, 2, \ldots, n\}$;
let $\{I_k\}_{k \in K}$ a partition of I into $r + 1$ classes.
A *posynomial* is a sum of positive monomials

$$G_k : \mathbb{R}^m_{++} \mapsto \mathbb{R}_{++} : t \mapsto \sum_{i \in I_k} C_i \prod_{j=1}^{m} t_j^{a_{ij}}$$

defined by data $a_{ij} \in \mathbb{R}$ and $C_i \in \mathbb{R}_{++}$

Example: $G(t_1, t_2, t_3) = 2\frac{t_1^2}{t_2} + 3\sqrt{t_2} + \frac{1}{3}\frac{t_2^{2/3}}{t_1 t_3^3}$

Many applications, especially in engineering
(optimizing design parameters, modelling power laws)

Primal problem

Optimize m variables in vector $t \in \mathbb{R}^m_{++}$

$$\inf \; G_0(t) \quad \text{s.t.} \quad G_k(t) \leq 1 \;\; \forall k \in K$$

Not convex: take $G_0(t) = \sqrt{t_1}$

Convexification

W.l.o.g. consider a linear objective and let

$$t_j = e^{y_j} \text{ for all } j \in \{1, 2, \ldots, m\}$$

$\Rightarrow$ we let

$$g_k : \mathbb{R}^m \mapsto \mathbb{R}_{++} : y \mapsto \sum_{i \in I_k} e^{a_i^T y - c_i}$$

with $c_i = -\log C_i \quad \Rightarrow \quad$ equivalence $g_k(y) = G_k(t)$

Convexified primal

Free variables $y \in \mathbb{R}^m$, data $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$

$$\sup b^T y \quad \text{s.t.} \quad g_k(y) \leq 1 \text{ for all } k \in K$$

(Lagrangean) dual

$$\begin{aligned} \inf \quad & c^T x + \sum_{k \in K} \sum_{\substack{i \in I_k \\ x_i > 0}} x_i \log \frac{x_i}{\sum_{i \in I_k} x_i} \\ \text{s.t.} \quad & Ax = b \text{ and } x \geq 0 \end{aligned}$$

Properties *[Duffin, Peterson and Zener, 1967]*

◇ Convex problem $\Rightarrow$ weak duality
◇ No duality gap !

The geometric cone

Definition ***[Glineur 99]***

Let $n \in \mathbb{N}$. Define $\mathcal{G}^n$ as

$$\mathcal{G}^n = \left\{ (x, \theta) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \mid \sum_{i=1}^{n} e^{-\frac{x_i}{\theta}} \leq 1 \right\}$$

with the convention $e^{-\frac{x_i}{0}} = 0$

Our goal: express geometric optimization in a conic form

Properties

- Special cases: $\mathcal{G}^0 = \mathbb{R}_+$ and $\mathcal{G}^1 = \mathbb{R}^2_+$
- $(x, \theta) \in \mathcal{G}^n$, $(x', \theta') \in \mathcal{G}^n$ and $\lambda \geq 0$
  $\Rightarrow \quad \lambda(x, \theta) \in \mathcal{G}^n \quad$ and $\quad (x + x', \theta + \theta') \in \mathcal{G}^n$
  $\Rightarrow \mathcal{G}^n$ is a *convex cone.*
- $\mathcal{G}^n$ is closed, solid and pointed
- The interior of $\mathcal{G}^n$ is ($\longrightarrow$ Slater condition)

$$\operatorname{int} \mathcal{G}^n = \left\{ (x, \theta) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++} \mid \sum_{i=1}^{n} e^{-\frac{x_i}{\theta}} < 1 \right\}$$

Dual cone

The dual cone $(\mathcal{G}^n)^*$ is given by

$$\left\{ (x^*, \theta^*) \in \mathbb{R}^n_+ \times \mathbb{R} \mid \theta^* \geq \sum_{x_i^* > 0} x_i^* \log \frac{x_i^*}{\sum_{i=1}^n x_i^*} \right\}$$

It is the epigraph of

$$f_n : \mathbb{R}^n_+ \mapsto \mathbb{R} : x \mapsto \sum_{x_i^* > 0} x_i^* \log \frac{x_i^*}{\sum_{i=1}^n x_i^*}$$

- ◇ Special cases: $(\mathcal{G}^0)^* = \mathbb{R}_+$ and $(\mathcal{G}^1)^* = \mathbb{R}^2_+$ (but $\mathcal{G}^n$ is not self-dual for $n > 1$)
- ◇ It is also convex, closed, solid and pointed.
- ◇ $((\mathcal{G}^n)^*)^* = \mathcal{G}^n$ (since $\mathcal{G}^n$ is closed).

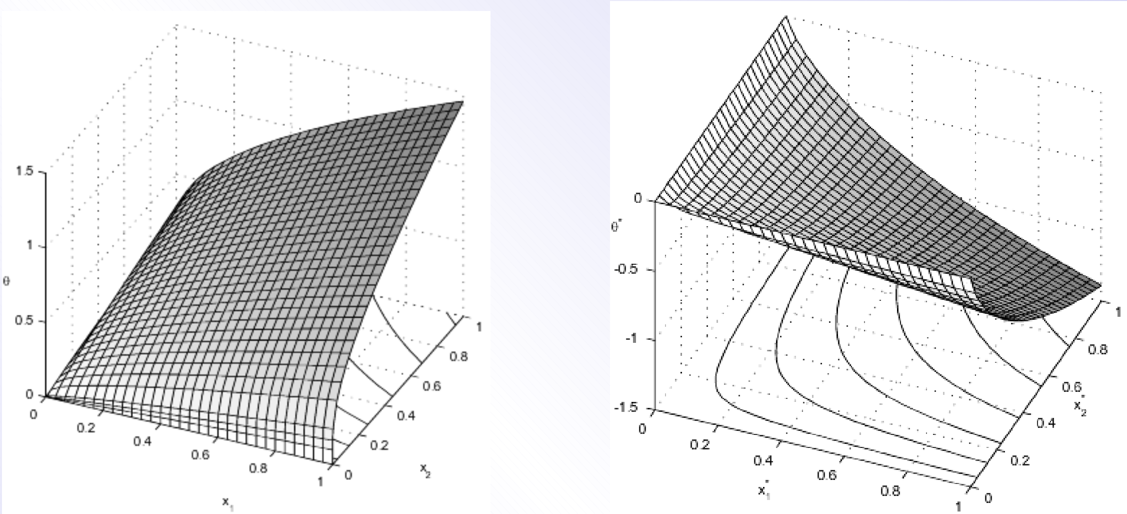

Figure 2: Boundary surfaces of the geometric cone $\mathcal{G}^2$ and its dual cone $(\mathcal{G}^2)^*$

◇ $\mathbb{R}_+^{n+1} \subseteq (\mathcal{G}^n)^*$ (since $\mathcal{G}^n \subseteq \mathbb{R}_+^{n+1}$)

◇ The interior of $(\mathcal{G}^n)^*$ is given by

$$\left\{ (x^*, \theta^*) \in \mathbb{R}_{++}^n \times \mathbb{R} \mid \theta^* > \sum_{i=1}^n x_i^* \log \frac{x_i^*}{\sum_{i=1}^n x_i^*} \right\}$$

Apply the general duality theory for conic primal-dual pairs, using our dual cones $\mathcal{G}^n$ and $(\mathcal{G}^n)^*$, to derive the duality properties of the geometric optimization primal-dual pair

Strategy diagram

$$
\begin{array}{ccccccc}
(PG) & \equiv & (CPG) & \stackrel{\text{Weak}}{\longleftrightarrow} & (CDG) & \equiv & (DG) \\
 & & \updownarrow^{*} & & \updownarrow & & \\
 & & (RPG) & \stackrel{\text{Strong}}{\longleftrightarrow} & (RDG) & & \\
 & & & & \uparrow & & \\
 & & & & \text{(Slater)} & &
\end{array}
$$

Formulation with $\mathcal{G}^n$ cone

Primal

$$\sup b^T y \quad \text{s.t.} \quad g_k(y) \leq 1 \text{ for all } k \in K$$

Introducing new variables $s_i = c_i - a_i^T y \ \forall i$ we get

$$\sup b^T y \quad \text{s.t.} \quad s = c - A^T y$$
$$\text{and} \quad \sum_{i \in I_k} e^{-s_i} \leq 1 \text{ for all } k \in K$$

$\Updownarrow$ (introducing *additional* v variables)

$$\sup b^T y \quad \text{s.t.} \quad \begin{pmatrix} A^T \\ 0 \end{pmatrix} y + \begin{pmatrix} s \\ v \end{pmatrix} = \begin{pmatrix} c \\ e \end{pmatrix}$$
$$\text{and} \quad (s_{I_k}, v_k) \in \mathcal{G}^{n_k} \text{ for all } k \in K$$

($e \equiv$ all-one vector, $n_k = \#I_k$)
This is a standard conic problem !
variables $(\tilde{y}, \tilde{s})$, data$(\tilde{A}, \tilde{b}, \tilde{c})$, cone K^* with

$$\tilde{y} = y, \ \tilde{s} = \begin{pmatrix} s \\ v \end{pmatrix}, \ \tilde{A} = \begin{pmatrix} A & 0 \end{pmatrix}, \ \tilde{b} = b,$$

$$\tilde{c} = \begin{pmatrix} c \\ e \end{pmatrix} \text{ and } K^* = \mathcal{G}^{n_1} \times \mathcal{G}^{n_2} \times \cdots \times \mathcal{G}^{n_r}$$

$\Rightarrow$ we can mechanically derive the dual !

$$\inf \begin{pmatrix} c \\ e \end{pmatrix}^T \begin{pmatrix} x \\ z \end{pmatrix} \quad \text{s.t.} \quad \begin{pmatrix} A & 0 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} = b$$
$$\text{and} \quad (x_{I_k}, z_k) \in (\mathcal{G}^{n_k})^* \ \forall k$$

$$\inf \begin{pmatrix} c \\ e \end{pmatrix}^T \begin{pmatrix} x \\ z \end{pmatrix} \quad \text{s.t.} \quad \begin{pmatrix} A & 0 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} = b$$

$$\text{and} \quad (x_{I_k}, z_k) \in (\mathcal{G}^{n_k})^* \ \forall k$$

$$\Leftrightarrow \inf c^T x + e^T z \quad \text{s.t.} \quad Ax = b,\ x_{I_k} \geq 0$$

$$\text{and} \quad z_k \geq \sum_{\substack{i \in I_k \\ x_i > 0}} x_i \log \frac{x_i}{\sum_{i \in I_k} x_i}$$

$$\Leftrightarrow \quad \inf \quad c^T x + \sum_{k \in K} \sum_{\substack{i \in I_k \\ x_i > 0}} x_i \log \frac{x_i}{\sum_{i \in I_k} x_i}$$

$$\text{s.t.} \quad Ax = b \text{ and } x \geq 0$$

Weak duality

y feasible for the primal, x is feasible for the dual

$$\Rightarrow \quad b^T y \leq c^T x + \sum_{k \in K} \sum_{\substack{i \in I_k \\ x_i > 0}} x_i \log \frac{x_i}{\sum_{i \in I_k} x_i} .$$

$$\Big(\sum_{i \in I_k} x_i\Big) e^{a_i^T y - c_i} = x_i \text{ for all } i \in I_k, k \in K$$

Proof *[Glineur 99]*

Weak duality theorem with conic primal-dual pair $\longrightarrow$ extend objective values to geometric primal-dual pair

Strong duality

Primal and dual feasible solutions $\Rightarrow$ zero duality gap (but attainment not guaranteed)

Proof *[Glineur 99]*

Provide a strictly feasible dual point

$$\Leftrightarrow \quad z_k > \sum_{i \in I_k} x_i \log \frac{x_i}{\sum_{i \in I_k} x_i} \textbf{ and } x_i > 0 \ \forall i$$

But the linear constraints $Ax = b$ may force $x_i = 0$ (for some i) at every feasible solution !

$\Rightarrow$ detect these zero x_i components and form a restricted primal-dual pair without these variables (which had no influence on the objective/constraints anyway)

Detection with a linear problem

$$\min\ 0 \quad \text{s.t.} \quad Ax = b \quad \text{and} \quad x \geq 0$$

Define $\mathcal{N}$ = set of indices i such that x_i is identically zero on the feasible region and $\mathcal{B}$ the set of the other indices. $(\mathcal{B}, \mathcal{N})$ is the optimal partition of this linear problem (Goldman-Tucker theorem)

Strategy

Remove variables x_i for all $i \in \mathcal{N}$

a. restricted primal-dual conic pair

b. strictly feasible dual solution

c. zero duality gap

Diagram

$$
\begin{array}{ccccccc}
(PG) & \equiv & (CPG) & \overset{\text{Weak}}{\longleftrightarrow} & (CDG) & \equiv & (DG) \\
 & & \updownarrow^{*} & & \updownarrow & & \\
 & & (RPG) & \overset{\text{Strong}}{\longleftrightarrow} & (RDG) & & \\
 & & & & \uparrow & & \\
 & & & & \text{(Slater)} & &
\end{array}
$$

(some technicalities needed to prove * equivalence)

Conlusion

$\Rightarrow$ the original primal optimum objective value is equal to the original dual optimum objective value

Application
Isotopic dating using geometric optimization

Interior-point methods

◇ Self-concordant functions

Conic optimization

◇ Formulation and duality

◇ Geometric optimization

◇ General framework: separable optimization

Applications

◇ Classification with ellipsoids and conic optimization

◇ Isotopic dating with geometric optimization

Introduction

(based on discussion with geology department, FPMs)

Radioactive decay

$$^{238}U \rightarrow {}^{206}Pb + 8\alpha \text{ and } {}^{235}U \rightarrow {}^{207}Pb + 7\alpha$$

Constant disintegration probability $\Rightarrow$ exponential decay

$$N(t) = N_0 e^{-\lambda t} \text{ and half-life } T = \frac{\log 2}{\lambda}$$

- ◇ $\lambda_{238} = 1.551 10^{-10} \text{year}^{-1}$ and $\lambda_{235} = 9.849 10^{-10} \text{year}^{-1}$
- ◇ $\frac{^{238}U}{^{235}U} = 137.9$ for all material nowadays
- ◇ We are able to measure $\frac{^{206}Pb}{^{207}Pb}$ for a sample

$\Rightarrow$ This is enough to date the sample !

Analysis

Let $t = 0 \Leftrightarrow$ formation of sample

$$^{238}U(t) = {}^{238}U(0)e^{-\lambda_{238}t}$$

$$\Downarrow$$

$$^{206}Pb(t) = {}^{238}U(0) - {}^{238}U(t) = {}^{238}U(t)(e^{\lambda_{238}t} - 1)$$

$$\Downarrow$$

$$\frac{^{206}Pb(t)}{^{207}Pb(t)} = \frac{^{238}U(t)}{^{235}U(t)} \frac{(e^{\lambda_{238}t} - 1)}{(e^{\lambda_{235}t} - 1)}$$

But we know both $\frac{^{206}Pb(t)}{^{207}Pb(t)}$ and $\frac{^{238}U(t)}{^{235}U(t)}$ for $t =$ now

Solve $\quad \dfrac{e^{\lambda_{238}t} - 1}{e^{\lambda_{235}t} - 1} = \rho \quad$ with $\quad \rho = \dfrac{^{206}Pb(t)}{^{207}Pb(t)} \dfrac{^{235}U(t)}{^{238}U(t)}$

Geometric optimization formulation

a. Equality $\longrightarrow$ Inequality with objective

$$\max t \quad \text{s.t.} \quad \frac{e^{\lambda_{238} t} - 1}{e^{\lambda_{235} t} - 1} \geq \rho$$

b. $\longrightarrow$ exponential form

$$\max t \quad \text{s.t.} \quad e^{\lambda_{238} t} \geq \rho\, e^{\lambda_{235} t} + (1 - \rho)$$

c. $\longrightarrow$ posynomial form $\equiv$ geometric optimization

$$\min e^{-t} \quad \text{s.t.} \quad 1 \geq \rho\, e^{(\lambda_{235} - \lambda_{238}) t} + (1 - \rho) e^{-\lambda_{238} t}$$

Example

When $\rho = 1/9$, one finds $t = 783$ million years

A general framework for separable convex optimization: Generalizing our conic formulations

(chapters 6–7)

Interior-point methods

◇ Self-concordant functions

Conic optimization

◇ Formulation and duality

◇ Geometric optimization

◇ General framework: separable optimization

Applications

◇ Classification with ellipsoids and conic optimization

◇ Isotopic dating with geometric optimization

Generalizing our framework

Comparing cones

$$\mathcal{G}^n = \left\{ (x, \theta) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \mid \sum_{i=1}^n e^{-\frac{x_i}{\theta}} \leq 1 \right\}$$

$$\mathcal{L}^p = \left\{ (x, \theta, \kappa) \in \mathbb{R}^n \times \mathbb{R}^2_+ \mid \sum_{i=1}^n \frac{|x_i|^{p_i}}{p_i \theta^{p_i - 1}} \leq \kappa \right\}$$

Variants

$$\mathcal{G}^n_2 = \left\{ (x, \theta, \kappa) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \mid \theta \sum_{i=1}^n e^{-\frac{x_i}{\theta}} \leq \kappa \right\}$$

$$\mathcal{L}^p = \left\{ (x, \theta, \kappa) \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+ \mid \theta \sum_{i=1}^n \frac{1}{p_i} \left| \frac{x_i}{\theta} \right|^{p_i} \leq \kappa \right\}$$

The separable cone *[Glineur 00]*

Consider a set of n scalar closed proper convex functions

$$f_i : \mathbb{R} \mapsto \mathbb{R}$$

and let

$$\mathcal{K}^f = \mathrm{cl}\left\{ (x, \theta, \kappa) \in \mathbb{R}^n \times \mathbb{R}_{++} \times \mathbb{R} \mid \theta \sum_{i=1}^{n} f_i(\frac{x_i}{\theta}) \leq \kappa \right\}$$

- $\mathcal{K}^f$ generalizes $\mathcal{L}^p$ and $\mathcal{G}_2^n$
- $\mathcal{K}^f$ is a closed convex cone
- $\mathcal{K}^f$ is solid and pointed

◇ $(x, \theta, \kappa) \in \operatorname{int} \mathcal{K}^f$ iff

$$x_i \in \operatorname{int} \operatorname{dom} f_i \text{ and } \theta \sum_{i=1}^n f_i(\frac{x_i}{\theta}) < \kappa$$

◇ The dual of $(\mathcal{K}^f)^*$ is defined by

$$\Big\{(x^*, \theta^*, \kappa^*) \in \mathbb{R}^n \times \mathbb{R}_{++} \times \mathbb{R} \mid \kappa^* \sum_{i=1}^n f_i^*(-\frac{x_i^*}{\kappa^*}) \leq \theta^* \Big\}$$

using the conjugate functions

$$f_i^* : x^* \mapsto \sup_{x \in \mathbb{R}^n} \{x^T x^* - f_i(x)\}$$

(also closed, proper and convex)

Separable optimization *[Glineur 00]*

Primal

$$\sup\ b^T y \quad \text{s.t.} \quad \sum_{i \in I_k} f_i(c_i - a_i^T y) \leq d_k - f_k^T y \quad \forall k \in K$$

Dual

$$\inf\ \psi(x, z) = c^T x + d^T z + \sum_{k \in K | z_k > 0} z_k \sum_{i \in I_k} f_i^*\left(-\frac{x_i}{z_k}\right)$$

$$- \sum_{k \in K | z_k = 0} \inf_{x^*_{I_k} \in \mathrm{dom}\, f_{I_k}} x_{I_k}^T x^*_{I_k}$$

$$\text{s.t. } Ax + Fz = b \text{ and } z \geq 0 \ .$$

◇ Justification for conventions when $\theta = 0$

◇ Mix different types of constraints within problems

Some other examples

◇

$$f : x \mapsto \begin{cases} -\sqrt{a^2 - x^2} & \text{if } |x| \leq a \\ +\infty & \text{if } |x| > a \end{cases}$$

$$f^* : x^* \mapsto a\sqrt{1 + x^{*2}}$$

(square roots, circles and ellipses)

◇

$$f : x \mapsto \begin{cases} -\frac{1}{p}x^p & \text{if } x \geq 0 \\ +\infty & \text{if } x < 0 \end{cases} \quad 0 < p < 1$$

$$f^* : x^* \mapsto \begin{cases} -\frac{1}{q}(-x^*)^q & \text{if } x^* < 0 \\ +\infty & \text{if } x^* \geq 0 \end{cases} \quad -\infty < q < 0$$

(*CES* functions in production and consumer theory)

◇

$$f : x \mapsto \begin{cases} -\frac{1}{2} - \log x & \text{if } x > 0 \\ +\infty & \text{if } x \leq 0 \end{cases}$$

$$f^* : x^* \mapsto \begin{cases} -\frac{1}{2} - \log(-x^*) & \text{if } x^* < 0 \\ +\infty & \text{if } x^* \geq 0 \end{cases}$$

(with property that $f^*(x^*) = f(-x^*)$)

Conclusions
Summary and perspectives

Contributions

Interior-point methods

- ◇ Overview of self-concordancy theory
- ◇ Discussion over different definitions
- ◇ Optimal complexity of short-step method
- ◇ Improvement of useful Lemma

Approximations

- ◇ Approximation of geometric optimization with l_p-norm optimization

Conic optimization

- New convex cones to model
  a. geometric optimization
  b. l_p-norm optimization
- Simplified proofs of their duality properties
- New framework of separable optimization

Applications

- Classification using semidefinite optimization
- Isotopic dating using geometric optimization

Research directions

Interior-point methods

◇ Replace self-concordancy conditions
by single condition involving complexity $\kappa\sqrt{\nu}$

Conic optimization

◇ Duality properties of separable optimization

◇ Self-concordant barrier for separable optimization

◇ Implementation of interior-point methods

Thank you for your attention