Model of internal force

Let us derive a simple model for a force arising in a rigid bar when the end points of the bar are shifted in the space in different directions. At this moment the actual reason for the shifts is not important. However, we assume that the shifts are both small, so in all situations we want to get linear dependencies.

Let us consider a bar positioned in a d-dimensional real (column) vector space R^d. Of course, the only reasonable values are $d = 1, 2, 3$. However, in order to treat all cases simultaneously we use the abstract notation. In the space R^d we introduce the standard inner product of two vectors:

$$\langle x, y\rangle = \sum_{i=1}^{d} x^{(i)} y^{(i)}, \quad x, y \in R^d,$$

and the Euclidean norm

$$\|x\| = \langle x, x\rangle^{1/2} = \left[\sum_{i=1}^{d} \left(x^{(i)}\right)^2\right]^{1/2},$$

which we use in order to measure distances in R^d.

The initial parameters of our model are as follows:

- *Positions* of end points are given by two vectors p_1 and $p_2 \in R^d$.
- *Cross-section* of the bar S.
- *Density* of the material of the bar ρ.
- *Young modulus* of the material of the bar E.

From this parameters we can derive the following information:

- *Length* of the bar $L = \|p_1 - p_2\|$.
- *Volume* of the bar $V = LS$.
- *Mass of the bar* $\mu = \rho V$.

Let us assume that the end points are shifted:

$$p_1 \to p_1 + v_1, \quad p_2 \to p_2 + v_2.$$

In accordance to one-dimensional Hooke law the force f arising in the end point of the bar is defined as follows:

$$f = \frac{\Delta L}{L} ES = \frac{\Delta L}{L^2} EV = \frac{\Delta L}{\rho L^2} E\mu, \tag{1}$$

where ΔL is a elongation of the bar. Let us show how this law works in R^k. Assume that we are interested in a force g_{12} which acts on the first node along the bar in the direction of the second node. Of course, $g_{21} = -g_{12}$. Denote

$$\Delta = v_1 - v_2.$$

Then $\Delta L = \|p_1 - p_2 + \Delta\| - L$. Let us derive the vector form of the law (1). Clearly, the only vector equation compatible with (1) is

$$g_{12} = -(\|p_1 - p_2 + \Delta\| - L) \cdot \frac{p_1 - p_2 + \Delta}{\|p_1 - p_2 + \Delta\|} \cdot \frac{E\mu}{L^2}. \tag{2}$$

Let us make this dependence linear in Δ.

$$\|p_1 - p_2 + \Delta\| - L = \frac{\|p_1 - p_2 + \Delta\|^2 - L^2}{\|p_1 - p_2 + \Delta\| + L} = \frac{2\langle \Delta, p_1 - p_2\rangle + \|\Delta\|^2}{\|p_1 - p_2 + \Delta\| + L} \approx \frac{1}{L}\langle \Delta, p_1 - p_2\rangle,$$

$$\frac{p_1 - p_2 + \Delta}{\|p_1 - p_2 + \Delta\|} \approx \frac{1}{L}(p_1 - p_2).$$

Thus, we get the following expressions for the forces:

$$g_{12} = -\frac{E\mu}{L^4}(p_1 - p_2)\langle p_1 - p_2, v_1 - v_2\rangle.$$

It is convenient to introduce the following notation:

$$A_{12} = \frac{E}{L^2} \cdot \frac{(p_1 - p_2)(p_1 - p_2)^T}{\|p_1 - p_2\|^2}.$$

Note that $A_{12} = A_{21}$. Then the forces arising in the nodes p_1 and p_2 are as follows:

$$\begin{aligned} g_{12} &= \mu A_{12} v_2 - \mu A_{12} v_1, \\ g_{21} &= \mu A_{21} v_1 - \mu A_{21} v_2. \end{aligned} \tag{3}$$