Rigidity for lattices in Lie groups

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0. Introduction

An important feature of the theory of discrete subgroups of Lie groups is the interplay between properties of the ambient group and properties of one (some) of its discrete subgroups.
This interplay takes many different forms, and here we will be concerned with the so-called rigidity phenomena.

Let us start with the simplest example: $\mathbb{Z} \leq (\mathbb{R},+)$.
Even though $\mathbb{Z}$ is quite small compared to $\mathbb{R}$, it turns out any continuous (locally bounded) homomorphism $\mathbb{R} \xrightarrow{\varphi} \mathbb{R}$ is uniquely determined by its image $\varphi(\mathbb{Z})$ on $\mathbb{Z}$ (this is a direct consequence of the solution of Cauchy's equation). In view of this, one could say the inclusion $\mathbb{Z} \leq \mathbb{R}$ is rigid, or simply $\mathbb{Z}$ is a rigid subgroup of $\mathbb{R}$.
In fact, more is true. If $\varphi: \mathbb{R} \to G$ is a continuous morphism to a connected, real Lie group G, then again φ is determined by $\varphi(\mathbb{Z})$.
Indeed, $\varphi(1)$ lies in the image of the exponential map $\mathfrak{g} \to G$ by assumption, say $\varphi(1) = \exp V_1$, $V_1 \in \mathfrak{g}$. Then $\varphi(r) = \exp(r V_1)$ is forced for every $r \in \mathbb{R}$.

In fact, even more is true: given any morphism $\varphi_1 : \mathbb{Z} \to G$ to a connected, real Lie group G, whose image intersects $(\exp \mathfrak{g}) - \{e\}$, there exists a unique map $\varphi : \mathbb{R} \to G$ which extends φ_1 on a subgroup $n\mathbb{Z}$ of finite index: $\varphi(r.n) = \exp(r.v_n)$ if $\varphi_1(n) = \exp v_n \neq e$.

(The condition is automatic when exp is surjective, e.g. when G is compact. The passage to a finite-index subgroup is necessary, as $\begin{pmatrix} -a & \\ & -1/a \end{pmatrix} \notin \exp \mathfrak{sl}_2(\mathbb{R})$ for $a>0$, $1 \mapsto \begin{pmatrix} -a & \\ & -1/a \end{pmatrix}$ defines a morphism $\varphi_1 : \mathbb{Z} \to SL_2(\mathbb{R})$, $\varphi(2) = \begin{pmatrix} a^2 & \\ & \frac{1}{a^2} \end{pmatrix} \in \exp \mathfrak{sl}_2(\mathbb{R})$ but

$$\varphi(1) = \begin{pmatrix} a & \\ & 1/a \end{pmatrix} \neq \varphi_1(1) \quad)$$

Analogous rigidity theorems have been proven for lattices in general Lie groups; we will survey them here. Their proofs go well beyond the basic properties of the exp map, though.

1. Recap on lattices in Lie groups

Definition: (lattice, uniform)

Let G be a (locally compact) topological group. A discrete subgroup $\Gamma \leq G$ is called a lattice in G if the quotient space G/Γ can be endowed with a finite, G-invariant, regular measure μ.
A lattice $\Gamma \leq G$ is called (non-)uniform (or (non-) cocompact) if G/Γ is (not) compact.

Better properties/rigidity hold for higher rank real Lie groups; so we recall

Definition: ($\mathbb{R}$-rank)

Let G be a real Lie group. The real rank $\operatorname{rank}_{\mathbb{R}} G$ of G is

$$\max \{ d \in \mathbb{N} \mid \exists \text{ an embedding of } G_m(\mathbb{R})_+^d \hookrightarrow G \}$$

Here $G_m(\mathbb{R})_+$ is the multiplicative group $(\mathbb{R}^\times)_+$ (seen as a Lie group).
We say G (or $\Gamma \leq G$ irreducible lattice) is of higher rank if $\operatorname{rank}_{\mathbb{R}} G \geq 2$ and (often implicitly) G is a semi-simple Lie group.

Examples:

(i) $\mathbb{Z}^n \leq \mathbb{R}^n$ is a lattice. It is uniform, as $\mathbb{R}^n/\mathbb{Z}^n$ is the n-torus.

(ii) If $\Gamma \leq \operatorname{Isom} H^2(\mathbb{R})$ is the stabilizer of a regular tiling of $H^2(\mathbb{R})$ with tiles of finite hyperbolic area, then Γ is a lattice.

(iii) $SL_n(\mathbb{Z})$ is a lattice in $SL_n(\mathbb{R})$. It is not uniform.
Discreteness is trivial, finiteness of the covolume follows from reduction theory. $\begin{pmatrix} a & & & \\ & a^{-1} & & \\ & & 1 & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \to \infty$ in the quotient $\frac{SL_n(\mathbb{R})}{SL_n(\mathbb{Z})}$.

(iv) Let $L = \mathbb{Q}(\sqrt{d})$, with $d=2$.
Let h be the $L/\mathbb{Q}$-hermitian n-form:

$$h(x,y) = x_1\sigma(y_1) + \cdots + x_n\sigma(y_n) \qquad \begin{array}{l} x,y \in L^n \\ \sigma(\sqrt{d}) \mapsto -\sqrt{d} \end{array}$$

Then $G = SU_h(\mathbb{R}) \cong SL_n(\mathbb{R})$ since $L/\mathbb{Q}$ splits over $\mathbb{R}$ (since $d \geq 0$).

It turns out $\Gamma = SU_h(\mathbb{Q}) \cap M_{2n}(\mathbb{Z})$ (intersection taken by seeing $SU_h(\mathbb{Q}) \hookrightarrow M_{2n}(\mathbb{Q})$)
is a lattice in G. It is uniform when $n \leq 3$, Because then $x_1^2 + x_2^2 + x_3^2$ is still anisotropic over $\mathbb{Q}_2$, so h is anisotropic over $\mathbb{Q}$.

The last two examples are prototypes of arithmetic lattices.

Properties: Let Γ be a higher rank lattice in G semisimple.
(i) Γ has property (T) (hence also property (HFA), hence also (FAb)).
(ii) Γ is F_∞ (hence also FP_∞). In particular, Γ is finitely presented. (This point doesn't require higher rank. Moreover, Γ has F iff it is a torsion-free lattice.)
(iii) If $N \triangleleft \Gamma$, then either N is central or Γ/N is finite (i.e. $\frac{\Gamma}{Z(\Gamma)}$ is just infinite). This is Margulis' celebrated normal subgroup theorem.
(iv) Γ is "arithmetic" (Margulis' arithmeticity theorem).

Remark: (Margulis' NST proof strategy)

To prove his NST, Margulis used a peculiar (but since-then-used-and-abused) strategy: given $N \trianglelefteq \Gamma$ not central, he showed that

(i) Γ/N has (T) (this is inherited from Γ)

(ii) Γ/N is amenable. (this is deduced using Margulis' factor theorem)

Theorem: (Margulis' factor theorem)

Let G be a semisimple real Lie group of higher rank, $B \leq G$ Borel.
Let $\Gamma \leq G$ be an irreducible lattice, and $N \trianglelefteq \Gamma$ not central.
Suppose that Γ acts on a non-empty compact metrizable space X and that we are given an (essentially) Γ-equivariant measurable map

$$\varphi: G/B \longrightarrow X$$ (used to endow X with φ_*(quasi-invariant meas. of G/B))

Then the action of Γ on X extends (almost everywhere) to an action of G on X for which φ is (essentially) G-equivariant.
In consequence, the Γ-space X is measurably isomorphic to G/P for some parabolic subgroup $P \supseteq B$.

The existence of a Γ-equivariant map φ to $\mathrm{Prob}(X)$ is guaranteed by the fact B is amenable: look at $L^\infty(G, \mathrm{Prob}(X))^\Gamma$ and pick a B-invariant measure m_B on it. Then average $\int_{L^\infty(G,\mathrm{Prob}(X))^\Gamma} f \, dm_B(f) = \varphi$ to find a B-invariant, Γ-equivariant map $\varphi: G \longrightarrow \mathrm{Prob}(X)$.

($L^\infty(G, \mathrm{Prob}(X))^\Gamma$ is compact since G/Γ has finite measure and $\mathrm{Prob}(X)$ is compact.)

From here, Margulis' factor theorem implies that $\mathrm{Prob}(X)$ is Γ-equivariantly measurably isomorphic to G/P (with left mult.)
If X is any compact, metrizable space on which N acts trivially, surely N acts trivially on $\mathrm{Prob}(X)$, so N acts essentially trivially on G/P.

But N acts continuously on G/P, and open sets have positive measure, so N acts trivially. Let K be the kernel $\bigcap gPg^{-1}$.
If $K \neq G$, there is a simple factor H of G commuting with K.
Then $K, \Gamma \subseteq N_G(N)$, hence $\overline{K \cdot \Gamma} = G \subseteq N_G(N)$ and $N \trianglelefteq G$.
Since N is discrete, it must be central.
Otherwise, it must be that $K = G = P$, $G/P = \{\text{pt mass}\}$, and the image of this point in $\mathrm{Prob}(X)$ is a measure fixed by Γ. This shows that Γ/N is amenable.

2. Some classical rigidity results

Theorem: (Weil's local rigidity)

Let G be a semisimple Lie group not isogenic to $SL_2(\mathbb{R})$ nor $SL_2(\mathbb{C})$. Let $\Gamma \leq G$ be an irreducible lattice, generated by the set $S \subseteq \Gamma$. There exists a neighborhood U of $e \in G$ such that for every morphism $\rho : \Gamma \to G$ with $\rho(S) \subseteq U$, there exists $g \in G$ for which $\rho(\gamma) = g\gamma g^{-1}$ $\forall \gamma \in \Gamma$.

Remark: This implies that the conjugacy class of Γ in $\mathrm{Sub}(G)$ is open. "Chabauty local rigidity"

Theorem: (Mostow-Prasad (strong) rigidity)

Let G, G' be semisimple real Lie groups, both adjoint & without compact factors, and assume G is not isogenic to $SL_2(\mathbb{R})$. Let $\Gamma \leq G$, $\Gamma' \leq G'$ be irreducible lattices. If $\rho : \Gamma \xrightarrow{\sim} \Gamma'$ is a group isomorphism there exists an analytic isomorphism $\tilde{\rho} : G \xrightarrow{\sim} G'$ such that $\tilde{\rho}|_{\Gamma} = \rho$.

Strong rigidity implies notably that the fundamental group of a finite-volume hyperbolic n-manifold $(n \geq 3)$ determines the manifold uniquely.

Theorem: (Margulis' superrigidity)

Let G, G' be semisimple Lie groups, with G' adjoint and G, G' without compact factors, and assume that $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$. Let $\Gamma \leq G$ be an irreducible lattice. Let $\rho : \Gamma \to G'$ be a morphism such that $\rho(\Gamma)$ is not contained in a proper, connected Lie subgroup of G'. Then ρ extends to an analytic morphism $\tilde{\rho} : G \to G'$.

3. The Stuck-Zimmer theorem

Theorem: (Stuck-Zimmer)

Let G be a semisimple Lie group without compact factors. Assume that $\mathrm{rank}_{\mathbb{R}} G \geq 2$ and that G has property (T). Then every irreducible Borel p.m.p. action of G is either essentially free or essentially transitive.

Here, irreducible means that every non-central normal subgroup of G acts ergodically.
We will state a "subgroup version" of the Stuck-Zimmer theorem.

Definition: ($\mathrm{Sub}(G)$, IRS, URS)

Let G be a topological group. Then the set $\mathrm{Sub}(G)$ of closed subgroups admits a topology with basis

$$\mathcal{O}(U_1, \dots, U_n, K) := \{ H \in \mathrm{Sub}(G) : H \cap U_i \neq \emptyset \text{ for } i=1,\dots,n,\ H \cap K = \emptyset \}$$

called the Chabauty (or Fell for closed subsets) topology.
$\mathrm{Sub}(G)$ is compact with this topology.
An invariant random subgroup (IRS) of G is a probability measure on $\mathrm{Sub}(G)$ which is invariant for the conjugation action $G \curvearrowright \mathrm{Sub}(G)$. A uniformly recurrent subgroup is a minimal, closed G-subset of $\mathrm{Sub}(G)$.

Examples:

(i) $\mathrm{Sub}(\mathbb{R}) \cong [0,\infty]$ with $\alpha\mathbb{Z} \mapsto \alpha$, $\mathbb{R} \mapsto 0$, $\{0\} \mapsto \infty$

(ii) $\mu \in \mathrm{IRS}(G)$ is a Dirac mass iff $\mathrm{supp}\,\mu = \{N\}$ for some $N \trianglelefteq G$.

(iii) If $\Gamma \leq G$ is a lattice, then the finite measure on G/Γ can be pushed to an IRS μ_Γ, supported on $\{g\Gamma g^{-1} \mid g \in G\}$

If now $N \trianglelefteq \Gamma$, then again the same works. The resulting IRS is denoted $\mu_{N \trianglelefteq \Gamma}$, and is supported on $\{gNg^{-1} \mid g \in G\}$.

(iv) If $G \curvearrowright (X,\nu)$ is p.m.p., then it is known (Varadarajan) that G_x is closed a.e. Moreover, the map $X \to \mathrm{Sub}(G)$: $x \mapsto G_x$ is measurable. Hence one can push ν to an IRS in G.

Theorem: (Stuck-Zimmer for IRS)

Let G be a semisimple Lie group without compact factors. Assume that $\mathrm{rank}_{\mathbb{R}} G \geq 2$ and G has property (T). Then every non-atomic irreducible IRS in G is of the form μ_Γ for some lattice $\Gamma \leq G$.

Corollary: Margulis' normal subgroup theorem.

Proof: Let $N \trianglelefteq \Gamma \overset{\text{lattice}}{\leq} G$. Then $\mu_{N\trianglelefteq\Gamma}$ is an ergodic measure, since it is supported on a single orbit. Thus either $\mu_{N\trianglelefteq\Gamma}$ is atomic (then N is central) or N is a lattice (and then $[\Gamma : N] < \infty$).

Note: $\mu_{N\trianglelefteq\Gamma}$ is irreducible precisely because Γ is.

Corollary: (Stuck-Zimmer for actions of Γ)

Let G be as in the original S-Z theorem, and let $\Gamma \leq G$ be an irreducible lattice. Any ergodic, faithful, p.m.p. action $\Gamma \curvearrowright (X,\mu)$ is either essentially transitive or essentially free.