Let $\Gamma$ be a countable group.

Left regular representation

$$\lambda : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$$

$$(\lambda_g \xi)(h) = \xi(g^{-1}h)$$

$$\lambda_g \delta_h = \delta_{gh}$$

↳ linear span of $\{\lambda_g \mid g \in \Gamma\}$ is a $*$-subalgebra of $B(\ell^2(\Gamma))$

→ $\|\cdot\|$-closure : reduced group $C^*$-alg

$C^*_r(\Gamma)$

→ SOT-closure : group von Neumann algebra.

$L(\Gamma)$

↳ Remark: equiv. WOT-closure, or

$\{\lambda_g \mid g \in \Gamma\}''$

<u>Fact</u>: $L(\Gamma)$ is a factor

$$\Longleftrightarrow$$

$\Gamma$ is icc

<u>Proof</u>:

$\Leftarrow$ If $\Gamma$ icc and $T \in Z(L(\Gamma))$,

then, since $T\delta_e \in \ell^2(\Gamma)$ constant on conjugacy classes, $T\delta_e = \alpha\delta_e$, $\alpha \in \mathbb{C}$,

i.e. $T = \alpha \mathbb{1}$.

$\Rightarrow$ If $\Gamma$ not icc, say $C$ finite conj. cl., then

$$\sum_{g \in C} \lambda_g \in Z(L(\Gamma))$$

<u>Remark</u>: Also results for when $C^*_r(\Gamma)$ is simple [Kalantar, Kennedy '17]

<u>In general</u>

- $C^*_r(G)$ remembers more than $L(\Gamma)$
- $L(\Gamma)$ forgets everything for amenable groups
- $L(\Gamma)$ sometimes remembers everything e.g. $(\mathbb{Z}/2\mathbb{Z}) \wr (\Gamma \times \Gamma)$ $\Gamma$ icc hyperbolic
- All icc (T) groups superrigid? (Connes '82)

Note that

$$\tau(x) = \langle x\delta_e, \delta_e \rangle$$

defines a functional on both $C^*_r(\Gamma)$ and $L(\Gamma)$.

It is a normal, faithful, tracial state
$L(\Gamma)$

- $\tau(x^*x) \geq 0 \quad \forall x$
  $= 0 \iff x = 0$
- $\tau(\lambda_e) = 1$
- $\tau(xy) = \tau(yx)$
- continuous in WOT/SOT on unit ball of $\ell(\Gamma)_1$

We say $L(\Gamma)$ is a finite von Neumann algebra.

Fact: A factor has at most 1 tracial state.

Classification of factors

- type $I_n$, $n \in \mathbb{N} \cup \{\infty\}$
  $B(\mathcal{H})$
- type $II_1$ if admits tracial state
- type $II_\infty$ if admits semi-finite trace
- type $III$ : others
  $\rightsquigarrow III_\lambda$ for $\lambda \in [0,1]$

<u>Part 2</u>: Functions on groups & GNS

Trace $\tau$ on $L(\Gamma)$ restricts to a map $\tau : \Gamma \to \mathbb{C}$ with nice properties.

Let $G$ be a topological group.

We call a cont. function

$$\varphi : G \to \mathbb{C}$$

- <u>positive definite</u> (of positive type)

  if
$$\sum_{i,j=1}^{n} c_i \bar{c}_j \varphi(g_j^{-1} g_i) \geq 0 \qquad \forall c_1, \dots, c_n \in \mathbb{C}, \; g_1, \dots, g_n \in G$$

- <u>central</u> if
$$\varphi(g h g^{-1}) = \varphi(h) \qquad \forall g, h \in G$$

- <u>normalized</u> if
$$\varphi(e) = 1$$

- <u>indecomposable</u> if it is an extremal point of the convex space of PD, C, N functions.

<u>Note</u>: PD + C functions are called <u>characters</u> (or traces).

## Examples

1) $G \to \mathbb{C} : g \mapsto 1$

2) continuous homomorphisms $G \to \mathbb{C}$
"unitary characters"

3) characters of f.d. reps : $\pi : G \to \mathcal{U}(n)$

$$\chi : G \to \mathbb{C} : g \mapsto \frac{1}{n} \mathrm{Tr}(\pi(g))$$

4) matrix coefficients

$$\pi : G \to \mathcal{U}(B(\mathcal{H})) \quad \text{unirep}$$

For any $\xi, \eta \in \mathcal{H}$, define

$$\varphi_{\pi,\xi,\eta}(g) = \langle \pi(g)\xi, \eta \rangle \rightsquigarrow \text{PD } \underline{\text{iff}}\ \xi = \eta$$

$\hookrightarrow$ recall $\tau(g) = \langle \lambda_g \delta_e, \delta_e \rangle$

## GNS construction

Start: positive definite function

$$\varphi : G \to \mathbb{C}$$

with $G$ discrete

Consider

$$\mathbb{C}[G] = \{ f : G \to \mathbb{C} \mid \overset{\text{cont.}}{\text{finite}} \text{ support} \}$$

It is a unital $*$-algebra with

$$(f_1 * f_2)(x) = \sum_{y \in G} f_1(xy^{-1}) f_2(y)$$

$$f^*(x) = \overline{f(x^{-1})}.$$

Note that a pd function $\varphi : G \to \mathbb{C}$ defines a "positive" functional

$$\varphi : \mathbb{C}[G] \to \mathbb{C},$$

where positive means

$$\varphi(f^* f) \geq 0 \qquad \forall f \in \mathbb{C}[G]$$

Then

$$\langle f_1, f_2 \rangle = \varphi(f_2^* f_1)$$

$$\left( = \sum_{x,y \in G} f_1(x) \overline{f_2(y)} \, \varphi(y^{-1}x) \right)$$

defines a positive Hermitian form on $\mathbb{C}[G]$.

The set

$$N_\varphi = \{ f \in \mathbb{C}[G] \mid \langle f, f \rangle_\varphi = 0 \}$$

is a closed subspace + left ideal by CS.

$\Rightarrow$ Define $\mathcal{H}_\varphi = \overline{\mathbb{C}[G]/N_\varphi}^{\langle \cdot, \cdot \rangle_\varphi}$

Hilbert space completion.

The action of $G$ on $\mathbb{C}[G]$ by

$$g \cdot f(h) = f(g^{-1} h)$$

extends to a unitary operator on $\mathcal{H}_\varphi$.

So we get a unirep

$$\pi_\varphi : G \to \mathcal{U}(B(\mathcal{H}_\varphi))$$

Finally, set $\xi_\varphi = [\delta_e]$.

= cyclic vector, i.e.

$$\overline{\pi(G)\,\xi_\varphi} = \mathcal{H}_\varphi .$$

Then

$$\varphi(g) = \langle \pi_\varphi(g)\,\xi_\varphi,\ \xi_\varphi \rangle_\varphi .$$

$\rightsquigarrow (\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ is the GNS triple

<u>Note</u>: if we started with $\varphi = \delta_e$, we get left regular representation.

↳ Here we got ~~a~~ discrete vNa with a trace!

In general: if $\varphi$ is central,

$N_\varphi$ is a two-sided ideal, and the right ~~right~~ action

$$f(h)^g = f(hg)$$

also extends to a bounded operator

$$\rho_\varphi(g) \text{ on } \mathcal{H}_\varphi$$

In fact, $\pi_\varphi(G)$ & $\rho_\varphi(G)$ commute and

$$x \mapsto \langle x\xi_\varphi, \xi_\varphi \rangle$$

defines a traces on the vN- $\pi_\varphi(G)'' =: M_\varphi$

Moreover, $M_\varphi$ is a factor iff $\varphi$ is extremal/indecomposable.

## More examples

5) Let $\Gamma \curvearrowright (X,\mu)$ be a pmp action.
Not free. Then

$$g \mapsto \mu\{x \mid g\cdot x = x\}$$

is a character

6) Generalizing (but also not), ~~where~~ IRS

## Invariant Random Subgroup

If $G$ is a lc group, and

$$\mathrm{Sub}(G) = \{H < G \mid H \text{ is closed subgroup}\}$$

5) $H < G$ open subgroup,

$\mathbb{1}_H$ characteristic function of $H$.

$\hookrightarrow$ GNS $\rightsquigarrow$ quasi-regular representation of $G$ in $\ell^2(G/H)$

(or any $\chi : H \to \mathbb{C}$ unitary character on $G \backslash H$)

---

This is a compact Haus. top. space [Fell, '62] with the Chabauty topology, generated by

$\mathcal{U}(K, V_1, \ldots, V_n) =$

$$\{ H \in Sub(G) \mid H \cap K = \emptyset, \; H \cap V_i \neq \emptyset \text{ for } i = 1, \ldots, n \}$$

where $K$ is compact subsets of $G$.
$V_i$ are open

Note that $G$ acts on $Sub(G)$ by conjugation. by homeomorphisms

An $\underline{\text{IRS}}$ of $G$ is a $G$-invariant probability measure on $\mathrm{Sub}(G)$.

$\Gamma$ is discrete

$\rightsquigarrow$ whenever $\nu$ is an IRS of $\Gamma$,

$$\varphi_\nu : \Gamma \to \mathbb{C} :$$

$$\varphi_\nu(g) = \nu\{H \in \mathrm{Sub}(G) \mid g \in H\}$$

is a normalized character of $\Gamma$.

Eg's of IRS

1) $N \trianglelefteq G$

2) $\Gamma < G$ lattice, $\mu_{G/\Gamma}$ $G$-invariant regular prob. measure on $G/\Gamma$. Consider

$$\Phi : G/\Gamma \to \mathrm{Sub}(G). \quad g\Gamma \mapsto g\Gamma g^{-1}.$$

$\Phi_*(\mu) \in \mathrm{IRS}(G)$

3) $G \curvearrowright (X,\mu)$ pmp

$$\Phi : X \to \mathrm{Sub}(G) : \quad x \mapsto G_x \quad \text{stabilizer}$$

$\Phi_*(\mu) \in \mathrm{IRS}$

All IRS's arise in this way:

[Abért, Glasner, Virág '14] discrete groups

[Abért, Bergeron, Biringer, Gelander, Nikolov, Raimbault, Samet '17] ~~lcsz~~ lcsc groups

## Part 3: Rigidity

Margulis' superrigidity '75:

$\Gamma < G$ lattice in simple real Lie grp $G$ with finite centre & $\mathbb{R}$-rank$(G) \geq 2$.

$H$ another simple real Lie group with finite centre.

Let $\pi : \Gamma \to H$ homomorphism that has Zariski-dense image $\pi(\Gamma)$. Then

- $H$ is compact

or

- $\exists$ finite index subgrp $\Lambda < \Gamma$ s.t. $\pi|_\Lambda$ extends to cont. hom $G \to H$

Connes conjectured: similar result when regarding $\Gamma < \mathcal{U}(L(\Gamma))$ if $\Gamma$ has (T)?

$\to$ Peterson '13 solves this, Boutonnet, Houdayer '19 generalizes.

First result in this direction by Bekka '07

Theorem 1

Let $n \geq 3$, $M$ a finite factor and $\Gamma = SL_n(\mathbb{Z})$

$$\pi : \Gamma \to \mathcal{U}(M)$$

a group homomorphism. Assume $\pi(\Gamma)'' = M$.

Then either

1) $M$ is finite dimensional

or

2) there is a subgroup $\Lambda < \Gamma$ of finite index s.t. $\pi|_\Lambda$ extends to a normal homomorphism $L(\Lambda) \to M$.

$\hookrightarrow$ taking $\Gamma = PSL_n(\mathbb{Z})$,

$\Gamma = \Lambda$ in 2

Behind this is a character rigidity result.
indecomp.

Theorem 2

Let $\varphi$ be a character of $SL_n(\mathbb{Z})$.
Then either

1) it is the character of a f.d. unirep

or

2) it is the trivial extension of a character of the centre $SL_n(\mathbb{Z})$.

$\hookrightarrow$ Concretely: if $n$ odd: 2) says

$\varphi = \delta_e$

Theorem 1 from Theorem 2

n odd

If $M$ is not f.d., consider

$$\varphi = \tau_M \circ \pi .$$

$\Rightarrow \varphi = \delta_e$

$\Rightarrow$ $\pi$ extends to $M_\varphi = L(\Gamma)$

Theorem 2 from Theorem 1

Assume $\varphi : \Gamma \to \mathbb{C}$ not character of fd unirep. Consider $M_\varphi$ a $\text{II}_1$ factor

$\pi . \Gamma \to \mathcal{U}(M_e)$ extends to isomorphism

$\pi : L(\Gamma) \to M_\varphi$ & $\varphi = \tau \circ \lambda$

But extremal characters are in bijection with finite factorial reps, so $\varphi = \delta_e$.