Stationary induction, reduction to the non-commutative Nevo–Zimmer theorem for G

Abstract

These are the notes for a presentation of about 1 hour in the joint seminar with Louvain-la-Neuve. This is the fifth lecture in the series.

1 Recap

The setting is as in the previous lectures. G is a connected simple Lie group with trivial center and rank $\mathrm{rk}_{\mathbb{R}}(G) \geq 2$. Then we can choose a maximal compact subgroup $K \subset G$ and a maximal cocompact amenable subgroup $P \subset G$ (minimal parabolic subgroup) such that $G = KP$. In this case, we denote $\nu_P \in \mathrm{Prob}(G/P)$ the unique K-invariant (Borel) probability measure.

When μ is a K-invariant measure on G, then the Poisson boundary of (G, μ) is given by $(G/P, \nu_P)$. We can use uniqueness to see that ν_P is μ-stationary. Indeed, the measure $\mu * \nu_P$ is K-invariant and therefore it must equal ν_P.

We recall a few concepts from the previous lecture in the following definition.

Definition 1.1. Let Λ be a countable discrete group. Then $PD_1(\Lambda)$ denotes the convex compact space of all the positive definite functions $\varphi\colon \Lambda \to \mathbb{C}$ such that $\varphi(e) = 1$. Λ acts on $PD_1(\Lambda)$ by conjugation, and we write $\mathrm{Char}(\Lambda) = PD_1(\Lambda)^\Lambda$, so $\varphi \in \mathrm{Char}(\Lambda)$ if $\varphi(\lambda x \lambda^{-1}) = \varphi(x)$ for all $x, \lambda \in \Lambda$.

If H is an lcsc group and μ is a probability measure on H, then a bounded function $f \in L^\infty(H, \mu)$ is called *harmonic* if

$$(\mu * f)(g) = \int_H f(gh) d\mu(h) = f(g)$$

for every $g \in H$. We denote by $\mathrm{Har}^\infty(H, \mu)$ the space of bounded harmonic functions.

If $H \curvearrowright M$ is an action on a von Neumann algebra, a state ψ on M is called μ-stationary if $(\mu * \psi)(x) = \int_H (h \cdot \psi)(x) d\mu(h) = \psi(x)$ for every $x \in M$. In that case we call (M, ψ) an (H, μ)-von Neumann algebra. When M is abelian, this corresponds precisely to an (H, μ)-space (B, ν).

Lemma 1.2. *Suppose that (M, ψ) is an (H, μ)-von Neumann algebra. Then we get a well-defined (normal ucp) H-equivariant map*

$$\theta\colon M \to \mathrm{Har}^\infty(H, \mu) : \quad \theta_x(h) = (h \cdot \psi)(x). \tag{1.1}$$

In particular Lemma 1.2 above applies to abelian von Neumann algebras, i.e. measure spaces (B, ν) such that ν is μ-stationary. Recall that the *Poisson boundary* (B, ν) is the (H, μ) space such that the map (1.1) is a bijection.

Proof. It is straightforward to check that θ is H-equivariant. Given $x \in M$ we check that

$\theta_x \in L^\infty(H,\mu)$ is harmonic. Indeed, using that $\mu * \psi = \psi$, we get that

$$\begin{aligned}(\mu * \theta_x)(g) &= \int_H \theta_x(gh)d\mu(h) \\ &= \int_H (gh\cdot\psi)(x)d\mu_h \\ &= \int_H (h\cdot\psi)(g^{-1}\cdot x)d\mu(h) \\ &= (\mu*\psi)(g^{-1}\cdot x) = \psi(g^{-1}\cdot x) \\ &= (g\cdot\psi)(x) = \theta_x(g).\end{aligned}$$

□

Definition 1.3. A probability measure μ_0 on Γ is called a *Furstenberg measure* if the following three conditions hold.

1. $\mu_0(\gamma) > 0$ for every $\gamma \in \Gamma$.
2. The Poisson boundary of (Γ, μ_0) equals $(G/P, \nu_P)$.

In particular we have that $\mu_0 * \nu_P = \nu_P$.

Let Λ be a countable discrete group. If μ is a probability measure on Λ we say that $\varphi \in PD_1(\Lambda)$ is a μ-character if $(\mu * \varphi) = \varphi$, i.e.

$$\sum_{\lambda\in\Lambda} \mu(\lambda)\varphi(\lambda^{-1}x\lambda) = \varphi(x),$$

for every $x \in \Lambda$. Clearly, and character on Λ is a μ-character on Λ.

Theorem 1.4 (Theorem C)**.** *Let $\Gamma \subset G$ be a lattice and let μ_0 be a Furstenberg measure on Γ. Then the following hold.*

1. *Any μ_0-character on Γ is a genuine character.*
2. *Any extreme point $\varphi \in \mathrm{Char}(\Gamma)$ either gives rise to a finite dimensional representation or $\varphi = \delta_e$.*

Theorem 1.5 (Theorem D)**.** *Let $\Gamma \subset G$ be a lattice and let μ_0 be a Furstenberg measure on Γ. Let (M,ψ) be an ergodic (Γ,μ_0)-von Neumann algebra. Then precisely one of the following holds.*

1. *Either ψ is Γ-invariant.*
2. *Or there exists a proper parabolic subgroup $P \subset Q \subset G$ with $Q \neq G$ and a state preserving Γ-equivariant normal unital $*$-embedding*

   $$L^\infty(G/Q,\nu_Q) \to (M,\psi),$$

   where ν_Q denotes the unique K-invariant Borel probability measure on G/Q.

Our first goal is prove Theorem C using Theorem D!

Remark 1.6. We will apply Theorem D to the more specific situation where $\pi\colon \Gamma \to \mathcal{U}(M)$ is a unitary representation and $\sigma\colon \Gamma \curvearrowright M$ is given by $\sigma_\gamma(x) = \mathrm{Ad}(\pi(\gamma))(x)$. Then, if we happen to be in situation 2 of Theorem D, it follows from Lemma 2.2 that $\psi\circ\pi = \delta_e$. If time permits, I will try to give an idea of why this is true.

2 Deduce Theorem D from Theorem C

Proof. 1. The space $\mathrm{Char}_{\mu_0}(\Gamma)$ of all μ_0-characters on Γ is a nonempty compact convex, so by the Krein–Milman theorem, it suffices to prove the statement for an extreme point $\varphi \in \mathrm{Char}_{\mu_0}(\Gamma)$.

So let $\varphi \in \mathrm{Char}_{\mu_0}(\Gamma)$ be an extreme point and let $(\pi, \mathcal{H}, \xi)$ denote the corresponding GNS triple. Set $M = \pi(\Gamma)'' \subset B(\mathcal{H})$ and let ψ be the state on M given by $\psi(x) = \langle x\xi, \xi\rangle$. Recall that using this notation we get that

$$\varphi = \psi \circ \pi. \tag{2.1}$$

Let $\sigma \colon \Gamma \curvearrowright M$ be the action given by conjugation, i.e. $\sigma_\gamma(x) = \pi(\gamma)x\pi(\gamma^{-1})$ for $x \in M$. Then we make the following observations that follow directly from (2.1).

1. The normal state ψ on M is μ_0-stationary.
2. Our choses state φ is conjugation invariant iff ψ is Γ-invariant.

Basically, since $\varphi \in \mathrm{Char}_{\mu_0}(\Gamma)$ is an extreme point, we get that the action $\Gamma \curvearrowright M$ is ergodic. And so we are in the setting of Theorem D. If φ is not conjugation invariant, then ψ is not Γ-invariant and therefore we are in the second situation of Theorem D. By Remark 1.6 we get that $\varphi = \psi \circ \pi = \delta_e$ is conjugation invariant anyway.

2. Suppose that $\varphi \in \mathrm{Char}(\Gamma)$ is an extreme point. Denote by $(\pi, \mathcal{H}, \xi)$ the GNS representation associated to φ. Again define $M = \pi(\Gamma)'' \subset B(\mathcal{H})$ together with the faithful normal state (!) $\psi(x) = \langle x\xi, \xi\rangle$. Then M is a factor, because φ is an extreme point. Let $J \colon \mathcal{H} \to \mathcal{H}$ be the conjugate unitary given by $J(x\xi) = x^*\xi$ so that $JMJ = M'$. Define actions

$$\alpha \colon \Gamma \curvearrowright L^\infty(G/P) \mathbin{\overline{\otimes}} B(\mathcal{H}) : \quad \alpha_\gamma = \sigma_\gamma \otimes \mathrm{Ad}(J\pi(\gamma)J)$$
$$\beta \colon \Gamma \curvearrowright L^\infty(G/P) \mathbin{\overline{\otimes}} B(\mathcal{H}) : \quad \beta_\gamma = \mathrm{Ad}(1 \otimes \pi(\gamma)).$$

The actions α and β commute, and thus we obtain a well-defined action

$$\beta \colon \Gamma \curvearrowright N = (L^\infty(G/P) \mathbin{\overline{\otimes}} B(\mathcal{H}))^\alpha. \tag{2.2}$$

We can think of N as bounded measurable function $f \colon G/P \to B(\mathcal{H})$ satisfying the equivariance property $f(\gamma \cdot w) = \mathrm{Ad}(J\pi(\gamma)J)f(w)$ for every $\gamma \in \Gamma$ and a.e. $w \in G/P$. Thus we can identify $N \cong (L^\infty(G/\Gamma) \mathbin{\overline{\otimes}} B(\mathcal{H}))^P$. Since P is amenable, we conclude that N is an amenable von Neumann algebra. Also note that $1 \otimes M \subset N$. Consider the normal state $\Psi \in N_*$ given by

$$\Psi \colon N \to \mathbb{C} : f \mapsto \int_{G/P} \langle f(w)\xi, \xi\rangle d\nu_P(w).$$

Then Ψ is μ_0-stationary, which follows from the stationarity of ν_P. Furthermore, the action $\beta \colon \Gamma \curvearrowright N$ is ergodic. Indeed, we may view an element $f \in N^\beta$ as a Γ-equivariant function $G/P \to M' = JMJ$. Then we need to show that f is essentially constant to a scalar multiple of the identity. Define the measurable function

$$F \colon G/P \times G/P \to \mathbb{R} : \quad F(w_1, w_2) = \|Jf(w_1)J - Jf(w_2)J\|_2,$$

where $\|\cdot\|_2$ denotes the norm induced by the trace ψ. By Γ-equivariance of f we get that F is Γ-invariant. Since the Poisson boundary is doubly ergodic, we get that F is essentially constant. Then we conclude that f is essentially constant, say $f(w) = x \in M'$ for a.e. $w \in G/P$. Again by equivariance we get that x commutes with $J\pi(\Gamma)J$, so that $x \in M \cap M' = \mathbb{C}1$, which is what we needed to show.

Now we are again in the setting of Theorem D. If Ψ is not β-invariant, we get that $\Psi \circ (1 \otimes \pi) = \delta_e$ by Remark 1.6. Note that Ψ restricts to $1 \otimes \psi$ on $1 \otimes M$. Then it follows that $\psi \circ \pi = \delta_e$, i.e. ψ is $\varphi = \delta_e$.

If Ψ is Γ-invariant, then we show that $1 \otimes M = N$, so that M is amenable and has property (T). Then it follows that M is finite dimensional. Take $f \in N$, then one can show that

$$\Psi(\beta_\gamma^{-1}(f)) \int_{G/P} \langle f(\gamma \cdot w)\xi, \xi \rangle d\nu_P(\omega).$$

Thus writing θ for the isometric bijection

$$\theta \colon L^\infty(G/P, \nu_P) \to \mathrm{Har}^\infty(\Gamma, \mu_0) : \quad \theta_h(g) = \int_{G/P} h(g \cdot w) d\nu_P(w).$$

we get that $\Psi(\beta_\gamma^{-1}(f)) = \theta_h(\gamma)$, where $h \in L^\infty(G/P)$ is given by $h(w) = \langle f(w)\xi, \xi \rangle$. By invariance of Ψ we conclude that h is essentially constant. Since $1 \mathbin{\overline{\otimes}} M \subset N$, we deduce that for every $f \in N$ and every $a, b \in M$ we get that $(1 \otimes b^*) f (1 \otimes a) \in N$ and so the map $w \mapsto \langle f(w) a\xi, b\xi \rangle$ is essentially constant. By density of $M\xi \subset \mathcal{H}$ we get that $w \mapsto f(w)$ is essentially constant, and equal to some element $x \in B(\mathcal{H})$ say. By the equivariance of f we get again that $x \in M$, and so we get the desired conclusion $f \in 1 \otimes M$. □

3 On Lemma 2.2

Suppose that $\Gamma \to \mathcal{U}(M)$ is a unitary representation, so that the action $\sigma \colon \Gamma \curvearrowright M$ given by $\sigma_\gamma = \mathrm{Ad}(\pi(\gamma))$ makes (M, ψ) into an ergodic (Γ, μ_0)-von Neumann algebra. Assume moreover there exists a proper parabolic subgroup $P \subset Q \subset P$ and a state preserving unital $*$-embedding $L^\infty(G/Q, \nu_Q) \to M$.

Let $A \subset M$ be the separable unital C^*-algebra generated by $\pi(\Gamma)$ and $\theta(C(G/Q))$. Then A is globally Γ-invariant and the state $\psi\big|_A$ is μ_0-stationary. Then we saw in the previous lecture that there exists an essentially unique map $G/P \to \mathcal{S}(A) : w \mapsto \psi_w$ such that

$$\psi = \int_{G/P} \psi_w d\nu_P(w). \tag{3.1}$$

Denote $p_Q \colon G/P \to G/Q$ the measure preserving factor map. The state $\psi \colon \theta$ is μ_0-stationary, and thus it must equal ν_Q. Then we get that

$$\int_{G/P} \psi_w \circ \theta d\nu_P(w) = \psi \circ \theta = \delta_Q = \int_{G/P} \delta_{p_Q(w)} d\nu_P(w).$$

By uniqueness of boundary maps we get that $\psi_w \circ \theta = \delta_{p_Q(w)}$ for a.e. $w \in G/P$. As $\delta_{p_Q(w)}$ is multiplicative, we get that $\theta(C(G/Q))$ lies in the multiplicative domain of ψ_w for a.e. $w \in G/P$.

Pick $\gamma \neq e$. Since the action $\Gamma \curvearrowright G/Q$ is essentially free and since $p_Q \colon G/P \to G/Q$ is pmp we get that $\gamma \cdot p_Q(w) \neq p_Q(w)$ for a.e. G/P. Pick $w \in G/P$ such that $\gamma \cdot p_Q(w) \neq p_Q(w)$ and such that $\theta(C(G/Q))$ is in the multiplicative domain of ψ_w and write $y = \pi_Q(w)$. Take $f \in C(G/Q)$ such that $f(y) = 1$ and $f(\gamma \cdot y) = 0$. Then we get that

$$\begin{aligned} \psi_w(\pi(\gamma)) &= f(y)\psi_w(\pi(\gamma) = \psi_w(\theta(f)\pi(\gamma)) = \psi_w(\pi(\gamma)\pi(\gamma^{-1})\theta(f)\pi(\gamma)) \\ &= \psi_w(\pi(\gamma)\theta(\gamma^{-1} \cdot (f))) = \psi_w(\pi(\gamma)) f(\gamma \cdot y) = 0. \end{aligned}$$

Since this holds for a.e. $w \in G/P$ we conclude from (3.1) that $\psi \circ \pi = \delta_e$.