Proof of the non-commutative Nevo–Zimmer theorem, Part I

1 A recap

Throughout these notes we work with a connected simple Lie group G with trivial center and $\mathrm{rk}_{\mathbb{R}}(G) \geq 2$. We fix throughout a maximal compact subgroup K of G and a maximal cocompact amenable subgroup $P \subseteq G$ such that

$$G = KP \ .$$

For any parabolic subgroup $P \subseteq Q \subseteq G$ we let $\nu_Q \in \mathrm{Prob}(G/Q)$ denote the unique K-invariant Borel probability measure on G/Q. For any lcsc group H we let m_H denote the Haar measure, and recall that $\nu_Q \circ \pi^{-1} \sim m_G$ where $\pi : G \to G/Q$.

Lemma 1.1. *Let $\mu \in Prob(G)$ be a probability measure that is equivalent to the Haar measure and K-invariant. Let $\mathcal{A}$ be any separable G-C^*-algebra and $\phi \in \mathcal{S}(\mathcal{A})$ any μ-stationary state, i.e. $\mu * \phi = \phi$.*
Then there exists an essentially unique G-equivariant measurable map $\beta_\phi : G/P \to \mathcal{S}(\mathcal{A}) : w \mapsto \phi_w$ that satisfies

$$\phi = \int_{G/P} \phi_w \, \mathrm{d}\nu_P(w) \ .$$

Remark 1.2. This result is proved by Boutonnet and Houdayer for general locally compact second countable Hausdorff (lcsc) groups H, where $(G/P, \nu_P)$ is replaced by the (H, μ)-Poisson Boundary. Furstenberg proved (in 1963) that in this setup the (G, μ)-Poisson Boundary is $(G/P, \nu_P)$.

Definition 1.3. Let H be a lcsc group and $\mu \in \mathrm{Prob}(H)$ be equivalent to the Haar measure. Let $\mathcal{M}$ be any H-von Neumann algebra and $\varphi \in \mathcal{M}_*$ any normal state. We say $(\mathcal{M}, \varphi)$ is an ergodic (H, μ)-von Neumann algebra if

- the action $\sigma : H \curvearrowright \mathcal{M}$ is ergodic (i.e. $\mathcal{M}^H = \mathbb{C}I$), and
- φ is μ-stationary, i.e. $\mu * \phi = \phi$.

Lemma 1.4. *If $(\mathcal{M}, \varphi)$ is an ergodic (H, μ)-von Neumann algebra then φ is automatically faithful.*

Proof. Let q denote the support projection of φ, then

$$1 = \varphi(q) = (\mu * \varphi)(q) = \int_H \varphi(\sigma_h^{-1}(q)) \, \mathrm{d}\mu(h) \ ,$$

but $0 \leq \varphi(\sigma_h^{-1}(q)) \leq 1$ for all $h \in H$, and hence we must have $\varphi(\sigma_h^{-1}(q)) = 1$ μ-a.e. Since the function $h \to \varphi(\sigma_h^{-1}(q))$ is continuous and μ has full support (by equivalence with the Haar measure), we get $\varphi(\sigma_h^{-1}(q)) = 1$ for all $h \in H$, so $q \in \mathcal{M}^H = \mathbb{C}I$. Since q is a projection, $q = 1$ and hence φ is faithful. □

2 Setup for Theorem (E)

Theorem 2.1 (Theorem E)**.** *Let $\mu \in Prob(G)$ be any K-invariant Borel probability measure which is equivalent to the Haar measure. Let $(\mathcal{M}, \varphi)$ be any ergodic (G, μ)-von Neumann algebra. Then the following dichotomy holds*

- *Either φ is G-invariant,*
- *Or there exists a proper parabolic subgroup $P \subseteq Q \subseteq G$ and a G-equivariant normal $*$-embedding $\Theta : L^\infty(G/Q, \nu_Q) \to \mathcal{M}$ such that $\varphi \circ \Theta = \nu_Q$.*

Standing assumption: We assume throughout the talks that φ is not G-invariant.

Lemma 2.2. *For the proof we may assume that $\mathcal{M} \subseteq B(H)$ where H is a separable Hilbert space such that there exists $\xi \in H$ with $\varphi(x) = \langle x\xi, \xi\rangle$ for all $x \in \mathcal{M}$.*

Proof. Since φ is not G-invariant there exist a $x \in \mathcal{M}$ with $G \ni g \to \varphi(\sigma_g^{-1}(x))$ not constant. A connected Lie group is second countable, so we can find a dense set $S \subseteq G$, and then the countable set

$$\mathrm{span}_{\mathbb{Q}+i\mathbb{Q}}\{a_1 \cdots a_n \mid a_i \in Sx \cup Sx^*\ , n \in \mathbb{N}\}$$

is a weakly dense subset of the von Neumann algebra $\mathcal{M}_{sep}$ generated by Gx. It follows that the GNS representation (H, ρ, ξ) of the state $\varphi|_{\mathcal{M}_{sep}}$ satisfies that H is separable. Since $\varphi|_{\mathcal{M}_{sep}}$ is normal and faithful $\rho : \mathcal{M}_{sep} \to \rho(\mathcal{M}_{sep})$ is a vNA isomorphism, and by choice then $(\rho(\mathcal{M}_{sep}), \langle \cdot\xi, \xi\rangle)$ is still an ergodic (G, μ)-von Neumann algebra with $\langle \cdot\xi, \xi\rangle$ not G-invariant. If the Theorem is true for $\rho(\mathcal{M}_{sep})$, the map $\rho^{-1} \circ \Theta$ will give the desired embedding. □

Lemma 2.3. *We can pick a globally G-invariant σ-weakly dense separable unital C^*-algebra $\mathcal{A} \subseteq \mathcal{M}$ such that $G \curvearrowright \mathcal{A}$ is norm-continuous.*

Proof. The set $\{x \in \mathcal{M} \mid g \mapsto \sigma_g(x) \text{ is norm continuous}\}$ contains for every $x \in \mathcal{M}$ and $f \in C_c(G)$ the element $\int_G f(g)\sigma_g(x)\mathrm{d}m_g(g)$, and a well chosen net of such f supported around $e \in G$ would converge σ-weakly to x, proving that the set is σ-weakly dense in $\mathcal{M}$. Since $\mathcal{M}$ has separable predual (H is separable) we can find a countable dense set $\mathcal{C}$ in this set. Since G is second countable $\mathcal{A} := C^*(\bigcup_g \sigma_g(\mathcal{C}))$ is separable. □

For simplicity we assume throughout that $G = \mathrm{SL}_3(\mathbb{R})$, and we define subgroups of G by

$$P = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}, \overline{P} = \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix}, V = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, \overline{V} = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix}, S = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}.$$

Lemma 2.4. *The map $\overline{V} \times P \to G : (\overline{v}, p) \mapsto \overline{v}p$ is a homeomorphism onto an open m_G-conull subset of G, under which $m_G|_{\overline{V}P}$ is equivalent to $m_{\overline{V}} \times m_P$.*

Proof. If $\overline{v}_1 p_1 = \overline{v}_2 p_2$ then $\overline{v}_2^{-1}\overline{v}_1 = p_2 p_1^{-1} \in \overline{V} \cap P = \{Id\}$, proving that the map is injective. By considering the multiplication of matrices in $\overline{V}$ and P it is clear this map is a homeomorphism onto its image. If $A = (a_{i,j})_{i,j=1}^3 \in G$ has $a_{1,1} = 0$ it does not lie in $\overline{V}P$. If $a_{1,1} \neq 0$ then

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{-a_{1,2}}{a_{1,1}} & 1 & 0 \\ \frac{-a_{1,3}}{a_{1,1}} & 0 & 1 \end{pmatrix} \cdot A = \begin{pmatrix} a_{1,1} & * & * \\ 0 & a_{2,2} - a_{2,1}\frac{a_{1,2}}{a_{1,1}} & * \\ 0 & * & * \end{pmatrix} := B$$

and this can not lie in $\overline{V}P$ if $a_{2,2}a_{1,1} - a_{2,1}a_{1,2} = 0$. If $a_{2,2}a_{1,1} - a_{2,1}a_{1,2} \neq 0$ there is similarly a $\overline{v} \in \overline{V}$ such that $\overline{v}B \in P$. Hence the $A \in G \setminus \overline{V}P$ are exactly the ones satisfying $a_{1,1} = 0$ and $a_{2,2}a_{1,1} - a_{2,1}a_{1,2} = 0$ which is a closed set of measure 0. The last statement follows by noticing that for every $K \subseteq P$, $U \mapsto m_G(UK)$ defines a left invariant measure on $\overline{V}$, so $m_G(UK) = c_K m_{\overline{V}}(U)$, where $K \mapsto c_K$ will give a right invariant measure on P. □

We now use Lemma 1.1 on the separable C*-algebra $\mathcal{A}$ and the μ-stationary state $\varphi|_A$, to get a G-equivariant measurable map $\beta : G/P \to \mathcal{S}(\mathcal{A}) : w \mapsto \phi_w$ as in that Lemma. Set $\psi = \phi_{eP}$, by G-equivariance ψ is a P-invariant state on $\mathcal{A}$ s.t.

$$\varphi|_{\mathcal{A}} = \int_{G/P} \psi \circ \sigma_g^{-1} \, \mathrm{d}\nu_P(gP) \,.$$

Let $(\pi_\psi, H_\psi, \xi_\psi)$ be the GNS triple associated with $(\mathcal{A}, \psi)$, set $\mathcal{N} = \pi_\psi(\mathcal{A})''$, and denote by ψ the normal state $\langle \cdot \xi_\psi, \xi_\psi \rangle$ on $\mathcal{N}$.
Since ψ is P-invariant, standard GNS-representation arguments gives a continuous action $\sigma^{\mathcal{N}} : P \curvearrowright \mathcal{N}$. We assume throughout for simplicity that ψ is a faithful state on $\mathcal{N}$.

2.1 Step 1: G-equivariantly embed $\mathcal{M}$ into $\mathrm{Ind}_P^G(\mathcal{N})$.

By Lemma 2.4 the restriction of the quotient map $G \xrightarrow{\pi} G/P$ gives a injection $\overline{V} \to G/P$, and since $\nu_P \sim m_G \circ \pi^{-1}$ and $\pi^{-1}(\overline{V}/P) = \overline{V}P$ its image is ν_P-conull. Since

$$m_{\overline{V}}(U) = 0 \iff m_G(UP) = 0 \iff m_G(\pi^{-1}(\pi(U))) = 0 \iff \nu_P(\pi(U)) = 0$$

we see that $\nu_{\overline{V}} := \nu_P \circ \pi \sim m_{\overline{V}}$. Now let $\tau : G/P \to G$ denote a section satisfying $\tau(\overline{v}P) = \overline{v}$ for $\overline{v} \in \overline{V}$, which is then measurable up to a null set.
Recall from seminar 6 that we can identify the system $G \curvearrowright \mathrm{Ind}_P^G(\mathcal{N})$ with the action $\tilde{\sigma} : G \curvearrowright L^\infty(G/P)\overline{\otimes}\mathcal{N}$ given by

$$\tilde{\sigma}_g(F)(w) = \sigma^{\mathcal{N}}_{c_\tau(g,g^{-1}w)}(F(g^{-1}w)) \text{ for } F \in L^\infty(G/P)\overline{\otimes}\mathcal{N},\, g \in G \text{ and } w \in G/P.$$

Using our measure space isomorphism we get a von Neumann algebra isomorphism

$$B : \; L^\infty(G/P)\overline{\otimes}\mathcal{N} \xrightarrow{\sim} L^\infty(\overline{V})\overline{\otimes}\mathcal{N} \;:\; B(f)(\overline{v}) = f(\overline{v}P) \,.$$

Since

$$\begin{aligned} c_\tau(p, p^{-1}\overline{v}P) = \tau(\overline{v}P)^{-1}p\tau(p^{-1}\overline{v}P) &= \begin{cases} \overline{v}^{-1}pp^{-1}\overline{v} & \text{if } p \in \overline{V} \\ \tau(\overline{v}P)^{-1}p\tau(p^{-1}\overline{v}pP) & \text{if } p \in S \end{cases} \\ &= \begin{cases} e & \text{if } p \in \overline{V} \\ p & \text{if } p \in S \,, \end{cases} \end{aligned}$$

we can then transport the G-action on $L^\infty(G/P)\overline{\otimes}\mathcal{N}$ to an action on $L^\infty(\overline{V})\overline{\otimes}\mathcal{N}$, and get a very concrete formula for the action of $\overline{P} = S\overline{V}$:

$$\begin{aligned} B(\tilde{\sigma}_p(f))(\overline{v}) = \sigma^{\mathcal{N}}_{c_\tau(p,p^{-1}\overline{v}P)}(f(p^{-1}\overline{v}P)) &= \begin{cases} f(p^{-1}\overline{v}P) & \text{if } p \in \overline{V} \\ \sigma_p^{\mathcal{N}}(f(p^{-1}\overline{v}pP)) & \text{if } p \in S \end{cases} \\ &= \begin{cases} B(f)(p^{-1}\overline{v}) & \text{if } p \in \overline{V} \\ \sigma_p^{\mathcal{N}}(B(f)(p^{-1}\overline{v}p)) & \text{if } p \in S \end{cases} \end{aligned} \tag{2.1}$$

Lemma 2.5 (Lemma 3.1)**.** *The map $\iota : \mathcal{A} \to L^\infty(G/P)\overline{\otimes}\mathcal{N}$ given by $\iota(a)(w) = \pi_\psi(\sigma^{-1}_{\tau(w)}(a))$ for $a \in \mathcal{A}$ and $w \in G/P$ extends to a well defined G-embedding*[1] *$\iota : \mathcal{M} \to L^\infty(G/P)\overline{\otimes}\mathcal{N}$ such that $(\nu_p \otimes \psi) \circ \iota = \varphi$.*

[1]This is Sam's shorthand terminology for a G-equivaraint unital normal injective $*$-homomorphism

Proof. Since $\psi \in \mathcal{N}_*$ is P-invariant we get for $a \in \mathcal{A}$ that

$$(\nu_p \otimes \psi)(\iota(a)) = \int_{G/P} \psi(\sigma_{\tau(w)}^{-1}(a))\, \mathrm{d}\nu_P(w) = \int_{G/P} \psi(\sigma_g^{-1}(a))\, \mathrm{d}\nu_P(gP) = \varphi(a)\,.$$

A straightforward computation which can be found int he notes reveals $\iota(\sigma_g(a)) = \tilde{\sigma}_g(\iota(a))$ for $a \in \mathcal{A}$, hence by continuity it suffices to prove that ι extends to a normal unital $*$-embedding. Since $1_{G/P}$ implements ν_P on $L^2(G/P, \nu_P)$ and ξ_ψ implements ψ on H_ψ, then $1_{G/P} \otimes \xi_\psi$ implements $\nu_p \otimes \psi$ on $H' = L^2(G/P, \nu_P) \otimes H_\psi$. Let p be the projection onto $\overline{\iota(\mathcal{A}) 1_{G/P} \otimes \xi_\psi}$, then $(\iota, PH', 1_{G/P} \otimes \xi_\psi)$ is a GNS-representation for $\varphi|_{\mathcal{A}}$. By Lemma 2.2, Lemma 2.3 and uniqueness of the GNS triple this implies that $\mathcal{A} \ni a \to \iota(a)p \in B(H')$ is unitarily equivalent to the inclusion map $\mathcal{A} \hookrightarrow \mathcal{M}$, and hence we get a normal unital $*$-isomorphism $\mathcal{M} \to \iota(\mathcal{A})''p$. If $f \in L^\infty(G/P, \mathcal{N})$ satisfies $f\iota(a)1_{G/P} \otimes \xi_\psi = 0$ for all $a \in \mathcal{A}$, then

$$f(w)\pi_\psi(\sigma_{\tau(w)}^{-1}(a))\xi_\psi = 0$$

for almost all $w \in G/P$. By continuity and separability of $\mathcal{A}$, we get $f(w)\pi_\psi(\sigma_{\tau(w)}^{-1}(a))\xi_\psi = 0$ for all $a \in \mathcal{A}$ and a.e. $w \in G/P$. Since ξ_ψ is cyclic this proves that the map $\iota(\mathcal{A})'' \to \iota(\mathcal{A})''p : f \mapsto fp$ is a normal unital $*$-isomorphism. ◻

2.2 Step 2: Claim 3.2

Recall that

$$\phi|_{\mathcal{A}} = \int_{G/P} \psi \circ \sigma_g^{-1}\, \mathrm{d}\nu_P(gP)\,.$$

with ψ P-invariant. Since ϕ is not G-invariant, and $\mathcal{A}$ is dense in $\mathcal{M}$, this implies that ψ is not G-invariant. Since G can not contain a dense open subgroup, then $G = \langle \overline{V}, P \rangle$, and it follows that ψ can not be $\overline{V}$-invariant. Write

$$E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & 0 & 1 \end{pmatrix},\ E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & * & 1 \end{pmatrix}$$

then $\overline{V} = \langle E_{21}, E_{31}, E_{32} \rangle$, and hence we can assume WLG that ψ is not E_{32}-invariant. Set

$$V_0 = \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \overline{V}_0 = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & 0 & 1 \end{pmatrix},\ \overline{U}_0 = E_{32},\ s = \begin{pmatrix} 1/4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

then the map $\overline{V}_0 \times \overline{U}_0 \to \overline{V} : (v, u) \mapsto vu$ is a homeomorphism. Now

$$s \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} s^{-1} = \begin{pmatrix} 1 & a/8 & b/8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ s^{-1} \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix} s = \begin{pmatrix} 1 & 0 & 0 \\ a/8 & 1 & 0 \\ b/8 & 0 & 1 \end{pmatrix}$$

and s commutes with $\overline{U_0}$. The isomorphism $L^2(\overline{V}_0) \otimes L^2(\overline{U}_0) \simeq L^2(\overline{V}_0 \times \overline{U}_0) \simeq L^2(\overline{V})$ implements an isomorphism $L^\infty(\overline{V}) = L^\infty(\overline{V}_0) \overline{\otimes} L^\infty(\overline{U}_0)$.

Lemma 2.6 (Claim 3.2)**.** *Let $a \in \mathcal{A} \subseteq \mathcal{M}$. For every $n \in \mathbb{N}$ define $a_n = \frac{1}{n+1} \sum_{k=0}^{n} \sigma_s^k(a) \in \mathcal{A}$. Then*

- $\exists!$ *ψ-preserving faithful conditional expectation $E_s : \mathcal{N} \to \mathcal{N}^s$.*

□ *For every* $(\overline{v}_0,\overline{u}_0)\in\overline{V}$, $\iota(a_n)(\overline{v}_0,\overline{u}_0)\to E_s(\iota(a)(\overline{e},\overline{u}_0))$ *strongly in* $\mathcal{N}$.

△ *Defining the bounded measurable* $f:\overline{V}\to\mathcal{N}^s$ *by* $f((\overline{v}_0,\overline{u}_0))=E_s(\iota(a)(\overline{e},\overline{u}_0))$, *then* $f\in\iota(\mathcal{M})$. *Thus there exists a unique* $a_\infty\in\mathcal{M}$ *with* $\iota(a_\infty)=f$ *and* $a_n\to a_\infty$ *strongly in* $\mathcal{M}$.

Proof. ○: Since $\sigma:P\curvearrowright\mathcal{N}$ is ψ-preserving, we get that $\psi=\psi\circ\sigma_s$. If α is the modular automorphism group for ψ, this implies ψ satisfies the KMS condition for $\sigma_s^{-1}\circ\alpha\circ\sigma_s$, so by faithfulness of ψ, $\sigma_s^{-1}\circ\alpha\circ\sigma_s=\alpha$. Hence α leaves $\mathcal{N}^s$ invariant, and the existence of E_s follows from Tomita-Takesaki Theory. It can be checked that $E_s(a)=PaP$ where P is the orthogonal projection onto $\{\xi\in H_\psi\mid U\xi=\xi\}$ where $\sigma_s=\mathrm{Ad}U$.

□: Fix $(\overline{v}_0,\overline{u}_0)\in\overline{V}$ and $\varepsilon>0$. Since $G\curvearrowright\mathcal{A}$ is norm-continuous and $s^{-k}\overline{v}_0^{-1}s^k\to 0$ for $k\to\infty$, there exists a k_0 such that

$$\begin{aligned}\|\iota(\sigma_s^k(a))(\overline{v}_0,\overline{u}_0)-\iota(\sigma_s^k(a))(\overline{e},\overline{u}_0)\| &= \|\pi_\psi(\sigma_{\overline{u}_0}^{-1}(\sigma_{\overline{v}_0}^{-1}(\sigma_s^k(a))))-\pi_\psi(\sigma_{\overline{u}_0}^{-1}(\sigma_s^k(a)))\|\\ &\le \|\sigma_{s^{-k}\overline{v}_0^{-1}s^k}(a)-a\|\le\varepsilon\end{aligned}$$

for all $k\ge k_0$. Since $\varepsilon>0$ was arbitrary, we conclude

$$\|\iota(a_n)(\overline{v}_0,\overline{u}_0)-\iota(a_n)(\overline{e},\overline{u}_0)\|=\left\|\frac{1}{n+1}\sum_{k=1}^{n}(\iota(\sigma_s^k(a))(\overline{v}_0,\overline{u}_0)-\iota(\sigma_s^k(a))(\overline{e},\overline{u}_0))\right\|\to 0 \tag{2.2}$$

as $n\to\infty$. Since s commutes with $\overline{u}_0$ we get

$$\sigma_s^k(\iota(a)(\overline{e},\overline{u}_0))=\sigma_s^k(\pi_\psi(\sigma_{\overline{u}_0}^{-1}(a)))=\pi_\psi(\sigma_{\overline{u}_0}^{-1}(\sigma_s^k(a)))=\iota(\sigma_s^k(a))(\overline{e},\overline{u}_0)\ .$$

Let U be the unitary implementing σ_s, i.e. $UaU^*=\sigma_s(a)$, then $U\xi_\psi=\xi_\psi$ since the action $\sigma:s^{\mathbb{Z}}\curvearrowright\mathcal{N}$ is ψ-preserving. Hence von Neumann's ergodic theorem implies that

$$\begin{aligned}&\left\|\frac{1}{n+1}\sum_{k=0}^{n}\sigma_s^k(\iota(a)(\overline{e},\overline{u}_0))-E_s(\iota(a)(\overline{e},\overline{u}_0))\right\|_\psi\\ &=\left\|\frac{1}{n+1}\sum_{k=0}^{n}U^k\iota(a)(\overline{e},\overline{u}_0)\xi_\psi-P\iota(a)(\overline{e},\overline{u}_0)\xi_\psi\right\|^2\to 0 \text{ for } n\to\infty.\end{aligned}$$

Combining this with (2.2) we obtain that

$$\lim_n\|\iota(a_n)(\overline{v}_0,\overline{u}_0)-E_s(\iota(a)(\overline{e}_0,\overline{u}_0))\|_\psi=0\ ,$$

and since ψ is a faithful state, we get strong convergence.

△: Set $f:\overline{V}\to\mathcal{N}^s:(\overline{v}_0,\overline{u}_0))\mapsto E_s(\iota(a)((\overline{e},\overline{u}_0)))$. By Lebesgue's dominated convergence we get

$$\|\iota(a_n)-f\|_{\nu_P\otimes\psi}^2=\int_{\overline{V}}\|\iota(a_n)((\overline{v}_0,\overline{u}_0))-f((\overline{v}_0,\overline{u}_0))\|_\psi^2\,\mathrm{d}\nu_{\overline{V}}((\overline{v}_0,\overline{u}_0))\to 0 \text{ for } n\to\infty.$$

This implies that $\iota(a_n)\to f$ strongly in $L^\infty(\overline{V})\overline{\otimes}\mathcal{N}$. This implies that $f\in\iota(\mathcal{M})$, and since ι is an embedding we get a unique $a_\infty=\iota^{-1}(f)\in\mathcal{M}$ such that $a_n\to a_\infty$. □