Lattices of minimal covolume in $\mathrm{SL}_n(\mathbb{R})$

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Abstract

The objective of this paper is to determine the lattices of minimal covolume in $\mathrm{SL}_n(\mathbb{R})$, for $n \geq 3$. The answer turns out to be the simplest one: $\mathrm{SL}_n(\mathbb{Z})$ is, up to automorphism, the unique lattice of minimal covolume in $\mathrm{SL}_n(\mathbb{R})$. In particular, lattices of minimal covolume in $\mathrm{SL}_n(\mathbb{R})$ are non-uniform when $n \geq 3$, contrasting with Siegel's result for $\mathrm{SL}_2(\mathbb{R})$. This answers for $\mathrm{SL}_n(\mathbb{R})$ the question of Lubotzky: is a lattice of minimal covolume typically uniform or not?

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0 Introduction

0.1 A brief history

The study of lattices of minimal covolume in SL_n originated with Siegel's work [Sie45b] on $\mathrm{SL}_2(\mathbb{R})$, which in turn can be traced back to Hurwitz's work [Hur92]. Siegel showed that in $\mathrm{SL}_2(\mathbb{R})$, a lattice of minimal covolume is given by the $(2,3,7)$-triangle group. He raised the question to determine which lattices attain minimum covolume in groups of isometries of higher-dimensional hyperbolic spaces. For $\mathrm{SL}_2(\mathbb{C})$, which acts on hyperbolic 3-space, the minimum among non-uniform lattices was established by Meyerhoff [Mey85]; among all lattices in $\mathrm{SL}_2(\mathbb{C})$, the minimum was exhibited more recently by Gehring, Marshall and Martin [GM09, MM12], and is attained by a uniform lattice.

Lubotzky established the analogous result [Lub90] for $\mathrm{SL}_2\left(\mathbb{F}_q((t^{-1}))\right)$, where this time $\mathrm{SL}_2\left(\mathbb{F}_q[t]\right)$ attains the smallest covolume. Lubotzky observed that in this case, as opposed to the $(2,3,7)$-triangle group in $\mathrm{SL}_2(\mathbb{R})$, the lattice of minimal covolume is not uniform; he then asked whether, for a lattice of minimal covolume in a semi-simple Lie group, the typical situation is to be uniform, or not.

Progress has been made on this question, and Salehi Golsefidy showed [SG09] that for most Chevalley groups G of rank at least 2, $G\left(\mathbb{F}_q[t]\right)$ is the unique (up to isomorphism) lattice of minimal covolume in $G\left(\mathbb{F}_q((t^{-1}))\right)$. Salehi Golsefidy also obtained [SG13] that for most simply connected almost simple groups over $\mathbb{F}_q((t^{-1}))$, a lattice of minimal covolume will be non-uniform (provided Weil's conjecture on Tamagawa numbers holds).

On the other side of the picture, when the rank is 1, Belolipetsky and Emery [Bel04, BE12] determined the lattices of minimal covolume among arithmetic lattices in $\mathrm{SO}(n,1)(\mathbb{R})$ ($n \geq 4$) and showed that they are non-uniform. For $\mathrm{SU}(n,1)(\mathbb{R})$, Emery and Stover [ES14] determined the lattices of minimal covolume among the non-uniform arithmetic ones, but to the best of the author's knowledge, this has not been compared to the uniform arithmetic ones in this case. Unfortunately, in the rank 1 case, it is not known whether a lattice of minimal covolume is necessarily arithmetic.

The above results give a partial answer to the question of Lubotzky in these two respective situations. In this paper, we intend to contribute to the question for $\mathrm{SL}_n(\mathbb{R})$. We show that for $n \geq 3$, up to automorphism, the non-uniform lattice $\mathrm{SL}_n(\mathbb{Z})$ is the unique lattice of minimal covolume in $\mathrm{SL}_n(\mathbb{R})$.

0.2 Outline

The goal of the present paper is to prove the following theorem.

Theorem. *Let $n \geq 3$ and let Γ be a lattice of minimal covolume for some (any) Haar measure in $\mathrm{SL}_n(\mathbb{R})$. Then $\sigma(\Gamma) = \mathrm{SL}_n(\mathbb{Z})$ for some (algebraic) automorphism σ of $\mathrm{SL}_n(\mathbb{R})$.*

The argument relies in an indispensable way on the important work of Prasad [Pra89] and Borel and Prasad [BP89] (there will be multiple references to results contained in these two articles). We will proceed as follows.

We start with a lattice Γ of minimal covolume in $\mathrm{SL}_n(\mathbb{R})$. Using Margulis' arithmeticity theorem and Rohlfs' maximality criterion, we find a number field k, an archimedean place v_0 and a simply connected absolutely almost simple k-group G for which Γ is identified with the normalizer of a principal arithmetic subgroup Λ in $G(k_{v_0})$. The latter means that there is a collection of parahoric subgroups $\{P_v\}_{v\in V_f}$ such that Λ consists precisely of the elements of $G(k)$ whose image in $G(k_v)$ lies in P_v for all $v \in V_f$. This allows us to express the covolume of Γ as $\mu(G(k_{v_0})/\Gamma) = [\Gamma : \Lambda]^{-1}\, \mu(G(k_{v_0})/\Lambda)$.

The factor $\mu(G(k_{v_0})/\Lambda)$ can be computed using Prasad's volume formula [Pra89], and the result depends on the arithmetics of k and of the parahorics P_v, as well as on the quasi-split inner form of G.

On the other hand, the index $[\Gamma : \Lambda]$ can be controlled using techniques developed by Rohlfs [Roh79], and Borel and Prasad [BP89]. The bound depends namely on the first Galois cohomology group of the center of G and on its action on the types of the parahorics P_v.

Once we have an estimate on the covolume of Γ, we can compare it to the covolume of $\mathrm{SL}_n(\mathbb{Z})$ in $\mathrm{SL}_n(\mathbb{R})$. We argue that for the former not to exceed the latter, it must be that k is $\mathbb{Q}$, G is an inner form of SL_n, and all the parahorics are hyperspecial. This is carried out in sections 4-6.

Finally, using local-global techniques, we conclude that Γ must be the image of $\mathrm{SL}_n(\mathbb{Z})$ under some automorphism of $\mathrm{SL}_n(\mathbb{R})$.

0.3 Acknowledgements

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0.4 Notation and preliminaries

The contents of the paper will assume familiarity with the theory of algebraic groups, Bruhat-Tits theory and basic number theory. We refer the reader to [PR94] for an exposition of some of these topics and a more complete list of the available literature.

As much as possible, we will follow the notation adopted by Borel and Prasad in [Pra89] and [BP89].

- $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ respectively denote the sets of strictly positive natural, rational, real and complex numbers. For p a place or a prime, $\mathbb{Q}_p$ denotes the field of p-adic numbers and $\mathbb{Z}_p$ its ring of p-adic integers. $\mathbb{F}_p$ denotes the finite field with p elements.

- In what is to follow, we will fix a number field k of degree m, and V, V_∞ and V_f will always denote the set of places, archimedean places and non-archimedean places of k. We will always normalize each non-archimedean place v so that $\operatorname{im} v = \mathbb{Z}$.
- For $v \in V$, k_v will denote the v-adic completion of k. For $v \in V_f$, $\widehat{k}_v$ is the maximal unramified extension of k_v, $\mathfrak{f}_v$ denotes the residue field of k at v and $q_v = \#\mathfrak{f}_v$ is the cardinality of the latter.
- $\mathbb{A}_k$ denotes the ring of adeles of k, and the adeles of $\mathbb{Q}$ will be abbreviated $\mathbb{A}$.
- When working with the adele points $G(\mathbb{A}_k)$ (or variations of them, e.g. finite adeles) of an algebraic group G, we will freely identify $G(k)$ with its image in $G(\mathbb{A}_k)$ under the diagonal embedding, and vice-versa.
- For l a finite extension of k, we denote D_l the absolute value of the discriminant of l (over $\mathbb{Q}$) and $\mathfrak{d}_{l/k}$ the relative discriminant of l over k; h_l is the class number of l. The units of l will be denoted by U_l, and the subgroup of roots of unity in l by $\mu(l)$.
- G will be a simply connected absolutely almost simple group (of type A_r) defined over k. We denote $r = n-1$ its absolute rank, and for $v \in V_f$, r_v is its rank over $\widehat{k}_v$.
- $\mathscr{G}$ denotes the quasi-split inner k-form of G, l will denote its splitting field.
- SU_n denotes the special unitary group defined over $\mathbb{R}$ associated to the positive-definite hermitian form $|z_1| + \cdots + |z_n|$ on $\mathbb{C}^n$. Its group $\mathrm{SU}_n(\mathbb{R})$ of real points is the usual special unitary group, the unique compact connected simply connected almost simple Lie group of type A_{n-1}.
- ζ denotes Riemann's zeta function.
- For $n \in \mathbb{Z}$, we set $\tilde{n} = 1$ or 2 if n is respectively odd or even.
- For $x \in \mathbb{R}$, $\lceil x \rceil$ denotes the ceiling of x, that is the smallest integer n such that $n \geq x$.
- V_n will denote the quantity $\prod_{i=1}^{n-1} \frac{i!}{(2\pi)^{i+1}}$.

1 The setting

On SL_n, we pick a left-invariant exterior form ω_0 of highest degree which is defined over $\mathbb{Q}$. The form ω_0 induces a left-invariant form on $\mathrm{SL}_n(\mathbb{R})$, also to be denoted ω_0, which in turn induces a left-invariant form on $\mathrm{SU}_n(\mathbb{R})$ through their common Lie algebra. Let $c_0 \in \mathbb{R}$ be such that $\mathrm{SU}_n(\mathbb{R})$ has volume 1 for the Haar measure determined in this way by $c_0\omega_0$; we denote μ_0 the Haar measure given by $c_0\omega_0$ on $\mathrm{SL}_n(\mathbb{R})$.

Computing the covolume of $\mathrm{SL}_n(\mathbb{Z})$ goes back to Siegel [Sie45a], and for this particular measure, it is given by

$$\mu_0(\mathrm{SL}_n(\mathbb{R}) \,/\, \mathrm{SL}_n(\mathbb{Z})) = \left(\prod_{i=1}^{r} \frac{i!}{(2\pi)^{i+1}} \right) \cdot \prod_{i=2}^{n} \zeta(i).$$

(To obtain this, one can for example use [Pra89, thm. 3.7]; see §2 below. For the lattice $\Lambda = \mathrm{SL}_n(\mathbb{Z})$, one can take $P_v = \mathrm{SL}_n(\mathbb{Z}_v)$, so that $e(P_v) = \frac{(q_v-1)q_v^{n^2-1}}{\prod_{i=0}^{n-1}(q_v^n-q_v^i)} = \prod_{i=2}^{n} \frac{1}{1-q_v^{-i}}$ and $\prod_{v\in V_f} e(P_v) = \prod_{i=2}^{n} \zeta(i)$.)

Let Γ be a lattice of minimal covolume for μ_0 in $\mathrm{SL}_n(\mathbb{R})$ (the existence of such a lattice can be obtained using the Kazhdan-Margulis theorem, see for example [Wan72]); in particular, Γ is a maximal lattice. By Margulis' arithmeticity theorem [Mar91, Ch. IX §1.5] and Rohlfs' maximality criterion [BP89, prop. 1.4] combined, there is a number field k, a place $v_0 \in V_\infty$, a simply connected absolutely almost simple group G defined over k, and a parahoric subgroup P_v of $G(k_v)$ for each $v \in V_f$, such that:

(i) $k_{v_0} = \mathbb{R}$

(ii) there is an isomorphism $\iota : \mathrm{SL}_n \to G$ defined over k_{v_0} (in particular, $\mathrm{SL}_n(\mathbb{R}) \cong G(k_{v_0})$)

(iii) $G(k_v)$ is compact for any archimedean place $v \neq v_0$

(iv) the collection $\{P_v\}_{v\in V_f}$ is coherent, i.e. $\prod_{v\in V_\infty} G(k_v) \times \prod_{v\in V_f} P_v$ is an open subgroup of the adele group $G(\mathbb{A}_k)$

(v) $\iota(\Gamma)$ is the normalizer of the lattice $\Lambda = G(k)\cap\iota(\Gamma)$ in $G(k_{v_0})$, and $\Lambda = G(k)\cap\prod_{v\in V_f} P_v$ is the *principal arithmetic subgroup* determined by the collection $\{P_v\}_{v\in V_f}$.

This already imposes the signature of k and of the splitting field l of the quasi-split inner form $\mathscr{G}$ of G. Indeed, we have $k_v \cong \mathbb{R}$ for $v \in V_\infty - \{v_0\}$, otherwise $G(k_v) \cong \mathrm{SL}_n(\mathbb{C})$ is not compact; hence k is totally real. Note that in fact, for each $v \in V_\infty - \{v_0\}$, $G(k_v)$ is isomorphic to $\mathrm{SU}_n(\mathbb{R})$, the unique compact connected simply connected almost simple Lie group of type A_{n-1}.

Recall that since G is of type A, either $l = k$ or l is a quadratic extension of k. Regardless, if $v \in V_\infty - \{v_0\}$, it may not be that l embeds into k_v: indeed, if this happens, then $\mathscr{G}$ splits over k_v, and thus G would be an inner k_v-form of SL_n. This prohibits $G(k_v)$ from being compact, as inner k_v-forms of SL_n are isotropic when $n \geq 3$. Thus, in the former case, when G is an inner k-form, it must be that $V_\infty - \{v_0\}$ is empty, i.e. $l = k = \mathbb{Q}$. In the latter case, when G is an outer k-form, for each $v \in V_\infty - \{v_0\}$ the real embedding $k \to k_v$ extends to two (conjugate) complex embeddings of l. On the other hand, G, hence $\mathscr{G}$, splits over k_{v_0}, thus l embeds in k_{v_0}. Combined, we see in this case that the signature of l is $(2, m-1)$.

On G, we pick a left-invariant exterior form ω of highest degree which is defined over k. The form ω induces a left-invariant form on $G(k_{v_0})$, also to be denoted ω, which in turn induces a left-invariant form on $\mathrm{SU}_n(\mathbb{R})$ through their common Lie algebra. Let $c \in \mathbb{R}$ be such that $\mathrm{SU}_n(\mathbb{R})$ has volume 1 for the Haar measure determined in this way by $c\omega$; we denote μ the Haar measure determined by $c\omega$ on $G(k_{v_0})$. By construction, μ agrees with the measure induced from μ_0 through the isomorphism ι. In what follows, we will freely identify $\mathrm{SL}_n(\mathbb{R})$ with $G(k_{v_0})$, Γ with its image $\iota(\Gamma)$ and μ_0 with μ. With this, we have

$$\mu_0(\mathrm{SL}_n(\mathbb{R})/\Gamma) = \mu(G(k_{v_0})/\Gamma) = [\Gamma : \Lambda]^{-1}\, \mu(G(k_{v_0})/\Lambda).$$

2 Prasad's volume formula

We fix a left-invariant exterior form ω_{qs} defined over k on the quasi-split inner k-form $\mathscr{G}$ of G. As before, ω_{qs} induces for each $v \in V_\infty$ an invariant form on $\mathscr{G}(k_v)$, and in turn on any maximal compact subgroup of $\mathscr{G}(\mathbb{C})$ through their common Lie algebra. (Note again that such a maximal compact subgroup can be identified with $\mathrm{SU}_n(\mathbb{R})$.) For each $v \in V_\infty$, we choose $c_v \in k_v$ such that the corresponding maximal compact subgroup has measure 1 for the Haar measure determined in this way by $c_v\omega_{qs}$.

Let $\varphi : G \to \mathscr{G}$ be an isomorphism, defined over some Galois extension K of k, such that $\varphi^{-1} \circ {}^\gamma\varphi$ is an inner automorphism of G for all γ in the Galois group of K over k. Then φ induces an invariant form $\omega^* = \varphi^*(\omega_{qs})$ on G, defined over k. Once again, ω^* induces for each $v \in V_\infty$ a form on $G(k_v)$ and then a form on any maximal compact subgroup of $G(\mathbb{C})$ through their Lie algebras. It turns out [Pra89, §3.5] that the volume of any such maximal compact subgroup for the Haar measure determined in this way by $c_v\omega^*$ is 1. This implies in particular that the Haar measure determined on $G(k_{v_0})$ by $c_{v_0}\omega^*$ is actually the measure μ that we constructed earlier.

For each $v \in V_\infty$, we endow $G(k_v)$ with the Haar measure μ_v determined by $c_v\omega^*$. As we observed, $\mu_{v_0} = \mu$, and for $v \in V_\infty - \{v_0\}$, $G(k_v)$ is compact, hence $\mu_v(G(k_v)) = 1$ by definition of μ_v. The product $G_\infty = \prod_{v\in V_\infty} G(k_v)$ is then endowed with the product measure $\mu_\infty = \prod_{v\in V_\infty} \mu_v$. The lattice Λ embeds diagonally in G_∞; we will abusively denote its image by Λ as well. If F is a fundamental domain for Λ in $G(k_{v_0})$, then $F_\infty = F \times \prod_{v\in V_\infty-\{v_0\}} G(k_v)$ is a fundamental domain for Λ in G_∞. Therefore

$$\mu_\infty(G_\infty/\Lambda) = \mu_\infty(F_\infty) = \mu_{v_0}(F) \cdot \prod_{v\in V_\infty-\{v_0\}} \mu(G(k_v)) = \mu(G(k_{v_0})/\Lambda).$$

Using this observation, the main result from [Pra89] allows us to compute

$$\mu(G(k_{v_0})/\Lambda) = D_k^{\frac{1}{2}\dim G}(D_l/D_k^{[l:k]})^{\frac{1}{2}\mathfrak{s}(\mathscr{G})}\left(\prod_{i=1}^{r}\frac{i!}{(2\pi)^{i+1}}\right)^{[k:\mathbb{Q}]}\prod_{v\in V_f} e(P_v). \tag{V}$$

Here, l is the splitting field of the quasi-split inner k-form $\mathscr{G}$ of G (l is k or a quadratic extension of k), $r = n-1$ is the absolute rank of G, $\mathfrak{s}(\mathscr{G}) = 0$ if $\mathscr{G}$ is split, otherwise $\mathfrak{s}(\mathscr{G}) = \frac{1}{2}r(r+3)$ if r is even or $\mathfrak{s}(\mathscr{G}) = \frac{1}{2}(r-1)(r+2)$ if r is odd, and $e(P_v) = \frac{q_v^{(\dim\overline{M}_v+\dim\overline{\mathscr{M}}_v)/2}}{\#\overline{M}_v(\mathfrak{f}_v)}$ is the inverse of the volume of P_v for a particular measure. We refer to [Pra89] for the unexplained notation (in the present setting, $S = V_\infty$ consists only of real places).

3 An upper bound on the index

For the convenience of the reader, we briefly recollect the upper bound on the index $[\Gamma : \Lambda]$ developed by Borel and Prasad. The complete exposition, proofs and references are to be found in [BP89, §2 & §5] (in the present setting, $\mathscr{S} = \{v_0\}$, $G' = G$, $\Gamma' = \Gamma$, etc.).

For each place $v \in V_f$, we fix a maximal k_v-split torus T_v of G; we also fix an Iwahori subgroup I_v of $G(k_v)$ such that the chamber in the affine building of $G(k_v)$ fixed by I_v is contained in the apartment corresponding to T_v. We denote by Δ_v the basis determined by I_v of the affine root system of $G(k_v)$ relative to T_v.

The group $\mathrm{Aut}(G(k_v))$, hence also the adjoint group $\overline{G}(k_v)$, acts on Δ_v; we denote by $\xi_v : \overline{G}(k_v) \to \mathrm{Aut}(\Delta_v)$ the corresponding morphism. Let Ξ_v be the image of ξ_v.

Let C be the center of G and $\varphi : G \to \overline{G}$ the natural central isogeny, so that there is an exact sequence of algebraic groups

$$1 \to C \to G \xrightarrow{\varphi} \overline{G} \to 1.$$

This sequence gives rise to long exact sequences (of pointed sets), which we store in the following commutative diagram ($v \in V$).

$$\begin{array}{ccccccccccc} 1 & \longrightarrow & C(k) & \longrightarrow & G(k) & \xrightarrow{\varphi} & \overline{G}(k) & \xrightarrow{\delta} & \mathrm{H}^1(k,C) & \longrightarrow & \mathrm{H}^1(k,G) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & C(k_v) & \longrightarrow & G(k_v) & \xrightarrow{\varphi} & \overline{G}(k_v) & \xrightarrow{\delta_v} & \mathrm{H}^1(k_v,C) & \longrightarrow & \mathrm{H}^1(k_v,G) \end{array} \qquad (\mathrm{L}_v)$$

When $v \in V_f$, we have that $\mathrm{H}^1(k_v, G) = 1$ by a result of Kneser [Kne65] and thus δ_v induces an isomorphism

$$\overline{G}(k_v)/\varphi(G(k_v)) \cong \mathrm{H}^1(k_v, C).$$

Recall that ξ_v is trivial on $\varphi(G(k_v))$. Thus ξ_v induces a map $\mathrm{H}^1(k_v, C) \to \Xi_v$, which we abusively denote by ξ_v as well.

Let $\Delta = \prod_{v \in V_f} \Delta_v$, $\Xi = \bigoplus_{v \in V_f} \Xi_v$ and $\Theta = \prod_{v \in V_f} \Theta_v$, where $\Theta_v \subset \Delta_v$ is the type of the parahoric P_v associated to Λ. Ξ acts on Δ componentwise, and we denote by Ξ_{Θ_v} the stabilizer of Θ_v in Ξ_v and Ξ_Θ the stabilizer of Θ in Ξ. The morphisms ξ_v induce a map

$$\xi : \mathrm{H}^1(k, C) \to \Xi : c \mapsto \xi(c) = (\xi_v(c_v))_{v \in V_f}$$

where c_v denotes the image of c in $\mathrm{H}^1(k_v, C)$. With this, we define

$$\begin{aligned} \mathrm{H}^1(k,C)_\Theta &= \{c \in \mathrm{H}^1(k,C) \mid \xi(c) \in \Xi_\Theta\} \\ \mathrm{H}^1(k,C)'_\Theta &= \{c \in \mathrm{H}^1(k,C)_\Theta \mid c_{v_0} = 1\} \\ \mathrm{H}^1(k,C)_\xi &= \{c \in \mathrm{H}^1(k,C) \mid \xi(c) = 1\}. \end{aligned}$$

Borel and Prasad [BP89, prop. 2.9] use the exact sequence due to Rohlfs

$$1 \to C(k_{v_0})/(C(k) \cap \Lambda) \to \Gamma/\Lambda \to \delta(\overline{G}(k)) \cap \mathrm{H}^1(k,C)'_\Theta \to 1.$$

Since $k_{v_0} = \mathbb{R}$, $C(k_{v_0}) = \{1\}$ or $\{1, -1\}$ depending whether n is odd or even. In particular, it follows that $C(k_{v_0}) = C(k) \cap \Lambda$ and $\Gamma/\Lambda \cong \delta(\overline{G}(k)) \cap \mathrm{H}^1(k,C)'_\Theta$. Also, it is clear that the kernel of ξ restricted to $\delta(\overline{G}(k)) \cap \mathrm{H}^1(k, C)'_\Theta$ is contained in $\delta(\overline{G}(k)) \cap \mathrm{H}^1(k, C)_\xi$, implying that $\#\left(\delta(\overline{G}(k)) \cap \mathrm{H}^1(k, C)'_\Theta\right) \leq \#\left(\delta(\overline{G}(k)) \cap \mathrm{H}^1(k, C)_\xi\right) \cdot \prod_{v \in V_f} \#\Xi_{\Theta_v}$, and in turn,

$$[\Gamma : \Lambda] \leq \#\left(\delta(\overline{G}(k)) \cap \mathrm{H}^1(k, C)_\xi\right) \cdot \prod_{v \in V_f} \#\Xi_{\Theta_v} \leq \# \mathrm{H}^1(k, C)_\xi \cdot \prod_{v \in V_f} \#\Xi_{\Theta_v}. \tag{I}$$

In the next two subsections, we try to control the size of $\delta(\overline{G}(k)) \cap \mathrm{H}^1(k, C)_\xi$. We distinguish the case where G is an inner k-form of SL_n from the case G is an outer k-form. For the former, we follow the argument of [BP89, prop. 5.1]. In the latter, we will adapt to our setting a refinement of the bounds of Borel and Prasad due to Mohammadi and Salehi Golsefidy [MSG12, §4]. Except for minor modifications, all the material in this section can be found in these two sources.

3.1 The inner case

Although in the inner case we have already established that $k = \mathbb{Q}$, we will discuss it for an arbitrary (totally real) field k, as this will be useful to treat the outer case as well. Let us thus assume G is an inner k-form, i.e. (by the classification) G is isomorphic to $\mathrm{SL}_{n'} \mathfrak{D}$ for some central division algebra $\mathfrak{D}$ over k of index $d = n/n'$. Similarly, over k_v, G is isomorphic to $\mathrm{SL}_{n_v} \mathfrak{D}_v$ for some central division algebra $\mathfrak{D}_v$ over k_v of index $d_v = n/n_v$. The center C of G is isomorphic to μ_n, the kernel of the map $\mathrm{GL}_1 \to \mathrm{GL}_1 : x \mapsto x^n$, and thus for any field extension K of k, $\mathrm{H}^1(K, C)$ may (and will in this paragraph) be identified with $K^\times / K^{\times n}$ (where $K^{\times n} = \{x^n \mid x \in K^\times\}$). With this identification, the canonical map $\mathrm{H}^1(k, C) \to \mathrm{H}^1(k_v, C)$ corresponds to the canonical map $k^\times / k^{\times n} \to k_v^\times / k_v^{\times n}$.

The action of $\mathrm{H}^1(k_v, C)$ on Δ_v can be described as follows: Δ_v is a cycle of length n_v, on which $\overline{G}(k_v)$ acts by rotations, i.e. Ξ_v can be identified with $\mathbb{Z}/n_v\mathbb{Z}$. The action of $\mathrm{H}^1(k_v, C)$ is then given by the morphism

$$k_v^\times / k_v^{\times n} \to \mathbb{Z}/n_v\mathbb{Z} : x \mapsto v(x) \mod n_v.$$

From this description, we see that $x \in k_v^\times / k_v^{\times n}$ acts trivially on Δ_v precisely when $v(x) \in n_v\mathbb{Z}$; in particular, if G splits over k_v, x acts trivially if and only if $v(x) \in n\mathbb{Z}$. We can form the exact sequence

$$1 \to k_n / k^{\times n} \to \mathrm{H}^1(k, C)_\xi \xrightarrow{(v)_{v \in V_f}} \bigoplus_{v \in V_f} \mathbb{Z}/n\mathbb{Z},$$

where $k_n = \{x \in k^\times \mid v(x) \in n\mathbb{Z} \text{ for all } v \in V_f\}$. By the above, the image of $\mathrm{H}^1(k, C)_\xi$ lies in the subgroup $\bigoplus_{v \in V_f} n_v\mathbb{Z}/n\mathbb{Z}$. Let T be the set of places $v \in V_f$ where G does not split over k_v, i.e. for which $n_v \neq n$. Then the exact sequence yields

$$\# \mathrm{H}^1(k, C)_\xi \leq \#(k_n / k^{\times n}) \cdot \prod_{v \in T} d_v.$$

The proof of [BP89, prop. 0.12] shows that $\#(k_n / k^{\times n}) \leq h_k \tilde{n} n^{[k:\mathbb{Q}]-1}$, where $\tilde{n} = 1$ or 2 if n is respectively odd or even. In the case $k = \mathbb{Q}$, which will be of interest later, it is indeed clear that $\#(\mathbb{Q}_n / \mathbb{Q}^{\times n}) = \tilde{n}$.

3.2 The outer case

Second, we assume G is an outer k-form. The centers of G and of the quasi-split inner form $\mathscr{G}$ of G are k-isomorphic, hence there is an exact sequence

$$1 \to C \to \mathrm{R}_{l/k}(\mu_n) \xrightarrow{N} \mu_n \to 1, \tag{1}$$

where μ_n denotes the kernel of the map $\mathrm{GL}_1 \to \mathrm{GL}_1 : x \mapsto x^n$ as above, $\mathrm{R}_{l/k}$ denotes the restriction of scalars from l to k, and N is (induced by) the norm map of l/k. The long exact sequence associated to it yields

$$1 \to \mu_n(k)/N(\mu_n(l)) \to \mathrm{H}^1(k, C) \to l_0/l^{\times n} \to 1 \tag{2}$$

where $l_0/l^{\times n}$ denotes the kernel of the norm map $N : l^\times/l^{\times n} \to k^\times/k^{\times n}$. The Hasse principle for simply connected groups allows us to write

$$\begin{array}{ccccc} \overline{G}(k) & \xrightarrow{\quad\delta\quad} & \mathrm{H}^1(k,C) & \longrightarrow & \mathrm{H}^1(k,G) \\ \downarrow & & \downarrow & & \downarrow \wr \\ \prod_{v\in V_\infty} \overline{G}(k_v) & \xrightarrow{(\delta_v)_v} & \prod_{v\in V_\infty} \mathrm{H}^1(k_v,C) & \longrightarrow & \prod_{v\in V_\infty} \mathrm{H}^1(k_v,G). \end{array} \tag{3}$$

If n is odd, we can make the following simplifications: $\mu_n(k) = \{1\}$ and thus $\mathrm{H}^1(k, C) \cong l_0/l^{\times n}$ in (2); using the analogous sequence for k_v, we also have $\mathrm{H}^1(k_v, C) \cong \{1\}$ for $v \in V_\infty$. Thus, in (3), we read that δ is surjective and conclude $\delta(\overline{G}(k)) \cong l_0/l^{\times n}$.

If n is even, a weaker conclusion holds provided l has at least one complex place, i.e. if $V_\infty \neq \{v_0\}$. Indeed, if $v_1 \in V_\infty - \{v_0\}$, so that $l \otimes_k k_{v_1} = \mathbb{C}$, then $(l \otimes_k k_{v_1})^\times/(l \otimes_k k_{v_1})^{\times n} = \{1\}$ and the long exact sequences associated to (1) read

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \mathrm{H}^1(k,C) & \longrightarrow & l_0/l^{\times n} & \longrightarrow & 1 \\ & & \downarrow \wr & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \{\pm 1\} & \xrightarrow{\sim} & \mathrm{H}^1(k_{v_1},C) & \longrightarrow & 1 & \longrightarrow & 1. \end{array} \tag{4}$$

The first row splits, and thus we may identify $\mathrm{H}^1(k, C) \cong \{\pm 1\} \oplus l_0/l^{\times n}$; then $l_0/l^{\times n}$ is precisely the kernel of the canonical map $\mathrm{H}^1(k, C) \to \mathrm{H}^1(k_{v_1}, C)$. Now since the adjoint map $G(k_{v_1}) \to \overline{G}(k_{v_1})$ is surjective (recall that $G(k_{v_1}) \cong \mathrm{SU}_n(\mathbb{R})$), we have in (L_{v_1}) that the image of $\overline{G}(k)$ in $\mathrm{H}^1(k_{v_1}, C)$ is trivial, hence $\delta(\overline{G}(k)) \subset l_0/l^{\times n}$.

If n is even and $V_\infty = \{v_0\}$, then $k = \mathbb{Q}$. We have, for each $v \in V_f$,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \frac{\mu(k)}{N(\mu(l))} & \longrightarrow & \mathrm{H}^1(k,C) & \longrightarrow & l_0/l^{\times n} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \frac{\mu(k_v)}{N(\mu_n(l\otimes k_v))} & \longrightarrow & \mathrm{H}^1(k_v,C) & \longrightarrow & \frac{\ker(N: l\otimes k_v \to k_v)}{(l\ \otimes k_v)^{\times n}} & \longrightarrow & 1. \end{array}$$

We observe that $\mu(k)/N(\mu(l))$ ($\cong \{\pm 1\}$) acts trivially on Δ_v for every $v \in V_f$ (see for example [MSG12, §4]), hence the action factors through $l_0/l^{\times n}$. Thus $\#\,\mathrm{H}^1(k, C)_\xi = 2 \cdot \# l_\xi / l^{\times n}$, where $l_\xi/l^{\times n} = \{x \in l_0/l^{\times n} \mid \xi(x) = 1\}$, so that the bound we establish below will hold with an extra factor $\tilde{n}$ in the case $k = \mathbb{Q}$.

It remains to understand the action of $l_0/l^{\times n}$ on Δ. Let $x \in l$ and let

$$(x) = \prod_{\mathfrak{P}} \mathfrak{P}^{i_{\mathfrak{P}}} \overline{\mathfrak{P}}^{i_{\overline{\mathfrak{P}}}} \cdot \prod_{\mathfrak{p}'} \mathfrak{p}'^{i_{\mathfrak{p}'}} \cdot \prod_{\mathfrak{P}''} \mathfrak{P}''^{i_{\mathfrak{P}''}}$$

be the unique factorization of the fractional ideal of l generated by x, where $(\mathfrak{P}, \overline{\mathfrak{P}})$ (resp. $\mathfrak{p}'$, $\mathfrak{P}''$) runs over the set of primes of l that lie over primes of k that split over l (resp. over inert primes of k, over ramified primes of k). When $x \in l_0$, $N(x) \in k^{\times n}$ and thus n divides $i_{\mathfrak{P}} + i_{\overline{\mathfrak{P}}}$, $2i_{\mathfrak{p}'}$ and $i_{\mathfrak{P}''}$.

Observe that $v \in V_f$ splits over l if and only if l embeds into k_v, that is, if and only if ($\mathscr{G}$ splits over k_v and) G is an inner k_v-form of SL_n. In particular, at such a place v, G is isomorphic to $\mathrm{SL}_{n_v} \mathfrak{D}_v$ for some central division algebra $\mathfrak{D}_v$ over k_v of index $d_v = n/n_v$. In [MSG12, §4], it is shown that when v splits as $\mathfrak{P}\overline{\mathfrak{P}}$ over l, the action of $x \in l_0$ is analogous to the inner case described in 3.1, hence x acts trivially on Δ_v if and only if n divides $d_v i_{\mathfrak{P}}$ (and thus n also divides $d_v i_{\overline{\mathfrak{P}}}$), i.e. $v_{\mathfrak{P}}(x) = 0 \mod n_v$ (and $v_{\overline{\mathfrak{P}}}(x) = 0 \mod n_v$). When v is inert, say v corresponds to $\mathfrak{p}'$, then x acts trivially on Δ_v if and only if n divides $i_{\mathfrak{p}'}$ [MSG12, §4].

Let T be the set of places $v \in V_f$ such that v splits over l and G is not split over k_v, and let T^l be a subset of the finite places of l consisting of precisely one extension of each $v \in T$, so that restriction to k defines a bijection from T^l to T. By the discussion above, we can form an exact sequence

$$1 \to (l_n \cap l_0)/l^{\times n} \to l_\xi/l^{\times n} \xrightarrow{(w)_{w\in T^l}} \bigoplus_{w \in T^l} \mathbb{Z}/n\mathbb{Z},$$

where $l_n = \{x \in l^\times \mid w(x) \in n\mathbb{Z}$ for each normalized finite place w of $l\}$ and $l_\xi/l^{\times n} = \{x \in l_0/l^{\times n} \mid \xi(x) = 1\}$. Moreover, the image of $l_\xi/l^{\times n}$ lies in the subgroup $\bigoplus_{w\in T^l} n_v\mathbb{Z}/n\mathbb{Z}$. Thus, if we assume $k \neq \mathbb{Q}$ (so that we may identify $\delta(\overline{G}(k))$ with a subgroup of $l_0/l^{\times n}$),

$$\#\left(\delta(\overline{G}(k)) \cap \mathrm{H}^1(k, C)_\xi\right) \leq \#\left(l_\xi/l^{\times n}\right) \leq \#\left((l_n \cap l_0)/l^{\times n}\right) \cdot \prod_{v \in T} d_v.$$

We get the concrete bound on the index

$$[\Gamma : \Lambda] \leq h_l \tilde{n}^m n \cdot \prod_{v\in T} d_v \cdot \prod_{v \in V_f} \#\Xi_{\Theta_v}$$

by combining this with (I) and lemma A.1. If $k = \mathbb{Q}$, we have instead

$$[\Gamma : \Lambda] \leq h_l \tilde{n}^{m+1} n \cdot \prod_{v\in T} d_v \cdot \prod_{v \in V_f} \#\Xi_{\Theta_v}.$$

4 The field k is $\mathbb{Q}$

We set $m = [k : \mathbb{Q}]$ and as before, $n = r + 1$. The purpose of this section is to show that $k = \mathbb{Q}$, i.e. $m = 1$.

We start by recalling that if P_v is special (in particular, if it is hyperspecial), i.e. Θ_v consists of a single special (resp. hyperspecial) vertex of Δ_v, then Ξ_{Θ_v} is trivial. Regardless of the type Θ_v, we have $\#\Xi_{\Theta_v} \le \tilde{n}$ unless G is an inner k_v-form of SL_n (say $G \cong SL_{n_v}(\mathfrak{D}_v)$), in which case $\#\Xi_{\Theta_v} \le \#\Delta_v = n_v$, where $n_v - 1$ is the rank of G over k_v. (For example, this can be seen explicitly on all the possible relative local Dynkin diagrams Δ_v for $G(k_v)$, enumerated in [Tit79, §4] or [MSG12, §2]. In the inner case, the Dynkin diagram is a cycle on which the adjoint group acts as rotations.)

By a result of Kneser [Kne65], G is quasi-split over the maximal unramified extension $\widehat{k}_v$ of k_v for any $v \in V_f$. This means that over $\widehat{k}_v$, G is isomorphic to $\mathscr{G}$. The quasi-split k-forms of simply connected absolutely almost simple groups of type A_{n-1} are well understood [Tit66]: either $\mathscr{G} \cong \mathrm{SL}_n$, or $\mathscr{G} \cong \mathrm{SU}_{n,l}$, the special unitary group associated to the split hermitian form on l^n, where l is a quadratic extension of k equipped with the canonical involution (incidentally, l is the splitting field of $\mathrm{SU}_{n,l}$, in accordance with the notation introduced). Thus, over $\widehat{k}_v$, only these two possibilities arise for G. (Nonetheless, $\mathscr{G}$ might split over $\widehat{k}_v$; in fact, it does so except at finitely many places.) In particular, the rank r_v of G over $\widehat{k}_v$ is either r, or the ceiling of $r/2$.

4.1 The inner case

The case where G is an inner k-form of SL_n (i.e. when $l = k$) has been treated in section 1. We observed that if G is an inner k_v-form of SL_n for some $v \in V_\infty$, then $G(k_v)$ cannot be compact. This forced $V_\infty = \{v_0\}$ and thus $k = \mathbb{Q}$.

4.2 The outer case

Here we settle the case where G is an outer k-form of SL_n, i.e. when $[l : k] = 2$. We observed in section 1 that l has two real embeddings (extending $k \to k_{v_0}$) and $m-1$ pairs of conjugate complex embeddings. Suppose that $m > 1$.

Let T be the finite set of places $v \in V_f$ such that v splits over l and G is not split over k_v. Then, according to section 3.2, we have

$$[\Gamma : \Lambda] \le h_l \tilde{n}^m n \cdot \prod_{v \in T} d_v \cdot \prod_{v \in V_f} \#\Xi_{\Theta_v}$$

where $\tilde{n} = 1$ or 2 if n is odd or even, and h_l denotes the class number of l. Combined with (V), we find (abbreviating $V_n = \prod_{i=1}^{n-1} \frac{i!}{(2\pi)^{i+1}}$)

$$\mu(G(k_{v_0})/\Gamma) \ge \tilde{n}^{-m} n^{-1} h_l^{-1} D_k^{\frac{n^2-1}{2}} (D_l/D_k^2)^{\frac{1}{2}\mathfrak{s}(\mathscr{G})} V_n^m \cdot \prod_{v \in T} d_v^{-1} \cdot \prod_{v \in V_f} \#\Xi_{\Theta_v}^{-1} \cdot \prod_{v \in V_f} e(P_v).$$

We use [Pra89, prop. 2.10, rem. 2.11] and the observations made at the begining of section 4 to study the local factors of the right-hand side.

(i) If $v \in T$, then we use $e(P_v) \geq (q_v - 1)q_v^{(n^2 - n^2 d_v^{-1} - 2)/2}$ to obtain $d_v^{-1} \cdot \#\Xi_{\Theta_v}^{-1} \cdot e(P_v) \geq n^{-1} \cdot (q_v - 1)q_v^{n^2/4-1} > 1$ when $n \geq 4$. When $n = 3$, then $d_v = 3$ and we also have $d_v^{-1} \cdot \#\Xi_{\Theta_v}^{-1} \cdot e(P_v) \geq n^{-1} \cdot (q_v - 1)q_v^{n^2/3-1} > 1$ (lemma A.2).

(ii) If $v \notin T$ but P_v is special, then $\#\Xi_{\Theta_v} = 1$ and $e(P_v) > 1$, thus $\#\Xi_{\Theta_v}^{-1} \cdot e(P_v) > 1$.

(iii) If $v \notin T$, P_v is not special and G is not split over k_v, then we use that $e(P_v) \geq (q_v + 1)^{-1} q_v^{r_v + 1}$ to obtain $\#\Xi_{\Theta_v}^{-1} \cdot e(P_v) \geq \tilde{n}^{-1} \cdot (q_v + 1)^{-1} q_v^{\lceil (n-1)/2 \rceil + 1} > 1$ (lemma A.3).

(iv) If $v \notin T$, P_v is not special but G splits over k_v, then P_v is properly contained in a hyperspecial parahoric H_v. There is a canonical surjection $H_v \to \mathrm{SL}_n(\mathfrak{f}_v)$, under which the image of P_v is the proper parabolic subgroup $\overline{P}_v$ of $\mathrm{SL}_n(\mathfrak{f}_v)$ whose type consists of the vertices belonging to the type of P_v in the Dynkin diagram obtained by removing the vertex corresponding to H_v in the affine Dynkin diagram of $G(k_v)$. In particular, it follows that $[H_v : P_v] = [\mathrm{SL}_n(\mathfrak{f}_v) : \overline{P}_v]$ and we may compute using lemma A.14
$$e(P_v) = [H_v : P_v] \cdot e(H_v) > [H_v : P_v] > q^{n-1}.$$
Hence $\#\Xi_{\Theta_v}^{-1} \cdot e(P_v) > n^{-1} q^{n-1} > 1$.

Multiplying all the factors together, we have that
$$\prod_{v \in T} d_v^{-1} \cdot \prod_{v \in V_f} \#\Xi_{\Theta_v}^{-1} \cdot \prod_{v \in V_f} e(P_v) > 1$$
and we can thus write
$$\mu(G(k_{v_0})/\Gamma) > \tilde{n}^{-m} n^{-1} h_l^{-1} D_k^{\frac{n^2-1}{2}} (D_l / D_k^2)^{\frac{1}{2}\mathfrak{s}(\mathscr{G})} V_n^m. \tag{5}$$

Recall that D_l/D_k^2 is the norm of the relative discriminant $\mathfrak{d}_{l/k}$ of l over k; in particular, D_l/D_k^2 is a positive integer. Note also that $\mathfrak{s}(\mathscr{G}) \geq 5$ if $n \geq 3$. We combine this with two number-theoretical bounds: from the results in [BP89, §6], we use that
$$h_l^{-1} D_l \geq \frac{1}{100} \left(\frac{12}{\pi} \right)^{2m};$$
from Minkowski's geometry of numbers (see for example [Sam70, §4.3]), we recall (k is totally real)
$$D_k^{\frac{1}{2}} \geq \frac{m^m}{m!}.$$
Altogether, we obtain
$$\begin{aligned} \mu(G(k_{v_0})/\Gamma) &> \frac{1}{100 \tilde{n}^m} \left(\frac{12}{\pi} \right)^{2m} D_k^{\frac{n^2-5}{2}} (D_l / D_k^2)^{\frac{1}{2}\mathfrak{s}(\mathscr{G}) - 1} V_n^m n^{-1} \\ &\geq \frac{1}{100 \tilde{n}^m} \left(\frac{12}{\pi} \right)^{2m} \left(\frac{m^m}{m!} \right)^{n^2-5} V_n^m n^{-1}. \end{aligned} \tag{6}$$

We consider the function $M : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ defined by

$$M(m,n) = \frac{1}{100\tilde{n}^m} \left(\frac{12}{\pi}\right)^{2m} \left(\frac{m^m}{m!}\right)^{n^2-5} \left(\prod_{i=1}^{n-1} \frac{i!}{(2\pi)^{i+1}}\right)^{m-1} n^{-1}.$$

(As V_n appears once as a factor in the covolume of $\mathrm{SL}_n(\mathbb{Z})$, we dropped its exponent above by one.) The function M is strictly increasing in both variables, provided $m \geq 2$ and $n \geq 6$ (lemma A.4). In consequence, if $m \geq 2$, $n \geq 9$,

$$\frac{\mu(G(k_{v_0})/\Gamma)}{\mu(\mathrm{SL}_n(\mathbb{R})/\,\mathrm{SL}_n(\mathbb{Z}))} > \frac{M(m,n)}{\prod_{i=2}^{n} \zeta(i)} > \frac{M(2,9)}{\prod_{i=2}^{\infty} \zeta(i)} > 1,$$

(cf. lemma A.13) and Γ is not of minimal covolume.

In a similar manner, we would like to show that m cannot be large. To this end, Odlyzko's bounds on discriminants [Odl76, table 4] are well-suited. We have

$$D_k^{\frac{1}{2}} > A^m \cdot E, \text{ with } A = 29.534^{\frac{1}{2}} \text{ and } E = e^{-4.13335}.$$

Combining with (6), we obtain

$$\mu(G(k_{v_0})/\Gamma) > \frac{1}{100\tilde{n}^m} \left(\frac{12}{\pi}\right)^{2m} (A^m E)^{n^2-5} V_n^m n^{-1}.$$

We consider the function $M' : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ defined by

$$M'(m,n) = \frac{1}{100\tilde{n}^m} \left(\frac{12}{\pi}\right)^{2m} (A^m E)^{n^2-5} \left(\prod_{i=1}^{n-1} \frac{i!}{(2\pi)^{i+1}}\right)^{m-1} n^{-1}.$$

M' is also strictly increasing in both variables, provided $m \geq 4$ and $n \geq 4$ (lemma A.6). This means that if $m \geq 6$, $n \geq 4$,

$$\frac{\mu(G(k_{v_0})/\Gamma)}{\mu(\mathrm{SL}_n(\mathbb{R})/\,\mathrm{SL}_n(\mathbb{Z}))} > \frac{M'(m,n)}{\prod_{i=2}^{n} \zeta(i)} > \frac{M'(6,4)}{\prod_{i=2}^{\infty} \zeta(i)} > 1,$$

(cf. table A.7 and lemma A.13) and Γ is not of minimal covolume.

We may thus restrict our attention to the range $4 \leq n \leq 8$ and $2 \leq m \leq 5$ (we will treat the case $n = 3$ with a separate argument at the end of this section). By further sharpening our estimates on the discriminant, we will show that all these values are excluded as well, forcing $m = 1$.

From the bound (6) and the estimate $\mu(G(k_{v_0})/\Gamma) \leq \mu(\mathrm{SL}_n(\mathbb{R})/\,\mathrm{SL}_n(\mathbb{Z})) < 2.3 \cdot V_n$ (A.13), we deduce an upper bound on the discriminant of k:

$$\begin{aligned} D_k &< \left(230\tilde{n}^m \left(\frac{\pi}{12}\right)^{2m} (D_l/D_k^2)^{1-\frac{1}{2}\mathfrak{s}(\mathscr{G})} V_n^{1-m} n\right)^{\frac{2}{n^2-5}} \qquad (7) \\ &\leq \left(230\tilde{n}^m \left(\frac{\pi}{12}\right)^{2m} V_n^{1-m} n\right)^{\frac{2}{n^2-5}} =: C(m,n). \end{aligned}$$

As can be seen by comparing the values of C (table A.8) with the smallest discriminants (table A.9), this bound already rules out $n \geq 7$. We use these two tables to obtain information about D_k. A lower bound on D_k in turn will give us a bound on the relative discriminant: using (6) again,

$$D_l/D_k^2 < \left(230\tilde{n}^m \left(\frac{\pi}{12}\right)^{2m} D_k^{\frac{5-n^2}{2}} V_n^{1-m} n\right)^{\frac{2}{s(\mathscr{G})-2}}. \tag{8}$$

We proceed to rule out all values of m. In what follows, unless specified otherwise, any bound on D_k is obtained using (7), (A.8) or (A.9), and any upper bound on D_l/D_k^2 using (8). Claims made on the existence of a field l satisfying certain conditions are always made with the underlying assumption that l is a quadratic extension of k of signature $(2, m-1)$.

$\underline{m=5}$ gives $14641 \leq D_k \leq 15627$ (and $n = 4$). A quick look in the online database of number fields [JR14] shows[1] that there is only one such field (with $D_k = 14641$). Now for l, Odlyzko's bound [Odl76, table 4] reads

$$D_l > (29.534)^2 \cdot (14.616)^8 \cdot e^{-8.2667} \geq 4.66756 \cdot 10^8$$

and in particular, we compute that $D_l/D_k^2 \geq 2.177$ (hence $D_l/D_k^2 \geq 3$). On the other hand, (8) yields

$$D_l/D_k^2 < 1.271,$$

ruling out this case.

$\underline{m=4}$ gives $725 \leq D_k \leq 1741$ (and $n = 4$). A quick look in the database [JR14] shows that there are three fields satisfying this requirement, with discriminants respectively 725, 1125, 1600.

(i) If $D_k = 1600$, then $D_l/D_k^2 < 1.365$, hence $D_l = D_k^2 = 2560000$. But, as observed in the database, there are no fields l of signature $(2,3)$ with $D_l \leq 3950000$.

Unfortunately, the database has no complete records for fields with signature $(2,3)$ and discriminants past 3950000. We will thus need to refine our bounds to be able to treat the two other possible values for D_k. First, we go back to our bound on the class number h_l: as in [BP89, §6], we use Zimmert's bound $R_l \geq 0.04 \cdot e^{2\cdot 0.46+(m-1)\cdot 0.1}$ on the regulator of l along with the Brauer-Siegel theorem (with $s = 2$) to deduce

$$h_l \leq 100 \cdot e^{-0.82-0.1\cdot m} \cdot (2\pi)^{-2m} \cdot \zeta(2)^{2m} \cdot D_l \leq 29.523 \cdot \left(\frac{\pi}{12}\right)^8 \cdot D_l.$$

Using this, we may rewrite the bound (8) as

$$D_l/D_k^2 < \left(67.9029\tilde{n}^4 \left(\frac{\pi}{12}\right)^8 D_k^{\frac{5-n^2}{2}} V_n^{-3} n\right)^{\frac{2}{s(\mathscr{G})-2}}.$$

[1] The database [JR14] provides a certificate of completeness for certain queries. All allusions made here refer to searches that are proven complete. However, it is important to note that in [JR14], class numbers are computed assuming the generalized Riemann hypothesis (the rest of the data being unconditional). The class numbers referred to in this paper were therefore all verified using PARI/GP's `bnfcertify` command. A PARI/GP script of this process is available on the author's page (`math.ucsd.edu/~fthilman/`).

(ii) If $D_k = 1125$, then our new bound yields $D_l/D_k^2 \leq 2$, hence $D_l \leq 2D_k^2 = 2531250$ and this is ruled out by the database.

(iii) If $D_k = 725$, then our new bound yields $D_l/D_k^2 \leq 11$, hence $D_l \leq 11D_k^2 = 5781875$. Selmane [Sel99] has computed all fields of signature $(2,3)$ that possess a proper subfield and have discriminant $D_l \leq 6688609$. It turns out that among those, only the field with discriminant -5781875 can be an extension of k. As observed in the online database, this field has class number 1. Substituting this information in (5), we see that the right-hand side exceeds $2.3 \cdot V_n$.

<u>$m = 3$</u> gives $49 \leq D_k \leq 194$ (and $n = 4$ or 5). A quick look in the database [JR14] shows that there are four fields satisfying this requirement, with discriminants respectively 49, 81, 148, 169.

(i) If $D_k = 169$, then $D_l/D_k^2 < 1.661$ hence $D_l = D_k^2 = 28561$. There are no fields l with $D_l \leq 28000$.

(ii) If $D_k = 148$, then $D_l/D_k^2 \leq 2$. There are no fields l with $D_l/148^2 = 1$ or 2.

(iii) If $D_k = 81$, then $D_l/D_k^2 \leq 24$. An extensive search in the database shows that this can only be satisfied by one field l, with discriminant $D_l = 81^2 \cdot 17$. It has class number $h_l = 1$, hence we may substitute this information in (5) and compute that the right-hand side exceeds $2.3 \cdot V_n$.

(iv) If $D_k = 49$, then $D_l/D_k^2 \leq 155$. An extensive search in the database shows that there are 6 fields l satisfying this condition. They correspond to $D_l/D_k^2 = 13, 29, 41, 64, 97$ or 113, and all have class number 1. Then, in (5), the right-hand side again exceeds $2.3 \cdot V_n$ (note that it suffices to check this for the smallest value of D_l/D_k^2).

<u>$m = 2$</u> gives $5 \leq D_k \leq 21$ (and $4 \leq n \leq 6$). It is well known (and can be observed in the database [JR14]) that there are 6 fields satisfying this requirement, with discriminants respectively 5, 8, 12, 13, 17, 21. From (8), we see that $D_l/D_k^2 \leq 214$, 38, 8, 6, 2, 1 respectively.

(i) If $D_k = 21$ or 17, we observe that $D_l \leq 578$. There are no fields with $D_l \leq 578$ that can be extensions of k in these cases.

(ii) If $D_k = 13$, then the database exhibits only one possible field l with $D_l = 13^2 \cdot 3$. This field has trivial class group, and using this information in (5), we see that the right-hand side exceeds $2.3 \cdot V_n$.

(iii) If $D_k = 12$, then there are again no fields with $D_l \leq 8D_k^2$.

(iv) If $D_k = 8$, then there are 11 candidates l with $D_l \leq 38 \cdot 8^2$, and all have trivial class group. The one with smallest relative discriminant has $D_l/D_k^2 = 7$. For this field (hence for all of them), the right-hand side of (5) is again too large.

(v) If $D_k = 5$, there are 25 candidates l with $D_l \leq 214 \cdot 5^2$, and all have trivial class group. The one with smallest relative discriminant has $D_l = 11$. This field (hence all of them) is one more time excluded by (5).

It remains to deal with the case $n = 3$. First, we proceed as above, using lemma A.6, $M'(16,3) \simeq 4.6751...$, and $\zeta(2)\cdot\zeta(3) < 1.97731$ to see that

$$\frac{\mu(G(k_{v_0})/\Gamma)}{\mu(\mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z}))} > \frac{M'(m,3)}{\zeta(2)\cdot\zeta(3)} > 1$$

provided $m \geq 16$. Hence we may restrict our attention to the range $2 \leq m \leq 15$.

Unfortunately, this bound on the degree of k is too large to allow us to work with a number field database. Of course, the reason this bound is large is that the powers of D_k and D_l appearing in (5) are very small. In turn, the bound we used for the class number h_l was very greedy in terms of D_l, aggravating the situation. In fact, we can use (5) and one of Odlyzko's bounds [Odl76] for D_l to obtain a lower bound on h_l:

$$h_l \geq \frac{D_k^{-1} D_l^{\frac{5}{2}} V_3^{m-1}}{3\cdot\zeta(2)\cdot\zeta(3)} \geq \frac{D_l^2 V_3^{m-1}}{3\cdot\zeta(2)\cdot\zeta(3)} > \frac{(25.465^2\cdot 13.316^{2m-2}\cdot e^{-7.0667})^2\cdot V_3^{m-1}}{3\cdot\zeta(2)\cdot\zeta(3)}. \tag{9}$$

We record the values of this bound in table A.10 (for small values of m, we used the actual minimum for D_l to obtain this lower bound for h_l).

To solve this issue, we use the following trick. The Hilbert class field L of l has degree $[L:\mathbb{Q}] = 2mh_l$, signature $(2h_l,(m-1)h_l)$ and discriminant $D_L = D_l^{h_l}$. Hence, when the class number is large, we can use Odlyzko's bounds [Odl76] for D_L in order to improve our bounds on D_l. Namely, we have

$$D_l = D_L^{\frac{1}{h_l}} > 60.015^2\cdot 22.210^{2m-2}\cdot e^{\frac{-80.001}{h_l}}.$$

We record this bound for D_l in table A.11.

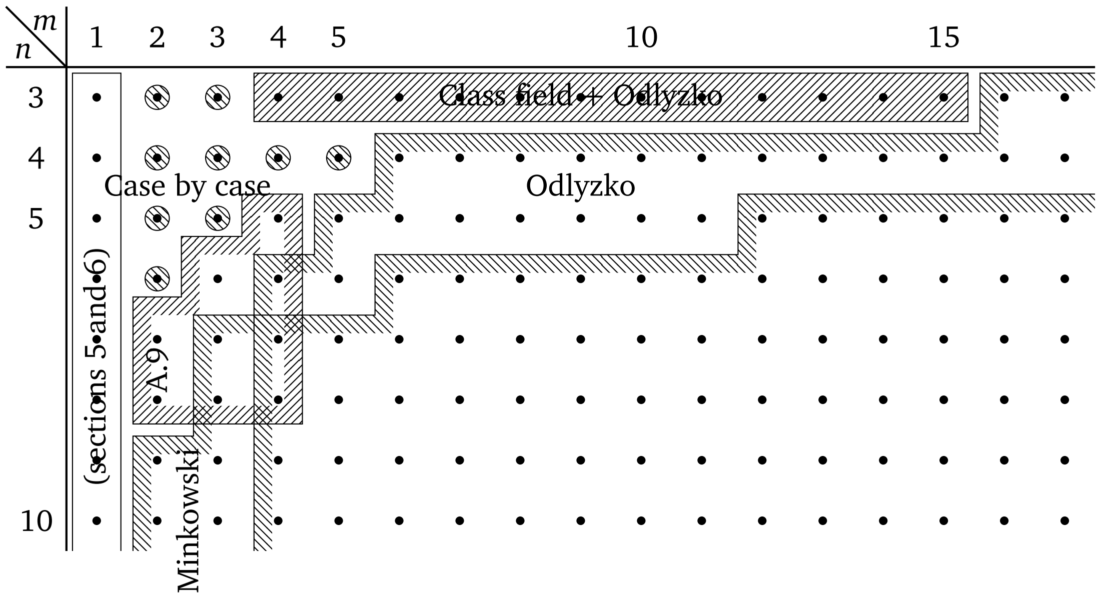
Now using $D_l \geq D_k^2$, we may rewrite (6) as

$$\zeta(2)\cdot\zeta(3)\cdot V_3 > \mu(G(k_{v_0})/\Gamma) > \frac{1}{300}\left(\frac{12}{\pi}\right)^{2m} D_l\cdot V_3^m$$

and check that this inequality contradicts the bound in table A.11 as soon as $m \geq 4$. For $m = 3$ and $m = 2$, the bound reads respectively $D_l \leq 4578732$ and $D_l \leq 13643$.

Finally, to treat the remaining two cases, we can use the online database [JR14]. If $m = 3$, we observe that all fields of signature $(2,2)$ with discriminant $D_l \leq 4578732$ have class number either $h_l = 1$ or $h_l = 2$; this contradicts (9) and table A.10. Similarly, if $m = 2$, we observe in the database that all fields of signature $(2,1)$ with discriminant $D_l \leq 13643$ also have class number either $h_l = 1$ or $h_l = 2$. This is again a contradiction to (9) and table A.10.

Remark. Below is a summary of the various discriminant bounds that were used in this section to exclude a given pair (m,n) from giving rise to a lattice of minimal covolume.



5 G is an inner form of SL_n

The purpose of this section is to show that G is an inner k-form of SL_n, i.e. that $\mathscr{G}$ splits over k. Let us thus suppose, for contradiction, that $[l:k] > 1$.

We have shown in section 4 that $k = \mathbb{Q}$, so that the bounds (5) and (6) obtained in 4.2 can be adapted as follows: (the extra factor $\tilde{n}$ is due to the correction in the index bound when $k = \mathbb{Q}$, cf. section 3.2)

$$\begin{aligned}\mu(G(k_{v_0})/\Gamma) &> \tilde{n}^{-2} n^{-1} h_l^{-1} D_l^{\frac{1}{2}\mathfrak{s}(\mathscr{G})} V_n \\ &\geq \frac{1}{100\tilde{n}^2}\left(\frac{12}{\pi}\right)^2 D_l^{\frac{1}{2}\mathfrak{s}(\mathscr{G})-1} V_n n^{-1}.\end{aligned}$$

First, let us assume that $h_l \neq 1$. Since l is totally real, this implies $D_l \geq 40$. Note that $\mathfrak{s}(\mathscr{G}) \geq \frac{1}{2}(r^2 + r - 2) = \frac{1}{2}(n^2 - n - 2)$. Therefore

$$\mu(G(k_{v_0})/\Gamma) > \frac{1}{100\tilde{n}^2}\left(\frac{12}{\pi}\right)^2 40^{\frac{1}{4}(n^2-n-6)} V_n n^{-1}.$$

We consider the function $N : \mathbb{N} \to \mathbb{R}$ defined by

$$N(n) = \frac{1}{100\tilde{n}^2}\left(\frac{12}{\pi}\right)^2 40^{\frac{1}{4}(n^2-n-6)} n^{-1}.$$

The function N is strictly increasing, provided $n \geq 2$ (lemma A.12). In consequence, if $n \geq 4$, then $N(n) \geq N(4) \simeq 2.30692...$ and thus

$$\frac{\mu(G(k_{v_0})/\Gamma)}{\mu(\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z}))} > \frac{N(n)}{\prod_{i=2}^{n}\zeta(i)} > \frac{N(4)}{\prod_{i=2}^{\infty}\zeta(i)} > 1,$$

hence Γ is not of minimal covolume. For $n = 3$ we notice that $\mathfrak{s}(\mathscr{G}) = 5$, so that

$$\mu(G(k_{v_0})/\Gamma) > \frac{1}{300}\left(\frac{12}{\pi}\right)^2 40^{\frac{3}{2}} \cdot V_3 > 12.3035 \cdot V_3$$

and Γ is not of minimal covolume.

Second, if $h_l = 1$, then at least $D_l \geq 5$ and we may consider the function $N' : \mathbb{N} \to \mathbb{R}$ defined by

$$N'(n) = \tilde{n}^{-2} n^{-1} 5^{\frac{1}{4}(n^2-n-2)}.$$

The function N' is strictly increasing (lemma A.12) and $N'(4) \simeq 3.49385...$, thus

$$\frac{\mu(G(k_{v_0})/\Gamma)}{\mu(\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z}))} > \frac{N(n)}{\prod_{i=2}^{n} \zeta(i)} > \frac{N(4)}{\prod_{i=2}^{\infty} \zeta(i)} > 1,$$

and Γ is not of minimal covolume. For $n = 3$, we use again that $\mathfrak{s}(\mathscr{G}) = 5$ to see that

$$\mu(G(k_{v_0})/\Gamma) > \frac{1}{3} \cdot 5^{\frac{5}{2}} \cdot V_3 > 18.6338 \cdot V_3$$

and Γ is not of minimal covolume. This forces $l = k$ and G to be an inner form.

6 The parahorics P_v are hyperspecial and G splits at all places

So far, we have established that $k = l = \mathbb{Q}$ and G is an inner k-form of SL_n; thus, G is isomorphic to $\mathrm{SL}_{n'} \mathfrak{D}$ for some central division algebra $\mathfrak{D}$ over k of index $d = n/n'$. Similarly, over k_v, G is isomorphic to $\mathrm{SL}_{n_v} \mathfrak{D}_v$ for some central division algebra $\mathfrak{D}_v$ over k_v of index $d_v = n/n_v$. Recall that T is the finite set of places $v \in V_f$ where G does not split over k_v, and let T' be the finite set of places $v \in V_f$ where P_v is not a hyperspecial parahoric; of course, $T \subset T'$. The goal of this section is to show that T' is empty.

According to section 3.1, we have

$$\# \mathrm{H}^1(k, C)_\xi \leq \tilde{n} \cdot \prod_{v \in T} d_v,$$

with $d_v \geq 2$ if $v \in T$. Also, as we noted at the begining of section 4,

$$\#\Xi_{\Theta_v} \leq n_v \text{ if } v \in T, \quad \#\Xi_{\Theta_v} \leq r + 1 = n \text{ if } v \in T', \quad \#\Xi_{\Theta_v} = 1 \text{ otherwise.}$$

Combined with (V) and (I), we obtain

$$\begin{aligned} \mu(G(k_{v_0})/\Gamma) &\geq \tilde{n}^{-1} V_n \cdot \prod_{v \in T} d_v^{-1} \cdot \prod_{v \in T} n_v^{-1} \cdot \prod_{v \in T'-T} n^{-1} \cdot \prod_{v \in V_f} e(P_v) \\ &= \tilde{n}^{-1} V_n \cdot \prod_{v \in T'} n^{-1} \cdot \prod_{v \in V_f} e(P_v). \end{aligned} \tag{10}$$

Recall that for any $v \in V_f$, $e(P_v) > 1$. If $v \in T$, then according to [Pra89, remark 2.11], we have

$$e(P_v) \geq (q_v - 1) q_v^{\frac{1}{2}(n^2 - n^2 d_v^{-1} - 2)} \geq (q_v - 1) q_v^{\frac{1}{4}n^2 - 1}.$$

Now if T is not empty, then by looking at the Hasse invariant of $\mathfrak{D}$, it appears that $d_v \geq 2$ for at least two (finite) places. This means that T has at least two elements, and using lemma A.2, we see that if $n \geq 4$,

$$\prod_{v \in T} n^{-1} e(P_v) \geq (n^{-1}(2-1) \cdot 2^{\frac{1}{4}n^2 - 1}) \cdot (n^{-1}(3-1) \cdot 3^{\frac{1}{4}n^2 - 1}) \geq 27.$$

If $n = 3$, then actually $d_v = 3$ for at least two (finite) places, and

$$\prod_{v \in T} n^{-1} e(P_v) \geq (n^{-1}(2-1) \cdot 2^{\frac{1}{3}n^2 - 1}) \cdot (n^{-1}(3-1) \cdot 3^{\frac{1}{3}n^2 - 1}) = 8.$$

In particular, it is clear from (10) that Γ is not of minimal covolume. Hence it must be that T is empty and G splits everywhere.

On the other hand, if $v \in T' - T$, then P_v is properly contained in a hyperspecial parahoric H_v. As discussed previously, there is a canonical surjection $H_v \to \mathrm{SL}_n(\mathfrak{f}_v)$, under which the image of P_v is the proper parabolic subgroup $\overline{P}_v$ of $\mathrm{SL}_n(\mathfrak{f}_v)$ whose type consists of the vertices belonging to the type of P_v in the Dynkin diagram obtained by removing the vertex corresponding to H_v in the affine Dynkin diagram of $G(k_v)$. In particular, it follows that $[H_v : P_v] = [\mathrm{SL}_n(\mathfrak{f}_v) : \overline{P}_v]$ and thus using lemma A.14,

$$e(P_v) = [H_v : P_v] \cdot e(H_v) \geq q_v^{n-1} \cdot e(H_v).$$

Of course, as G splits everywhere, we have that $e(H_v)$ is equal to the corresponding factor $e(\mathrm{SL}_n(\mathbb{Z}_v)) = \prod_{i=2}^{n} \frac{1}{1 - q_v^{-i}}$ for $\mathrm{SL}_n(\mathbb{Q}_v)$. In consequence,

$$\frac{\mu(G(k_{v_0})/\Gamma)}{\mu(\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z}))} \geq \frac{\tilde{n}^{-1} \prod_{v \in T'} n^{-1} \cdot \prod_{v \in V_f} e(P_v)}{\prod_{v \in V_f} e(\mathrm{SL}_n(\mathbb{Z}_v))} \geq \tilde{n}^{-1} \prod_{v \in T'} (n^{-1} q_v^{n-1}) \geq 1$$

with equality only if $n = 4$, $T' = \{2\}$ and $\#\Xi_{\Theta_2} = 4$. Notice however that this bound is rather rough; by examining the types of the parahorics carefully, one obtains much better bounds. For example, to achieve $\#\Xi_{\Theta_v} = n$, P_v must be an Iwahori subgroup, in which case $[H_v : P_v] \geq q_v^{(n^2 - n)/2}$ in lemma A.14. This rules out the equality case above and thus T' must be empty as well.

7 Conclusion

As we have shown in section 6, G splits over k_v for all $v \in V_f$ and thus for all $v \in V$. As before, let $\mathfrak{D}$ be a central division algebra over k $(= \mathbb{Q})$ of degree d such that $G \cong \mathrm{SL}_{n'}(\mathfrak{D})$ over k. Now since G splits at all places, we have for any $v \in V$ that $G(k_v) \cong \mathrm{SL}_n(k_v)$, or in other words, that the group of elements of reduced norm 1 in $\mathrm{M}_{n'}(\mathfrak{D}) \otimes_k k_v$ is isomorphic to $\mathrm{SL}_n(k_v)$. This implies that $\mathrm{M}_{n'}(\mathfrak{D}) \otimes_k k_v \cong \mathrm{M}_n(k_v)$, i.e. $\mathfrak{D}_v = \mathfrak{D} \otimes_k k_v$ splits over k_v. It then follows from the Albert–Brauer–Hasse–Noether theorem that $\mathfrak{D} = k$ and in turn $G(k) \cong \mathrm{SL}_n(k)$ and G is split over k. From hereon, we will thus identify G with SL_n through this isomorphism, to be denoted η.

Since each parahoric P_v is hyperspecial, for each $v \in V_f$ there exists $g_v \in \mathrm{GL}_n(\mathbb{Q}_v)$ such that $g_v P_v g_v^{-1} = \mathrm{SL}_n(\mathbb{Z}_v)$. As the family $\{P_v\}$ is coherent, we may assume that $g_v = 1$ except for finitely many $v \in V_f$. In this way, $g = (1, (g_v)_{v \in V_f})$ determines an element of the adele group $\mathrm{GL}_n(\mathbb{A})$. The class group of GL_n over $\mathbb{Q}$ is trivial [PR94, ch. 8], therefore

$$\mathrm{GL}_n(\mathbb{A}) = (\mathrm{GL}_n(\mathbb{R}) \times \prod_{v \in V_f} \mathrm{GL}_n(\mathbb{Z}_v)) \cdot \mathrm{GL}_n(\mathbb{Q}),$$

and we can write $g = (1, (g'_v h)_{v \in V_f})$ for $g'_v \in \mathrm{GL}_n(\mathbb{Z}_v)$ and $h \in \mathrm{GL}_n(\mathbb{Q})$. In consequence, $hP_v h^{-1} = g_v'^{-1} \mathrm{SL}_n(\mathbb{Z}_v) g'_v = \mathrm{SL}_n(\mathbb{Z}_v)$, and thus

$$h\Lambda h^{-1} = h\,\mathrm{SL}_n(\mathbb{Q}) h^{-1} \cap \prod_{v \in V_f} hP_v h^{-1} = \mathrm{SL}_n(\mathbb{Q}) \cap \prod_{v \in V_f} \mathrm{SL}_n(\mathbb{Z}_v) = \mathrm{SL}_n(\mathbb{Z}).$$

In turn, $h\Gamma h^{-1} = \mathrm{SL}_n(\mathbb{Z})$, as $\mathrm{SL}_n(\mathbb{Z})$ (or equivalently Λ) is its own normalizer in $\mathrm{SL}_n(\mathbb{R})$. One way to obtain this fact is using Rohlfs' exact sequence (see section 3). Indeed, clearly $C(k_{v_0}) = C(k) \cap \Lambda$, and on the other hand, since Λ is given by hyperspecial parahorics, we may identify

$$\mathrm{H}^1(k, C)'_\Theta = \{x \in \mathbb{Q}^\times / \mathbb{Q}^{\times n} \mid v(x) \in n\mathbb{Z} \text{ for } v \in V_f, \text{ and } x \in \mathbb{R}^{\times n}\} = \{1\}.$$

Hence Γ/Λ is trivial as claimed.

Finally, retracing our identifications, we find that $\mathrm{SL}_n(\mathbb{Z})$ is the image of Γ under the automorphism $\sigma : \mathrm{SL}_n(\mathbb{R}) \xrightarrow{\iota} G(k_{v_0}) \xrightarrow{\eta} \mathrm{SL}_n(\mathbb{R}) \xrightarrow{c_h} \mathrm{SL}_n(\mathbb{R})$ of $\mathrm{SL}_n(\mathbb{R})$ (here c_h denotes conjugation by h). This concludes the proof of the

Theorem. *Let $n \geq 3$ and let Γ be a lattice of minimal covolume for some (any) Haar measure in $\mathrm{SL}_n(\mathbb{R})$. Then $\sigma(\Gamma) = \mathrm{SL}_n(\mathbb{Z})$ for some (algebraic) automorphism σ of $\mathrm{SL}_n(\mathbb{R})$.*

A Appendix: Bounds for sections 4 through 6

A.1 Lemma. *Let k be a totally real number field of degree m and let l be a quadratic extension of k of signature $(2m_1, m_2)$, so that $m = m_1 + m_2$. Let $n \in \mathbb{N}$ and set $l_0 = \{x \in l^\times \mid N_{l/k}(x) \in k^{\times n}\}$ and $l_n = \{x \in l^\times \mid w(x) \in n\mathbb{Z}$ for each normalized finite place w of $l\}$. Then*

$$\#\left((l_n \cap l_0)/l^{\times n}\right) \leq \#\left(\mu(l)/\mu(l)^n\right) \cdot \tilde{n}^{m-1} n^{m_1} \cdot \#\mathscr{C}_n,$$

where $\mu(l)$ is the group of roots of unity of l, $\tilde{n} = 1$ or 2 depending if n is odd or even, and $\mathscr{C}_n$ is the n-torsion subgroup of the class group $\mathscr{C}$ of l.

Moreover, if $N_{l/k}$ is surjective from U_l onto $U_k/\{\pm 1\}$, then

$$\#\left((l_n \cap l_0)/l^{\times n}\right) \leq \#\left(\mu(l)/\mu(l)^n\right) \cdot n^{m_1} \cdot \#\mathscr{C}_n.$$

Proof. According to [BP89, prop. 0.12], there is an exact sequence

$$1 \to U_l/U_l^n \to l_n/l^{\times n} \to \mathscr{C}_n \to 1,$$

where U_l denotes the group of units of the ring of integers of l, and $\mathscr{C}_n$ is the n-torsion subgroup of the class group $\mathscr{C}$ of l. Intersecting with $l_0/l^{\times n}$ yields

$$\#\left((l_n \cap l_0)/l^{\times n}\right) \leq \#\left((U_l \cap l_0)/U_l^n\right) \cdot \#\mathscr{C}_n.$$

Dirichlet's units theorem [Sam70, §4.4] states that U_l is the internal direct product $F_l \times \mu(l)$ of F_l, the free abelian subgroup of U_l (of rank $2m_1 + m_2 - 1$) generated by some system of fundamental units, and $\mu(l)$, the subgroup of roots of unity in $l^\times$. Since $\mu(l) \subset l_0$, we also have that $U_l \cap l_0$ is the internal direct product of $F_l \cap l_0$ and $\mu(l)$. Additionally, it is clear that under this identification, U_l^n corresponds to the subgroup $F_l^n \times \mu(l)^n$ of $(F_l \cap l_0) \times \mu(l)$. In consequence,

$$\#\left((U_l \cap l_0)/U_l^n\right) = \#\left((F_l \cap l_0)/F_l^n\right) \cdot \#\left(\mu(l)/\mu(l)^n\right),$$

and it remains to study $(F_l \cap l_0)/F_l^n$; to this end, we switch to additive notation.

We write L for the free abelian group $U_l/\mu(l)$ (canonically isomorphic to F_l) in additive notation, and M for its free subgroup $U_k/\{\pm 1\}$ (of rank $m-1$) consisting of units lying in k. The norm $N_{l/k}$ induces a map $N : L \to M$, and in turn a map $L/nL \to M/nM$ also denoted by N, whose kernel L_0/nL corresponds precisely to $(F_l \cap l_0)/F_l^n$. In other words, the sequence

$$0 \to L_0/nL \to L/nL \xrightarrow{N} M/nM$$

is exact. It is clear that $\#(L/nL) = n^{2m_1+m_2-1}$ and $\#(M/nM) = n^{m-1}$. If N is surjective, then it follows that $\#(L_0/nL) = n^{m_1}$. In any case, we have $2M \subset N(L)$ hence we may write

$$\#\left(\frac{N(L)+nM}{nM}\right) = \#\left(\frac{N(L)+nM}{2M+nM}\right) \cdot \#\left(\frac{2M+nM}{nM}\right).$$

As $2M + nM = \tilde{n}M$, we have $\#\left(\frac{2M+nM}{nM}\right) = \left(\frac{n}{\tilde{n}}\right)^{m-1}$ and the lemma follows. ❑

A.2 Lemma. *The function* $\mathbb{N}\times\mathbb{N}\to\mathbb{R}$ *defined by* $E(n,q)=n^{-1}\cdot(q-1)q^{n^2/4-1}$ *is increasing in both* n *and* q *provided* $n,q\geq 2$*. In consequence,* $n^{-1}\cdot(q-1)q^{n^2/4-1}>1$ *provided* $n\geq 4$*. Similarly,* $n^{-1}\cdot(q-1)q^{n^2/3-1}>1$ *provided* $n\geq 3$*.*

Proof. We compute, for $n,q\geq 2$,

$$\frac{E(n,q+1)}{E(n,q)}=\frac{q(q+1)^{\frac{1}{4}n^2-1}}{(q-1)q^{\frac{1}{4}n^2-1}}=\frac{q^2(q+1)^{\frac{1}{4}n^2}}{(q^2-1)q^{\frac{1}{4}n^2}}>1.$$

and

$$\frac{E(n+1,q)}{E(n,q)}=\frac{n}{n+1}\cdot q^{\frac{1}{4}(2n+1)}\geq\frac{2}{3}\cdot 2^{\frac{5}{4}}>1.$$

Thus E is strictly increasing in n and q if $n,q\geq 2$, and $E(4,2)=2$. The proof of the second inequality is analogous. ❏

A.3 Lemma. *Let* $n,q\in\mathbb{N}$ *with* $q\geq 2$*. Then* $\tilde{n}^{-1}\cdot(q+1)^{-1}q^{\lceil(n+1)/2\rceil}>1$ *provided* $n\geq 3$*.*

Proof. Observe that $E(n,q)=\frac{q^{\lceil(n+1)/2\rceil}}{(q+1)\tilde{n}}$ is increasing in n and strictly increasing in q, as

$$\frac{E(n+1,q)}{E(n,q)}=\frac{\tilde{n}}{\widetilde{n+1}}q^{2-\tilde{n}}\geq 1$$

and

$$\frac{E(n,q+1)}{E(n,q)}=\frac{(q+1)(q+1)^{\lceil(n+1)/2\rceil}}{(q+2)q^{\lceil(n+1)/2\rceil}}=\frac{(q^2+2q+1)(q+1)^{\lceil(n+1)/2\rceil-1}}{(q^2+2q)q^{\lceil(n+1)/2\rceil-1}}>1.$$

Finally $E(3,2)=\frac{4}{3}$. ❏

A.4 Lemma. *The function* $M:\mathbb{N}\times\mathbb{N}\to\mathbb{R}$ *defined by*

$$M(m,n)=\frac{1}{100\tilde{n}^m}\left(\frac{12}{\pi}\right)^{2m}\left(\frac{m^m}{m!}\right)^{n^2-5}\left(\prod_{i=1}^{n-1}\frac{i!}{(2\pi)^{i+1}}\right)^{m-1}n^{-1}$$

(where $\tilde{n}=1$ *or* 2 *if* n *is odd or even) is strictly increasing in both* m *and* n*, provided* $m\geq 2$ *and* $n\geq 6$*.*

Proof. For F a function of two integer variables m and n, we denote $\partial_m F$ (resp. $\partial_n F$) the function defined by $\partial_m F(m,n)=\frac{F(m+1,n)}{F(m,n)}$ (resp. $\partial_n F(m,n)=\frac{F(m,n+1)}{F(m,n)}$). In order to show that M increases in m (resp. in n), we intent to show that $\partial_m M>1$ (resp. $\partial_n M>1$).

We have

$$\partial_m M(m,n)=\frac{144}{\pi^2\tilde{n}}\left(\frac{(m+1)^m}{m^m}\right)^{n^2-5}\cdot\prod_{i=1}^{n-1}\frac{i!}{(2\pi)^{i+1}}$$

$$\partial_n M(m,n)=\left(\frac{\tilde{n}}{\widetilde{n+1}}\right)^m\cdot\frac{n}{n+1}\cdot\left(\frac{m^m}{m!}\right)^{2n+1}\left(\frac{n!}{(2\pi)^{n+1}}\right)^{m-1}$$

and thus

$$\partial_m(\partial_n M)(m,n) = \partial_n(\partial_m M)(m,n) = \frac{\tilde{n}}{\widetilde{n+1}} \cdot \left(\frac{(m+1)^m}{m^m}\right)^{2n+1} \cdot \frac{n!}{(2\pi)^{n+1}}.$$

Now if $m \geq 2$ and $n \geq 4$, then $\frac{(m+1)^m}{m^m} \geq \frac{9}{4}$ and we have

$$\partial_m(\partial_n M)(m,n) \geq \frac{1}{2}\left(\frac{9}{4}\right)^{2n+1} \frac{n!}{(2\pi)^{n+1}} = \frac{9}{16\pi} \cdot \left(\frac{81}{16\pi}\right)^n \cdot \frac{n!}{2^n} \geq \frac{9}{16\pi} \cdot \left(\frac{81}{16\pi}\right)^4 > 1.$$

This means that provided $m \geq 2$ and $n \geq 4$, $\partial_m M$ increases in n and $\partial_n M$ increases in m.

Finally, assuming $m \geq 2$ and $n \geq 6$ respectively, we have

$$\partial_m M(m,6) = \frac{144}{2\pi^2}\left(\frac{(m+1)^m}{m^m}\right)^{31} \cdot \prod_{i=1}^{5} \frac{i!}{(2\pi)^{i+1}} \geq \frac{144}{2\pi^2}\left(\frac{9}{4}\right)^{31} \cdot \prod_{i=1}^{5} \frac{i!}{(2\pi)^{i+1}} > 1$$

$$\partial_n M(2,n) = \left(\frac{\tilde{n}}{\widetilde{n+1}}\right)^2 \frac{n}{n+1} \cdot 2^{2n+1} \cdot \frac{n!}{(2\pi)^{n+1}} \geq \frac{3}{14} \cdot 2^n \frac{n!}{\pi^{n+1}} \geq \frac{3}{14} \cdot 2^6 \cdot \frac{6!}{\pi^7} > 1$$

hence $\partial_m M(m,n) > 1$ and $\partial_n M(m,n) > 1$ provided $m \geq 2$ and $n \geq 6$, completing the proof. ❑

A.5 Table. The table below contains some values of the function M from lemma A.4.

(n,m)	1	2	3	4	5	6	7	8
2	0.0364756	0.00337012	0.000276781	0.0000215771	1.63315×10^{-6}	1.21281×10^{-7}	8.88761×10^{-9}	6.44933×10^{-10}
3	0.0486342	0.00231876	0.000177084	0.0000166585	1.76356×10^{-6}	2.01469×10^{-7}	2.42731×10^{-8}	3.04153×10^{-9}
4	0.0182378	0.000214239	9.19392×10^{-6}	6.99962×10^{-7}	7.37412×10^{-8}	9.57798×10^{-9}	1.43998×10^{-9}	2.41175×10^{-10}
5	0.0291805	0.000860260	0.000267434	0.000235765	0.000375160	0.000873531	0.00265357	0.00980934
6	0.0121585	0.000715847	0.00162363	0.0185268	0.528020	27.1489	2107.97	221884.
7	0.0208432	0.0374453	11.9823	37981.0	4.41409×10^{8}	1.18530×10^{13}	5.71337×10^{17}	4.24155×10^{22}
8	0.00911891	0.556912	35451.1	4.88495×10^{10}	3.84324×10^{17}	9.29477×10^{24}	4.92580×10^{32}	4.65827×10^{40}
9	0.0162114	685.655	2.23863×10^{11}	3.83726×10^{21}	6.20398×10^{32}	4.26138×10^{44}	8.04066×10^{56}	3.19899×10^{69}
10	0.00729513	306071.	9.29184×10^{17}	3.98641×10^{32}	2.82701×10^{48}	1.22281×10^{65}	1.87055×10^{82}	7.27033×10^{99}
11	0.0132639	1.40574×10^{10}	1.27888×10^{28}	4.91209×10^{48}	5.79785×10^{70}	6.22507×10^{93}	3.12510×10^{117}	4.89869×10^{141}

A.6 Lemma. *The function $M' : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ defined by*

$$M'(m,n) = \frac{1}{100\tilde{n}^m}\left(\frac{12}{\pi}\right)^{2m} (A^m E)^{n^2-5} \left(\prod_{i=1}^{n-1} \frac{i!}{(2\pi)^{i+1}}\right)^{m-1} n^{-1}$$

(where $\tilde{n} = 1$ or 2 if n is odd or even, and $A = 29.534^{\frac{1}{2}}$, $E = e^{-4.13335}$) is strictly increasing in both m and n, provided $m \geq 4$ and $n \geq 4$. Moreover, $M'(m,n)$ is strictly increasing in m provided $n \geq 3$.

Proof. In order to show that M' increases in m (resp. in n), we intend to show that $\partial_m M > 1$ (resp. $\partial_n M > 1$); the notation is as in lemma A.4.

We have

$$\partial_m M'(m,n) = \frac{144}{\pi^2 \tilde{n}} \cdot A^{n^2-5} \cdot \prod_{i=1}^{n-1} \frac{i!}{(2\pi)^{i+1}}$$

$$\partial_n M'(m,n) = \left(\frac{\tilde{n}}{\widetilde{n+1}}\right)^m (A^m E)^{2n+1} \left(\frac{n!}{(2\pi)^{n+1}}\right)^{m-1} \left(\frac{n}{n+1}\right)$$

and thus

$$\partial_m(\partial_n M')(m,n) = \partial_n(\partial_m M')(m,n) = \frac{\tilde{n}}{\widetilde{n+1}} \cdot A^{2n+1} \cdot \frac{n!}{(2\pi)^{n+1}}.$$

As clearly $A^2 > 2\pi$, we have (if $n \geq 3$)

$$\partial_m(\partial_n M')(m,n) > \frac{1}{2} \cdot A \cdot \frac{n!}{2\pi} > 1.$$

This means that $\partial_m M'$ increases in n and $\partial_n M'$ increases in m. Assuming respectively $m \geq 1$ and $n \geq 4$, we have

$$\begin{aligned} \partial_m M'(m,3) &= \frac{144}{\pi^2} \cdot A^4 \cdot \frac{2}{(2\pi)^5} > 1 \\ \partial_n M'(4,n) &\geq \frac{1}{2^4} \cdot \left(A^4 E\right)^{2n+1} \cdot \frac{(n!)^3}{(2\pi)^{3n+3}} \cdot \frac{4}{5} \\ &\geq \frac{1}{2^4} \cdot \left(A^4 E\right)^9 \cdot \frac{(6!)^3}{(2\pi)^{21}} \cdot \frac{4}{5} > 1 \end{aligned}$$

hence $\partial_m M'(m,n) > 1$ and $\partial_n M'(m,n) > 1$ provided $m \geq 4$ and $n \geq 4$. Moreover, $\partial_m M'(m,n) > 1$ if $n \geq 3$, completing the proof. ❑

A.7 Table. The table below contains some values of the function M' from lemma A.6.

(n,m)	1	2	3	4	5	6	7	8
2	0.418729	0.0142379	0.000484124	0.0000164615	5.59732×10^{-7}	1.90323×10^{-8}	6.47149×10^{-10}	2.20047×10^{-11}
3	2.80041×10^{-6}	7.27880×10^{-6}	0.0000189190	0.0000491740	0.000127813	0.000332209	0.000863474	0.00224433
4	3.99708×10^{-14}	2.79970×10^{-11}	1.96100×10^{-8}	0.0000137356	0.00962086	6.73878	4720.08	3.30611×10^{6}
5	1.84711×10^{-23}	2.62212×10^{-16}	3.72231×10^{-9}	0.0528412	750123.	1.06486×10^{13}	1.51165×10^{20}	2.14591×10^{27}
6	1.68676×10^{-35}	2.85139×10^{-23}	4.82016×10^{-11}	81.4827	1.37743×10^{14}	2.32849×10^{26}	3.93621×10^{38}	6.65400×10^{50}
7	4.80891×10^{-49}	1.09207×10^{-29}	2.48000×10^{-10}	5.63189×10^{9}	1.27896×10^{29}	2.90442×10^{48}	6.59571×10^{67}	1.49783×10^{87}
8	2.65506×10^{-65}	6.66279×10^{-38}	1.67200×10^{-10}	4.19583×10^{17}	1.05293×10^{45}	2.64229×10^{72}	6.63074×10^{99}	1.66396×10^{127}
9	4.52005×10^{-83}	1.88536×10^{-45}	7.86407×10^{-8}	3.28019×10^{30}	1.36821×10^{68}	5.70695×10^{105}	2.38043×10^{143}	9.92906×10^{180}
10	1.47804×10^{-103}	1.08376×10^{-54}	7.94662×10^{-6}	5.82681×10^{43}	4.27247×10^{92}	3.13276×10^{141}	2.29707×10^{190}	1.68431×10^{239}
11	1.48182×10^{-125}	3.59121×10^{-63}	0.870337	2.10928×10^{62}	5.11187×10^{124}	1.23887×10^{187}	3.00243×10^{249}	7.27644×10^{311}

A.8 Table. The table below contains some values of $C(m,n) = \left(230\tilde{n}^m \left(\frac{\pi}{12}\right)^{2m} V_n^{1-m} n\right)^{\frac{2}{n^2-5}}$.

(n,m)	1	2	3	4	5	6	7	8
3	6.87691	125.979	2307.81	42276.9	774473.	1.41876×10^7	2.59904×10^8	4.76120×10^9
4	2.40966	21.6241	194.053	1741.42	15627.4	140239.	1.25850×10^6	1.12937×10^7
5	1.54762	8.80582	50.1044	285.090	1622.14	9229.86	52517.2	298819.
6	1.40247	6.73460	32.3393	155.292	745.707	3580.86	17195.1	82570.5
7	1.23838	4.82334	18.7864	73.1708	284.992	1110.01	4323.37	16839.0
8	1.20619	4.19700	14.6037	50.8142	176.811	615.221	2140.69	7448.64
9	1.13928	3.44306	10.4054	31.4468	95.0368	287.215	868.006	2623.24

A.9 Table. The table below contains the absolute value of the smallest discriminant D_k of a totally real number field of degree m (see for example [Voi08] or [JR14]).

m	1	2	3	4	5	6	7	8
$\min D_k$	1	5	49	725	14641	300125	20134393	282300416

A.10 Table. The tables below contains some values of $H(m) = \frac{(A^2B^{2m-2}E)^2 V_3^{m-1}}{3\cdot\zeta(2)\cdot\zeta(3)}$ for $A = 25.465$, $B = 13.316$, $E = e^{-7.0667}$ if $m \geq 5$, and otherwise $H(m)$ is obtained from (9) using the smallest discriminant for the signature $(2, m-1)$ (see [JR14, Sel99]).

m	2	3	4	5	6	7	8	9
$H(m)$	2.603	5.527	26.39	87.71	563.2	3616.4	23222.2	149118.

m	10	11	12	13	14	15
$H(m)$	9.58×10^5	6.15×10^6	3.95×10^7	2.54×10^8	1.63×10^9	1.05×10^{10}

A.11 Table. The table below contains some values of $60.015^2 \cdot 22.210^{2m-2} \cdot e^{\frac{-80.001}{H(m)}}$, where $H(m)$ is as in table A.10.

m	2	3	4	5	6	7	8
$D_l >$	8.05×10^{-8}	454.01	2.08×10^{10}	8.57×10^{13}	9.13×10^{16}	5.08×10^{19}	2.55×10^{22}

m	9	10	11	12	13	14	15
$D_l >$	1.26×10^{25}	6.23×10^{27}	3.07×10^{30}	1.52×10^{33}	7.48×10^{35}	3.69×10^{38}	1.82×10^{41}

A.12 Lemma. *The function $N : \mathbb{N} \to \mathbb{R}$ defined by*

$$N(n) = \frac{1}{100\tilde{n}^2} \left(\frac{12}{\pi}\right)^2 40^{\frac{1}{4}(n^2-n-6)} n^{-1}.$$

(where $\tilde{n} = 1$ or 2 if n is odd or even) is strictly increasing provided $n \geq 2$. The same holds for

$$N'(n) = \tilde{n}^{-2} n^{-1} 5^{\frac{1}{4}(n^2-n-2)}.$$

Proof. We compute

$$\frac{N(n+1)}{N(n)} = \frac{\tilde{n}^2}{\widetilde{n+1}^2} \cdot 40^{\frac{1}{2}n} \cdot \frac{n}{n+1} \geq \frac{1}{4} \cdot 40 \cdot \frac{2}{3} > 1.$$

The proof for N' is analogous. ❑

A.13 Lemma.

$$\prod_{i=2}^{\infty} \zeta(i) < 2.3$$

Proof. We have

$$\ln \prod_{i=9}^{\infty} \zeta(i) = \sum_{i=9}^{\infty} \ln(1 + (\zeta(i) - 1)) \leq \sum_{i=9}^{\infty} (\zeta(i) - 1) = \sum_{i=9}^{\infty} \sum_{j=2}^{\infty} \frac{1}{j^i}$$
$$= \sum_{j=2}^{\infty} \frac{1}{j^9} \sum_{i=0}^{\infty} \frac{1}{j^i} = \sum_{j=2}^{\infty} \frac{1}{j^9} \frac{j}{j-1} \leq 2 \sum_{j=2}^{\infty} \frac{1}{j^9} = 2(\zeta(9) - 1);$$

hence $\prod_{i=2}^{\infty} \zeta(i) \leq \exp(2\zeta(9) - 2) \cdot \prod_{i=2}^{8} \zeta(i) < 2.3$ ❑

A.14 Lemma. *Let P be a parabolic subgroup of $\mathrm{SL}_n(\mathbb{F}_q)$ and let $n_1, n_2, \dots, n_k$ be integers such that the complement of the type θ of P in the Dynkin diagram of $\mathrm{SL}_n(\mathbb{F}_q)$ consists of $k - \#\theta$ connected components of respectively $n_1 - 1, n_2 - 1, \dots, n_{k-\#\theta} - 1$ vertices and $n_{k-\#\theta+1} = n_{k-\#\theta+2} = \cdots = n_k = 1$. Then $[\mathrm{SL}_n(\mathbb{F}_q) : P] \geq q^{\frac{1}{2}(n^2 - \sum_{i=1}^k n_i^2)}$. In particular, if P is a proper parabolic subgroup, then $[\mathrm{SL}_n(\mathbb{F}_q) : P] \geq q^{n-1}$.*

Proof. Without loss of generality, we may assume that P contains the subgroup B of upper triangular matrices and that elements of P are of the form

$$\left(\begin{array}{cc|c|c|cccc|c}
\multicolumn{2}{c|}{\multirow{2}{*}{n_1}} & * & \cdots & * & * & * & & * \\
 & & * & \cdots & * & * & * & & * \\
\hline
0 & 0 & n_2 & \cdots & * & * & * & & * \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \cdots & & & & & * \\
0 & 0 & 0 & \cdots & & n_{k-1} & & & * \\
0 & 0 & 0 & \cdots & & & & & * \\
\hline
0 & 0 & 0 & \cdots & 0 & 0 & 0 & & n_k
\end{array}\right)$$

where n_i indicates a block in $\mathrm{GL}_{n_i}(\mathbb{F}_q)$, $*$ indicates an arbitrary entry in $\mathbb{F}_q$, and the determinant of the whole matrix is 1. Hence

$$\#P = \frac{\prod_{j=1}^{n_1-1}(q^{n_1} - q^j) \cdots \prod_{j=1}^{n_k-1}(q^{n_k} - q^j) \cdot q^{\frac{1}{2}(n^2 - \sum_{i=1}^k n_i^2)}}{q-1}$$

and

$$\begin{aligned}
\frac{\#\,\mathrm{SL}_n(\mathbb{F}_q)}{\#P} &= \frac{\prod_{j=0}^{n-1}(q^n - q^j)}{\prod_{j=0}^{n_1-1}(q^{n_1} - q^j) \cdots \prod_{j=0}^{n_k-1}(q^{n_k} - q^j) \cdot q^{\frac{1}{2}(n^2 - \sum_{i=1}^k n_i^2)}} \\
&= \frac{q^{\frac{n(n-1)}{2}} \cdot \prod_{j=1}^{n}(q^j - 1)}{q^{\frac{1}{2}(n^2 - \sum_{i=1}^k n_i^2)} q^{\frac{1}{2}\sum_{i=1}^k n_i(n_i - 1)} \cdot \prod_{j=1}^{n_1}(q^j - 1) \cdots \prod_{j=1}^{n_k}(q^j - 1)} \\
&= \frac{\prod_{j=1}^{n}(q^j - 1)}{\prod_{j=1}^{n_1}(q^j - 1) \cdots \prod_{j=1}^{n_k}(q^j - 1)} \\
&= q^{\frac{1}{2}(n(n-1) - \sum_{i=1}^k n_i(n_i - 1))} \cdot \frac{\prod_{j=1}^{n_1}(q^j - 1) \cdot \prod_{j=1}^{n_2}(q^j - q^{-n_1}) \cdots \prod_{j=1}^{n_k}(q^j - q^{-\sum_{i=1}^{k-1} n_i})}{\prod_{j=1}^{n_1}(q^j - 1) \cdots \prod_{j=1}^{n_k}(q^j - 1)}.
\end{aligned}$$

Of course, $n(n-1) - \sum_{i=1}^k n_i(n_i - 1) = (n^2 - \sum_{i=1}^k n_i^2)$. Now the ratio in the right-hand side is clearly greater then 1, as, taken in order, each factor in the numerator is bigger than the corresponding one in the denominator.

Finally, we observe that if P is proper, $k \geq 2$ and $n^2 - \sum_{i=1}^k n_i^2 \geq 2n_1 n_2 \geq 2(n-1)$. ❑

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