Appendices

A Land quality improvement

Let us consider that $\alpha$ solely indicates land quality, and assume that the LP household has the possibility to invest in land quality improvement so as to become a HP household. This investment costs a fix amount, denoted by $\theta(w^* - c)$, with $\theta > 0$. At the beginning of each period, the LP household decides whether or not to invest in quality upgrade with the bequest inherited. Obviously, the investment will be made if and only if (i) it improves the household income, and (ii) there is enough bequest to finance the associated cost. It is further assumed that

$$b_0 = \frac{\sigma}{1-\sigma} \left( \alpha - \frac{1}{2} - x_m \right) < \theta(w^* - c)$$

in order to capture the idea that the LP household is always constrained in the closed economy in improving land quality. In other words, quality upgrade may become financially feasible only with migration and remittances sent home. Let the indicator function $I_t \in \{0, 1\}$ denote this upgrade decision. The income maximization problem for a LP household in the open economy is

$$\max_{\{0 \leq m_t \leq 1, I_t \in \{0, 1\}\}} \quad \frac{y_t}{\frac{1}{1-I_t}} = \left[ (1-I_t)\alpha + I_t\bar{x} \right] \left( 1 - m_t \right) - \frac{(1-m_t)^2}{2} + (w^* - c)m_t - I_t\theta(w^* - c)$$

s.t.

$$\begin{align*}
&b_t \geq m_t c + I_t\theta(w^* - c) \\
&b_0 = \frac{\sigma}{1-\sigma} \left( \alpha - \frac{1}{2} - x_m \right)
\end{align*}$$

(10)

Naturally, given $\theta$ fixed and investment costs are not prohibitively expensive, the more productive the HP household is, the more likely the LP household will invest in land quality improvement. Therefore, it is the most interesting to look at the low inequality case, where $\bar{x} \in [\alpha, \alpha_0]$ and $m_t = \frac{b_t}{c}$. The quality investment condition is

$$I_t = 1 \quad \Leftrightarrow \quad \left. \frac{y_t}{\left| I_t=1, m_t = \frac{b_t - \theta(w^* - c)}{c} \right.} \right| > \left. \frac{y_t}{\left| I_t=0, m_t = \frac{b_t}{c} \right.} \right| .$$

The necessary and sufficient conditions for the inequality to hold are respectively

$$\theta(w^* - c) < c \left[ \sqrt{(1 + w^* - \bar{x})^2 + 2(\bar{x} - \alpha)} - (1 + w^* - \bar{x}) \right] ;$$

(11)

$$\left( 1 - \frac{b_t}{c} \right) > \frac{\theta(w^* - c)}{2} + c(w^* - \bar{x}) \frac{\bar{x} - \alpha}{\theta(w^* - c) - c}.$$

(12)

Condition (11) states that the cost of investment cannot be prohibitively expensive to the degree that it rules out any incentive to invest. Moreover, Condition (12) says that the remaining size of household, or those non-migrant family members who work on own farm, cannot be too small so that the land quality upgrade benefits a sufficiently large scale of agricultural production to justify the cost of investment.
With $b_0$ given by the steady state level of wealth in the close economy, an assumption made in the original model, along with Assumption (9), inequalities at period 1 remain the same as in Section 4 since there are not yet remittances sent home to help relax liquidity constraint. If the LP household never overcomes the liquidity constraint or their remaining household members are too few to justify the investment in improving land quality (i.e., $I_t = 0$, ∀t ≥ 0), then the evolution of inter-household inequalities will not be changed at all by the possibility of land upgrade. In fact, even if $I_t = 1$, it will mean that the fall of inequality is sped up from period $t$ when inequality is already on the decline. In the case of an inverse U-shaped relationship between migration and income inequality and $I_t = 1$ occurs when income inequality is on the rise, quality investment works to downsize the bump in Fig. 2. Finally at the steady state, inequalities vanish entirely. In summary, the qualitative evolution of inequalities remains the same with or without the possibility of land quality investment: the inter-household income inequality may still be characterized by a Kuznets curve, which rises in the short run but declines eventually.²

B Proof for Lemma 1

We know that at the steady state $b_{r+1} = b_r$ for Eq. 3, which has two roots. However, only the larger one

$$b_{ss}^m = c \left[ 1 + \left( w^* - \alpha - \frac{c}{\sigma} \right) + \sqrt{\left( w^* - \alpha - \frac{c}{\sigma} \right)^2 + 2 \left( w^* - x_m - \frac{c}{\sigma} \right)} \right]$$

is stable. From our formulation of the utility function, it is clear that wealth should be always positive, and it is straightforward to show that this stable steady state is indeed positive. In the meantime, it should also satisfy the binding liquidity constraint:

$$c \left[ 1 + \left( w^* - \alpha - \frac{c}{\sigma} \right) + \sqrt{\left( w^* - \alpha - \frac{c}{\sigma} \right)^2 + 2 \left( w^* - x_m - \frac{c}{\sigma} \right)} \right] < c$$

$$\Leftrightarrow w^* - x_m < \frac{c}{\sigma} \quad \text{(or} \quad \sigma < \frac{c}{w^* - x_m})$$

Armed with this condition, now we can prove that this stable steady state is a real solution, that is, $(w^* - \alpha - \frac{c}{\sigma})^2 + 2 \left( w^* - x_m - \frac{c}{\sigma} \right) \geq 0$. Rewrite the inequality

¹This will require a low enough $\theta$ so that the LP household can already afford land quality improvement in the short run.

²Of course, we can also imagine that both the LP and the HP households may invest to upgrade land quality to, say $\bar{\alpha} > \bar{\alpha}$. In this case, we may have a twin-peaked income inequality dynamics, where the second peak begins when the HP household overcomes its constraint to improve land quality.
above as \( \left[ \frac{c}{\sigma} - (w^* - \alpha) \right]^2 \geq 2 \left[ \frac{c}{\sigma} - (w^* - x_m) \right] \). With the assumption \( x_m < \alpha - \frac{1}{2} \), it is sufficient to show that

\[
\left[ \frac{c}{\sigma} - \left( w^* - \left( x_m + \frac{1}{2} \right) \right) \right]^2 \geq 2 \left[ \frac{c}{\sigma} - (w^* - x_m) \right]
\]

\[\Leftrightarrow \left[ \frac{c}{\sigma} - \left( w^* - x_m \right) - \frac{1}{2} \right]^2 \geq 0 \]

Since the last inequality always holds, we conclude that \( b^m \) is indeed real.

**C Computations for \( \bar{\alpha}_0 \) and \( \bar{\alpha}_1 \) in Lemma 2**

We derive \( \bar{\alpha}_1 \) simply from \( c (1 + w^* - c - \bar{\alpha}) = \bar{b}_1 = \frac{\sigma}{1 - \sigma} (\bar{\alpha} - \frac{1}{2} - x_m) \). Similarly but with more complexity, we derive \( \bar{\alpha}_0 \) from \( c (1 + w^* - c - \bar{\alpha}) = \bar{b}_{ss} = \frac{\sigma}{1 - \sigma} \left[ \frac{(1 + w^* - c - \bar{\alpha})^2}{2} + \bar{\alpha} - \frac{1}{2} - x_m \right] \). This equality leads to

\[
w^* - c - \bar{\alpha}_0 = \frac{1 - \sqrt{1 - \frac{2\sigma}{c(1-\sigma)} \left[ \frac{\sigma}{c(1-\sigma)} (w^* - c - x_m) - 1 \right]}}{\frac{\sigma}{c(1-\sigma)}}
\]

We eliminate the other root that is certainly positive because \( w^* - c - \bar{\alpha}_0 < 0 \) in this scenario, but we still have to check if this root is negative. We find that with \( w^* - x_m < \frac{c}{\sigma} \), the condition we derive in Lemma 1, it is indeed negative. After some rearrangements, we have \( \bar{\alpha}_0 \) as shown in Lemma 2.

**D Proof for Proposition 2**

D.1 Higher-medium inequality case

In order to know how the income inequality changes in the short run, we write

\[
\Gamma_1^y - \Gamma_0^y = \frac{1}{2} \left( 1 + w^* - c - \bar{\alpha} \right)^2 + \bar{\alpha} - \frac{1}{2} - \frac{b_1}{c} \left( w^* - \bar{\alpha} - c + 1 - \frac{b_1}{2c} \right)
\]

\[
\bar{\alpha} - \frac{1}{2}
\]

whose sign is the same as that of the following equation:

\[
f(\bar{\alpha}) = \frac{(1 + w^* - c - \bar{\alpha})^2}{2 (\bar{\alpha} - \frac{1}{2})} - \frac{\frac{b_1}{c} \left( w^* - \bar{\alpha} - c + 1 - \frac{b_1}{2c} \right)}{\bar{\alpha} - \frac{1}{2}}
\]

Obviously as \( \bar{\alpha} \to (1 + w^* - c) \), the upper bound of \( \bar{\alpha} \), we have \( f(\bar{\alpha}) < 0 \).
However, \( f(\alpha) \) does not always stay negative. To illustrate, we set \( \alpha = \alpha_{\text{min}} = \frac{c(1-\sigma)(1+w^*-c)+\sigma(\frac{1}{2}+x_m)}{c(1-\sigma)+\sigma}. \) We know that to determine the sign of \( f(\alpha_{\text{min}}) \) is equivalent to look at the sign of the equation below.

\[
g(\alpha) = \frac{\left( w^*-c-x_m+\frac{1}{2} \right)^2}{1 + \frac{c(1-\sigma)}{\sigma}} - \left( \alpha - \frac{1}{2} - x_m \right) \left( w^*-c + \frac{1}{2} + \frac{\sigma x_m}{c(1-\sigma)} \right) \\
\cdot \left[ 1 + w^*-\alpha - c - \frac{\sigma \left( \alpha - \frac{1}{2} - x_m \right)}{2c(1-\sigma)} \right]
\]

where \( \max \left[ 1, \left( x_m + \frac{1}{2} \right) \right] < \alpha < w^*-c. \) In the case where \( x_m > \frac{1}{2}, \) we have \( g(\alpha) > 0 \) as \( \alpha \to (x_m + \frac{1}{2}), \) the lower bound of \( \alpha. \) It therefore implies that \( \lim_{\alpha \to (x_m+\frac{1}{2})} f(\alpha_{\text{min}}) > 0. \)

Because we have shown above that the sign of \( f(\alpha) \) varies with the parameter values, we conclude that the effect of migration and remittances on income equality is ambiguous in the short run.

The proof for the long-run income inequality can be found in the proof for the next case.

D.2 Lower-medium inequality case

To know whether the income inequality increases or decreases in the short run, we have to investigate the sign of

\[
\Gamma_1^\gamma - \Gamma_0^\gamma = \frac{\alpha - \frac{1}{2} + \frac{1}{2} \left( w^*-\alpha - c + 1 - \frac{b}{2c} \right)}{\alpha - \frac{1}{2} + \frac{1}{2c} \left( w^*-\alpha - c + 1 - \frac{b}{2c} \right)} - \alpha - \frac{1}{2} \frac{1}{\alpha - \frac{1}{2}}
\]

Since we have derived the wealth levels at \( t = 1, \) after some rearrangements, we know it is equivalent to look at the sign of

\[
f(\alpha, \alpha) = \frac{(\alpha - \frac{1}{2} - x_m) \left( w^*-\alpha - c + 1 - \frac{\sigma(\alpha-\frac{1}{2}-x_m)}{2c(1-\sigma)} \right)}{\alpha - \frac{1}{2}} \\
- \left( \alpha - \frac{1}{2} - x_m \right) \left( w^*-\alpha - c + 1 - \frac{\sigma(\alpha-\frac{1}{2}-x_m)}{2c(1-\sigma)} \right)
\]

Obviously \( \lim_{\alpha \to (x_m+\frac{1}{2})} f(\cdot) > 0. \) However, \( f(\cdot) \) does not always stay positive. For example, when we set \( \alpha = w^*-c, \) we obtain

\[
f(\alpha) = \left( w^*-c - \alpha \right) \left[ 1 + \frac{\sigma}{2c(1-\sigma)} - \frac{x_m}{\alpha - \frac{1}{2}} \right]
\]

\[
- \left( \frac{\sigma x_m^2}{2c(1-\sigma)(\alpha-\frac{1}{2})(w^*-c-\frac{1}{2})} - \frac{x_m}{\alpha - \frac{1}{2}} \right]
\]

Studying the properties of \( f(\alpha) \), we first find that it is a strictly concave function, i.e., \( \frac{\partial f(\alpha)}{\partial \alpha} < 0, \forall \alpha > \frac{1}{2}. \) Second, \( f(\alpha) = 0 \) has two solutions: \( \alpha = w^*-c \) and \( \alpha = \frac{1}{2} + x_m \left[ \frac{2c(1-\sigma)(w^*-c+\frac{1}{2})+\sigma x_m}{2c(1-\sigma)+\sigma(w^*-c-\frac{1}{2})} \right]. \) When we choose a parameter set such that the second root
is smaller than the first one, both roots are then smaller than \( \alpha_0 \). So, this strictly concave function \( f(\alpha) \) is negative within the range of \( \alpha \), which implies that the income inequality is reduced in the short run.

In order to confirm that the income inequality is reduced in the long run, we need to prove that

\[
\Gamma_y^{\alpha} \leq \Gamma_0^y = \frac{\frac{1}{2} (w^* - c + \alpha - x_m)(w^* - c + x_m)}{\alpha - \frac{1}{2}} \quad \forall \alpha \in \mathbb{R}
\]

The rest of the proof is very similar to the proof for the wealth inequality in the long run. We can also rearrange this inequality so that it suffices to demonstrate that the function \( f(\alpha_{min}) < 0 \), which in turn is equivalent to show the following to be true:

\[
g(\alpha) = \left( \alpha - \frac{1}{2} \right) \left( 1 - B + \frac{\sigma x_m}{c(1 - \sigma)} \right) - \left( B + w^* - c - \frac{1}{2} \right) \left( \frac{\sigma x_m}{c(1 - \sigma)} + 1 + w^* - \alpha - \frac{c}{\sigma} + A \right) < 0.
\]

We find that \( \frac{\partial^2 g(\alpha)}{\partial \alpha^2} > 0 \), \( \forall \alpha \in \mathbb{R} \), or \( g(\cdot) \) is a convex function in \( \alpha \). Moreover, \( g(\frac{1}{2}) < g(\alpha_{min}) = 0 \) while \( \frac{1}{2} < x_m + \frac{1}{2} < \alpha < w^* - c < \alpha_{min} \). Thus, we conclude that \( g(\alpha) < 0 \), and so we have proven that income inequality is reduced in the long run. Since this result is valid for \( w^* - c + B \leq \alpha < 1 + w^* - c \), the proof above also applies to the higher-medium inequality case.

### D.3 Low inequality case

Since in the short run, the income dynamics is the same for the low and the lower-medium inequality cases, we can follow the same steps in the last case to show that the income inequality may rise in the short run (i.e., \( \lim_{\alpha \to (x_{m} + \frac{1}{2})} f(\cdot) > 0 \)). However, the effect is not certainly positive. For example, when \( \alpha = w^* - c \), we obtain

\[
f(\alpha) = \left[ \alpha - (w^* - c) \right] \left[ 1 + \frac{\sigma}{2c(1 - \sigma)} \left( \frac{x_m}{\alpha - \frac{1}{2}} \right)(w^* - c - \frac{1}{2}) \right] - \frac{\sigma x_m^2}{2c(1 - \sigma)(\alpha - \frac{1}{2})(w^*-c-\frac{1}{2})} \quad > 0.
\]

Studying the properties of \( f(\alpha) \), we first find that it is a strictly convex function, i.e., \( \frac{\partial^2 f(\alpha)}{\partial \alpha^2} > 0 \), \( \forall \alpha > \frac{1}{2} \). Second, \( f(\alpha) = 0 \) has two solutions: \( \alpha = w^* - c \) and \( \alpha = \frac{1}{2} + x_m \left[ \frac{2c(1 - \sigma)(w^* - c + \frac{1}{2}) + \sigma x_m}{2c(1 - \sigma) + \sigma(x_m - c - \frac{1}{2})} \right] \left( > \frac{1}{2} + x_m \right) \). We can choose a set of parameter values such that the second root is smaller than the first one. So, we can ensure that the strictly convex function \( f(\alpha) \) is negative, which implies that the income inequality is reduced in the short run, for all \( \alpha \) located within these two roots.
To prove that income inequality is reduced in the long run, we firstly rewrite $y_{ss}^m$ as $x_m + \left(\frac{1}{\sigma} - 1\right)\bar{b}_{ss}^m$, so it is equivalent to demonstrate that

$$\frac{x_m + \left(\frac{1}{\sigma} - 1\right)\bar{b}_{ss}^m}{x_m + \left(\frac{1}{\sigma} - 1\right)\bar{b}_{ss}^m} < \frac{\bar{a} - \frac{1}{2}}{\bar{a} - \frac{1}{2}} \quad \text{or} \quad \frac{x_m + \left(\frac{1}{\sigma} - 1\right)\bar{b}_{ss}^m}{\bar{a} - \frac{1}{2}} < \frac{x_m + \left(\frac{1}{\sigma} - 1\right)\bar{b}_{ss}^m}{\bar{a} - \frac{1}{2}}$$

Suppose $\bar{a} = \alpha$, the right hand side (RHS) of the second inequality is equal to the left hand side (LHS). Thus, it suffices to show that $\frac{\partial LHS}{\partial \alpha} < \frac{\partial RHS}{\partial \alpha}$ holds for all $\bar{a} > \alpha$. Indeed,

$$\frac{\partial LHS}{\partial \alpha} = -\left\{ \frac{c(\frac{1}{\sigma} - 1)}{\bar{a} - \frac{1}{2}} \left[ 1 + \frac{w^* - \bar{a} - \frac{\sigma}{c}}{\sqrt{(w^* - \bar{a} - \frac{\sigma}{c})^2 + 2(w^* - x_m - \frac{\sigma}{c})}} \right] + \frac{x_m + \left(\frac{1}{\sigma} - 1\right)\bar{b}_{ss}^m}{(\bar{a} - \frac{1}{2})^2} \right\} < 0.$$

**E Proofs for Proposition 3**

E.1 Higher-medium inequality case

In order to confirm that the wealth inequality is reduced in the short run, we need to prove that

$$\Gamma_2^b - \Gamma_1^b = \frac{\sigma}{\bar{a} - \frac{1}{2} - x_m + \frac{b_m}{c}(w^* - \alpha + 1 - \frac{b_m}{2c})} - \frac{\bar{a} - \frac{1}{2} - x_m}{\bar{a} - \frac{1}{2} - x_m} < 0.$$  

After some rearrangements, we know that it is equivalent to demonstrate the following to be true.

$$\frac{(1 + w^* - c - \bar{a})^2}{2(\bar{a} - \frac{1}{2} - x_m)} - \frac{\sigma}{c(1 - \sigma)} \left[ 1 + w^* - c - \alpha - \frac{\sigma(\alpha - \frac{1}{2} - x_m)}{2c(1 - \sigma)} \right] < 0.$$

Denoting the left-hand side of the inequality as $f(\bar{a})$, one finds that $\frac{\partial f(\bar{a})}{\partial \bar{a}} \leq 0$, $\forall \bar{a}$. In other words, $f(\bar{a})$ reaches its maximum when $\bar{a} = \bar{a}_{min} = \frac{c(1 - \sigma)(1 + w^* - c) + \sigma}{2(c(1 - \sigma) + \sigma)}$, the minimal value of $\bar{a}$. Thereby, it is sufficient to show that $f(\bar{a}_{min}) < 0$. Again after some rearrangements, we find that to determine the sign of $f(\bar{a}_{min})$ is equivalent to know the sign of

$$g(\alpha) = (1 + w^* - c - \alpha) \left( 1 - 2 \left[ \frac{c(1 - \sigma)}{\sigma} + 1 \right] + \left( \alpha - \frac{1}{2} - x_m \right) \left( 2 + \frac{\sigma}{c(1 - \sigma)} \right) \right).$$

where $g(\alpha)$ monotonically increases in $\alpha$, $\forall c$, $\sigma$, $\alpha$. By assumption, $\frac{1}{2} + x_m < \alpha < w^* - c$; thus, as long as we can prove $g(w^* - c) < 0$, it immediately implies that $g(\alpha) < 0$, $\forall \alpha$. With

$$g(w^* - c) = \left( 2 + \frac{\sigma}{c(1 - \sigma)} \right) \cdot \left( w^* - x_m - \frac{c}{\sigma} - \frac{1}{2} \right).$$
and \( w^* - x_m < \frac{c}{\sigma} \) by Lemma 1, \( g(w^* - c) \) is indeed negative. Hence, we have proven that wealth inequality is reduced in the short run.

The proof for the long-run wealth inequality can be found in the proof for the next case.

E.2 Lower-medium inequality case

The proof for the short-run wealth inequality is identical to that of the low inequality case (see below).

In order to confirm that the wealth inequality is reduced in the long run, we need to prove that

\[
\Gamma^b_{ss} - \Gamma^b_1 = \frac{\sigma}{\lambda} \left[ \frac{1}{2} (1 + w^* - c - \bar{\alpha})^2 + \bar{\alpha} - \frac{1}{2} - x_m + \bar{b}^m_{ss} \right] - \frac{\sigma}{\lambda} \left[ \frac{1}{2} - x_m \right] < 0
\]

After some rearrangements, we know that it is equivalent to demonstrate the following to be true.

\[
\frac{(1 + w^* - c - \bar{\alpha})^2}{2(\bar{\alpha} - \frac{1}{2} - x_m)} + 1 - \frac{c(1-\sigma)}{\sigma} \left( 1 + w^* - \bar{\alpha} - \frac{c}{\sigma} + A \right) \left( \frac{\sigma}{1-\sigma} \right) (w^* - c - x_m) < 0
\]

where \( A = \sqrt{(w^* - \bar{\alpha} - \frac{c}{\sigma})^2 + 2(w^* - x_m - \frac{c}{\sigma})} \).

Denoting the left-hand side of the inequality as \( f(\bar{\alpha}) \), one finds that \( \frac{\partial f(\bar{\alpha})}{\partial \bar{\alpha}} \leq 0 \), \( \forall \bar{\alpha} \in ]\bar{\alpha}, (1 + w^* - c)[ \). In other words, \( f(\bar{\alpha}) \) reaches its maximum when \( \bar{\alpha} = \bar{\alpha}_{min} = w^* - c + B \), where \( B = \frac{c(1-\sigma)}{\sigma} \left\{ 1 - \frac{2\sigma}{c(1-\sigma)} \left( \frac{\sigma}{c(1-\sigma)} (w^* - c - x_m) - 1 \right) \right\} \). Thereby, it is sufficient to demonstrate that \( f(\bar{\alpha}_{min}) < 0 \). Again after some rearrangements, we find that to determine the sign of \( f(\bar{\alpha}_{min}) \) is equivalent to know the sign of

\[
g(\bar{\alpha}) = \left( \frac{1}{2} - x_m \right) (1-B) - \left( B + w^* - c - x_m - \frac{1}{2} \right) \left( 1 + w^* - \bar{\alpha} - \frac{c}{\sigma} + A \right)
\]

We find that \( \frac{\partial^2 g(\bar{\alpha})}{\partial \bar{\alpha}^2} > 0 \), \( \forall \bar{\alpha} \in \mathbb{R} \), or \( g(\cdot) \) is a convex function in \( \bar{\alpha} \). Moreover, \( g(x_m + \frac{1}{2}) < g(\bar{\alpha}_{min}) = 0 \) while \( x_m + \frac{1}{2} < \bar{\alpha} < w^* - c < \bar{\alpha}_{min} \). Thus, we conclude that \( g(\bar{\alpha}) < 0 \), and so we have proven that wealth inequality is reduced in the long run. Since this result is valid for \( w^* - c + B \leq \bar{\alpha} < 1 + w^* - c \), the proof above also applies to the higher-medium inequality case.

E.3 Low inequality case

In order to know whether the wealth inequality is reduced in the short run, we look at the sign of \( \Gamma^b_2 - \Gamma^b_1 \). After some computation, we have

\[
\Gamma^b_2 - \Gamma^b_1 = \frac{-\sigma (\bar{\alpha} - \frac{1}{2} - x_m) (\bar{\alpha} - \frac{1}{2} - x_m)}{c(1-\sigma)} \cdot \left[ 1 + \frac{\sigma}{2c(1-\sigma)} \right] \cdot (\bar{\alpha} - \bar{\alpha})
\]

which is obviously negative and indicates that the wealth inequality is indeed reduced in the short run.
To demonstrate that the wealth inequality is also reduced in the long run, let us first write

\[ b_{t+1} = f(b_t, \alpha) = \sigma \left[ \left( \alpha - \frac{1}{2} - x_m \right) + \frac{b_t}{c} \left( w^* - \alpha + 1 - \frac{b_t}{2c} \right) \right] \]

with \( b_{ss} = f(b_{ss}, \alpha) \), which implies \( b_{ss} = b(\alpha) \). Thus, we can define that at \( b = b_{ss} \),

\[ g(b, \alpha) = f(b(\alpha), \alpha) - b(\alpha) = 0 \]

From the function form of \( f(\cdot) \), we know \( \frac{\partial g}{\partial \alpha} \neq 0 \). This allows us to apply the implicit function theorem and obtain

\[ \frac{db(\alpha)}{d\alpha} = -\frac{\frac{\partial g}{\partial \alpha}}{\frac{\partial g}{\partial b}} = \frac{\partial f}{\partial \alpha} \frac{1}{1 - \frac{\partial f}{\partial b}} \] (13)

Given \( \overline{\alpha} > \alpha \), we need to prove that \( \Gamma^b_{ss} < \Gamma^b_i = \frac{\sigma - \frac{1}{2} - x_m}{\overline{\alpha} - \frac{1}{2} - x_m} \). That is,

\[ \frac{f[b(\overline{\alpha}), \overline{\alpha}]}{f[b(\alpha), \alpha]} < \frac{f[0, \alpha]}{f[0, \overline{\alpha}]} \quad \text{or} \quad \frac{f[b(\overline{\alpha}), \overline{\alpha}]}{f[0, \overline{\alpha}]} < \frac{f[b(\alpha), \alpha]}{f[0, \alpha]} \]

Suppose \( \overline{\alpha} = \alpha \), the right hand side (RHS) of the second inequality is equal to the left hand side (LHS). Hence, in order to prove our speculation to be true, it suffices to show that \( \frac{\partial \text{LHS}}{\partial \alpha} < \frac{\partial \text{RHS}}{\partial \alpha} = 0 \) holds for all \( \overline{\alpha} > \alpha \). In other words, we need to demonstrate that

\[ \frac{f[0, \overline{\alpha}]}{f[b(\overline{\alpha}), \overline{\alpha}]} \cdot \frac{\partial f[b(\overline{\alpha}), \overline{\alpha}]}{\partial \overline{\alpha}} < \frac{\partial f[0, \overline{\alpha}]}{\partial \overline{\alpha}} \]

\[ \Rightarrow \quad \frac{\sigma \left( \overline{\alpha} - \frac{1}{2} - x_m \right) \cdot \sigma \left( 1 - \frac{\overline{b}_{ss}}{c} \right)}{1 - \frac{c}{2c} \left( w^* - \overline{\alpha} + 1 - \frac{\overline{b}_{ss}}{c} \right)} < \sigma \quad \text{(using Eq. 13)} \]

\[ \Rightarrow \quad \sigma \left[ \left( \overline{\alpha} - \frac{1}{2} - x_m \right) + \frac{\overline{b}_{ss}}{c} \left( w^* - \overline{\alpha} + 1 - \frac{\overline{b}_{ss}}{c} \right) \right] < \overline{b}_{ss} + \overline{\alpha} \overline{b}_{ss} \frac{2c^2}{c} \left( \overline{\alpha} - \frac{1}{2} - x_m \right) \]

\[ \Rightarrow \quad f(\overline{b}_{ss}, \overline{\alpha}) - \overline{b}_{ss} - \frac{\sigma \overline{b}_{ss}^2}{2c^2} < \frac{\sigma \overline{b}_{ss}}{c} \left( \overline{\alpha} - \frac{1}{2} - x_m \right) \]

\[ \Rightarrow \quad -\frac{\sigma \overline{b}_{ss}^2}{2c^2} < \frac{\sigma \overline{b}_{ss}}{c} \left( \overline{\alpha} - \frac{1}{2} - x_m \right). \]

Since the last inequality always holds, we have proven our case.

**F Gini coefficient**

Following the definition of the Gini coefficient (Fig. 4) that it is the ratio of the area between the Lorenz curve and the uniform distribution of income to the whole area under the uniform distribution, we have

\[ G_t = 1 - \frac{1}{2} \cdot \rho^2 \cdot \bar{y}_t \left( 1 - m^*_t \right)^2 + \frac{1}{2} \cdot \bar{y}_t \left( 1 - \overline{m}_t \right)^2 + \rho \cdot \bar{y}_t \left( 1 - m^*_t \right) \left( 1 - \overline{m}_t \right) \]

\[ \frac{1}{2} \left[ \rho \left( 1 - m^*_t \right) + (1 - \overline{m}_t) \right] \cdot \left[ \rho \cdot \bar{y}_t \left( 1 - m^*_t \right) + \bar{y}_t \left( 1 - \overline{m}_t \right) \right] \]
Define $\phi_t = \frac{1 - m^*_t}{1 - m^*_0}$. After some computations, we obtain the Gini coefficient shown in Footnote 19.

**G Proof for Proposition 4**

We look at the scenario where the HP household is liquidity constrained (so $m^*_t = \frac{\bar{m}_t}{c}$) and the labor market equilibrium changes from $CS$ in the closed economy to $OS$ in the open economy, or where there are always some LP family members working on their own farm. We choose to examine this type of transition because it is when $\bar{\alpha}$ and $\alpha$ are the closest amongst all the possible transitions of equilibria.

In order to know whether $\Gamma_{ss} > \Gamma_0$, we examine the sign of

$$
\Gamma_{ss} - \Gamma_0 = \frac{\bar{y}_{ss}}{y_{ss}} - \frac{\bar{y}_0}{y_0} = \frac{(1 - \sigma)\bar{b}_{ss}^m + x_m}{(1 - \sigma)\bar{b}_{ss}^m + x_m} - \frac{(1 - \sigma)\bar{b}_1 + x_m}{(1 - \sigma)\bar{b}_1 + x_m}
$$

which is equivalent to know the sign of $\Psi$

$$
\Psi = \bar{b}_{ss}^m \cdot \bar{b}_1 \cdot \left( \frac{\bar{b}_{ss}^m}{\bar{b}_{ss}^m} - \frac{\bar{b}_1}{\bar{b}_1} \right) + \left( \frac{x_m \sigma}{1 - \sigma} \right) \left( \frac{\bar{b}_{ss}^m - \bar{b}_{ss}^m}{1 - \sigma} \right) - \left( \frac{x_m \sigma}{1 - \sigma} \right) \left( \frac{\bar{b}_1}{1 - \sigma} \right)
$$

with

$$
\bar{b}_1 = \frac{\sigma}{1 - \sigma} \left[ \bar{\alpha} - \frac{1}{2} - x_m + \frac{\rho^2 (\bar{\alpha} - \alpha)^2}{2(1 + \rho)^2} \right] \quad \text{and} \quad \bar{b}_1 = \frac{\sigma}{1 - \sigma} \left[ \frac{\alpha - 1}{2} - x_m + \frac{(\bar{\alpha} - \alpha)^2}{2(1 + \rho)^2} \right].
$$
Next, we derive $\overline{b}_{ss}$ as a function of $\underline{b}_{ss}$ from the LP household’s wealth accumulation at its steady state:

$$
\underline{b}_{ss} = \sigma \left[ \alpha \left( 1 - \frac{\alpha - \alpha + \underline{b}_{ss} - \underline{b}_{ss}}{1 + \rho} \right) - \frac{1}{2} \left( 1 - \frac{\alpha - \alpha + \underline{b}_{ss} - \underline{b}_{ss}}{1 + \rho} \right) \right]^{2} + \left( \frac{\alpha - \alpha + \underline{b}_{ss} - \underline{b}_{ss}}{1 + \rho} \right) \left( \alpha - 1 + \frac{\alpha - \alpha + \underline{b}_{ss} - \underline{b}_{ss}}{1 + \rho} \right) 
$$

$$
+ \frac{b_{ss}}{c} (w^* - c) + b_{ss} - x_m 
$$

Then, we obtain

$$
\frac{\overline{b}_{ss}}{\underline{b}_{ss}} - \overline{b}_1 = (1 + \rho) \sqrt{1 + 2 \cdot \left( \frac{c}{\sigma} - w^* + \alpha - 1 \right) \cdot \frac{b_{ss}}{c} - 2 \left( \alpha - x_m - \frac{1}{2} \right)} 
$$

With the two relationships we just obtained above, we can now rewrite $\Psi$:

$$
\Psi = c(1 + \rho) \left( \alpha - 1 + \frac{\alpha - \alpha}{2(1 + \rho)} \right) + \sqrt{\frac{\overline{b}_{ss}}{c} - 2 \cdot \left( \frac{c}{\sigma} - w^* + \alpha - 1 \right) \cdot \frac{b_{ss}}{c} - 2 \left( \alpha - x_m - \frac{1}{2} \right)} 
$$

Finally,

$$
\lim_{\alpha \to \overline{\alpha}} \Psi = c(1 + \rho) \left( \alpha - 1 + \frac{\alpha - \alpha}{2(1 + \rho)} \right) + \sqrt{\frac{\overline{b}_{ss}}{c} - 2 \cdot \left( \frac{c}{\sigma} - w^* + \alpha - 1 \right) \cdot \frac{b_{ss}}{c} - 2 \left( \alpha - x_m - \frac{1}{2} \right)} > 0
$$

Hence, we have proven that when $\overline{\alpha}$ is sufficiently close to $\alpha$, the income inequality is increased in the long run, i.e., $\Gamma^y_0 < \Gamma^y_{ss}$. 

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In Fig. 5, where we choose identical parameter values to those in Fig. 3, we actually observe that $\Psi$ is positive, or the long-run income inequality is increased, for a good range of $\bar{\alpha}$. In comparison, the wealth inequality is always decreased, i.e., $\Gamma^b > \Gamma^b_{ss}$.