Using hyperbolic systems of balance laws for modeling, control and stability analysis of physical networks

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Abstract

The operation of many physical networks having an engineering relevance may be represented by hyperbolic systems of balance laws in one space dimension. By using a Lyapunov stability analysis, it can be shown that the exponential stability of the equilibrium is guaranteed if the boundary conditions are dissipative. This essential property is helpful for solving the associated control problem of designing control laws at the network junctions in order to stabilize the system. This control design issue is illustrated with an application to water level control in hydraulic networks.

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1. Introduction.

The operation of many physical networks having an engineering relevance may be represented by hyperbolic systems of balance laws in one space dimension. Among the potential applications we have in mind, we mention for instance hydraulic networks (for irrigation or navigation), electric line networks, road traffic networks or gas pipeline networks.

In each of these applications, the network is represented by a directed graph with edges and nodes. Along the edges, the dynamics of the concerned physical quantities are modelled by hyperbolic partial differential equations (PDEs) under the form of so-called 2×2 systems of balance laws. The nodes of the graph represent the physical junctions between some of the edges of the network. The mechanisms occuring at these junctions are modelled by algebraic relations that determine the boundary conditions of the PDEs. They generally depend on network physical constraints but, in many instances, they can also be assigned by using appropriate control devices (like hydraulic gates in open channels or traffic lights in road networks).

In this paper, our main concern is to discuss the exponential stability of the classical solutions for such physical network systems. The analysis relies on Lyapunov functions. Essentially, we shall see that the time derivative of the Lyapunov function can be made negative if the boundary conditions are dissipative. There is therefore an underlying control problem which is the problem of designing the control laws at the network junctions in order to make the corresponding boundary conditions dissipative.

The content of the paper is summarized as follows.

In Section 2 we give the basic definition of an hyperbolic 2×2 system of balance laws over a finite space interval.

In Section 3 we give three physical examples of such systems : the telegrapher equations for electrical lines, the Saint-Venant equations for hydraulic channels, the Aw-Rascle equations for road traffic.

In section 4, we give the general equations of **networks** represented by a set of n hyperbolic 2×2 systems of balance laws.

In section 5, the purpose is to present the *characteristic form* of hyperbolic 2×2 systems of balance laws and its linearisation with respect to a steady-state equlibrium.

In Section 6, we present a Lyapunov function used for analysing the asymptotic convergence of the classical solutions of the linearised system under linear boundary conditions. This analysis leads to the concept of *dissipative boundary conditions* that guarantee the exponential stability of the system.

In Section 7, we present an explicit characterisation of sufficient dissipative boundary conditions which guarantee the system exponential stability in the case where the considered balance laws are almost conservative.

In Section 8, it is shown that, for hydraulic networks described by Saint-Venant equations, the dissipative boundary conditions may be sufficient to guarantee the stability for systems of balance laws which deviate widely from their corresponding conservation laws.

In Section 9, we address the implications of the previous stability properties for the design of boundary feedback control for physical networks when the boundary conditions are assigned by control devices that are used for network regulation and stabilisation.The particular example of the level control problem in hydraulic networks is treated in details.

In Section 10, we explain how the linear Lyapunov stability analysis developed in the previous sections can be extended to the case of *nonlinear* hyperbolic systems of balance laws.

Finally, in Section 11, we conclude the paper with bibliographical notes for the readers who would like to go deeper in the subject.

2. Hyperbolic 2×2 systems of balance laws.

In this section we give the basic definition of an hyperbolic 2×2 system of balance laws over a finite space interval as it will be used throughout the paper. Let Ω be a non-empty connected open set in $\mathbb{R}^2$. A 2×2 system of balance laws is a system of PDEs of the form

$$\partial_t \mathbf{y} + \partial_x \mathbf{f}(\mathbf{y}) + \mathbf{g}(\mathbf{y}) = \mathbf{0} \qquad t \in [0, +\infty) \qquad x \in [0, L] \tag{1}$$

where

- t and x are the two independent variables: a time variable $t \in [0, +\infty)$ and a space variable $x \in [0, L]$ over a finite interval;
- $\mathbf{y} \triangleq (y_1 \quad y_2)^T : [0, +\infty) \times [0, L] \to \Omega$ is the vector of the two dependent variables called densities;
- $\mathbf{f} \in C^1(\Omega, \mathbb{R}^2)$ is the vector of the flux densities;
- $\mathbf{g} \in C^1(\Omega, \mathbb{R}^2)$ is the vector of source terms.

Since $\mathbf{f}$ is a C^1 function, system (1) can be written under the form of a quasi-linear system of PDEs

$$\partial_t \mathbf{y} + \mathbf{A}(\mathbf{y}) \partial_x \mathbf{y} + \mathbf{g}(\mathbf{y}) = \mathbf{0}$$

with the jacobian matrix $\mathbf{A}(\mathbf{y}) \triangleq \partial \mathbf{f} / \partial \mathbf{y}$. The system (1) is strictly **hyperbolic** if $\mathbf{A}(\mathbf{y})$ has two distinct real eigenvalues $a_1(\mathbf{y}) \neq a_2(\mathbf{y})$ $\forall \mathbf{y} \in \Omega$.

In the special case where there are no source terms (i.e. $\mathbf{g}(\mathbf{y}) = \mathbf{0}$ $\forall \mathbf{y} \in \Omega$), the system (1) reduces to

$$\partial_t \mathbf{y} + \partial_x \mathbf{f}(\mathbf{y}) = \mathbf{0} \tag{2}$$

which constitutes a so-called hyperbolic 2×2 system of **conservation laws**.

3. Examples.

In this section we shall give three physical examples of such systems. The first example is the well-known telegrapher equation for the modelling of electric lines. It is a very simple linear system. The two other examples are nonlinear. They are presented here because they are particularly well suited for illustrating the theoretical developments of the paper : the first one is the well-known Saint-Venant equation for modelling the water flow in open channels [9]; the second one is the Aw-Rascle equation for modelling the road traffic flow [1].

The telegrapher equation

This equation, which is a simplification of the Maxwell equations, describes the propagation of an electric signal along an electrical line with inductance L, capacitance C, resistance R and admittance G under the form of a 2×2 system of balance laws

$$\partial_t \begin{pmatrix} I \\ V \end{pmatrix} + \partial_x \begin{pmatrix} -L^{-1}V \\ -C^{-1}I \end{pmatrix} + \begin{pmatrix} RL^{-1}I \\ GC^{-1}V \end{pmatrix} = \mathbf{0}$$

with $I(t,x)$ the current and $V(t,x)$ the voltage at time t and location x along the line. From this equation, we have

$$\mathbf{y} = \begin{pmatrix} I \\ V \end{pmatrix} \qquad \mathbf{A}(\mathbf{y}) = \begin{pmatrix} 0 & -L^{-1} \\ -C^{-1} & 0 \end{pmatrix} \qquad \mathbf{g}(\mathbf{y}) = \begin{pmatrix} RL^{-1}I \\ GC^{-1}V \end{pmatrix}.$$

The system is hyperbolic since the matrix $\mathbf{A}$ has two distinct real eigenvalues $a_{1,2} = \pm(\sqrt{LC})^{-1}$.

The Saint-Venant equation

This equation, which is a simplification of the Navier-Stokes equation, describes the water propagation in a prismatic channel with rectangular cross-section and constant slope as follows :

$$\partial_t \begin{pmatrix} H \\ V \end{pmatrix} + \partial_x \begin{pmatrix} HV \\ \frac{1}{2}V^2 + gH \end{pmatrix} + \begin{pmatrix} 0 \\ g[CV^2H^{-1} - S] \end{pmatrix} = \mathbf{0} \tag{3}$$

with $H(t,x)$ the water height and $V(t,x)$ the water velocity at time t and location x along the channel. g is the gravity constant, C a friction parameter and S the canal slope. From this equation we have :

$$\mathbf{y} = \begin{pmatrix} H \\ V \end{pmatrix} \qquad \mathbf{A}(\mathbf{y}) = \begin{pmatrix} V & H \\ g & V \end{pmatrix} \qquad \mathbf{g}(\mathbf{y}) = \begin{pmatrix} 0 \\ g[CV^2H^{-1} - S] \end{pmatrix}.$$

The eigenvalues of the matrix $\mathbf{A}(\mathbf{y})$ are :

$$a_1(\mathbf{y}) = V + \sqrt{gH} \quad \text{and} \quad a_2(\mathbf{y}) = V - \sqrt{gH}.$$

The system is hyperbolic when the so-called Froude's number $Fr = V/\sqrt{gH} < 1$. In such a case, the flow in the channel is said to be *fluvial* or *subcritical*.

The Aw-Rascle equation

The Aw-Rascle equation is a basic fluid model for the description of road traffic dynamics. It is directly given here in the quasi-linear form. The model is as follows:

$$\partial_t \begin{pmatrix} \rho \\ V \end{pmatrix} + \begin{pmatrix} V & \rho \\ 0 & V + \rho V_o'(\rho) \end{pmatrix} \partial_x \begin{pmatrix} \rho \\ V \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{V - V_o(\rho)}{\tau} \end{pmatrix} = \mathbf{0} \tag{4}$$

with $\rho(t,x)$ the traffic density and $V(t,x)$ the speed of the vehicles at time t and location x along the road. The function $V_o(\rho)$ is the preferential speed function : it is a decreasing function that represents the relation, in the average, beween the speed of the vehicles and the traffic density (the higher the density, the lower the speed of the vehicles). The constant parameter τ is a positive time constant. The eigenvalues of the matrix $\mathbf{A}(\mathbf{y})$ are :

$$a_1(\mathbf{y}) = V \quad \text{and} \quad a_2(\mathbf{y}) = V + \rho V_o'(\rho).$$

The system is hyperbolic.

4. Networks of hyperbolic 2×2 systems of balance laws.

We now consider physical networks (e.g. irrigation or road networks). We assume that the topology of the network is represented by a graph. The edges of the graph represent the physical links (i.e the canals or the roads) having dynamics expressed by n hyperbolic 2×2 systems of balance laws

$$\partial_t \mathbf{y}_i + \mathbf{A}_i(\mathbf{y_i})\partial_x \mathbf{y}_i + \mathbf{g}_i(\mathbf{y}_i) = \mathbf{0} \quad i = 1, \ldots, n \quad \text{with} \quad \mathbf{y}_i \triangleq \begin{pmatrix} y_i \\ y_{n+i} \end{pmatrix} \tag{5}$$

or in a compact matrix form

$$\partial_t \mathbf{Y} + \mathbf{F}(\mathbf{Y})\partial_x \mathbf{Y} + \mathbf{G}(\mathbf{Y}) = \mathbf{0} \tag{6}$$

with the notation $\mathbf{Y} \triangleq (y_1, y_2, \ldots, y_n, y_{n+1}, \ldots, y_{2n})^T$ and the maps $\mathbf{F}$ and $\mathbf{G}$ defined accordingly.

The nodes of the graph represent the physical junctions between the links of the network. The physical interconnection mechanisms that occur at the junctions are supposed to be described by a set of static "junction models" that fix the boundary conditions of the PDEs (5). At this stage, it is not necessary to give a more precise formulation of these junction models. The issue will be addressed in Section 9.

5. Steady-state, characteristic form and linearisation.

The main purpose of this section is to present the so-called characteristic form of hyperbolic 2×2 systems of balance laws and its linearisation with respect to a steady-state equlibrium.

Steady-state : A steady-state solution (or equilibrium) for each system (5) is a constant solution $\mathbf{y}_i(t,x) = \mathbf{y}_i^* \; \forall t \in [0, +\infty)$, $\forall x \in [0, L]$ which satisfies the condition $\mathbf{g}_i(\mathbf{y}_i^*) = 0$.

It is a well known property (e.g. [11], [15]) that any system of the form (5) can be transformed into a (diagonalised) characteristic form by using appropriate characteristic coordinates. This means that, for any equilibrium $\mathbf{y}_i^*$, there exists a change of coordinates $\boldsymbol{\xi}_i = \Phi_i(\mathbf{y}_i)$ such that

1. $\Phi_i(\mathbf{y}_i^*) = \mathbf{0}$
2. the Jacobian matrix of Φ_i diagonalises the coefficient matrix $\mathbf{A}_i(\mathbf{y}_i)$ in Ω:

$$\Phi_i'(\mathbf{y}_i)\mathbf{A}_i(\mathbf{y}_i) = \mathbf{D}_i(\mathbf{y}_i)\Phi_i'(\mathbf{y}_i) \;\; \mathbf{y}_i \in \Omega \quad \text{with} \quad \mathbf{D}_i(\mathbf{y}_i) \triangleq \operatorname{diag}\{a_i(\mathbf{y}_i), a_{n+i}(\mathbf{y}_i)\}.$$

$(a_i(\mathbf{y}_i), a_{n+i}(\mathbf{y}_i)$ denote the two eigenvalues of the matrix $\mathbf{A}_i(\mathbf{y}_i))$.

Finding the change of coordinates $\boldsymbol{\xi}_i(\mathbf{y}_i)$ requires to find a solution of the first order partial differential equation $\Phi_i'(\mathbf{y}_i)\mathbf{A}_i(\mathbf{y}_i) = \mathbf{D}_i(\mathbf{y}_i)\Phi_i'(\mathbf{y}_i)$. As it is shown by Lax in [11, pages 34-35], this partial differential equation can be reduced to the integration of *ordinary* differential equations. Moreover, in many cases, these ordinary differential equations can be explicitly solved by using separation of variables, homogeneity or symmetry properties. Below we shall give the explicit solutions for the Saint-Venant and the Aw-Rascle equations. See also [15, Tome I, pages 146-147, page 152] for other examples.

In the coordinates $\boldsymbol{\xi}_i \triangleq (\xi_i, \xi_{n+i})^T$, each system (5) can then be rewritten in the **characteristic form**:

$$\partial_t \boldsymbol{\xi}_i + \mathbf{C}_i(\boldsymbol{\xi}_i)\partial_x \boldsymbol{\xi}_i + \mathbf{h}_i(\boldsymbol{\xi}_i) = \mathbf{0} \tag{7}$$

$$\text{with } \mathbf{C}_i(\boldsymbol{\xi}_i) \triangleq \mathbf{D}_i(\Phi_i^{-1}(\boldsymbol{\xi}_i)) \text{ and } \mathbf{h}_i(\boldsymbol{\xi}_i) \triangleq \mathbf{\Phi_i'}(\Phi_i^{-1}(\boldsymbol{\xi}_i)\mathbf{g_i}(\Phi_i^{-1}(\boldsymbol{\xi}_i)).$$

The network model (6) is then written in a more compact characteristic matrix form

$$\partial_t \boldsymbol{\xi} + \mathbf{C}(\boldsymbol{\xi})\partial_x \boldsymbol{\xi} + \mathbf{h}(\boldsymbol{\xi}) = \mathbf{0} \tag{8}$$

with the notation $\boldsymbol{\xi} \triangleq (\xi_1, \xi_2, \ldots, \xi_n, \xi_{n+1}, \ldots, \xi_{2n})$ and the corresponding definitions for the maps $\mathbf{C}$ and $\mathbf{h}$:

$$\mathbf{C}(\boldsymbol{\xi}) \triangleq \operatorname{diag}\{c_1(\xi_1, \xi_{n+1}), c_2(\xi_2, \xi_{n+2}), \ldots, c_{n+1}(\xi_1, \xi_{n+1}), \ldots, c_{2n}(\xi_n, \xi_{2n})\}$$
$$\mathbf{h}(\boldsymbol{\xi}) = (h_1(\xi_1, \xi_{n+1}), h_2(\xi_2, \xi_{n+2}), \ldots, h_{n+1}(\xi_1, \xi_{n+1}), \ldots, h_{2n}(\xi_n, \xi_{2n}))^T .$$

By definition of the coordinate transformation, the equilibrium of the characteristic form is zero (i.e. $\boldsymbol{\xi}_i^* = \Phi_i(\mathbf{y}_i^*) = \mathbf{0}\ \forall i$). It follows that the linearised characteristic form about the equilibrium is the hyperbolic 2×2 system of linear balance laws given by

$$\partial_t \boldsymbol{\xi} + \boldsymbol{\Lambda}\partial_x \boldsymbol{\xi} + \mathbf{B}\boldsymbol{\xi} = \mathbf{0} \quad \text{with} \quad \boldsymbol{\Lambda} \triangleq \mathbf{C}(\mathbf{0}) \quad \text{and} \quad \mathbf{B} \triangleq \mathbf{h}'(\mathbf{0}). \tag{9}$$

In the sequel, we shall use the notation $\boldsymbol{\Lambda} \triangleq \operatorname{diag}\{\lambda_i, i = 1, \ldots, 2n\}$. The λ_i eigenvalues are called *characteristic velocities.*

Examples

- For the Saint-Venant equation (3) an equilibrium is a constant state H^*, V^* that verifies the relation
$$SH^* = C(V^*)^2.$$
The linearisation of the Saint-Venant equations around the equilibrium is given by equation (9) with the following characteristic state variables (ξ_1, ξ_2), characteristic velocities λ_1, λ_2 and matrix $\mathbf{B}$:
$$\xi_1 = (V - V^*) + (H - H^*)\sqrt{\frac{g}{H^*}} \qquad \xi_2 = (V - V^*) - (H - H^*)\sqrt{\frac{g}{H^*}}$$
$$\lambda_2 = V^* - \sqrt{gH^*} < 0 < \lambda_1 = V^* + \sqrt{gH^*}$$
$$\mathbf{B} \triangleq \begin{pmatrix} \gamma & \delta \\ \gamma & \delta \end{pmatrix} \text{ with } \gamma = \frac{gS}{2}\Big(\frac{2}{V^*} - \frac{1}{\sqrt{gH^*}}\Big) > 0 \text{ and } \delta = \frac{gS}{2}\Big(\frac{2}{V^*} + \frac{1}{\sqrt{gH^*}}\Big) > 0$$

- For the Aw-Rascle equation (4) an equilibrium is a constant state ρ^*, V^* that verifies the relation
$$V^* = V_o(\rho^*).$$
The linearisation of the Aw-Raxcle equation around the equilibrium is given by equation (9) with the following characteristic state variables (ξ_1, ξ_2), characteristic velocities λ_1, λ_2 and matrix $\mathbf{B}$:
$$\xi_1 = V - V^* - V_o'(\rho^*)(\rho - \rho^*) \qquad \xi_2 = V - V^*$$
$$0 < \lambda_2 = V^* + \rho^* V_o'(\rho^*) < \lambda_1 = V^*$$
$$\mathbf{B} \triangleq \begin{pmatrix} \gamma & \delta \\ \gamma & \delta \end{pmatrix} \text{ with } \gamma = -\frac{V_o'(\rho^*)}{\tau} > 0 \text{ and } \delta = \frac{V^*}{\tau} > 0.$$

6. Lyapunov stability analysis of the linearised system.

In this section, we present a Lyapunov function which can be used for analysing the asymptotic convergence of the classical solutions of the linearised system (9) under linear boundary conditions of the general form

$$\mathbf{K}_0\boldsymbol{\xi}(t,0) + \mathbf{K}_1\boldsymbol{\xi}(t,L) = \mathbf{0}, \quad t \in [0,+\infty) \tag{10}$$

We consider the Cauchy problem

$$\begin{aligned}
&\partial_t\boldsymbol{\xi} + \boldsymbol{\Lambda}\partial_x\boldsymbol{\xi} + \mathbf{B}\boldsymbol{\xi} = \mathbf{0} \;\; t \in [0,+\infty), \quad x \in (0,L), && \text{(11a)}\\
&\mathbf{K}_0\boldsymbol{\xi}(t,0) + \mathbf{K}_1\boldsymbol{\xi}(t,L) = \mathbf{0}, \quad t \in [0,+\infty), && \text{(11b)}\\
&\boldsymbol{\xi}(0,x) = \boldsymbol{\xi}^o(x), \quad x \in (0,L). && \text{(11c)}
\end{aligned}$$

This Cauchy problem is well-posed (see e.g. [5, Section 2.1 and Section 2.3]). This means that for any initial condition $\boldsymbol{\xi}^o \in L^2((0,L);\mathbb{R}^{2n})$ and for every $T>0$, there exists $C(T)>0$ such that a solution $\boldsymbol{\xi}(t,x) \in C^0([0,+\infty);L^2((0,1);\mathbb{R}^{2n}))$ exists, is unique and satisfies

$$\|\boldsymbol{\xi}(t,\cdot)\|_{L^2((0,1);\mathbb{R}^{2n})} \leqslant C(T)\|\boldsymbol{\xi}^o\|_{L^2((0,1);\mathbb{R}^{2n})}, \; \forall t \in [0,T].$$

Our concern is to analyse the exponential stability of the system (9)-(10) according to the following definition.

Definition 1. The system (9)-(10) is exponentially stable (in L^2-norm) if there exist $\nu>0$ and $C>0$ such that, for every initial condition $\boldsymbol{\xi}^0(x) \in L^2((0,L);\mathbb{R}^{2n})$ the solution to the Cauchy problem (11) satisfies

$$\|\boldsymbol{\xi}(t,\cdot)\|_{L^2((0,L);\mathbb{R}^{2n})} \leqslant Ce^{-\nu t}\|\boldsymbol{\xi}^o\|_{L^2((0,1);\mathbb{R}^{2n})}.$$

■

The following candidate Lyapunov function is defined:

$$V = \int_0^L \boldsymbol{\xi}^T\mathbf{P}(x)\boldsymbol{\xi}dx \tag{12}$$

where the weighting matrix $\mathbf{P}(x)$ is defined as follows: $\mathbf{P}(x) \triangleq \mathrm{diag}\{p_ie^{-\sigma_i\mu x}, i=1,\dots,2n\}$, with $\mu>0$, $p_i>0$ positive real numbers and $\sigma_i = \mathrm{sign}(\lambda_i)$.

The time derivative of V along the solutions of (11) is

$$\begin{aligned}
\dot{V} &= \int_0^L \Big(\partial_t\boldsymbol{\xi}^T\mathbf{P}(x)\boldsymbol{\xi} + \boldsymbol{\xi}^T\mathbf{P}(x)\partial_t\boldsymbol{\xi}\Big)\,dx\\
&= -\int_0^L \Big(\partial_x\boldsymbol{\xi}^T\boldsymbol{\Lambda}\mathbf{P}(x)\boldsymbol{\xi} + \boldsymbol{\xi}^T\mathbf{P}(x)\boldsymbol{\Lambda}\partial_x\boldsymbol{\xi} + \boldsymbol{\xi}^T\mathbf{B}^T\mathbf{P}(x)\boldsymbol{\xi} + \boldsymbol{\xi}^T\mathbf{P}(x)\mathbf{B}\boldsymbol{\xi}\Big)\,dx\\
&= -\int_0^L \partial_x(\boldsymbol{\xi}^T\mathbf{M}(x)\boldsymbol{\xi})dx - \int_0^L \boldsymbol{\xi}^T\Big(\mathbf{B}^T\mathbf{P}(x) + \mathbf{P}(x)\mathbf{B}\Big)\boldsymbol{\xi}\,dx
\end{aligned}$$

with the positive diagonal matrix $\mathbf{M}(x) \triangleq \mathrm{diag}\{p_i|\lambda_i|e^{-\sigma_i\mu x}, i=1,\dots,2n\}$. Integrating by parts, we obtain:

$$\begin{aligned}
\dot{V} &= -\int_0^L \partial_x \Big[\boldsymbol{\xi}^T \mathbf{M}(x)\boldsymbol{\xi}\Big]\, dx - \int_0^L \boldsymbol{\xi}^T \Big(\mu\mathbf{M}(x) + \mathbf{B}^T\mathbf{P}(x) + \mathbf{P}(x)\mathbf{B}\Big)\boldsymbol{\xi}\, dx \\
&= -\Big[\boldsymbol{\xi}^T \mathbf{M}(x)\boldsymbol{\xi}\Big]_0^L - \int_0^L \boldsymbol{\xi}^T \Big(\mu\mathbf{M}(x) + \mathbf{B}^T\mathbf{P}(x) + \mathbf{P}(x)\mathbf{B}\Big)\boldsymbol{\xi}\, dx \\
&= -\Big[\boldsymbol{\xi}^T(t,L)\mathbf{M}(L)\boldsymbol{\xi}(t,L) - \boldsymbol{\xi}^T(t,0)\mathbf{M}(0)\boldsymbol{\xi}(t,0)\Big] - \int_0^L \boldsymbol{\xi}^T \Big(\mu\mathbf{M}(x) + \mathbf{B}^T\mathbf{P}(x) + \mathbf{P}(x)\mathbf{B}\Big)\boldsymbol{\xi}\, dx.
\end{aligned}$$

The system (9)-(10) is exponentially stable if this function $\dot{V}$ is negative definite. We have thus shown the following result.

Theorem 1. The system (9)-(10) is exponentially stable if there exist $\mu > 0$ and $p_i > 0$ $i = 1, \ldots, 2n$ such that

C1. The boundary quadratic form $\boldsymbol{\xi}^T(t,0)\mathbf{M}(0)\boldsymbol{\xi}(t,0) - \boldsymbol{\xi}^T(t,L)\mathbf{M}(L)\boldsymbol{\xi}(t,L)$ is positive definite under the constraint of the linear boundary condition $\mathbf{K}_0\boldsymbol{\xi}(t,0) + \mathbf{K}_1\boldsymbol{\xi}(t,L) = \mathbf{0}$;

C2. The matrix $\mu\mathbf{M}(x) + \mathbf{B}^T\mathbf{P}(x) + \mathbf{P}(x)\mathbf{B}$ is positive definite $\forall x \in (0, L)$. ■

Boundary conditions that satisfy condition C1 are called **Dissipative Boundary Conditions**. Condition C1 is satisfied if and only if the leading principal minors of order $> 4n$ of the matrix

$$\begin{pmatrix} \mathbf{0} & \mathbf{K}_0 & \mathbf{K}_1 \\ -\mathbf{K}_0^T & \mathbf{M}(0) & \mathbf{0} \\ -\mathbf{K}_1^T & \mathbf{0} & -\mathbf{M}(L) \end{pmatrix}$$

are strictly positive (see [18]).

7. Dissipative boundary conditions for systems of (almost) conservative laws.

In this section, we will present a variant of Theorem 1 with an explicit characterisation of a sufficient dissipative boundary condition which guarantees the system exponential stability in the case where $\|\mathbf{B}\|$ is sufficiently small or, in more intuitive terms, when the considered balance laws are almost conservative. We know that $\boldsymbol{\Lambda}$ is a diagonal matrix with non-zero real diagonal entries. Without loss of generality, possibly through a permutation of the state variables, we may assume that

$$\boldsymbol{\Lambda} = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_{2n}\}, \qquad \lambda_i > 0 \;\; \forall i \in \{1, \ldots, m\}, \qquad \lambda_i < 0 \;\; \forall i \in \{m+1, \ldots, 2n\}. \tag{13}$$

We introduce the notations

$$\begin{aligned}
&\boldsymbol{\xi}^+ = (\xi_1, \ldots, \xi_m) \;\text{ and }\; \boldsymbol{\xi}^- = (\xi_{m+1}, \ldots, \xi_{2n}) \;\text{ such that }\; \boldsymbol{\xi} = (\boldsymbol{\xi}^{+T}, \boldsymbol{\xi}^{-T}) \\
&\boldsymbol{\Lambda}^+ = \mathrm{diag}\{\lambda_1, \ldots, \lambda_m\} \;\text{ and }\; \boldsymbol{\Lambda}^- = \mathrm{diag}\{|\lambda_{m+1}|, \ldots, |\lambda_{2n}|\} \;\text{ such that }\; \boldsymbol{\Lambda} = \mathrm{diag}\{\boldsymbol{\Lambda}^+, -\boldsymbol{\Lambda}^-\}.
\end{aligned}$$

With these notations, the linear hyperbolic system (9) is writtem

$$\partial_t \begin{pmatrix} \boldsymbol{\xi}^+ \\ \boldsymbol{\xi}^- \end{pmatrix} + \begin{pmatrix} \boldsymbol{\Lambda}^+ & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{\Lambda}^- \end{pmatrix} \partial_x \begin{pmatrix} \boldsymbol{\xi}^+ \\ \boldsymbol{\xi}^- \end{pmatrix} + \mathbf{B}\boldsymbol{\xi} = \mathbf{0}. \tag{14}$$

The general linear boundary condition (10) is written in the specific form

$$\begin{pmatrix} \boldsymbol{\xi}^{+}(t,0) \\ \boldsymbol{\xi}^{-}(t,L)) \end{pmatrix} = \underbrace{\begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{pmatrix}}_{\mathbf{K}} \begin{pmatrix} \boldsymbol{\xi}^{+}(t,L) \\ \boldsymbol{\xi}^{-}(t,0) \end{pmatrix}. \tag{15}$$

Let $\mathcal{D}_p$ denote the set of diagonal $p \times p$ real matrices with strictly positive diagonal entries. We introduce the following norm for the matrix $\mathbf{K}$:

$$\rho(\mathbf{K}) \triangleq \inf \left\{ \|\Delta \mathbf{K} \Delta^{-1}\|, \Delta \in \mathcal{D}_{2n} \right\}$$

where $\| \ \|$ denotes the ususal matrix 2-norm. We have the following Theorem.

Theorem 2. If $\rho(\mathbf{K}) < 1$, there exists $\varepsilon > 0$ such that, if $\|\mathbf{B}\| < \varepsilon$, then the linear hyperbolic system (14)-(15) is exponentially stable.

Proof. With the above notations, the candidate Lyapunov function (12) is written

$$V = \int_0^L \left[(\boldsymbol{\xi}^{+T} P_0 \boldsymbol{\xi}^{+}) e^{-\mu x} + (\boldsymbol{\xi}^{-T} P_1 \boldsymbol{\xi}^{-}) e^{\mu x} \right] dx. \tag{16}$$

with $P_0 \in \mathcal{D}_m$, $P_1 \in \mathcal{D}_{2n-m}$ and $\mu > 0$. The time derivative of V is

$$\dot{V} = \dot{V}_1 + \dot{V}_2 \tag{17}$$

with

$$\begin{aligned} \dot{V}_1 &\triangleq -\left[\boldsymbol{\xi}^{+T} P_0 \boldsymbol{\Lambda}^{+} \boldsymbol{\xi}^{+} e^{-\mu x} \right]_0^L + \left[\boldsymbol{\xi}^{-T} P_1 \boldsymbol{\Lambda}^{-} \boldsymbol{\xi}^{-} e^{\mu x} \right]_0^L \\ \dot{V}_2 &\triangleq \int_0^L \boldsymbol{\xi}^T \left(\mu P(x) \boldsymbol{\Lambda} + \mathbf{B}^T P(x) + P(x) \mathbf{B} \right) \boldsymbol{\xi} \, dx \end{aligned}$$

and $P(x) \triangleq \operatorname{diag}\{P_0 e^{-\mu x}, P_1 e^{\mu x}\}$.

In order to prove that the boundary condition (15) is dissipative we will show that P_0, P_1 and μ can be selected such that $\dot{V}_1$ is a negative definite quadratic form. For this analysis, we introduce the following notations:

$$\boldsymbol{\xi}_0^{-}(t) \triangleq \boldsymbol{\xi}^{-}(t,0) \quad \boldsymbol{\xi}_1^{+}(t) \triangleq \boldsymbol{\xi}^{+}(t,L).$$

Using the boundary condition (15), we have

$$\begin{aligned} \dot{V}_1 &= -\left[\boldsymbol{\xi}^{+T} P_0 \boldsymbol{\Lambda}^{+} \boldsymbol{\xi}^{+} e^{-\mu x} \right]_0^L + \left[\boldsymbol{\xi}^{-T} P_1 \boldsymbol{\Lambda}^{-} \boldsymbol{\xi}^{-} e^{\mu x} \right]_0^L \\ &= -\left(\boldsymbol{\xi}_1^{+T} P_0 \boldsymbol{\Lambda}^{+} \boldsymbol{\xi}_1^{+} \, e^{-\mu L} + \boldsymbol{\xi}_0^{-T} P_1 \boldsymbol{\Lambda}^{-} \boldsymbol{\xi}_0^{-} \right) \\ &+ \left(\boldsymbol{\xi}_1^{+T} K_{00}^T + \boldsymbol{\xi}_0^{-T} K_{01}^T \right) P_0 \boldsymbol{\Lambda}^{+} \left(K_{00} \boldsymbol{\xi}_1^{+} + K_{01} \boldsymbol{\xi}_0^{-} \right) \\ &+ \left(\boldsymbol{\xi}_1^{+T} K_{10}^T + \boldsymbol{\xi}_0^{-T} K_{11}^T \right) P_1 \boldsymbol{\Lambda}^{-} \left(K_{10} \boldsymbol{\xi}_1^{+} + K_{11} \boldsymbol{\xi}_0^{-} \right) e^{\mu L}. \end{aligned}$$

Since $\rho(\mathbf{K}) < 1$ by assumption, there exist $D_0 \in \mathcal{D}_m$, $D_1 \in \mathcal{D}_{2n-m}$ and $\Delta \triangleq \operatorname{diag}\{D_0, D_1\}$ such that

$$\|\Delta \mathbf{K} \Delta^{-1}\| < 1. \tag{18}$$

The matrices P_0 and P_1 are selected such that $P_0\mathbf{\Lambda}^+ = D_0^2$ and $P_1\mathbf{\Lambda}^- = D_1^2$. We define $\mathbf{z}_0 \triangleq D_0\boldsymbol{\xi}_0^-$, $\mathbf{z}_1 \triangleq D_1\boldsymbol{\xi}_1^+$ and $\mathbf{z}^T \triangleq (\mathbf{z}_0^T, \mathbf{z}_1^T)$. Then, using inequality (18), we have

$$\begin{aligned}
&\left(\boldsymbol{\xi}_1^{+T} K_{00}^T + \boldsymbol{\xi}_0^{-T} K_{01}^T\right) P_0\mathbf{\Lambda}^+ \left(K_{00}\boldsymbol{\xi}_1^+ + K_{01}\boldsymbol{\xi}_0^-\right)\\
&+ \left(\boldsymbol{\xi}_1^{+T} K_{10}^T + \boldsymbol{\xi}_0^{-T} K_{11}^T\right) P_1\mathbf{\Lambda}^- \left(K_{10}\boldsymbol{\xi}_1^+ + K_{11}\boldsymbol{\xi}_0^-\right)\\
&= \|\Delta\mathbf{K}\Delta^{-1}\mathbf{z}\|^2 \; < \; \|\mathbf{z}\|^2 = \; \boldsymbol{\xi}_1^{+T} P_0\mathbf{\Lambda}^+\boldsymbol{\xi}_1^+ + \boldsymbol{\xi}_0^{-T} P_1\mathbf{\Lambda}^-\boldsymbol{\xi}_0^- .
\end{aligned}$$

It follows that μ can be taken sufficiently small such that $\dot{V}_1$ is a negative definite quadratic form.

Moreover, for any $\mu > 0$, there exist clearly two positive constants $\varepsilon > 0$ and $\alpha > 0$ such that

$$\|\mathbf{B}\| < \varepsilon \;\Rightarrow\; \dot{V}_2 \leqslant -\alpha V \;\Rightarrow\; \dot{V} = \dot{V}_1 + \dot{V}_2 \leqslant -\alpha V.$$

Consequently the solutions of the system (14)-(15) exponentially converge to $\mathbf{0}$ in L^2-norm. ∎

8. Application to the linearised Saint-Venant equation.

In the previous section, we have shown that for systems with small $\|\mathbf{B}\|$, the dissipative boundary condition $\rho(\mathbf{K}) < 1$ is a sufficient stability condition. However it must be emphasized that, in particular instances, this stability condition may be valid even if $\|\mathbf{B}\|$ is not small. This is true in particular for hydraulic networks described by Saint-Venant equations as long as the subcritical flow condition is satisfied as we shall illustrate in the present section. For the sake of simplicity, we consider the specific case of a single pool of an open-channel, but the property can be easily extended to networks of interconnected pools. As we have seen in Section 4, the linearised Saint-Venant equation in characteristic form is written as

$$\partial_t\xi_1 + \lambda_1\partial_x\xi_1 + \gamma\xi_1 + \delta\xi_2 = 0 \tag{19a}$$
$$\partial_t\xi_2 - |\lambda_2|\partial_x\xi_2 + \gamma\xi_1 + \delta\xi_2 = 0 \tag{19b}$$

$$\begin{aligned}
&\text{with } \; \lambda_2 = V^* - \sqrt{gH^*} < 0 < \lambda_1 = V^* + \sqrt{gH^*}\\
&\text{and } \; \gamma = \frac{gS}{2}\Big(\frac{2}{V^*} - \frac{1}{\sqrt{gH^*}}\Big) > 0 \;\; , \;\; \delta = \frac{gS}{2}\Big(\frac{2}{V^*} + \frac{1}{\sqrt{gH^*}}\Big) > 0.
\end{aligned}$$

We consider this system under simple boundary conditions of the form

$$\xi_1(t,0) = k_1\xi_2(t,0) \qquad\qquad \xi_2(t,L) = k_2\xi_1(t,L) \tag{20}$$

which are a special case of (15). With these notations, the Lyapunov function (16) is written

$$V = \int_0^L \left(\xi_1^2 p_1 e^{-\mu x} + \xi_2^2 p_2 e^{\mu x}\right) dx, \qquad p_1, p_2, \mu > 0.$$

After some calculations the time derivative of this Lyapunov function is

$$\begin{aligned}
\dot{V} = &- \Big[\lambda_1\xi_1^2 p_1 e^{-\mu x} - |\lambda_2|\xi_2^2 p_2 e^{\mu x}\Big]_0^L\\
&- \int_0^L (\xi_1 \;\; \xi_2) \begin{pmatrix} (\lambda_1\mu + 2\gamma)p_1 e^{-\mu x} & \delta p_1 e^{-\mu x} + \gamma p_2 e^{\mu x} \\ \delta p_1 e^{-\mu x} + \gamma p_2 e^{\mu x} & (|\lambda_2|\mu + 2\delta)p_2 e^{\mu x} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} dx.
\end{aligned}$$

Using the boundary conditions (20) with the notations $\xi_{1,2}(0) \triangleq \xi_{1,2}(t,0)$ and $\xi_{1,2}(L) \triangleq \xi_{1,2}(t,L)$, the stability condition relative to the first term of $\dot{V}$ becomes

$$\begin{aligned}
& -\Big[\lambda_1 \xi_1^2 p_1 e^{-\mu x} - |\lambda_2| \xi_2^2 p_2 e^{\mu x}\Big]_0^L \\
& = \lambda_1 p_1 \xi_1^2(L) e^{-\mu L} - |\lambda_2| p_2 \xi_2^2(L) e^{\mu L} - \lambda_1 p_1 \xi_1^2(0) + |\lambda_2| p_2 \xi_2^2(0) \\
& = \lambda_1 p_1 \xi_1^2(L) e^{-\mu L} - |\lambda_2| p_2 k_2^2 \xi_1^2(L) e^{\mu L} - \lambda_1 p_1 k_1^2 \xi_2^2(0) + |\lambda_2| p_2 \xi_2^2(0) \\
& = -\xi_1^2(L)\big[|\lambda_2| p_2 k_2^2 e^{\mu L} - \lambda_1 p_1 e^{-\mu L}\big] - \xi_2^2(0)\big[\lambda_1 p_1 k_1^2 - |\lambda_2| p_2\big] \\
& < 0
\end{aligned}$$

$$\iff \quad \boxed{k_1^2 k_2^2 e^{2\mu L} < \frac{\lambda_1}{|\lambda_2|}\frac{p_1}{p_2} k_1^2 < 1.} \tag{21}$$

The stability condition relative to the second term of $\dot{V}$ is

$$\begin{pmatrix} (\lambda_1 \mu + 2\gamma) p_1 e^{-\mu x} & \delta p_1 e^{-\mu x} + \gamma p_2 e^{\mu x} \\ \delta p_1 e^{-\mu x} + \gamma p_2 e^{\mu x} & (|\lambda_2|\mu + 2\delta) p_2 e^{\mu x} \end{pmatrix} > 0$$

$$\iff \quad \boxed{\begin{array}{c} \textbf{(a)} \quad \lambda_1 \mu + 2\gamma > 0 \qquad |\lambda_2|\mu + 2\delta > 0, \\ \\ \textbf{(b)} \quad p_1 p_2 (\lambda_1 \mu + 2\gamma)(|\lambda_2|\mu + 2\delta) - (\delta p_1 e^{-\mu x} + \gamma p_2 e^{\mu x})^2 > 0. \end{array}} \tag{22}$$

The goal is now to determine conditions on the parameters k_1, k_2, $p_1 > 0$, $p_2 > 0$ and $\mu > 0$ such that these sufficient stability conditions are satisfied.

Condition (22)-**(a)** is satisfied for any $\mu > 0$.

We are going to show that condition (22)-**(b)** is satisfied for sufficiently small $\mu > 0$ if the parameters p_1, p_2 are selected such that $\delta p_1 = \gamma p_2$. For any $p_1 > 0, p_2 > 0$ and $\mu > 0$, the term $(\delta p_1 e^{-\mu x} + \gamma p_2 e^{\mu x})^2$ in condition (22)-**(b)** is maximum either at $x = 0$ or at $x = L$.

- For $x = 0$, we have:

$$\begin{aligned}
& p_1 p_2 (\lambda_1 \mu + 2\gamma)(|\lambda_2|\mu + 2\delta) - (\delta p_1 + \gamma p_2)^2 \\
& = \mu^2 [p_1 p_2 \lambda_1 |\lambda_2|] + \mu [2 p_1 p_2 (\lambda_1 \delta + |\lambda_2| \gamma)] - (\delta p_1 - \gamma p_2)^2 \\
& = \mu^2 [p_1 p_2 \lambda_1 |\lambda_2|] + \mu [2 p_1 p_2 (\lambda_1 \delta + |\lambda_2| \gamma)] > 0
\end{aligned}$$

  for any $\mu > 0$.

- For $x = L$, we have:

$$\begin{aligned}
& p_1 p_2 (\lambda_1 \mu + 2\gamma)(|\lambda_2|\mu + 2\delta) - (\delta p_1 e^{-\mu L} + \gamma p_2 e^{\mu L})^2 \\
& = p_1 p_2 \Big[\mu^2 \lambda_1 |\lambda_2| + 2\mu(\lambda_1 \delta + |\lambda_2| \gamma)\Big] - (\delta p_1 e^{-\mu L} - \gamma p_2 e^{\mu L})^2 > 0
\end{aligned}$$

  for $\mu > 0$ sufficiently small (because the function $F(\mu) \triangleq (\delta p_1 e^{-\mu L} - \gamma p_2 e^{\mu L})^2$ is quadratic in μ since $F(0) = 0$ and $F'(0) = 0$ if $\delta p_1 = \gamma p_2$).

Finally, condition (21) is satisfied if $|k_1| < \sqrt{\dfrac{|\lambda_2|\delta}{\lambda_1\gamma}}$ and $|k_2| < \sqrt{\dfrac{\lambda_1\gamma}{|\lambda_2|\delta}}$.

We remark that this result is valid for arbitrarily large values of γ and δ (i.e. arbitrarily large values of $\|\mathbf{B}\|$) as long as the fluvial flow condition $V^*/\sqrt{gH^*} < 1$ is satisfied. Moreover, we remark also that $|k_1k_2| < 1$ which, in this special case, is equivalent to the sufficient dissipative boundary condition $\rho(\mathbf{K}) < 1$.

9. Junction models and boundary control design.

In Theorems 1 and 2, it has been shown that the exponential stability of the system can be guaranteed if the boundary conditions are dissipative. In this section, we intend to discuss the implications of this property for the design of boundary feedback control for physical networks. The mechanisms that occur at the junctions of physical networks are most often modelled by algebraic relations that determine the boundary conditions of the involved PDEs. These algebraic relations depend on inherent physical constraints on the network operation (like e.g. flow conservation conditions) but they can often also be partially assigned by control devices which may be used for network regulation and stabilisation. We thus consider the general model (6) of a network of 2×2 systems of balance laws

$$\partial_t\mathbf{Y} + \mathbf{F}(\mathbf{Y})\partial_x\mathbf{Y} + \mathbf{G}(\mathbf{Y}) = \mathbf{0} \quad t \in [0, +\infty) \;\; x \in [0, L] \tag{23}$$

and we assume that the junction models provide a set of $2n$ boundary conditions written under the general form

$$\mathbf{b}(\mathbf{Y}(0,t), \mathbf{Y}(L,t), \mathbf{U}(t)) = 0 \tag{24}$$

where $\mathbf{U}(t)$ is the vector of control inputs. There is no universal structure for the function $\mathbf{b}$ whose specific form depends on the specificities of each special case and of the implemented control devices. Rather than presenting a generic analysis of the control design issue, it is therefore more informative to treat a particular example in details.

Application to level control in open-channels.

In navigable rivers or irrigation channels (see e.g. [3]) the water is transported along the channel under the power of gravity through successive pools separated by automated gates that are used to regulate the water flow, as illustrated in Figures 1 and 2. We consider a channel with n pools

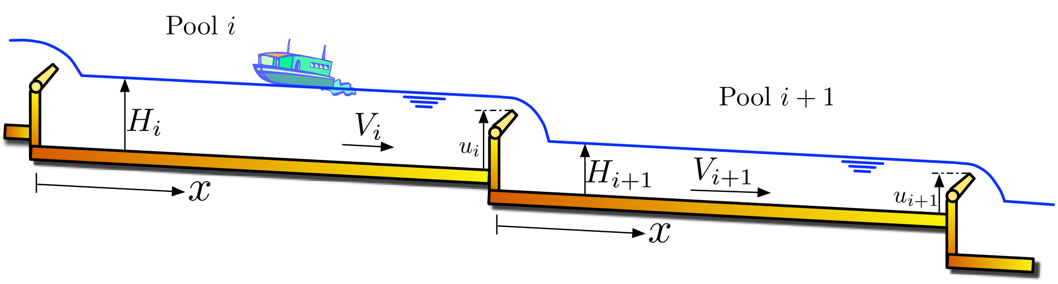

Figure 1: Lateral view of successive pools of an open-water channel with overflow gates.

Figure 2: Automated control gates in the Sambre river (Belgium). The left gate is in operation. The right gate is lifted for maintenance. (©L.Moens)

the dynamics of which being described by Saint-Venant equations:

$$\partial_t \begin{pmatrix} H_i \\ V_i \end{pmatrix} + \partial_x \begin{pmatrix} H_i V_i \\ \frac{1}{2}V_i^2 + gH_i \end{pmatrix} + \begin{pmatrix} 0 \\ g[CV_i^2 H_i^{-1} - S] \end{pmatrix} = \mathbf{0}, \qquad i = 1 \ldots, n. \tag{25}$$

In this model, for simplicity, we assume that all the pools have a rectangular section with the same width W. System (25) is subject to a set of $2n$ boundary conditions that are distributed into three subsets:

1) A first subset of $n-1$ conditions expresses the natural physical constraint of flow-rate conservation between the pools (the flow that exits pool i is (evidently) equal to the flow that enters pool $i+1$):

$$H_i(t,L)V_i(t,L) = H_{i+1}(t,0)V_{i+1}(t,0) \qquad i = 1,\ldots,n-1$$

2) A second subset of n boundary condition is made up of the equations that describe the gate operations. A standard gate model is given by the algebraic relation

$$H_i(t,L)V_i(t,L) = k_G\sqrt{\left[H_i(t,L) - u_i(t)\right]^3} \qquad i = 1,\ldots,n \tag{26}$$

where k_G is a positive constant coefficient and $u_i(t)$ denotes the weir elevation which is a control input (see Fig.1).

3) The last boundary condition imposes the value of the canal inflow rate that we denote $Q_0(t)$:

$$WH_1(t,0)V_1(t,0) = Q_0(t).$$

Depending on the application, $Q_0(t)$ may be viewed as a control input (in irrigation channels) or as a disturbance input (in navigable rivers).

Boundary control design.

From Section 5, we know that the characteristic state variables of system (25) are

$$\xi_i = (V_i - V_i^*) + (H_i - H_i^*)\sqrt{\frac{g}{H_i^*}} \qquad \xi_{n+i} = (V_i - V_i^*) - (H_i - H_i^*)\sqrt{\frac{g}{H_i^*}} \qquad i = 1, \dots, n. \quad (27)$$

Motivated by the stability analysis of the previous section, and in particular by the relation (20), we now assume that we want the boundary conditions to satisfy the following relations in characteristic coordinates:

$$\xi_{n+i}(t, L) = -k_i \xi_i(t, L) \qquad i = 1, \dots, n. \quad (28)$$

with k_i control tuning parameters. Then, eliminating ξ_i $(i = 1, \dots, 2n)$ between (26), (27) and (28) we get the following expressions for boundary feedback control laws that realise the target boundary conditions (28):

$$u_i = H_i(t, L) - \left[\frac{H_i(t, L)}{k_G}\left(\frac{1 - k_i}{1 + k_i}(H_i(t, L) - H_i^*)\sqrt{\frac{g}{H_i^*}} + \sqrt{\frac{SH_i^*}{C}}\right)\right]^{2/3} \qquad i = 1, \dots, n. \quad (29)$$

It can be seen that these control laws have the form of a state feedback. In addition, it must be emphasized that the implementation of the controls is particularly simple since only measurements of the water levels $H_i(t, L)$ at the gates are required. This means that the feedback implementation does not require neither level measurements inside the pools nor any velocity or flow rate measurements.

Closed loop stability analysis.

We consider the closed loop system with a constant inflow rate $Q_0(t) = Q^*$. We are going to explicit sufficient conditions on the control tuning parameters k_i that guarantee the dissipativity of the boundary conditions and therefore the exponential stability of the equilibrium according to Theorem 2. The first task is to express the linearisation of the boundary conditions in the form (15):

$$\begin{pmatrix} \boldsymbol{\xi}^+(t, 0) \\ \boldsymbol{\xi}^-(t, L)) \end{pmatrix} = \begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{pmatrix} \begin{pmatrix} \boldsymbol{\xi}^+(t, L) \\ \boldsymbol{\xi}^-(t, 0) \end{pmatrix}.$$

In the present application we have $\boldsymbol{\xi}^+ \triangleq (\xi_1, \dots, \xi_n)^T$ and $\boldsymbol{\xi}^- \triangleq (\xi_{n+1}, \dots, \xi_{2n})^T$. The matrix K_{10} is immediately given by the conditions (28):

$$K_{10} = \mathrm{diag}\{-k_i, i = 1, \dots, n\}.$$

Straightforward calculations show that matrices K_{00} and K_{01} and K_{11} are as follows:

$$K_{00} = 0 \qquad K_{01} = \mathrm{diag}\{\frac{\lambda_{n+i}}{\lambda_i}, i = 1, \dots, n\}$$

with the characteristic velocities (see Section 5)

$$\lambda_i \triangleq V_i^* + \sqrt{gH_i^*} \qquad \text{and} \qquad \lambda_{n+i} \triangleq V_i^* - \sqrt{gH_i^*}.$$

Finally, the matrix K_{11} is a $n \times n$ matrix with entries

$$K_{11}[i+1, i] = \frac{(\lambda_i - k_i \lambda_{n+i})}{\lambda_{i+1}}\sqrt{\frac{H_i^*}{H_{i+1}^*}} \quad \text{and} \quad 0 \text{ elsewhere.}$$

Then it can be checked that, in this special case, the dissipative boundary condition $\rho(\mathbf{K}) < 1$ is satisfied if the control tuning parameters k_i verify the conditions:

$$|k_i| < \frac{|\lambda_i|}{|\lambda_{n+i}|} \quad \forall i \in \{1, \ldots, n\}.$$

In physical terms, this means that the control gains k_i must be smaller than the ratio between the largest and the smallest characteristic velocity in each pool.

10. Lyapunov stability analysis of nonlinear systems.

In this section we shall briefly explain how the **linear** Lyapunov stability analysis of Section 7 can be extended to the case of the **nonlinear** hyperbolic system (8). Using the notations and definitions introduced in Section 7, the nonlinear hyperbolic system (8) in characteristic coordinates in a neighborhood of the origin is written

$$\begin{pmatrix} \partial_t \boldsymbol{\xi}^+ + \boldsymbol{\Lambda}^+(\boldsymbol{\xi}) \partial_x \boldsymbol{\xi}^+ \\ \partial_t \boldsymbol{\xi}^- - \boldsymbol{\Lambda}^-(\boldsymbol{\xi}) \partial_x \boldsymbol{\xi}^- \end{pmatrix} + \mathbf{h}(\boldsymbol{\xi}) = \mathbf{0} \tag{30}$$

and is considered under nonlinear boundary conditions of the form

$$\begin{pmatrix} \boldsymbol{\xi}^+(t,0) \\ \boldsymbol{\xi}^-(t,L)) \end{pmatrix} = \mathbf{H} \begin{pmatrix} \boldsymbol{\xi}^-(t,0) \\ \boldsymbol{\xi}^+(t,L)) \end{pmatrix} \tag{31}$$

with a nonlinear map $\mathbf{H} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$.

With Theorem 2 we have proved the convergence to zero of the solutions of the linear system (14)-(15) in $L^2(0,L)$-norm. Unfortunately the same Lyapunov function cannot be directly used to analyse the local syability in the nonlinear case. As we have emphasized in detail in [7], in order to extend the Lyapunov stability analysis to the nonlinear case, it is needed to prove a convergence in $H^2(0,L)$-norm. We therefore adopt the following definition of the (local) exponential stability of the steady-state solution $\boldsymbol{\xi}(t,x) \equiv 0$.

Definition 2. The equilibrium solution $\boldsymbol{\xi} \equiv 0$ of the nonlinear hyperbolic system (30)-(31) is exponentially stable (for the H^2-norm) if there exist $\delta > 0$, $\nu > 0$ and $C > 0$ such that, for every initial condition

$$\boldsymbol{\xi}(0,x) = \boldsymbol{\xi}^0(x) \in H^2((0,1), \mathbb{R}^n) \tag{32}$$

satisfying

$$\|\boldsymbol{\xi}^0\|_{H^2((0,1),\mathbb{R}^n)} \leqslant \delta,$$

the classical solution $\boldsymbol{\xi}$ to the Cauchy problem (30)–(31)–(32) satisfies

$$\|\boldsymbol{\xi}(t,\cdot)\|_{H^2((0,1),\mathbb{R}^n)} \leqslant C e^{-\nu t} \|\boldsymbol{\xi}^0\|_{H^2((0,1),\mathbb{R}^n)} \quad \forall t \in [0,+\infty). \tag{33}$$

■

The stability property may then be generalised as follows to the nonlinear case.

Theorem 3. If $\rho(\mathbf{H}'(0)) < 1$, there exists $\varepsilon > 0$ such that, if $\|\mathbf{h}'(0)\| < \varepsilon$, then the equilibrium $\boldsymbol{\xi} \equiv 0$ of the nonlinear hyperbolic system (30)–(31) is exponentially stable. ■

The proof of this theorem is much more complicated than its linear counterpart and can be established by using the approach followed in [6]. It makes use of an augmented Lyapunov function (see (16) for comparison) of the form

$$
\begin{aligned}
V = & \int_0^L \left[(\boldsymbol{\xi}^{+T} P_0 \boldsymbol{\xi}^{+}) e^{-\mu_1 x} + (\boldsymbol{\xi}^{-T} P_1 \boldsymbol{\xi}^{-}) e^{\mu_1 x} \right] dx \\
& + \int_0^L \left[(\partial_x \boldsymbol{\xi}^{+T} Q_0 \partial_x \boldsymbol{\xi}^{+}) e^{-\mu_2 x} + (\partial_x \boldsymbol{\xi}^{-T} Q_0 \partial_x \boldsymbol{\xi}^{-}) e^{\mu_2 x} \right] dx \\
& + \int_0^L \left[(\partial_{xx} \boldsymbol{\xi}^{+T} R_0 \partial_{xx} \boldsymbol{\xi}^{+}) e^{-\mu_3 x} + (\partial_{xx} \boldsymbol{\xi}^{-T} R_0 \partial_{xx} \boldsymbol{\xi}^{-}) e^{\mu_3 x} \right] dx
\end{aligned} \tag{34}
$$

with the weighting matrices

$$
\begin{aligned}
P_0 &= D_0^2(\boldsymbol{\Lambda}^+)^{-1} & P_1 &= D_1^2(\boldsymbol{\Lambda}^-)^{-1}, \\
Q_0 &= D_0^2(\boldsymbol{\Lambda}^+) & Q_1 &= D_1^2(\boldsymbol{\Lambda}^-), \\
R_0 &= D_0^2(\boldsymbol{\Lambda}^+)^3 & R_1 &= D_1^2(\boldsymbol{\Lambda}^-)^3.
\end{aligned}
$$

11. Concluding remarks and bibliographical notes.

In these lecture notes, we have been mainly concerned by the exponential stability analysis for hyperbolic systems of balance laws of the form

$$\partial_t \boldsymbol{\xi} + \mathbf{C}(\boldsymbol{\xi}) \partial_x \boldsymbol{\xi} + \mathbf{h}(\boldsymbol{\xi}) = \mathbf{0} \tag{35a}$$

$$\begin{pmatrix} \boldsymbol{\xi}^+(t,0) \\ \boldsymbol{\xi}^-(t,L)) \end{pmatrix} = \mathbf{H} \begin{pmatrix} \boldsymbol{\xi}^-(t,0) \\ \boldsymbol{\xi}^+(t,L)) \end{pmatrix} \tag{35b}$$

and its implications for the design of stabilising boundary feedback controllers. The main property is that the equilibrium of the system is exponentially stable if the dissipative boundary condition $\rho(\mathbf{H}'(0)) < 1$ holds. A first interesting remark is that the system (35) may be graphically represented as a feedback control system as shown in Fig.3. This emphasizes the underlying boundary

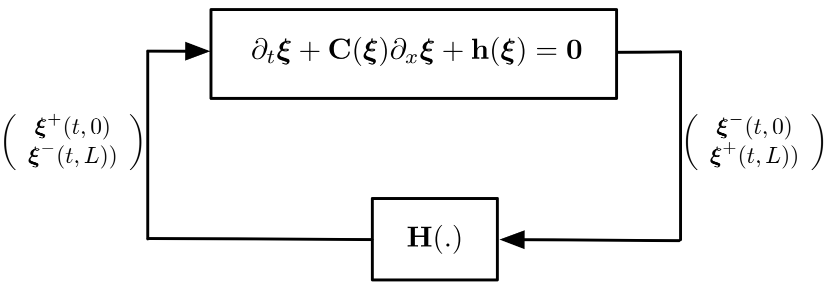

Figure 3: The hyperbolic system of balance laws (35) viewed as a a feedback control system.

control issue of our analysis and shows that the dissipativity condition can be interpreted as a kind of "small gain" stability condition for the feedback system.

The problem of analysing the asymptotic stability of the equilibrium $\boldsymbol{\xi} \equiv \mathbf{0}$ for systems of **conservation** laws $\partial_t \boldsymbol{\xi} + \mathbf{C}(\boldsymbol{\xi})\partial_x \boldsymbol{\xi} = \mathbf{0}$ has been considered in the literature for more than twenty years. To our knowledge, first results were published by Slemrod [16] and by Greenberg and Li [10] for the special case of 2×2 systems. A generalization to $n \times n$ systems was given by Li and his collaborators (see e.g. the textbook [17]). These results were based on a systematic use of direct estimates of the solutions and their derivatives along the characteristic curves. The exponential convergence (in $C^1(0, L)$-norm) of the solutions towards the equilibrium was established under a sufficient dissipative boundary condition [17, Theorem 1.3, page 173] formulated as follows: $\rho_s(|\mathbf{K}|) < 1$ (with $\mathbf{K} \triangleq \mathbf{H}'(0)$ in the nonlinear case) where $\rho_s(A)$ denotes the spectral radius of A and $|A|$ denotes the matrix whose elements are the absolute values of the elements of A. This approach has been applied for the control of networks of open channels in our previous paper [8] and by Leugering and Schmitt [12].

A different approach that uses the Lyapunov function (12) has been introduced in [7] in order to analyse the stability of systems of conservation laws under the same dissipative boundary condition $\rho_s(|\mathbf{K}|) < 1$. The Lyapunov function is related to a similar function used in [4] for the stabilization of the Euler equation of incompressible fluids. It is also similar to the Lyapunov function used in [19] to analyse the stability of a general class of *linear* symmetric hyperbolic systems. The contribution of [7] was to show how this kind of Lyapunov function can be extended to the form (34) in order to prove the exponential convergence of *nonlinear* systems of conservation laws in $H^2(0, L)$-norm. In addition of providing a more concise mathematical analysis, an advantage of having an explicit Lyapunov function is that it is a guarantee of robustness. Finally, in the recent papers [2] and [6], we have shown that this Lyapunov stability approach leads to a new explicit dissipative boundary condition $\rho(\mathbf{K}) < 1$ (introduced in Section 7) which is weaker since it can be shown that $\rho(\mathbf{K}) < \rho_s(|\mathbf{K}|)$ for certain $\mathbf{K}$ matrices (see [6] for details). All these results were established under static boundary conditions. However, for the linear case, the use of *dynamic* boundary conditions represented by ordinary differential equations has also been briefly addressed in [14] (including the use of PI-type feedback controllers).

In the present lecture notes, our main contribution has been to explain how this Lyapunov stability analysis can be further extended to the case of hyperbolic sytems of **balance** laws $\partial_t \boldsymbol{\xi} + \mathbf{C}(\boldsymbol{\xi})\partial_x \boldsymbol{\xi} + \mathbf{h}(\boldsymbol{\xi}) = \mathbf{0}$. In Theorem 1 we have first given a general implicit formulation of sufficient dissipative boundary conditions. Then in Theorems 2 and 3, we have shown that the explicit condition $\rho(\mathbf{K}) < 1$ also holds with a convergence in $H^2(0, L)$-norm for sytems of balance laws considered as perturbations of conservation laws. A variant of this property with convergence in $C^1(0, L)$-norm can also be found in the reference [13]. Finally, the most interesting and original result is given in Section 8 where we have given an example which shows that, in particular instances, the stability condition $\rho(\mathbf{K}) < 1$ may hold even in the case of balance laws with large source terms $\mathbf{h}(\boldsymbol{\xi})$ that deviate widely from the corresponding conservation laws.

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