# Chapter 10
# Exponential Stability of Semi-linear One-Dimensional Balance Laws

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**Abstract** Raman amplifiers and plug flow chemical reactors are typical examples of engineering systems that are conveniently represented by *semi-linear one-dimensional systems of balance laws*. The main goal of this chapter is to explain how a quadratic Lyapunov function can be used to prove the exponential stability of the steady state for this class of hyperbolic systems.

## 10.1 Introduction

The Lyapunov method is a well-established tool in stability analysis of dynamical systems. The principal merit of the method is that the actual solution (whether analytical or numerical) of the concerned system is not required. Meanwhile, the main drawback is that no systematic procedure exists for deriving Lyapunov functions and Laurent Praly is definitely one of the scientists who made the greatest contributions to their construction (see e.g., [3, 9–11, 14]). In this chapter, we bring a modest additional stone to this building. The main goal is to explain how a quadratic Lyapunov function can be used to prove the exponential stability of the steady state of *semi-linear one-dimensional hyperbolic systems of balance laws*. As a motivation, in the next section, we present some interesting physical examples of such systems.



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### *10.1.1 Raman Amplifiers*

Raman amplifiers are electro-optical devices that are used for compensating the natural power attenuation of laser signals transmitted along optical fibers in long distance communications. Their operation is based on the *Raman effect* which was discovered by [12]. The simplest implementation of Raman amplification in optical telecommunications is depicted in Fig. 10.1. The transmitted information is encoded by amplitude modulation of a laser signal with wavelength $\omega_s$. The signal is provided by an optical source at the channel input and received by a photo-detector at the output. A pump laser beam with wavelength $\omega_p$ is injected backward in the optical fiber. If the wavelengths are appropriately selected, the energy of the pump is transferred to the signal and produces an amplification that counteracts the natural attenuation. The dynamics of the signal and pump powers along the fiber are represented by the following system of two balance laws [4]:

$$\begin{aligned} &\partial_t S + \lambda_s\Big(\partial_x S + \alpha_s S - \beta_s SP\Big) = 0, \\ &\partial_t P - \lambda_p\Big(\partial_x P - \alpha_p P - \beta_p PS\Big) = 0, \end{aligned} \qquad t \in [0, +\infty), \quad x \in [0, L], \tag{10.1}$$

where $S(t,x)$ is the power of the transmitted signal, $P(t,x)$ is the power of the pump laser beam, $\lambda_s$ and $\lambda_p$ are the propagation group velocities of the signal and pump waves respectively, $\alpha_s$ and $\alpha_p$ are the attenuation coefficients per unit length, $\beta_s$ and $\beta_p$ are the amplification gains per unit length. All these positive constant parameters $\alpha_s$ and $\alpha_p$, $\beta_s$ and $\beta_p$, $\lambda_s$ and $\lambda_p$ are characteristic of the fiber material and dependent of the wavelengths $\omega_s$ and $\omega_p$.

As the input signal power and the launch pump power can be exogenously imposed, the boundary conditions are

$$S(t,0) = U_0, P(t,L) = U_L, \tag{10.2}$$

with constant inputs $U_0$ and $U_L$.

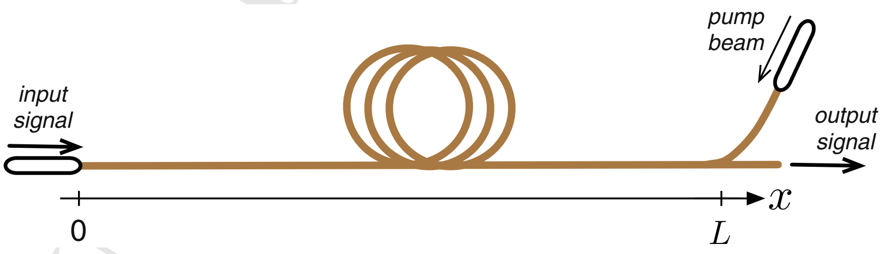


**Fig. 10.1** Optical communication with Raman amplification

### *10.1.2 Plug Flow Chemical Reactors*

A plug flow chemical reactor (PFR) is a tubular reactor where a liquid reaction mixture circulates. The reaction proceeds as the reactants travel through the reactor. Here, we consider the case of an horizontal PFR where a simple monomolecular reaction takes place

$$A \rightleftarrows B.$$

$A$ is the reactant species and $B$ is the desired product. The reaction is supposed to be exothermic and a jacket is used to cool the reactor. The cooling fluid flows around the wall of the tubular reactor. The dynamics of the PFR are described by the following system of balance laws:

$$\begin{aligned}
&\partial_t T_c - V_c \partial_x T_c - k_o(T_c - T_r) = 0,\\
&\partial_t T_r + V_r \partial_x T_r + k_o(T_c - T_r) - k_1 r(T_r, C_A, C_B) = 0,\\
&\partial_t C_A + V_r \partial_x C_A + r(T_r, C_A, C_B) = 0,\\
&\partial_t C_B + V_r \partial_x C_B - r(T_r, C_A, C_B) = 0,
\end{aligned} \tag{10.3}$$

where $t \in [0, +\infty)$, $x \in [0, L]$, $T_c(t, x)$ is the coolant temperature, $T_r(t, x)$ is the reactor temperature. The variables $C_A(t, x)$ and $C_B(t, x)$ denote the concentrations of the chemicals in the reaction medium. $V_c$ is the constant coolant velocity in the jacket, $V_r$ is the constant reactive fluid velocity in the reactor. The function $r(T_r, C_A, C_B)$ represents the reaction rate. A typical form of this function is

$$r(T_r, C_A, C_B) = (aC_A - bC_B) \exp\Big(-\frac{E}{RT_r}\Big),$$

where $a$ and $b$ are rate constants, $E$ is the activation energy and $R$ is the Boltzmann constant.

The system is subject to the following constant boundary conditions:

$$T_r(t, 0) = T_r^{\mathrm{in}}, \quad C_A(t, 0) = C_A^{\mathrm{in}}, \quad C_B(t, 0) = 0, \quad T_c(t, 0) = T_c^{\mathrm{in}}. \tag{10.4}$$

### *10.1.3 Chemotaxis*

Chemotaxis refers to the motion of certain living microorganisms (bacteria, slime molds, leukocytes ...) in response to the concentrations of chemicals. A simple model for one-dimensional chemotaxis, known as the Kac-Goldstein model, has been proposed in [5] in order to explain the spatial pattern formations in chemosensitive populations. Revisited in [6], this model, in its simplest form, is a system of two balance laws of the form

$$
\begin{aligned}
&\partial_t \varrho^+ + \gamma \partial_x \varrho^+ + \phi(\varrho^+, \varrho^-)(\varrho^- - \varrho^+) = 0, \\
&\partial_t \varrho^- - \gamma \partial_x \varrho^- + \phi(\varrho^+, \varrho^-)(\varrho^+ - \varrho^-) = 0,
\end{aligned}
\quad t \in [0, +\infty), \quad x \in [0, L], \tag{10.5}
$$

where $\varrho^+$ denotes the density of right-moving cells and $\varrho^-$ the density of left-moving cells. The function $\phi(\varrho^+, \varrho^-)$ is called the "turning function". The constant parameter $\gamma$ is the velocity of the cell motion. With the change of coordinates $\varrho \triangleq \varrho^+ + \varrho^-$, $q \triangleq \gamma(\varrho^+ - \varrho^-)$, we have the following alternative equivalent model:

$$
\begin{aligned}
&\partial_t \varrho + \partial_x q = 0, \\
&\partial_t q + \gamma^2 \partial_x \varrho - 2\phi\big(\frac{\varrho}{2} + \frac{q}{2\gamma}, \frac{\varrho}{2} - \frac{q}{2\gamma}\big) q = 0,
\end{aligned}
$$

where $\varrho$ is the total density and $q$ is a flux proportional to the difference of densities of right and left-moving cells. Remark that we have $q = \varrho V$ where

$$
V \triangleq \gamma \frac{\varrho^+ - \varrho^-}{\varrho^+ + \varrho^-}
$$

can be interpreted as the average group velocity of the moving cells.

Various possible turning functions are reviewed in [8]. A typical example is

$$
\phi(\varrho^+, \varrho^-) = \alpha \varrho^+ \varrho^- - \mu,
$$

where $\alpha$ and $\mu$ are positive constants.

A special case of interest (see, e.g., [7]) is when the cells are confined in the domain $[0, L]$. This situation may be represented by "no-flow boundary conditions" of the form

$$
\begin{aligned}
q(t, 0) &= \gamma\big(\varrho^+(t, 0) - \varrho^-(t, 0)\big) = 0, \\
q(t, L) &= \gamma\big(\varrho^+(t, L) - \varrho^-(t, L)\big) = 0.
\end{aligned} \tag{10.6}
$$

## 10.2 Exponential Stability of Semi-linear Hyperbolic Systems of Balance Laws

The examples given above are special cases of the general semi-linear hyperbolic system

$$
\mathbf{Y}_t + \Lambda \mathbf{Y}_x + G(\mathbf{Y}) = \mathbf{0}, \quad t \in [0, +\infty), \quad x \in [0, L], \tag{10.7}
$$

$$
\mathcal{B}\big(\mathbf{Y}(t, 0), \mathbf{Y}(t, L)\big) = \mathbf{0}, \quad t \in [0, +\infty), \tag{10.8}
$$

where

- $t$ and $x$ are the two independent variables: a time variable $t \in [0, +\infty)$ and a space variable $x \in [0, L]$ over a finite interval;

- $\mathbf{Y} : [0, +\infty) \times [0, L] \to \mathcal{Y}$ is the vector of state variables, with $\mathcal{Y}$ a nonempty connected open subset of $\mathbb{R}^n$;

- $\Lambda \in \mathcal{M}_{n,n}(\mathbb{R})$ is the diagonal matrix defined as

$$\Lambda \triangleq \begin{pmatrix} \Lambda^+ & 0 \\ 0 & -\Lambda^- \end{pmatrix} \quad \text{with} \quad \begin{cases} \Lambda^+ = \operatorname{diag}\left\{\lambda_1, \ldots, \lambda_m\right\}, \\ \Lambda^- = \operatorname{diag}\left\{\lambda_{m+1}, \ldots, \lambda_n\right\}, \end{cases} \tag{10.9}$$

  where $m \in [0, n]$ and $\lambda_i > 0 \ \forall i$;

- $G \in C^2(\mathcal{Y}, \mathbb{R}^n)$ is the vector of *source* terms;

- $\mathcal{B} \in C^2(\mathcal{Y} \times \mathcal{Y}, \mathbb{R}^n)$ is the vector of boundary conditions.

A steady state $\mathbf{Y}^*(x)$ is a solution of the ordinary differential equation $\Lambda \mathbf{Y}^*_x(x) + G(\mathbf{Y}^*(x)) = \mathbf{0}$ satisfying the boundary condition $\mathcal{B}\big(\mathbf{Y}^*(0), \mathbf{Y}^*(L)\big) = \mathbf{0}$.

We define the following change of coordinates:

$$\mathbf{Z}(t, x) \triangleq \mathbf{Y}(t, x) - \mathbf{Y}^*(x), \qquad \mathbf{Z} = (Z_1, \ldots, Z_n)^\mathsf{T}.$$

In the $\mathbf{Z}$ coordinates, the system (10.7), (10.8) is rewritten

$$\mathbf{Z}_t + \Lambda \mathbf{Z}_x + B(\mathbf{Z}, x) = \mathbf{0}, \tag{10.10}$$

$$\mathcal{B}\big(\mathbf{Z}(t, 0) + \mathbf{Y}^*(0), \mathbf{Z}(t, L) + \mathbf{Y}^*(L)\big) = \mathbf{0}, \tag{10.11}$$

where

$$B(\mathbf{Z}, x) \triangleq \Big[G(\mathbf{Z} + \mathbf{Y}^*(x)) - G(\mathbf{Y}^*(x))\Big].$$

Since $B(\mathbf{0}, x) = \mathbf{0}$ by definition of the steady state, it follows that there exists a matrix $M(\mathbf{Z}, x) \in \mathcal{M}_{n \times n}(\mathbb{R})$ such that (10.10) may be rewritten as

$$\mathbf{Z}_t + \Lambda \mathbf{Z}_x + M(\mathbf{Z}, x)\mathbf{Z} = \mathbf{0}, \tag{10.12}$$

with

$$M(\mathbf{0}, x) = \frac{\partial B}{\partial \mathbf{Z}}(\mathbf{0}, x).$$

In order to have a well-posed Cauchy problem, a basic requirement is that "at each boundary point the incoming information $\mathbf{Z}_{\text{in}}$ is determined by the outgoing information $\mathbf{Z}_{\text{out}}$" [13, Sect. 3], with the definitions



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$$\mathbf{Z}_{\text{in}}(t) \triangleq \begin{pmatrix} \mathbf{Z}^{+}(t,0) \\ \mathbf{Z}^{-}(t,L) \end{pmatrix} \quad \text{and} \quad \mathbf{Z}_{\text{out}}(t) \triangleq \begin{pmatrix} \mathbf{Z}^{+}(t,L) \\ \mathbf{Z}^{-}(t,0) \end{pmatrix}, \tag{10.13}$$

where $\mathbf{Z}^{+}$ and $\mathbf{Z}^{-}$ are defined as follows:

$$\mathbf{Z}^{+} = \begin{pmatrix} Z_1 \\ \vdots \\ Z_m \end{pmatrix}, \qquad \mathbf{Z}^{-} = \begin{pmatrix} Z_{m+1} \\ \vdots \\ Z_n \end{pmatrix}.$$

This means that the system (10.12) is subject to boundary conditions having the form

$$\mathbf{Z}_{\text{in}}(t) = \mathcal{H}\big(\mathbf{Z}_{\text{out}}(t)\big), \tag{10.14}$$

where the map $\mathcal{H} \in C^1(\mathbb{R}^n; \mathbb{R}^n)$.

Our concern is to analyze the exponential stability of the steady state $\mathbf{Z}(t,x) \equiv \mathbf{0}$ of the system (10.12) under the boundary condition (10.14) and under an initial condition

$$\mathbf{Z}(0,x) = \mathbf{Z}_{\text{o}}(x), \quad x \in [0,L]. \tag{10.15}$$

which satisfies the compatibility condition

$$\begin{pmatrix} \mathbf{Z}_{\text{o}}^{+}(0) \\ \mathbf{Z}_{\text{o}}^{-}(L) \end{pmatrix} = \mathcal{H}\begin{pmatrix} \mathbf{Z}_{\text{o}}^{+}(L) \\ \mathbf{Z}_{\text{o}}^{-}(0) \end{pmatrix}. \tag{10.16}$$

Let us first recall the following theorem on the well-posedness of the Cauchy problem (10.12), (10.14), (10.15).

**Theorem 1** *There exists $\delta_0 > 0$ such that, for every $\mathbf{Z}_o \in H^1((0,L);\mathbb{R}^n)$ satisfying*

$$\|\mathbf{Z}_o\|_{H^1((0,L);\mathbb{R}^n)} \leqslant \delta_0$$

*and the compatibility condition (10.16), the Cauchy problem (10.12), (10.14), (10.15) has a unique maximal classical solution*

$$\mathbf{Z} \in C^0([0,T), H^1((0,L);\mathbb{R}^n)) \tag{10.17}$$

*with $T \in (0,+\infty]$.*

*Moreover, if*

$$\|\mathbf{Z}(t,\cdot)\|_{H^1((0,L);\mathbb{R}^n)} \leqslant \delta_0, \ \forall t \in [0,T),$$

*then $T = +\infty$.*

A proof of this theorem is easily adapted from [1, Appendix B] by considering the special case of a constant matrix $\Lambda$ which allows to replace $H^2((0,L);\mathbb{R}^n)$ by $H^1((0,L);\mathbb{R}^n)$.

The definition of the exponential stability is as follows.

**Definition 1** The steady state $\mathbf{Z}(t,x) \equiv 0$ of the system (10.12), (10.14) is exponentially stable for the $H^1$-norm if there exist $\delta > 0$, $\nu > 0$ and $C > 0$ such that, for every $\mathbf{Z}_o \in H^1((0,L);\mathbb{R}^n)$ satisfying $\|\mathbf{Z}_o\|_{H^1((0,L);\mathbb{R}^n)} \leqslant \delta$ and the compatibility conditions (10.16), the solution $\mathbf{Z}$ of the Cauchy problem (10.12), (10.14), (10.15) is defined on $[0,+\infty) \times [0,L]$ and satisfies

$$\|\mathbf{Z}(t,.)\|_{H^1((0,L);\mathbb{R}^n)} \leq Ce^{-\nu t}\|\mathbf{Z}_o\|_{H^1((0,L);\mathbb{R}^n)}, \quad \forall t \in [0,+\infty). \tag{10.18}$$

Let us now define the matrix $\mathbf{K}$ as the linearization of the map $\mathcal{H}$ at the steady state

$$\mathbf{K} \triangleq \mathcal{H}'(\mathbf{0}).$$

We then have the following stability theorem.

**Theorem 2** *The steady state $\mathbf{Z}(t,x) \equiv \mathbf{0}$ of the system (10.12), (10.14) is exponentially stable for the $H^1$-norm if there exists a map $Q$ satisfying*

$$Q(x) \triangleq \mathbf{diag}\{Q^+(x), Q^-(x)\},$$
$$Q^+(x) \triangleq \mathbf{diag}\{q_1(x),\dots,q_m(x)\}, \quad Q^-(x) \triangleq \mathbf{diag}\{q_{m+1}(x),\dots,q_n(x)\},$$
$$q_i \in C^1([0,L];\mathbb{R}_+) \ \forall i.$$

*such that the following Matrix Inequalities hold:*

**(i)** *the matrix*

$$\begin{pmatrix} Q^+(L)\Lambda^+ & 0 \\ 0 & Q^-(0)\Lambda^- \end{pmatrix} - \mathbf{K}^\mathsf{T} \begin{pmatrix} Q^+(0)\Lambda^+ & 0 \\ 0 & Q^-(L)\Lambda^- \end{pmatrix} \mathbf{K} \tag{10.19}$$

*is positive semi-definite;*

**(ii)** *the matrix*

$$-Q'(x)\Lambda + Q(x)M(\mathbf{0},x) + M^\mathsf{T}(\mathbf{0},x)Q(x)$$

*is positive definite* $\forall x \in [0,L]$.

## 10.3 Proof in the Case Where $m = n$

For the clarity of the demonstration, we shall first prove the theorem in the special case where $m = n$, which means that the matrix $\Lambda$ is the positive diagonal matrix $\mathrm{diag}\{\lambda_1,\dots,\lambda_n\}$ with $\lambda_i > 0 \ \ \forall i = 1,\dots,n$. In that case, the boundary condition (10.14) and the compatibility conditions (10.16) are simply rewritten

$$\mathbf{Z}(t,0) = \mathcal{H}\Big(\mathbf{Z}(t,L)\Big), \tag{10.20}$$

$$\mathbf{Z}_{\mathrm{o}}(0) = \mathcal{H}\Big(\mathbf{Z}_{\mathrm{o}}(L)\Big). \tag{10.21}$$

Moreover, condition (i) of Theorem 2 is restated as

**(i-bis)** *the matrix $Q(L)\Lambda - \mathbf{K}^{\mathsf{T}}Q(0)\Lambda\mathbf{K}$ is positive semi-definite.*

For the stability analysis, we adopt the $H^1$ Lyapunov function candidate

$$\mathbf{V} \triangleq \mathbf{V}_1 + \mathbf{V}_2 \tag{10.22}$$

such that

$$\mathbf{V}_1 = \int_0^L \mathbf{Z}^{\mathsf{T}}Q(x)\mathbf{Z}\,dx, \tag{10.23}$$

$$\mathbf{V}_2 = \int_0^L \mathbf{Z}_t^{\mathsf{T}}Q(x)\mathbf{Z}_t\,dx, \tag{10.24}$$

where, by definition, the notation $\mathbf{Z}_t$ must be understood as

$$\mathbf{Z}_t \triangleq -\Lambda\mathbf{Z}_x - B(\mathbf{Z},x).$$

Let us remark that by (10.17) $\mathbf{V}$ is a continuous function of $t$. In order to prove Theorem 2, we temporarily assume that $\mathbf{Z}$ is of class $C^2$ on $[0,T]\times[0,L]$ and therefore that $\mathbf{V}$ is of class $C^1$ in $[0,T]$. Under this assumption (that will be relaxed later on) the first step of the proof is to compute the following estimates of $d\mathbf{V}_1/dt$ and $d\mathbf{V}_2/dt$.

*Estimate of $d\mathbf{V}_1/dt$*

The time derivative of $\mathbf{V}_1$ along the solutions of (10.12), (10.20) is[1]

$$\begin{aligned}\frac{d\mathbf{V}_1}{dt} &= \int_0^L 2\mathbf{Z}^{\mathsf{T}}Q(x)\mathbf{Z}_t dx\\ &= \int_0^L 2\mathbf{Z}^{\mathsf{T}}Q(x)\Big(-\Lambda\mathbf{Z}_x - B(\mathbf{Z},x)\Big)dx.\end{aligned}$$

Then, using integrations by parts, we get

$$\frac{d\mathbf{V}_1}{dt} = \mathcal{T}_{11} + \mathcal{T}_{12}, \tag{10.25}$$

with

[1]The notation $M^{\mathsf{T}}$ denotes the transpose of the matrix $M$.

$$\mathcal{T}_{11} \triangleq \Big[ -\mathbf{Z}^{\mathsf{T}} Q(x) \Lambda \mathbf{Z} \Big]_0^L, \tag{10.26}$$

$$\mathcal{T}_{12} \triangleq \int_0^L -\mathbf{Z}^{\mathsf{T}} Q'(x) \Lambda \mathbf{Z} - 2\mathbf{Z}^{\mathsf{T}} Q(x) B(\mathbf{Z}, x) dx. \tag{10.27}$$

From (10.26), we have

$$\mathcal{T}_{11} = -\mathbf{Z}^{\mathsf{T}}(t, L) Q(L) \Lambda \mathbf{Z}(t, L) + \mathbf{Z}^{\mathsf{T}}(t, 0) Q(0) \Lambda \mathbf{Z}(t, 0). \tag{10.28}$$

Let us introduce a notation in order to deal with estimates on "higher order terms". We denote by $\mathcal{O}(X; Y)$, with $X \geqslant 0$ and $Y \geqslant 0$, quantities for which there exist $C > 0$ and $\varepsilon > 0$, independent of $\mathbf{Z}$ and $\mathbf{Z}_t$, such that

$$(Y \leqslant \varepsilon) \Rightarrow (|\mathcal{O}(X; Y)| \leqslant CX).$$

Then from (10.28), using the boundary condition (10.20), we have

$$\mathcal{T}_{11} = -\mathbf{Z}^{\mathsf{T}}(t, L) \Big[ Q(L) \Lambda - \mathbf{K}^{\mathsf{T}} Q(0) \Lambda \mathbf{K} \Big] \mathbf{Z}(t, L) + \mathcal{O}(|\mathbf{Z}(t, L)|^3; |\mathbf{Z}(t, L)|), \tag{10.29}$$

and from (10.27) we have

$$\begin{aligned} \mathcal{T}_{12} = &- \int_0^L \mathbf{Z}^{\mathsf{T}} \Big[ - Q'(x)\Lambda + M^{\mathsf{T}}(\mathbf{0}, x) Q(x) + Q(x) M(\mathbf{0}, x) \Big] \mathbf{Z}\, dx \\ &+ \mathcal{O}\Big( \int_0^L |\mathbf{Z}|^3 dx; |\mathbf{Z}(t, .)|_0 \Big), \end{aligned} \tag{10.30}$$

where, for $f \in C^0([0, L]; \mathbb{R}^n)$, we denote $|f|_0 = \max\{|f(x)|;\, x \in [0, L]\}$.

*Estimate of $d\mathbf{V}_2/dt$*

By time differentiation of the system equations (10.12), (10.20), $\mathbf{Z}_t$ can be shown to satisfy the following hyperbolic dynamics:

$$\mathbf{Z}_{tt} + \Lambda \mathbf{Z}_{tx} + \frac{\partial B}{\partial \mathbf{Z}}(\mathbf{Z}, x)\mathbf{Z}_t = \mathbf{0}, \tag{10.31}$$

$$\mathbf{Z}_t(t, 0) = \mathcal{H}'(\mathbf{Z}(t, L))\mathbf{Z}_t(t, L). \tag{10.32}$$

The time derivative of $\mathbf{V}_2$ along the solutions of (10.12), (10.20), (10.31), (10.32) is

$$\begin{aligned} \frac{d\mathbf{V}_2}{dt} &= \int_0^L 2\mathbf{Z}_t^{\mathsf{T}} Q(x) (\mathbf{Z}_t)_t dx \\ &= \int_0^L 2\mathbf{Z}_t^{\mathsf{T}} Q(x) \Big( - \Lambda \mathbf{Z}_{tx} - \frac{\partial B}{\partial \mathbf{Z}}(\mathbf{Z}, x)\mathbf{Z}_t \Big) dx. \end{aligned}$$

Then, using integrations by parts, we get

$$\frac{d\mathbf{V}_2}{dt} = \mathcal{T}_{21} + \mathcal{T}_{22}, \tag{10.33}$$

with

$$\mathcal{T}_{21} \triangleq \Big[ -\mathbf{Z}_t^\mathsf{T} Q(x)\Lambda \mathbf{Z}_t \Big]_0^L, \tag{10.34}$$

$$\mathcal{T}_{22} \triangleq \int_0^L \mathbf{Z}_t^\mathsf{T} Q'(x)\Lambda\mathbf{Z}_t + 2\mathbf{Z}_t^\mathsf{T} Q(x)\Big(\frac{\partial B}{\partial \mathbf{Z}}(\mathbf{Z},x)\mathbf{Z}_t\Big)dx. \tag{10.35}$$

From (10.34), we have

$$\mathcal{T}_{21} = -\mathbf{Z}_t^\mathsf{T}(t,L)Q(L)\Lambda\mathbf{Z}_t(t,L) + \mathbf{Z}_t^\mathsf{T}(t,0)Q(0)\Lambda\mathbf{Z}_t(t,0). \tag{10.36}$$

Then, using the boundary condition (10.32), we get

$$\begin{aligned}\mathcal{T}_{21} = &-\mathbf{Z}_t^\mathsf{T}(t,L)\Big[Q(L)\Lambda - \mathbf{K}^\mathsf{T} Q(0)\Lambda\mathbf{K}\Big]\mathbf{Z}_t(t,L)\\ &+ \mathcal{O}(|\mathbf{Z}_t(t,L)|^2|\mathbf{Z}(t,L)|; |\mathbf{Z}(t,L)|).\end{aligned} \tag{10.37}$$

Moreover $\mathcal{T}_{22}$ is written

$$\begin{aligned}\mathcal{T}_{22} = &-\int_0^L \mathbf{Z}_t^\mathsf{T}\Big[-Q'(x)\Lambda + M^\mathsf{T}(\mathbf{0},x)Q(x) + Q(x)M(\mathbf{0},x)\Big]\mathbf{Z}_t\,dx\\ &+ \mathcal{O}\Big(\int_0^L |\mathbf{Z}_t|^2|\mathbf{Z}|dx; |\mathbf{Z}(t,.)|_0\Big).\end{aligned} \tag{10.38}$$

In the next lemma, we shall now use these estimates to show that the Lyapunov function exponentially decreases along the system trajectories.

**Lemma 1** *There exist positive real constants $\alpha$, $\beta$ and $\delta$ such that, for every $\mathbf{Z}$ such that $|\mathbf{Z}|_0 \leq \delta$, we have*

$$\frac{1}{\beta}\int_0^L (|\mathbf{Z}|^2 + |\mathbf{Z}_x|^2)dx \leqslant \mathbf{V} \leqslant \beta \int_0^L (|\mathbf{Z}|^2 + |\mathbf{Z}_x|^2)dx, \tag{10.39}$$

$$\frac{d\mathbf{V}}{dt} \leq -\alpha\mathbf{V}. \tag{10.40}$$

*Proof* Inequalities (10.39) follow directly from the definition of $\mathbf{V}$ and straightforward estimations.

Let us introduce the following compact matrix notations:

$$\mathcal{K} \triangleq Q(L)\Lambda - \mathbf{K}^\mathsf{T} Q(0)\Lambda\mathbf{K}, \tag{10.41}$$

$$\mathcal{L}(x) \triangleq -Q'(x)\Lambda + M^\mathsf{T}(\mathbf{0},x)Q(x) + Q(x)M(\mathbf{0},x). \tag{10.42}$$

Then it follows from (10.28), (10.30), (10.37), (10.38) that

$$\begin{aligned}\frac{d\mathbf{V}}{dt} = -\mathbf{Z}^{\mathsf{T}}(t,L)\mathcal{K}\,\mathbf{Z}(t,L) - \mathbf{Z}_t^{\mathsf{T}}(t,L)\mathcal{K}\,\mathbf{Z}_t(t,L)\\ + \mathcal{O}(|\mathbf{Z}(t,L)|(|\mathbf{Z}(t,L)|^2 + |\mathbf{Z}_t(t,L)|^2); |\mathbf{Z}(t,L)|)\\ - \int_0^L \Big(\mathbf{Z}^{\mathsf{T}}\mathcal{L}(x)\,\mathbf{Z} + \mathbf{Z}_t^{\mathsf{T}}\mathcal{L}(x)\,\mathbf{Z}_t\Big)dx\\ + \mathcal{O}(\int_0^L \Big((|\mathbf{Z}|^2| + |\mathbf{Z}_t|^2)|\mathbf{Z}|\Big)dx; |\mathbf{Z}(t,.)|_0).\end{aligned} \tag{10.43}$$

Then, by assumption **(i-bis)** of Theorem 2 and from (10.41), there exists $\delta_1 > 0$ such that if $|\mathbf{Z}(t,L)| < \delta_1$ then

$$\begin{aligned}-\mathbf{Z}^{\mathsf{T}}(t,L)\mathcal{K}\,\mathbf{Z}(t,L) - \mathbf{Z}_t^{\mathsf{T}}(t,L)\mathcal{K}\,\mathbf{Z}_t(t,L)\\ + \mathcal{O}(|\mathbf{Z}(t,L)|(|\mathbf{Z}(t,L)|^2 + |\mathbf{Z}_t(t,L)|^2); |\mathbf{Z}(t,L)|) \leqslant 0.\end{aligned} \tag{10.44}$$

Let us recall the following Sobolev inequality, see, e.g., [2]: for a function $\varphi \in C^1([0,L];\mathbb{R}^n)$, there exists $C_1 > 0$ such that

$$|\varphi|_0 \leqslant C_1 \int_0^L (|\varphi(x)|^2 + |\varphi'(x)|^2)dx. \tag{10.45}$$

Moreover, from (10.10) and (10.31), we know also that there exist $\delta_2 > 0$ and $C_2 > 0$ such that, if $|\mathbf{Z}(t,x)| + |\mathbf{Z}_t(t,x)| < \delta_2$, then

$$|\mathbf{Z}_t(t,x)| \leqslant C_2\big(|\mathbf{Z}(t,x)| + |\mathbf{Z}_x(t,x)|\big), \tag{10.46}$$

$$|\mathbf{Z}_x(t,x)| \leqslant C_2\big(|\mathbf{Z}(t,x)| + |\mathbf{Z}_t(t,x)|\big). \tag{10.47}$$

Using repeatedly, inequalities (10.45) to (10.47), it follows that there exists $\delta_3 > 0$ and $C_3 > 0$ such that, if $|\mathbf{Z}(t,.)|_0 < \delta_3$, then

$$\mathcal{O}(\int_0^L \Big((|\mathbf{Z}|^2| + |\mathbf{Z}_t|^2)|\mathbf{Z}|\Big)dx; |\mathbf{Z}(t,.)|_0\big) \leqslant C_3|\mathbf{Z}(t,.)|_0\mathbf{V}. \tag{10.48}$$

Using assumption (ii) of Theorem 2, there exists $\gamma > 0$ such that

$$-\int_0^L \Big(\mathbf{Z}^{\mathsf{T}}\mathcal{L}(x)\,\mathbf{Z} + \mathbf{Z}_t^{\mathsf{T}}\mathcal{L}(x)\,\mathbf{Z}_t\Big)dx \leqslant -2\gamma\mathbf{V}. \tag{10.49}$$

Finally it follows from (10.43), (10.44), (10.48) and (10.49) that, if $\delta < \min(\delta_1, \delta_3)$ is taken sufficiently small, then $\alpha > 0$ can be selected such that

$$\frac{d\mathbf{V}}{dt} = (-2\gamma + C_3|\mathbf{Z}(t,.)|_0)\mathbf{V} \leqslant -\alpha\mathbf{V},$$

for every $\mathbf{Z}(t,.)$ such that $|\mathbf{Z}(t,.)|_0 \leq \delta$. This concludes the proof of Lemma 1.

In this lemma, the estimates (10.39) and (10.40) were obtained under the assumption that $\mathbf{Z}$ is of class $C^2$ on $[0,T] \times [0,L]$. But the selection of $\alpha$ and $\beta$ does not depend on the $C^2$-norm of $\mathbf{Z}$: they depend only on the $C^0([0,T];H^1((0,L);\mathbb{R}^n))$-norm of $\mathbf{Z}$. Hence, using a classical density argument (see, e.g., [1, Comment 4.6]), the estimates (10.39) and (10.40) remain valid in the distribution sense if $\mathbf{Z}(.,.)$ is only of class $C^1$.

Let us now introduce

$$\varepsilon \triangleq \min\left\{\frac{\delta}{2C_1\beta}, \frac{\delta_0}{\beta}\right\}. \tag{10.50}$$

Note that $\beta \geqslant 1$ and therefore that $\delta \leqslant \delta_0$. Using Lemma 1, (10.45) and (10.50), for every $t \in [0,T]$

$$\left(\|\mathbf{Z}(t,.)\|_{H^1((0,L);\mathbb{R}^n)} \leqslant \varepsilon\right) \Longrightarrow \left(|\mathbf{Z}(t,.)|_0 \leq \frac{\delta}{2} \text{ and } \mathbf{V}(t) \leqslant \beta\varepsilon^2\right), \tag{10.51}$$

$$\begin{aligned}&\left(|\mathbf{Z}(t,.)|_0 \leq \delta \text{ and } \mathbf{V} \leqslant \beta\varepsilon^2\right)\\ &\qquad\Longrightarrow \left(|\mathbf{Z}(t,.)|_0 \leq \frac{\delta}{2} \text{ and } \|\mathbf{Z}(t,.)\|_{H^1((0,L);\mathbb{R}^n)} \leqslant \delta_0\right),\end{aligned} \tag{10.52}$$

$$\left(|\mathbf{Z}(t,.)|_0 \leq \delta\right) \Longrightarrow \left(\frac{d\mathbf{V}}{dt} \leqslant 0\right) \quad \text{in the distribution sense.} \tag{10.53}$$

Let $\mathbf{Z}_o \in H^1((0,L);\mathbb{R}^n)$ satisfy the compatibility condition (10.21) and

$$\|\mathbf{Z}_o\|_{H^1((0,L);\mathbb{R}^n)} < \varepsilon.$$

Let $\mathbf{Z} \in C^0([0,T^*),H^1((0,L);\mathbb{R}^n))$ be the maximal classical solution the Cauchy problem (10.12), (10.14), (10.15). Using implications (10.51) to (10.53) for $T \in [0,T^*)$, we get that

$$|\mathbf{Z}(t,\cdot)|_{H^1((0,L);\mathbb{R}^n)} \leqslant \delta_0,\ \forall t \in [0,T^*), \tag{10.54}$$

$$|\mathbf{Z}(t,\cdot)|_0 + |\mathbf{Z}_t(t,\cdot)|_0 \leqslant \delta,\ \forall t \in [0,T^*). \tag{10.55}$$

Using (10.54) and Theorem 1, we have that $T = +\infty$. Using Lemma 1 and (10.55), we finally obtain that



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$$\|\mathbf{Z}(t,\cdot)\|^2_{H^1((0,L);\mathbb{R}^n)} \leqslant \beta \mathbf{V}(t) \leqslant \beta \mathbf{V}(0) e^{-\alpha t} \leqslant \beta^2 \|\mathbf{Z}_{\mathrm{o}}\|^2_{H^1((0,L);\mathbb{R}^n)} e^{-\alpha t}.$$

This concludes the proof of Theorem 2.

## 10.4 Proof in the Case Where 0 < *M* < *N*

In this section, we explain the modifications of the proof that must be used to deal with the case $0 < m < n$. (Of course,the case $m = 0$ is equivalent to the case $m = n$ by considering $\mathbf{Z}(t, L - x)$ instead of $\mathbf{Z}(t, x)$.)

The major difference lies in functions $\mathcal{T}_{11}$ and $\mathcal{T}_{21}$ which are now written as follows:

$$\begin{aligned}
\mathcal{T}_{11} = &-\begin{pmatrix} \mathbf{Z}^+(t,L) \\ \mathbf{Z}^-(t,0) \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} Q^+(L)\Lambda^+ & 0 \\ 0 & Q^-(0)\Lambda^- \end{pmatrix} \begin{pmatrix} \mathbf{Z}^+(t,L) \\ \mathbf{Z}^-(t,0) \end{pmatrix} \\
&+ \begin{pmatrix} \mathbf{Z}^+(t,0) \\ \mathbf{Z}^-(t,L) \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} Q^+(0)\Lambda^+ & 0 \\ 0 & Q^-(L)\Lambda^- \end{pmatrix} \begin{pmatrix} \mathbf{Z}^+(t,0) \\ \mathbf{Z}^-(t,L) \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}_{21} = &-\begin{pmatrix} \mathbf{Z}_t^+(t,L) \\ \mathbf{Z}_t^-(t,0) \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} Q^+(L)\Lambda^+ & 0 \\ 0 & Q^-(0)\Lambda^- \end{pmatrix} \begin{pmatrix} \mathbf{Z}_t^+(t,L) \\ \mathbf{Z}_t^-(t,0) \end{pmatrix} \\
&+ \begin{pmatrix} \mathbf{Z}_t^+(t,0) \\ \mathbf{Z}_t^-(t,L) \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} Q^+(0)\Lambda^+ & 0 \\ 0 & Q^-(L)\Lambda^- \end{pmatrix} \begin{pmatrix} \mathbf{Z}_t^+(t,0) \\ \mathbf{Z}_t^-(t,L) \end{pmatrix}.
\end{aligned}$$

Using the boundary condition (10.14) and assumption **(i)** in these equations, it is then a straightforward exercise to verify that Theorem 2 can be established for the case $0 < m < n$ in a manner completely parallel to the one we have followed in the case $m = n$.

## 10.5 Conclusion

The main goal of this chapter was to explain how a quadratic Lyapunov function can be used to prove the exponential stability of the steady state of *semi-linear one-dimensional hyperbolic systems of balance laws*. Further stability results for hyperbolic systems of balance laws can be found in the textbook [1].

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## References


1. Bastin, G., Coron, J.M.: Stability and Boundary Stabilisation of One-Dimensional Hyperbolic Systems. Book in preparation (2015)
2. Brezis, H.: Analyse fonctionnelle, théorie et applications. Collection Mathématiques appliquées pour la maîtrise. Masson, Paris (1983)
3. Coron, J.M., Praly, L., Teel, A.: Feedback stabilization of nonlinear systems: sufficient conditions and Lyapunov and input-output techniques. In: Trends in control (Rome, 1995), pp. 293–348. Springer, Berlin (1995)
4. Dower, P., Farrel, P.: On linear control of backward pumped Raman amplifiers. In: Proceedings IFAC Symposium on System Identification, pp. 547–552. Newcastle, Australia (2006)
5. Goldstein, S.: On diffusion by discontinuous movements, and the telegraph equation. Q. J. Mech. Appl. Math. **4**, 129–156 (1951)
6. Kac, M.: A stochastic model related to the telegrapher's equation. Rocky Mt. J. Math. **4**, 497–509 (1956)
7. Lutscher, F.: Modeling alignment and movement of animals and cells. J. Math. Biol. **45**, 234–260 (2002)
8. Lutscher, F., Stevens, A.: Emerging patterns in a hyperbolic model for locally interacting cell systems. J. Nonlinear Sci. **12**, 619–640 (2002)
9. Mazenc, F., Praly, L.: Adding integrations, saturated controls, and stabilization for feedforward systems. IEEE Trans. Autom. Control **41**(11), 1559–1578 (1996). doi:10.1109/9.543995
10. Praly, L.: Une introduction à l'utilisation de fonctions de Lyapunov pour la stabilisation et l'atténuation de perturbations., chap. 2 in Commandes nonlinéaires. Lavoisier (2003)
11. Praly, L., Carnevale, D., Astolfi, A.: Dynamic versus static weighting of Lyapunov functions. IEEE Trans. Autom. Control **58**(6), 1557–1561 (2013). doi:10.1109/TAC.2012.2229813
12. Raman, C., Krishnan, K.: A new type of secondary radiation. Nature **121**, 501–502 (1928)
13. Russell, D.: Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions. SIAM Rev. **20**(4), 639–739 (1978)
14. Teel, A.R., Praly, L.: A smooth Lyapunov function from a class-$\mathcal{KL}$ estimate involving two positive semidefinite functions. ESAIM Control Optim. Calc. Var. **5**, 313–367 (electronic) (2000). doi:10.1051/cocv:2000113