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Exponential boundary feedback stabilization of a shock steady state for the inviscid Burgers equation

Georges Bastin

Department of Mathematical Engineering,
ICTEAM, University of Louvain,
Louvain-La-Neuve 1348, Belgium
georges.bastin@uclouvain.be

Jean-Michel Coron

Sorbonne Université, UPMC Université Paris 06,
UMR 7598, Laboratoire Jacques-Louis Lions,
F-75005, Paris, France

Institute for Theoretical Studies (ITS),
Eidgenössische Technische Hochschule Zürich,
Zürich 8092, Switzerland
coron@ann.jussieu.fr

Amaury Hayat

Sorbonne Université, UPMC Université Paris 06,
UMR 7598, Laboratoire Jacques-Louis Lions,
F-75005, Paris, France

Forschungsinstitut für Mathematik (FIM),
Eidgenössische Technische Hochschule Zürich,
Zürich 8092, Switzerland
hayat@ljll.math.upmc.fr

Peipei Shang*

School of Mathematical Sciences, Tongji University,
Shanghai, 200092, P. R. China

Sorbonne Université, UPMC Université Paris 06,
UMR 7598, Laboratoire Jacques-Louis Lions,
F-75005, Paris, France

Forschungsinstitut für Mathematik (FIM),
Eidgenössische Technische Hochschule Zürich,
Zürich 8092, Switzerland
shang@tongji.edu.cn

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*Corresponding author

In this paper, we study the exponential stabilization of a shock steady state for the inviscid Burgers equation on a bounded interval. Our analysis relies on the construction of an explicit strict control Lyapunov function. We prove that by appropriately choosing the feedback boundary conditions, we can stabilize the state as well as the shock location to the desired steady state in H^2-norm, with an arbitrary decay rate.

Keywords: Burgers equation; boundary feedback controls; shock steady state.

AMS Subject Classification: 35L02, 93D15, 35L67

1. Introduction

The problem of asymptotic stabilization for hyperbolic systems using boundary feedback control has been studied for a long time. We refer to the pioneer work due to Rauch and Taylor[38] and Russell[39] for linear coupled hyperbolic systems. The first important result of asymptotic stability concerning quasilinear hyperbolic equations was obtained by Slemrod[41] and Greenberg and Li.[19] These two works dealt with local dissipative boundary conditions. The result was established by using the method of characteristics, which allows to estimate the related bounds along the characteristic curves in the framework of C^1 solutions. Another approach to analyze the dissipative boundary conditions is based on the use of Lyapunov functions. Especially, Coron, Bastin and Andrea-Novel[13] used this method to study the asymptotic behavior of the nonlinear hyperbolic equations in the framework of H^2 solutions. In particular, the Lyapunov function they constructed is an extension of the entropy and can be made strictly negative definite by properly choosing the boundary conditions. This method has been later on widely used for hyperbolic conservation laws in the framework of C^1 solutions[11,20,21] or H^2 solutions[2,4,5,10,12,17,22] (see Ref. 3 for an overview of this method).

But all of these results concerning the asymptotic stability of nonlinear hyperbolic equations focus on the convergence to regular solutions, i.e. on the stabilization of regular solutions to a desired regular steady state. It is well known, however, that for quasilinear hyperbolic partial differential equations, solutions may break down in finite time when their first derivatives break up even if the initial condition is smooth.[29] They give rise to the phenomena of shock waves with numerous important applications in physics and fluid mechanics. Compared to classical case, very few results exist on the stabilization of less regular solutions, which requires new techniques. This is also true for related fields, as the optimal control problem.[9,37] For the problem of control and asymptotic stabilization of less regular solutions, we refer to Ref. 7 for the controllability of a general hyperbolic system of conservation laws, Refs. 6 and 35 for the stabilization in the scalar case and Refs. 7 and 14 for the stabilization of a hyperbolic system of conservation laws. In Refs. 6, 14 and 35, by using suitable feedback laws on both sides of the interval, one can steer asymptotically any initial data with sufficiently small total variations to any close constant steady states. All those results concern the boundary stabilization of constant steady states. In particular, as the target state is regular there is no need to stabilize any shock location. In this work, we will study the boundary stabilization of steady states with jump discontinuities for a scalar equation. We believe that our method can be applied to nonlinear hyperbolic systems as well. While preparing the revised version, our attention was drawn to a very recent work Ref. 36 studying a similar problem in the bounded variation (BV) norm. The method and the results are quite different and complementary to this work.

Hyperbolic systems have a wide application in fluid dynamics, and hydraulic jump is one of the best known examples of shock waves as it is frequently observed in open channel flow such as rivers and spillways. Other physical examples of shock waves can be found in road traffic or in gas transportation, with the water hammer phenomenon. In the literature, Burgers equation often appears as a simplification of the dynamical model of flows, as well as the most studied scalar model for transportation. Burgers turbulence has been investigated both analytically and numerically by many authors either as a preliminary approach to turbulence prior to an occurrence of the Navier–Stokes turbulence or for its own sake since the Burgers equation describes the formation and decay of weak shock waves in a compressible fluid.[26,32,44] From a mathematical point of view, it turns out that the study of Burgers equation leads to many of the ideas that arise in the field of nonlinear hyperbolic equations. It is therefore a natural first step to develop methods for the control of this equation. For the boundary stabilization problem of viscous Burgers equation, we refer to works by Krstic *et al.*[28,42] for the stabilization of regular shock-like profile steady states and Refs. 8 and 27 for the stabilization of null-steady-state. In Ref. 42, the authors proved that the shock-like profile steady states of the linearized unit viscous Burgers equation are exponentially stable when using high-gain "radiation" boundary feedback (i.e. static boundary feedback only depending on output measurements). However, they showed that there is a limitation in the decay rate achievable by radiation feedback, i.e. the decay rate goes to zero exponentially as the shock becomes sharper. Thus, they have to use another strategy (namely backstepping method) to achieve arbitrarily fast local convergence to arbitrarily sharp shock profiles. However, this strategy requires a kind of full-state feedback control, rather than measuring only the boundary data.

In this paper, we study the exponential asymptotic stability of a shock steady state of the Burgers equation in H^2-norm, which has been commonly used as a proper norm for studying the stability of hyperbolic systems (see e.g. Refs. 16, 24 and 43), as it enables to deal with Lyapunov functions that are integrals on the domain of quadratic quantities, which is relatively easy to handle. To that end, we construct an explicit Lyapunov function with a strict negative definite time derivative by properly choosing the boundary conditions. Though it has been shown in Ref. 15 that exponential stability in H^2-norm is not equivalent to C^1-norm, our result could probably be generalized to the C^1-norm for conservation laws by transforming the Lyapunov functions as in Refs. 11 and 20.

The first problem is to deal with the well-posedness of the corresponding initial boundary value problem (IBVP) on a bounded domain. The existence of the weak solution to the initial value problem (IVP) of Burgers equation was first studied by Hopf by using vanishing viscosity.[23] The uniqueness of the entropy solution was then studied by Oleinik.[34] One can refer to Ref. 29 for a comprehensive study of the well-posedness of hyperbolic conservation laws in piecewise continuous entropy solution case and also to Ref. 18 in the class of entropy BV functions. Although there are many results for the well-posedness of the IVP for hyperbolic conservation laws, the problem of IBVP is less studied due to the difficulty of handling the boundary condition. In Ref. 1, the authors studied IBVP but in the quarter plane, i.e. $x > 0, t > 0$. By requiring that the boundary condition at $x = 0$ is satisfied in a weak sense, they can apply the method introduced by LeFloch[30] and obtain the explicit formula of the solution. However, our case is more complicated since we consider the Burgers equation defined on a bounded interval.

The organization of the paper is the following. In Sec. 2, we formulate the problem and state our main results. In Sec. 3, we prove the well-posedness of the Burgers equation in the framework of piecewise continuously differentiable entropy solutions, which is one of the main results in this paper. Based on this well-posedness result, we then prove in Sec. 4 by a Lyapunov approach that for appropriately chosen boundary conditions, we can achieve the exponential stability in H^2-norm of a shock steady state with any given arbitrary decay rate and with an exact exponential stabilization of the desired shock location. This result also holds for the H^k-norm for any $k \geq 2$. In Sec. 5, we extend the result to a more general convex flux by requiring some additional conditions on the flux. Conclusion and some open problems are provided in Sec. 6. Finally, some technical proofs are given in Appendices A and B.

2. Problem Statement and Main Result

We consider the following nonlinear inviscid Burgers equation on a bounded domain:

$$y_t(t,x) + \left(\frac{y^2}{2}\right)_x (t,x) = 0 \tag{2.1}$$

with initial condition

$$y(0,x) = y_0(x), \quad x \in (0,L), \tag{2.2}$$

where $L > 0$ and boundary controls

$$y(t,0^+) = u_0(t), \quad y(t,L^-) = u_L(t). \tag{2.3}$$

In this paper, we will be exclusively concerned with the case where the controls $u_0(t) > 0$, $u_L(t) < 0$ have opposite signs and the state $y(t,\cdot)$ at each time t has a jump discontinuity as illustrated in Fig. 1. The discontinuity is a shock wave that occurs at position $x_s(t) \in (0,L)$. According to the Rankine–Hugoniot condition, the shock wave moves with the speed

$$\dot{x}_s(t) = \frac{y(t,x_s(t)^+) + y(t,x_s(t)^-)}{2} \tag{2.4}$$

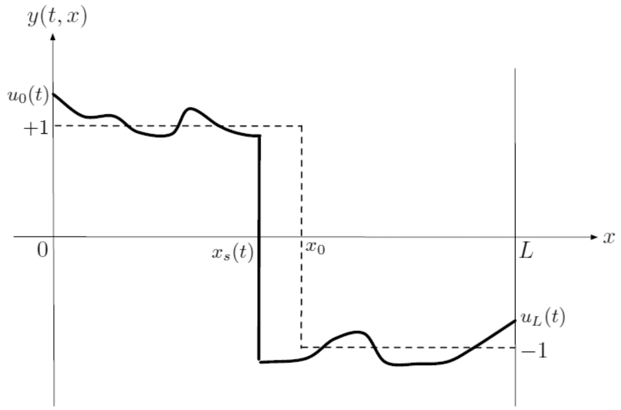

Fig. 1. Entropy solution to the Burgers equation with a shock wave.

which satisfies the Lax entropy condition[29]:

$$y(t, x_s(t)^+) < \dot{x}_s(t) < y(t, x_s(t)^-), \tag{2.5}$$

together with the initial condition

$$x_s(0) = x_{s0}. \tag{2.6}$$

Under a constant control $u_0(t) = -u_L(t) = 1$ for all t, for any $x_0 \in (0, L)$, the system (2.1), (2.3), (2.4) has a steady state (y^*, x_s^*) defined as follows:

$$\begin{aligned} y^*(x) &= \begin{cases} 1, & x \in [0, x_0), \\ -1, & x \in (x_0, L], \end{cases} \\ x_s^* &= x_0. \end{aligned} \tag{2.7}$$

These equilibria are clearly not isolated and, consequently, not asymptotically stable. Indeed, one can see that for any given equilibrium y^* satisfying (2.7), we can find initial data arbitrarily close to y^* which is also an equilibrium of the form (2.7). As the solution cannot be approaching the given equilibrium when t tends to infinity as long as the initial data is another equilibrium, this feature prevents any stability no matter how close the initial data is around y^*. With such open-loop constant control another problem could appear: any small mistake on the boundary control could result in a non-stationary shock moving far away from x_0. It is therefore relevant to study the boundary feedback stabilization of the control system (2.1), (2.3), (2.4).

In this paper, our main contribution is precisely to show how we can exponentially stabilize any of the steady states defined by (2.7) with boundary feedback controls of the following form:

$$\begin{aligned} u_0(t) &= k_1 y(t, x_s(t)^-) + (1 - k_1) + b_1(x_0 - x_s(t)), \\ u_L(t) &= k_2 y(t, x_s(t)^+) - (1 - k_2) + b_2(x_0 - x_s(t)). \end{aligned} \tag{2.8}$$

Here, it is important to emphasize that, with these controls, we are able not only to guarantee the exponential convergence of the solution $y(t,x)$ to the steady state y^* but also to exponentially stabilize the location of the shock discontinuity at the exact desired position x_0. In practice, if the system was used for instance to model gas transportation, the measures of the state around the shock could be obtained using sensors in the pipe. Note that if the control is applied properly, sensors would be only needed on a small region as the shock would remain located in a small region.

Before addressing the exponential stability issue, we first show that there exists a unique piecewise continuously differentiable entropy solution with $x_s(t)$ as its single shock for system (2.1)–(2.4), (2.6), (2.8) provided that y_0 and x_{s0} are in a small neighborhood of y^* and x_0, respectively.

For any given initial condition (2.2) and (2.6), we define the following zero-order compatibility conditions:

$$\begin{aligned} y_0(0^+) &= k_1 y_0(x_{s0}^-) + (1-k_1) + b_1(x_0 - x_{s0}),\\ y_0(L^-) &= k_2 y_0(x_{s0}^+) - (1-k_2) + b_2(x_0 - x_{s0}). \end{aligned} \tag{2.9}$$

Differentiating (2.9) with respect to time t and using (2.4), we get the following first-order compatibility conditions:

$$\begin{aligned} y_0(0^+)y_{0x}(0^+) &= k_1 y_0(x_{s0}^-)y_{0x}(x_{s0}^-) - k_1 y_{0x}(x_{s0}^-)\frac{y_0(x_{s0}^-)+y_0(x_{s0}^+)}{2}\\ &\quad + b_1\frac{y_0(x_{s0}^-)+y_0(x_{s0}^+)}{2},\\ y_0(L^-)y_{0x}(L^-) &= k_2 y_0(x_{s0}^+)y_{0x}(x_{s0}^+) - k_2 y_{0x}(x_{s0}^-)\frac{y_0(x_{s0}^-)+y_0(x_{s0}^+)}{2}\\ &\quad + b_2\frac{y_0(x_{s0}^-)+y_0(x_{s0}^+)}{2}. \end{aligned} \tag{2.10}$$

The first result of this paper deals with the well-posedness of system (2.1)–(2.4), (2.6), (2.8) and is stated in the following theorem.

Theorem 2.1. *For all* $T > 0$, *there exists* $\delta(T) > 0$ *such that, for every* $x_{s0} \in (0, L)$ *and* $y_0 \in H^2((0, x_{s0}); \mathbb{R}) \cap H^2((x_{s0}, L); \mathbb{R})$ *satisfying the compatibility conditions* (2.9)–(2.10) *and*

$$\begin{aligned} &|y_0 - 1|_{H^2((0,x_{s0});\mathbb{R})} + |y_0 + 1|_{H^2((x_{s0},L);\mathbb{R})} \leq \delta(T),\\ &|x_{s0} - x_0| \leq \delta(T), \end{aligned} \tag{2.11}$$

the system (2.1)–(2.4), (2.6), (2.8) *has a unique piecewise continuously differentiable entropy solution* $y \in C^0([0,T]; H^2((0,x_s(t));\mathbb{R}) \cap H^2((x_s(t),L);\mathbb{R}))$ *with* $x_s \in C^1([0,T];(0,L))$ *as its single shock. Moreover, there exists* $C(T)$ *such that the following estimate holds for all* $t \in [0,T]$:

$$\begin{aligned} &|y(t,\cdot) - 1|_{H^2((0,x_s(t));\mathbb{R})} + |y(t,\cdot) + 1|_{H^2((x_s(t),L);\mathbb{R})} + |x_s(t) - x_0|\\ &\quad \leq C(T)(|y_0 - 1|_{H^2((0,x_{s0});\mathbb{R})} + |y_0 + 1|_{H^2((x_{s0},L);\mathbb{R})} + |x_{s0} - x_0|). \end{aligned} \tag{2.12}$$

The proof of this result is given in Sec. 3.

Our next result deals with the exponential stability of the steady state (2.7) for the H^2-norm according to the following definition.

Definition 2.1. The steady state $(y^*, x_0) \in (H^2((0,x_0);\mathbb{R}) \cap H^2((x_0,L);\mathbb{R})) \times (0,L)$ of the system (2.1), (2.3), (2.4), (2.8) is exponentially stable for the H^2-norm with decay rate γ, if there exist $\delta^* > 0$ and $C > 0$ such that for any $y_0 \in H^2((0,x_{s0});\mathbb{R}) \cap H^2((x_{s0},L);\mathbb{R})$ and $x_{s0} \in (0,L)$ satisfying

$$\begin{aligned}
&|y_0 - y_1^*(0,\cdot)|_{H^2((0,x_{s0});\mathbb{R})} + |y_0 - y_2^*(0,\cdot)|_{H^2((x_{s0},L);\mathbb{R})} \leq \delta^*, \\
&|x_{s0} - x_0| \leq \delta^*
\end{aligned} \tag{2.13}$$

and the compatibility conditions (2.9)–(2.10), and for any $T > 0$ the system (2.1)–(2.4), (2.6), (2.8) has a unique solution $(y, x_s) \in C^0([0,T]; H^2((0,x_s(t));\mathbb{R}) \cap H^2((x_s(t),L);\mathbb{R})) \times C^1([0,T];\mathbb{R})$ and

$$\begin{aligned}
&|y(t,\cdot) - y_1^*(t,\cdot)|_{H^2((0,x_s(t));\mathbb{R})} + |y(t,\cdot) - y_2^*(t,\cdot)|_{H^2((x_s(t),L);\mathbb{R})} + |x_s(t) - x_0| \\
&\quad \leq Ce^{-\gamma t}(|y_0 - y_1^*(0,\cdot)|_{H^2((0,x_{s0});\mathbb{R})} + |y_0 - y_2^*(0,\cdot)|_{H^2((x_{s0},L);\mathbb{R})} + |x_{s0} - x_0|), \\
&\qquad \forall\, t \in [0,T).
\end{aligned} \tag{2.14}$$

In (2.13) and (2.14),

$$\begin{aligned}
y_1^*(t,x) &= y^*\left(x\frac{x_0}{x_s(t)}\right), \\
y_2^*(t,x) &= y^*\left(\frac{(x-L)x_0}{x_s(t)-L}\right).
\end{aligned} \tag{2.15}$$

Remark 2.1. At first glance it could seem peculiar to define y_1^* and y_2^* and to compare $y(t,\cdot)$ with these functions. However, the steady state y^* is piecewise H^2 with discontinuity at x_0, while the solution $y(t,x)$ is piecewise H^2 with discontinuity at the shock $x_s(t)$, which may be moving around x_0. Thus, to compare the solution y with the steady state y^* on the same space interval, it is necessary to define such functions y_1^* and y_2^*.

Remark 2.2. We emphasize here that the "exponential stability for the H^2-norm" is not the usual convergence of the H^2-norm of $y - y^*$ taken on $(0,L)$ as y and y^* do not belong to $H^2(0,L)$. This definition enables to define an exponential stability in H^2-norm for a function that has a discontinuity at some point and is regular elsewhere. Note that the convergence to 0 of the H^2-norm in the usual sense does not ensure the convergence of the shock location x_s to x_0. Thus, to guarantee that the state converges to the shock steady state, we have to take account of the shock location, which is explained in Definition 2.1.

Remark 2.3. Note that this definition of exponential stability only deals *a priori* with $t \in [0,T)$ for any $T > 0$. However, this together with Theorem 2.1 implies the global existence in time of the solution (y, x_s) and the exponential stability on $[0, +\infty)$. This is shown at the end of the proof of Theorem 4.1.

We can now state the main result of this paper.

Theorem 2.2. *Let $\gamma > 0$. If the following conditions hold*:

$$b_1 \in \left(\gamma e^{-\gamma x_0}, \frac{\gamma e^{-\gamma x_0}}{1 - e^{-\gamma x_0}}\right), \quad b_2 \in \left(\gamma e^{-\gamma(L-x_0)}, \frac{\gamma e^{-\gamma(L-x_0)}}{1 - e^{-\gamma(L-x_0)}}\right), \tag{2.16a}$$

$$k_1^2 < e^{-\gamma x_0}\left(1 - \frac{b_1}{\gamma}\left(b_1 \frac{1 - e^{-\gamma x_0}}{\gamma e^{-\gamma x_0}} + b_2 \frac{1 - e^{-\gamma(L-x_0)}}{\gamma e^{-\gamma(L-x_0)}}\right)\right), \tag{2.16b}$$

$$k_2^2 < e^{-\gamma(L-x_0)}\left(1 - \frac{b_2}{\gamma}\left(b_1 \frac{1 - e^{-\gamma x_0}}{\gamma e^{-\gamma x_0}} + b_2 \frac{1 - e^{-\gamma(L-x_0)}}{\gamma e^{-\gamma(L-x_0)}}\right)\right), \tag{2.16c}$$

then the steady state (y^*, x_0) of the system (2.1), (2.3), (2.4), (2.8) *is exponentially stable for the H^2-norm with decay rate $\gamma/4$.*

The proof of this theorem is given in Sec. 4.

Remark 2.4. One can actually check that for any $\gamma > 0$ there exist parameters b_1, b_2 and k_1, k_2 satisfying (2.16) as, for $b_1 = \gamma e^{-\gamma x_0}$ and $b_2 = \gamma e^{-\gamma(L-x_0)}$, one has

$$\begin{aligned} 1 - \frac{b_1}{\gamma}\left(b_1 \frac{1 - e^{-\gamma x_0}}{\gamma e^{-\gamma x_0}} + b_2 \frac{1 - e^{-\gamma(L-x_0)}}{\gamma e^{-\gamma(L-x_0)}}\right) &= 1 - e^{-\gamma x_0}(2 - e^{-\gamma x_0} - e^{-\gamma(L-x_0)}) \\ &= e^{-2\gamma x_0}(e^{\gamma x_0} - 1)^2 + e^{-\gamma L} > 0. \end{aligned} \tag{2.17}$$

Similarly, we get

$$\begin{aligned} 1 - \frac{b_2}{\gamma}&\left(b_1 \frac{1 - e^{-\gamma x_0}}{\gamma e^{-\gamma x_0}} + b_2 \frac{1 - e^{-\gamma(L-x_0)}}{\gamma e^{-\gamma(L-x_0)}}\right) \\ &= e^{-2\gamma(L-x_0)}(e^{\gamma(L-x_0)} - 1)^2 + e^{-\gamma L} > 0. \end{aligned} \tag{2.18}$$

Therefore, by continuity, there exist b_1 and b_2, satisfying condition (2.16a) such that there exist k_1 and k_2 satisfying (2.16b) and (2.16c). This implies that γ can be made arbitrarily large. And, from (2.16a)–(2.16c), we can note that for large γ the conditions on the k_i tend to

$$k_1^2 < e^{-\gamma x_0}, \quad k_2^2 < e^{-\gamma(L-x_0)}.$$

Remark 2.5. The result can also be generalized to H^k-norm for any integer $k \geq 2$ in the sense of Definition 2.1 by replacing H^2 with H^k. This can be easily done by just adapting the Lyapunov function defined below by (4.3)–(4.9) as was done in Secs. 4.5 and 6.2 of Ref. 3.

Remark 2.6. If we set $k_1 = k_2 = b_1 = b_2 = 0$, then from (2.8), $u_0(t) \equiv 1$ and $u_L(t) \equiv -1$. Thus it seems logical that the larger γ is, the smaller k_1 and k_2 are.

However, it could seem counter-intuitive that b_1 and b_2 have to tend to 0 when γ tends to $+\infty$, as if one sets $b_1 = 0$ and $b_2 = 0$, one cannot stabilize the location of the system just like in the constant open-loop control case. In other words, for any $\gamma > 0$ the prescribed feedback works while the limit feedback we obtain by letting $\gamma \to +\infty$ cannot even ensure the asymptotic stability of the system. The explanation behind this apparent paradox is that when γ tends to infinity, the Lyapunov function candidate used to prove Theorem 4.1 is not equivalent to the norm of the solution and cannot guarantee anymore the exponential decay of the solution in the H^2-norm. More precisely, one can see, looking at (4.69) and (4.71), that the hypothesis (4.16) of Lemma 4.1 does not hold anymore.

3. An Equivalent System with Shock-Free Solutions

Our strategy to analyze the existence and the exponential stability of the shock wave solutions to the scalar Burgers equation (2.1) is to use an equivalent 2×2 quasilinear hyperbolic system having shock-free solutions. In order to set up this equivalent system, we define the two following functions:

$$y_1(t,x) = y\left(t, x\frac{x_s(t)}{x_0}\right), \quad y_2(t,x) = y\left(t, L + x\frac{x_s(t) - L}{x_0}\right) \tag{3.1}$$

and the new state variables as follows:

$$\mathbf{z}(t,x) = \begin{pmatrix} z_1(t,x) \\ z_2(t,x) \end{pmatrix} = \begin{pmatrix} y_1(t,x) - 1 \\ y_2(t,x) + 1 \end{pmatrix}, \quad x \in (0, x_0). \tag{3.2}$$

The idea behind the definition of y_1, y_2 is to describe the behavior of the solution $y(t,x)$ before and after the moving shock, while studying functions on a time invariant interval. Observe indeed that the functions y_1 and y_2 in (3.1) correspond to the solution $y(t,x)$ on the time varying intervals $(0, x_s(t))$ and $(x_s(t), L)$ respectively, albeit with a time varying scaling of the space coordinate x which is driven by $x_s(t)$ and allows to define the new state variables (z_1, z_2) on the fixed time invariant interval $(0, x_0)$. The reason to rescale y_2 on $(0, x_0)$ instead of (x_0, L) is to simplify the analysis by defining state variables on the same space interval with the same direction of propagation.

Besides, from (3.2), the former steady state (y^*, x_0) corresponds now to the steady state $(\mathbf{z} = \mathbf{0}, x_s = x_0)$ in the new variables. With these new variables, the dynamics of (y, x_s) can now be expressed as follows:

$$\begin{aligned} &z_{1t} + \left(1 + z_1 - x\frac{\dot{x}_s}{x_0}\right) z_{1x}\frac{x_0}{x_s} = 0, \\ &z_{2t} + \left(1 - z_2 + x\frac{\dot{x}_s}{x_0}\right) z_{2x}\frac{x_0}{L - x_s} = 0, \\ &\dot{x}_s(t) = \frac{z_1(t,x_0) + z_2(t,x_0)}{2}, \end{aligned} \tag{3.3}$$

with the boundary conditions:

$$\begin{aligned} z_1(t,0) &= k_1 z_1(t,x_0) + b_1(x_0 - x_s(t)), \\ z_2(t,0) &= k_2 z_2(t,x_0) + b_2(x_0 - x_s(t)), \end{aligned} \tag{3.4}$$

and initial condition

$$\mathbf{z}(0,x) = \mathbf{z}^0(x), \quad x_s(0) = x_{s0}, \tag{3.5}$$

where $\mathbf{z}^0 = (z_1^0, z_2^0)^T$ and

$$\begin{aligned} z_1^0(x) &= y_0\left(x\frac{x_{s0}}{x_0}\right) - 1, \\ z_2^0(x) &= y_0\left(L + x\frac{x_{s0} - L}{x_0}\right) + 1. \end{aligned} \tag{3.6}$$

Furthermore, in the new variables, the compatibility conditions (2.9)–(2.10) are expressed as follows:

$$\begin{aligned} z_1^0(0) &= k_1 z_1^0(x_0) + b_1(x_0 - x_{s0}), \\ z_2^0(0) &= k_2 z_2^0(x_0) + b_2(x_0 - x_{s0}), \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} &(1 + z_1^0(0))z_{1x}^0(0)\frac{x_0}{x_{s0}} \\ &\quad = k_1\left(1 + z_1^0(x_0) - \frac{z_1^0(x_0) + z_2^0(x_0)}{2}\right) z_{1x}^0(x_0)\frac{x_0}{x_{s0}} + b_1\frac{z_1^0(x_0) + z_2^0(x_0)}{2}, \\ &(1 - z_2^0(0))z_{2x}^0(0)\frac{x_0}{L - x_{s0}} \\ &\quad = k_2\left(1 - z_2^0(x_0) + \frac{z_1^0(x_0) + z_2^0(x_0)}{2}\right) z_{2x}^0(x_0)\frac{x_0}{L - x_{s0}} + b_2\frac{z_1^0(x_0) + z_2^0(x_0)}{2}. \end{aligned} \tag{3.8}$$

Concerning the existence and uniqueness of the solution to the system (3.3)–(3.5), we have the following lemma.

Lemma 3.1. *For all $T > 0$, there exists $\delta(T) > 0$ such that, for every $x_{s0} \in (0, L)$ and $\mathbf{z}^0 \in H^2((0, x_0); \mathbb{R}^2)$ satisfying the compatibility conditions (3.7)–(3.8) and*

$$|\mathbf{z}^0|_{H^2((0,x_0);\mathbb{R}^2)} \leq \delta(T), \quad |x_{s0} - x_0| \leq \delta(T), \tag{3.9}$$

the system (3.3)–(3.5) has a unique classical solution $(\mathbf{z}, x_s) \in C^0([0,T]; H^2((0,x_0); \mathbb{R}^2)) \times C^1([0,T]; (0,L))$. Moreover, there exists $C(T)$ such that the following estimate holds for all $t \in [0,T]$

$$|\mathbf{z}(t,\cdot)|_{H^2((0,x_0);\mathbb{R}^2)} + |x_s(t) - x_0| \leq C(T)(|\mathbf{z}^0|_{H^2((0,x_0);\mathbb{R}^2)} + |x_{s0} - x_0|). \tag{3.10}$$

Proof. The proof of Lemma 3.1 is given in Appendix A. □

From this lemma, it is then clear that the proof of Theorem 2.1 follows immediately.

Proof of Theorem 2.1. The change of variables (3.1), (3.2) induces an equivalence between the classical solutions $(\mathbf{z}, x_s)$ of the system (3.3)–(3.5) and the entropy solutions with a single shock (y, x_s) of the system (2.1)–(2.4), (2.6), (2.8). Consequently, from (3.2) and provided $|\mathbf{z}^0|_{H^2((0,x_0);\mathbb{R}^2)}$ and $|x_{s0} - x_0|$ are sufficiently small, the existence and uniqueness of a solution with a single shock (y, x_s) to the system (2.1)–(2.4), (2.6), (2.8) satisfying the entropy condition (2.5) when (y_0, x_{s0}) is in a sufficiently small neighborhood of (y^*, x_0) follows directly from the existence and uniqueness of the classical solution $(\mathbf{z}, x_s)$ to the system (3.3)–(3.5) which is guaranteed by Lemma 3.1. □

Remark 3.1. Under the assumption in Lemma 3.1, if we assume furthermore that $\mathbf{z}^0 \in H^k((0, x_0); \mathbb{R}^2)$ with $k \geq 2$ satisfying the kth order compatibility conditions (see the definition in p. 143 of Ref. 3), then $(\mathbf{z}, x_s) \in C^0([0,T]; H^k((0, x_0); \mathbb{R}^2)) \times C^k([0,T]; \mathbb{R})$ and (3.10) still holds. This is a straightforward extension of the proof in Appendix A, thus we will not give the details of this proof here.

4. Exponential Stability for the H^2-Norm

This section is devoted to the proof of Theorem 2.2 concerning the exponential stability of the steady state of system (2.1), (2.3), (2.4), (2.8). Actually, on the basis of the change of variables introduced in the previous section, we know that we only have to prove the exponential stability of the steady state of the auxiliary system (3.3)–(3.4) according to the following theorem which is equivalent to Theorem 2.2.

Theorem 4.1. *For any* $\gamma > 0$, *if condition* (2.16) *on the parameters of the feedback holds, then there exist* $\delta^* > 0$ *and* $C > 0$ *such that for any* $\mathbf{z}^0 \in H^2((0, x_0); \mathbb{R}^2)$ *and* $x_{s0} \in (0, L)$ *satisfying*

$$|\mathbf{z}^0|_{H^2((0,x_0);\mathbb{R}^2)} \leq \delta^*, \quad |x_{s0} - x_0| \leq \delta^* \tag{4.1}$$

and the compatibility conditions (3.7)–(3.8), *and for any* $T > 0$ *the system* (3.3)–(3.5) *has a unique classical solution* $(\mathbf{z}, x_s) \in C^0([0,T]; H^2((0, x_0); \mathbb{R}^2)) \times C^1([0,T]; \mathbb{R})$ *such that*

$$\begin{aligned} &|\mathbf{z}(t, \cdot)|_{H^2((0,x_0);\mathbb{R}^2)} + |x_s(t) - x_0| \\ &\qquad \leq Ce^{-\gamma t/4}(|\mathbf{z}^0|_{H^2((0,x_0);\mathbb{R}^2)} + |x_{s0} - x_0|), \quad \forall\, t \in [0, T). \end{aligned} \tag{4.2}$$

When this theorem holds, we say that the steady state $(\mathbf{z} = \mathbf{0}, x_s = x_0)$ of the system (3.3)–(3.4) is exponentially stable for the H^2-norm with convergence rate $\gamma/4$. Recall that, from Remark 2.4, there always exist parameters such that (2.16) holds.

Before proving Theorem 4.1, let us give an overview of our strategy. We first introduce a Lyapunov function candidate V with parameters to be chosen. Then, in

Lemma 4.1, we give a condition on the parameters such that V is equivalent to the square of the H^2-norm of $\mathbf{z}$ plus the absolute value of $x_s - x_0$, which implies that proving the exponential decay of V with rate $\gamma/2$ is enough to show the exponential stability of the system with decay rate $\gamma/4$ for the H^2-norm. In Lemma 4.2, we show that in order to obtain Theorem 4.1, it is enough to prove that V decays along any solutions $(\mathbf{z}, x_s) \in C^3([0,T] \times [0,x_0]; \mathbb{R}^2) \times C^3([0,T]; \mathbb{R})$ with a density argument. Then in Lemma 4.3, we compute the time derivative of V along any C^3 solutions of the system and we give a sufficient condition on the parameters such that V satisfies a useful estimate along these solutions. Finally, we show that there exist parameters satisfying the sufficient condition of Lemma 4.3. This, together with Lemma 4.2, ends the proof of Theorem 4.1.

We now introduce the following candidate Lyapunov function which is defined for all $\mathbf{z} = (z_1, z_2)^T \in H^2((0,x_0); \mathbb{R}^2)$ and $x_s \in (0, L)$:

$$V(\mathbf{z}, x_s) = V_1(\mathbf{z}) + V_2(\mathbf{z}, x_s) + V_3(\mathbf{z}, x_s) + V_4(\mathbf{z}, x_s) + V_5(\mathbf{z}, x_s) + V_6(\mathbf{z}, x_s) \tag{4.3}$$

with

$$V_1(\mathbf{z}) = \int_0^{x_0} p_1 e^{\frac{-\mu x}{\eta_1}} z_1^2 + p_2 e^{\frac{-\mu x}{\eta_2}} z_2^2 dx, \tag{4.4}$$

$$V_2(\mathbf{z}, x_s) = \int_0^{x_0} p_1 e^{\frac{-\mu x}{\eta_1}} z_{1t}^2 + p_2 e^{\frac{-\mu x}{\eta_2}} z_{2t}^2 dx, \tag{4.5}$$

$$V_3(\mathbf{z}, x_s) = \int_0^{x_0} p_1 e^{\frac{-\mu x}{\eta_1}} z_{1tt}^2 + p_2 e^{\frac{-\mu x}{\eta_2}} z_{2tt}^2 dx, \tag{4.6}$$

$$V_4(\mathbf{z}, x_s) = \int_0^{x_0} \bar{p}_1 e^{\frac{-\mu x}{\eta_1}} z_1 (x_s - x_0)\, dx + \int_0^{x_0} \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} z_2 (x_s - x_0)\, dx + \kappa (x_s - x_0)^2, \tag{4.7}$$

$$V_5(\mathbf{z}, x_s) = \int_0^{x_0} \bar{p}_1 e^{\frac{-\mu x}{\eta_1}} z_{1t} \dot{x}_s \, dx + \int_0^{x_0} \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} z_{2t} \dot{x}_s \, dx + \kappa (\dot{x}_s)^2, \tag{4.8}$$

$$V_6(\mathbf{z}, x_s) = \int_0^{x_0} \bar{p}_1 e^{\frac{-\mu x}{\eta_1}} z_{1tt} \ddot{x}_s \, dx + \int_0^{x_0} \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} z_{2tt} \ddot{x}_s \, dx + \kappa (\ddot{x}_s)^2. \tag{4.9}$$

In (4.4)–(4.9), μ, p_1, p_2, $\bar{p}_1$, $\bar{p}_2$ are positive constants. Moreover

$$\eta_1 = 1, \quad \eta_2 = \frac{x_0}{L - x_0} \tag{4.10}$$

and

$$\kappa > 1. \tag{4.11}$$

Actually, in this section, we will need to evaluate $V(\mathbf{z}, x_s)$ only along the system solutions for which the variables $\mathbf{z}_t = (z_{1t}, z_{2t})$, $\mathbf{z}_{tt} = (z_{1tt}, z_{2tt})$, $\dot{x}_s$ and $\ddot{x}_s$ that appear in the definition of V can be well defined as functions of

$(\mathbf{z}, x_s) \in H^2((0,x_0);\mathbb{R}^2)\times(0,L)$ from the system (3.3)–(3.4) and their space derivatives. For example, z_{1t} and z_{2t} are defined as functions of $(\mathbf{z}, x_s)$ by

$$z_{1t} := -\left(1 + z_1 - x\frac{z_1(x_0)+z_2(x_0)}{2x_0}\right) z_{1x}\frac{x_0}{x_s}, \tag{4.12}$$

$$z_{2t} := -\left(1 - z_2 + x\frac{z_1(x_0)+z_2(x_0)}{2x_0}\right) z_{2x}\frac{x_0}{L-x_s}, \tag{4.13}$$

and z_{1tt} and z_{2tt} as functions of $(\mathbf{z}, x_s)$ by

$$\begin{aligned} z_{1tt} :=& -\left(1 + z_1 - x\frac{z_1(x_0)+z_2(x_0)}{2x_0}\right)(z_{1t})_x\frac{x_0}{x_s} \\ & - \left(z_{1t} - x\frac{z_{1t}(x_0)+z_{2t}(x_0)}{2x_0}\right) z_{1x}\frac{x_0}{x_s} - z_{1t}\frac{z_1(x_0)+z_2(x_0)}{2x_s}, \end{aligned} \tag{4.14}$$

$$\begin{aligned} z_{2tt} :=& -\left(1 - z_2 + x\frac{z_1(x_0)+z_2(x_0)}{2x_0}\right)(z_{2t})_x\frac{x_0}{L-x_s} \\ & + \left(z_{2t} - x\frac{z_{1t}(x_0)+z_{2t}(x_0)}{2x_0}\right) z_{2x}\frac{x_0}{L-x_s} + z_{2t}\frac{z_1(x_0)+z_2(x_0)}{2(L-x_s)}. \end{aligned} \tag{4.15}$$

The functions z_{1t} and z_{2t} which appear in (4.14) and (4.15) are supposed to be defined by (4.12) and (4.13), respectively.

Remark 4.1. When looking for a Lyapunov function to stabilize the state (z_1, z_2) in H^2-norm, the component $(V_1 + V_2 + V_3)$ can be seen as the most natural and easiest choice, as it is equivalent to a weighted H^2-norm by properly choosing the parameters. This kind of Lyapunov function, sometimes called *basic quadratic Lyapunov function*, is used for instance in Ref. 2 or Sec. 4.4 of Ref. 3. However, in the present case one needs to stabilize both the state $\mathbf{z}$ and the shock location x_s, which requires to add additional terms to the Lyapunov function in order to deal with x_s. Besides, as we have no direct control on x_s (observe that none of the terms of the right-hand side of (2.4), or equivalently of the third equation of (3.3), is a control), we need to add some coupling terms between the state $\mathbf{z}$ on which we have a control and the shock location x_s in the Lyapunov function. Thus, V_4 is designed to provide such coupling with the product of the component of $\mathbf{z}$ and x_s, while V_5 and V_6 are its analogous for the time derivatives terms (as V_2 and V_3 are the analogous of V_1 respectively for the first and second time derivative of $\mathbf{z}$).

We now state the following lemma, providing a condition on μ, p_1, p_2, $\bar{p}_1$ and $\bar{p}_2$ such that $V(\mathbf{z}, x_s)$ is equivalent to $(|\mathbf{z}|^2_{H^2((0,x_0);\mathbb{R}^2)} + |x_s - x_0|^2)$.

Lemma 4.1. *If*

$$\max(\Theta_1, \Theta_2) < 2, \tag{4.16}$$

where

$$\Theta_1 := \frac{\bar{p}_1^2}{p_1}\frac{\eta_1}{\mu}\left(1 - e^{\frac{-\mu x_0}{\eta_1}}\right), \quad \Theta_2 := \frac{\bar{p}_2^2}{p_2}\frac{\eta_2}{\mu}\left(1 - e^{-\frac{\mu x_0}{\eta_2}}\right), \tag{4.17}$$

there exists $\beta > 0$ *such that*

$$\beta(|\mathbf{z}|^2_{H^2((0,x_0);\mathbb{R}^2)} + |x_s - x_0|^2) \leq V \leq \frac{1}{\beta}(|\mathbf{z}|^2_{H^2((0,x_0);\mathbb{R}^2)} + |x_s - x_0|^2) \tag{4.18}$$

for any $(\mathbf{z}, x_s) \in H^2((0,x_0);\mathbb{R}^2) \times (0, L)$ *satisfying*

$$|\mathbf{z}|^2_{H^2((0,x_0);\mathbb{R}^2)} + |x_s - x_0|^2 < \beta^2. \tag{4.19}$$

Proof of Lemma 4.1. Let us start with

$$V_4 = \int_0^{x_0} \bar{p}_1 e^{\frac{-\mu x}{\eta_1}} z_1(x_s - x_0)dx + \int_0^{x_0} \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} z_2(x_s - x_0)dx + \kappa(x_s - x_0)^2. \tag{4.20}$$

Using Young's inequality we get

$$\begin{aligned} &-\frac{1}{2}\left(\int_0^{x_0} \bar{p}_1 e^{\frac{-\mu x}{\eta_1}} z_1\, dx\right)^2 - \frac{(x_s - x_0)^2}{2} - \frac{1}{2}\left(\int_0^{x_0} \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} z_2\, dx\right)^2 \\ &\quad - \frac{(x_s - x_0)^2}{2} + \kappa(x_s - x_0)^2 \\ &\quad \leq V_4 \leq \frac{1}{2}\left(\int_0^{x_0} \bar{p}_1 e^{\frac{-\mu x}{\eta_1}} z_1\, dx\right)^2 + \frac{(x_s - x_0)^2}{2} + \frac{1}{2}\left(\int_0^{x_0} \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} z_2\, dx\right)^2 \\ &\quad + \frac{(x_s - x_0)^2}{2} + \kappa(x_s - x_0)^2. \end{aligned} \tag{4.21}$$

Hence, using the Cauchy–Schwarz inequality and the expression of V_1 given in (4.4),

$$\begin{aligned} &p_1\left(1 - \frac{1}{2}\Theta_1\right)\int_0^{x_0} e^{\frac{-\mu x}{\eta_1}} z_1^2 dx + p_2\left(1 - \frac{1}{2}\Theta_2\right)\int_0^{x_0} e^{\frac{-\mu x}{\eta_2}} z_2^2\, dx + (x_s - x_0)^2(\kappa - 1) \\ &\quad \leq V_1 + V_4 \leq p_1\left(1 + \frac{1}{2}\Theta_1\right)\int_0^{x_0} e^{\frac{-\mu x}{\eta_1}} z_1^2 dx + p_2\left(1 + \frac{1}{2}\Theta_2\right)\int_0^{x_0} e^{\frac{-\mu x}{\eta_2}} z_2^2\, dx \\ &\quad + (x_s - x_0)^2(\kappa + 1), \end{aligned} \tag{4.22}$$

and similarly

$$\begin{aligned} &p_1\left(1 - \frac{1}{2}\Theta_1\right)\int_0^{x_0} e^{\frac{-\mu x}{\eta_1}} z_{1t}^2 dx + p_2(1 - \frac{1}{2}\Theta_2)\int_0^{x_0} e^{\frac{-\mu x}{\eta_2}} z_{2t}^2\, dx + (\dot{x}_s)^2(\kappa - 1) \\ &\quad \leq V_2 + V_5 \leq p_1\left(1 + \frac{1}{2}\Theta_1\right)\int_0^{x_0} e^{\frac{-\mu x}{\eta_1}} z_{1t}^2 dx + p_2\left(1 + \frac{1}{2}\Theta_2\right)\int_0^{x_0} e^{\frac{-\mu x}{\eta_2}} z_{2t}^2\, dx \\ &\quad + (\dot{x}_s)^2(\kappa + 1), \end{aligned} \tag{4.23}$$

and also

$$\begin{aligned} &p_1\left(1 - \frac{1}{2}\Theta_1\right)\int_0^{x_0} e^{\frac{-\mu x}{\eta_1}} z_{1tt}^2 dx + p_2\left(1 - \frac{1}{2}\Theta_2\right)\int_0^{x_0} e^{\frac{-\mu x}{\eta_2}} z_{2tt}^2\, dx + (\ddot{x}_s)^2(\kappa - 1) \\ &\quad \leq V_3 + V_6 \leq p_1\left(1 + \frac{1}{2}\Theta_1\right)\int_0^{x_0} e^{\frac{-\mu x}{\eta_1}} z_{1tt}^2 dx + p_2\left(1 + \frac{1}{2}\Theta_2\right)\int_0^{x_0} e^{\frac{-\mu x}{\eta_2}} z_{2tt}^2\, dx \\ &\quad + (\ddot{x}_s)^2(\kappa + 1). \end{aligned} \tag{4.24}$$

Hence, from (4.11), $\kappa > 1$ and (4.16) is satisfied, there exists $\sigma > 0$ such that

$$\sigma(|\mathbf{z}|^2_{H^2_t((0,x_0);\mathbb{R}^2)} + |x_s - x_0|^2) \leq V \leq \frac{1}{\sigma}(|\mathbf{z}|^2_{H^2_t((0,x_0);\mathbb{R}^2)} + |x_s - x_0|^2), \tag{4.25}$$

where, for a function $\mathbf{z} \in H^2((0,x_0);\mathbb{R}^2)$, $|\mathbf{z}|_{H^2_t((0,x_0);\mathbb{R}^2)}$ is defined by

$$|\mathbf{z}|_{H^2_t((0,x_0);\mathbb{R}^2)} = (|\mathbf{z}|^2_{L^2((0,x_0);\mathbb{R}^2)} + |\mathbf{z}_t|^2_{L^2((0,x_0);\mathbb{R}^2)} + |\mathbf{z}_{tt}|^2_{L^2((0,x_0);\mathbb{R}^2)})^{1/2}, \tag{4.26}$$

with $\mathbf{z}_t$ and $\mathbf{z}_{tt}$ defined as (4.12)–(4.15). Let us point out that from (4.12)–(4.15), there exists $C > 0$ such that

$$\frac{1}{C}|\mathbf{z}|_{H^2((0,x_0);\mathbb{R}^2)} \leq |\mathbf{z}|_{H^2_t((0,x_0);\mathbb{R}^2)} \leq C|\mathbf{z}|_{H^2((0,x_0);\mathbb{R}^2)}, \tag{4.27}$$

if $(|\mathbf{z}|^2_{H^2((0,x_0);\mathbb{R}^2)} + |x_s - x_0|^2) < 1/C$. It follows from (4.25) and (4.27) that $\beta > 0$ can be taken sufficiently small such that inequality (4.18) holds provided (4.19) is satisfied. This concludes the proof of Lemma 4.1. $\square$

Before proving Theorem 4.1, we introduce the following density argument, which shows that it is enough to prove the exponential decay of V along any C^3 solutions of the system.

Lemma 4.2. *Let V be a C^1 and nonnegative functional on $C^0([0,T]; H^2((0,x_0);\mathbb{R}^2)) \times C^1([0,T];\mathbb{R})$. If there exist $\delta > 0$ and $\gamma > 0$ such that for any $(\mathbf{z}, x_s) \in C^3([0,T] \times [0,x_0];\mathbb{R}^2) \times C^3([0,T];\mathbb{R})$ solution of* (3.3)–(3.4), *with associated initial condition $(\mathbf{z}^0, x_{s0})$ satisfying $|\mathbf{z}^0|_{H^2((0,x_0);\mathbb{R}^2)} \leq \delta$ and $|x_{s0} - x_0| \leq \delta$, one has*

$$\frac{dV(\mathbf{z}(t,\cdot), x_s(t))}{dt} \leq -\frac{\gamma}{2} V(\mathbf{z}(t,\cdot), x_s(t)), \tag{4.28}$$

then (4.28) *also holds in a distribution sense for any $(\mathbf{z}, x_s) \in C^0([0,T]; H^2((0,x_0);\mathbb{R}^2)) \times C^1([0,T];\mathbb{R})$ solution of* (3.3)–(3.4) *such that the associated initial condition $(\mathbf{z}^0, x_{s0})$ satisfies $|\mathbf{z}^0|_{H^2((0,x_0);\mathbb{R}^2)} < \delta$ and $|x_{s0} - x_0| < \delta$.*

Proof of Lemma 4.2. Let V be a C^1 and nonnegative functional on $C^0([0,T]; H^2((0,x_0);\mathbb{R}^2)) \times C^1([0,T];\mathbb{R})$ and let $(\mathbf{z}, x_s) \in C^0([0,T]; H^2((0,x_0);\mathbb{R}^2)) \times C^1([0,T];\mathbb{R})$ be solution of (3.3)–(3.4) with associated initial condition $|\mathbf{z}^0|_{H^2((0,x_0);\mathbb{R}^2)} \leq \delta$ and $|x_{s0} - x_0| \leq \delta$. Let $(\mathbf{z}^{0n}, x^n_{s0}) \in H^4((0,x_0);\mathbb{R}^2) \times (0,L)$, $n \in \mathbb{N}$ be a sequence of functions that satisfy the fourth-order compatibility conditions and

$$|\mathbf{z}^{0n}|_{H^2((0,x_0);\mathbb{R}^2)} \leq \delta, \quad |x^n_{s0} - x_0| \leq \delta, \tag{4.29}$$

such that $\mathbf{z}^{0n}$ converges to $\mathbf{z}^0$ in $H^2((0,x_0);\mathbb{R}^2)$ and x^n_{s0} converges to x_{s0}. From Remark 3.1, there exists a unique solution $(\mathbf{z}^n, x^n_s) \in C^0([0,T]; H^4((0,x_0);\mathbb{R}^2)) \times C^4([0,T];\mathbb{R})$ to (3.3)–(3.4) corresponding to the initial condition $(\mathbf{z}^{0n}, x^n_{s0})$ and for any $t \in [0,T]$, we have

$$|\mathbf{z}^n(t,\cdot)|_{H^2((0,x_0);\mathbb{R}^2)} + |x^n_s(t) - x_0| \leq C(T)(|\mathbf{z}^{0n}|_{H^2((0,x_0);\mathbb{R}^2)} + |x^n_{s0} - x_0|). \tag{4.30}$$

Hence, from (4.29) and the third equation of (3.3), the sequence $(\mathbf{z}^n, x_s^n)$ is bounded in $C^0([0,T]; H^2((0,x_0);\mathbb{R}^2)) \times C^1([0,T];\mathbb{R})$. By Corollary 4 of Ref. 40, we can extract a subsequence, which we still denote by $(\mathbf{z}^n, x_s^n)$ that converges to $(\mathbf{u}, y_s)$ in $(C^0([0,T]; C^1([0,x_0]; R^2)) \cap C^1([0,T]; C^0([0,x_0]; R^2))) \times C^1([0,T];\mathbb{R})$, which is a classical solution of (3.3)–(3.5). If we define

$$J(\mathbf{u}) = \begin{cases} +\infty & \text{if } \mathbf{u} \notin L^\infty((0,T); H^2((0,x_0);\mathbb{R}^2)), \\ |\mathbf{u}|_{L^\infty((0,T);H^2((0,x_0);\mathbb{R}^2))} & \text{if } \mathbf{u} \in L^\infty((0,T); H^2((0,x_0);\mathbb{R}^2)), \end{cases} \tag{4.31}$$

then J is lower semi-continuous and we have

$$J(\mathbf{u}) \leq \varliminf_{n\to+\infty} |\mathbf{z}^n|_{C^0([0,T];H^2((0,x_0);\mathbb{R}^2))}, \tag{4.32}$$

thus from (4.30) and the convergence of $(\mathbf{z}^{0n}, x_{s0}^n)$ in $H^2((0,x_0);\mathbb{R}^2) \times \mathbb{R}$, we have $J(\mathbf{u}) \in \mathbb{R}$ and $\mathbf{u} \in L^\infty((0,T); H^2((0,x_0);\mathbb{R}^2))$. Moreover, as $(\mathbf{u}, y_s)$ is a solution to (3.3)–(3.5), we get the extra regularity $\mathbf{u} \in C^0([0,T]; H^2((0,x_0);\mathbb{R}^2))$. Hence, from the uniqueness of the solution given by Lemma 3.1, $\mathbf{u} = \mathbf{z}$ and consequently $y_s = x_s$, which implies that $(\mathbf{z}^n, x_s^n)$ converges to $(\mathbf{z}, x_s)$ in $(C^0([0,T]; C^1([0,x_0];\mathbb{R}^2)) \cap C^1([0,T]; C^0([0,x_0];\mathbb{R}^2))) \times C^1([0,T];\mathbb{R})$. Now, we define $V^n(t) := V(\mathbf{z}^n(t,\cdot), x_s^n(t))$. Note that $V(t) = V(\mathbf{z}(t,\cdot), x_s(t))$ is continuous with time t and well-defined as, from Lemma 3.1, $\mathbf{z} \in C^0([0,T]; H^2((0,x_0);\mathbb{R}^2))$. As $(\mathbf{z}^n, x_s^n)$ belongs to $C^0([0,T]; H^4((0,x_0);\mathbb{R}^2)) \times C^4([0,T];\mathbb{R})$ and is thus C^3, and as it is a solution of (3.3)–(3.4) with initial condition satisfying (4.29), we have from (4.28)

$$\frac{dV^n}{dt} \leq -\frac{\gamma}{2} V^n, \tag{4.33}$$

thus V^n is decreasing on $[0,T]$. Therefore,

$$V^n(t) - V^n(0) \leq -\frac{\gamma t}{2} V^n(t), \quad \forall t \in [0,T], \tag{4.34}$$

which implies that

$$\left(1 + \frac{\gamma t}{2}\right) V^n(t) \leq V^n(0), \quad \forall t \in [0,T]. \tag{4.35}$$

Using the lower semi-continuity of J, by the continuity of V and the convergence of $(\mathbf{z}^{0n}, x_{s0}^n)$ in $H^2((0,x_0);\mathbb{R}^2) \times \mathbb{R}$, we have

$$\left(1 + \frac{\gamma t}{2}\right) V(t) \leq V(0), \quad \forall t \in [0,T]. \tag{4.36}$$

Note that instead of approximating $(\mathbf{z}^0, x_{s0})$, we could have approximated $(\mathbf{z}(s,\cdot), x_s(s))$ where $s \in [0,T)$ and follow the same procedure as above. Therefore, we have in fact for any $s \in [0,T)$

$$\left(1 + \frac{\gamma(t-s)}{2}\right) V(t) \leq V(s), \quad \forall t \in [s,T], \tag{4.37}$$

thus for any $0 \leq s < t \leq T$

$$\frac{V(t) - V(s)}{t - s} \leq -\frac{\gamma}{2} V(t), \tag{4.38}$$

which implies that (4.28) holds in the distribution sense. This ends the proof of Lemma 4.2. □

We now state our final lemma, which gives a sufficient condition so that V defined by (4.3)–(4.9) satisfies a useful estimate along any C^3 solutions.

Lemma 4.3. *Let V be defined by* (4.3)–(4.9). *If the matrix* $\mathbf{A}$ *defined by* (4.59)–(4.64) *is positive definite, then for any* $T > 0$, *there exists* $\delta_1(T) > 0$ *such that for any* $(\mathbf{z}, x_s) \in C^3([0,T] \times [0,x_0]; \mathbb{R}^2) \times C^3([0,T]; \mathbb{R})$ *solution of* (3.3)–(3.5) *satisfying* $|\mathbf{z}^0|_{H^2((0,x_0);\mathbb{R}^2)} \leq \delta_1(T)$ *and* $|x_{s0} - x_0| \leq \delta_1(T)$,

$$\begin{aligned}\frac{dV(\mathbf{z}(t,\cdot), x_s(t))}{dt}&\\ \leq -\frac{\mu}{2} V(\mathbf{z}(t,\cdot), x_s(t)) &+ O((|\mathbf{z}(t,\cdot)|_{H^2((0,x_0);\mathbb{R}^2)} + |x_s - x_0|)^3), \quad \forall t \in [0,T].\end{aligned} \tag{4.39}$$

Here and hereafter, $O(s)$ means that there exist $\varepsilon > 0$ and $C_1 > 0$, both independent of $\mathbf{z}$, x_s, T and $t \in [0,T]$, such that

$$(s \leq \varepsilon) \Rightarrow (|O(s)| \leq C_1 s).$$

To prove this lemma, we differentiate V with respect to time along any C^3 solutions and perform several estimates on the different components of V. For the sake of simplicity, for any $\mathbf{z} \in C^0([0,T]; H^2((0,x_0); \mathbb{R}^2))$, we denote from now on $|\mathbf{z}(t,\cdot)|_{H^2((0,x_0);\mathbb{R}^2)}$ by $|\mathbf{z}|_{H^2}$.

Proof of Lemma 4.3. Let V be given by (4.3)–(4.9) and $T > 0$. Let us assume that $(\mathbf{z}, x_s)$ is a C^3 solution to the system (3.3)–(3.5), with initial condition $|\mathbf{z}^0|_{H^2((0,x_0);\mathbb{R}^2)} \leq \delta_1(T)$ and $|x_{s0} - x_0| \leq \delta_1(T)$ respectively with $\delta_1(T) > 0$ to be chosen later on. Let us examine the different components of the Lyapunov function. We start by studying V_1, V_2 and V_3 which can be treated similarly as in Sec. 4.4 of Ref. 3. Differentiating V_1 along the solution $(\mathbf{z}, x_s)$ and integrating by parts, noticing (4.10), we have

$$\begin{aligned}\frac{dV_1}{dt} &= -2\int_0^{x_0} \left(p_1 e^{\frac{-\mu x}{\eta_1}} z_1 \left(1 + z_1 - x\frac{\dot{x}_s}{x_0}\right) \frac{x_0}{x_s} z_{1x} \right.\\ &\quad \left. + p_2 e^{\frac{-\mu x}{\eta_2}} z_2 \left(1 - z_2 + x\frac{\dot{x}_s}{x_0}\right) \frac{x_0}{L - x_s} z_{2x} \right) dx\\ &= -\mu V_1 - \left[p_1 e^{\frac{-\mu x}{\eta_1}} \frac{x_0}{x_s} z_1^2 + p_2 e^{\frac{-\mu x}{\eta_2}} \frac{x_0}{L - x_s} z_2^2 \right]_0^{x_0}\\ &\quad + O((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3).\end{aligned} \tag{4.40}$$

From (3.3), we have

$$z_{1tt} + \left(1 + z_1 - x\frac{\dot{x}_s}{x_0}\right) z_{1tx}\frac{x_0}{x_s} + \left(z_{1t} - x\frac{\ddot{x}_s}{x_0}\right) z_{1x}\frac{x_0}{x_s} + z_{1t}\frac{\dot{x}_s}{x_s} = 0,$$
$$z_{2tt} + \left(1 - z_2 + x\frac{\dot{x}_s}{x_0}\right) z_{2tx}\frac{x_0}{L - x_s} - \left(z_{2t} - x\frac{\ddot{x}_s}{x_0}\right) z_{2x}\frac{x_0}{L - x_s} - z_{2t}\frac{\dot{x}_s}{L - x_s} = 0. \tag{4.41}$$

Therefore, similarly to (4.40), we can obtain

$$\frac{dV_2}{dt} = -\mu V_2 - \left[p_1 e^{\frac{-\mu x}{\eta_1}}\frac{x_0}{x_s}z_{1t}^2 + p_2 e^{\frac{-\mu x}{\eta_2}}\frac{x_0}{L - x_s}z_{2t}^2\right]_0^{x_0} + O((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3). \tag{4.42}$$

From (4.41) and using (3.3), we get

$$z_{1ttt} + \left(1 + z_1 - x\frac{\dot{x}_s}{x_0}\right) z_{1ttx}\frac{x_0}{x_s} + 2\left(z_{1t} - x\frac{\ddot{x}_s}{x_0}\right) z_{1tx}\frac{x_0}{x_s} + \frac{\dot{x}_s}{x_s}\left(z_{1tt} + z_{1t}\frac{\dot{x}_s}{x_s}\right)$$
$$+ \left(z_{1tt} - x\frac{\dddot{x}_s}{x_0}\right) z_{1x}\frac{x_0}{x_s} + z_{1tt}\frac{\dot{x}_s}{x_s} + z_{1t}\frac{\ddot{x}_s x_s - (\dot{x}_s)^2}{x_c^2} = 0,$$
$$z_{2ttt} + \left(1 - z_2 + x\frac{\dot{x}_s}{x_0}\right) z_{2ttx}\frac{x_0}{L - x_s} - 2\left(z_{2t} - x\frac{\ddot{x}_s}{x_0}\right) z_{2tx}\frac{x_0}{L - x_s}$$
$$+ \frac{\dot{x}_s}{L - x_s}\left(-z_{2tt} + z_{2t}\frac{\dot{x}_s}{L - x_s}\right) - \left(z_{2tt} - x\frac{\dddot{x}_s}{x_0}\right) z_{2x}\frac{x_0}{L - x_s}$$
$$- z_{2tt}\frac{\dot{x}_s}{L - x_s} - z_{2t}\frac{\ddot{x}_s(L - x_s) + (\dot{x}_s)^2}{(L - x_s)^2} = 0. \tag{4.43}$$

Then differentiating V_3 along the system solutions and using (4.43), we have

$$\frac{dV_3}{dt} \leq -\left[p_1 e^{-\frac{\mu x}{\eta_1}}\frac{x_0}{x_s}(z_{1tt}^2)\left(1 + z_1 - x\frac{\dot{x}_s}{x_0}\right)\right]_0^{x_0}$$
$$- \left[p_2 e^{-\frac{\mu x}{\eta_2}}\frac{x_0}{L - x_s}z_{2tt}^2\left(1 - z_2 + x\frac{\dot{x}_s}{x_0}\right)\right]_0^{x_0} - \mu\min\left(\frac{x_0}{x_s}, \frac{L - x_0}{L - x_s}\right)V_3$$
$$- \mu\int_0^{x_0}\left(\frac{x_0}{x_s}p_1 e^{-\frac{\mu x}{\eta_1}}z_{1tt}^2 z_1 - \frac{L - x_0}{L - x_s}p_2 e^{-\frac{\mu x}{\eta_2}}z_{2tt}^2 z_2\right)dx$$
$$+ \mu\int_0^{x_0}\left(\frac{x_0}{x_s}p_1 e^{-\frac{\mu x}{\eta_2}}x z_{1tt}^2\frac{\dot{x}_s}{x_0} - \frac{L - x_0}{L - x_s}p_2 e^{-\frac{\mu x}{\eta_1}}x z_{2tt}^2\frac{\dot{x}_s}{x_0}\right)dx$$
$$- 3\int_0^{x_0}\left(p_1 e^{-\frac{\mu x}{\eta_1}}z_{1tt}^2\frac{\dot{x}_s}{x_s} - p_2 e^{-\frac{\mu x}{\eta_2}}z_{2tt}^2\frac{\dot{x}_s}{L - x_s}\right)dx$$
$$- \int_0^{x_0}\left(p_1 e^{-\frac{\mu x}{\eta_1}}z_{1tt}z_{1x}\frac{x_0}{x_s} - p_2 e^{-\frac{\mu x}{\eta_2}}z_{2tt}^2 z_{2x}\frac{x_0}{L - x_s}\right)dx$$

$$
\begin{aligned}
&-4\int_0^{x_0}\left(p_1e^{-\frac{\mu x}{\eta_1}}z_{1tt}\left(z_{1t}-x\frac{\ddot{x}_s}{x_0}\right)z_{1tx}\frac{x_0}{x_s}\right.\\
&\left.-p_2e^{-\frac{\mu x}{\eta_2}}z_{2tt}\left(z_{2t}-x\frac{\ddot{x}_s}{x_0}\right)z_{2tx}\frac{x_0}{L-x_s}\right)dx\\
&-2\int_0^{x_0}\left(p_1e^{-\frac{\mu x}{\eta_1}}z_{1tt}\left(z_{1tt}+z_{1t}\frac{\dot{x}_s}{x_s}\right)\frac{\dot{x}_s}{x_s}-p_2e^{-\frac{\mu x}{\eta_2}}z_{2tt}\right.\\
&\left.\times\left(z_{2tt}-z_{2t}\frac{\dot{x}_s}{L-x_s}\right)\frac{\dot{x}_s}{L-x_s}\right)dx-2\int_0^{x_0}\left(p_1e^{-\frac{\mu x}{\eta_1}}z_{1tt}z_{1t}\frac{\ddot{x}_sx_s-(\dot{x}_s)^2}{x_c^2}\right.\\
&\left.-p_2e^{-\frac{\mu x}{\eta_2}}z_{2tt}z_{2t}\frac{\ddot{x}_s(L-x_s)+(\dot{x}_s)^2}{(L-x_s)^2}\right)dx-2\int_0^{x_0}\left(p_1e^{-\frac{\mu x}{\eta_1}}z_{1tt}\left(z_{1tt}-x\frac{\dddot{x}_s}{x_0}\right)\right.\\
&\left.\times z_{1x}\frac{x_0}{x_s}-p_2e^{-\frac{\mu x}{\eta_2}}z_{2tt}\left(z_{2tt}-x\frac{\dddot{x}_s}{x_0}\right)z_{2x}\frac{x_0}{L-x_s}\right)dx.
\end{aligned}
\tag{4.44}
$$

Observe that, while previously all the cubic terms in $\mathbf{z}$ could be bounded by $|\mathbf{z}|_{H^2}^3$, here in the last line in (4.44) we have $\dddot{x}_s$ which is proportional to $\mathbf{z}_{tt}(t,x_0)$ and cannot be roughly bounded by the $|\mathbf{z}|_{H^2}$ norm. To overcome this difficulty, we transform these terms using Young's inequality and we get

$$
\begin{aligned}
&2\int_0^{x_0}\left(p_1e^{-\frac{\mu x}{\eta_1}}z_{1tt}\left(x\frac{\dddot{x}_s}{x_0}\right)z_{1x}\frac{x_0}{x_s}-p_2e^{-\frac{\mu x}{\eta_2}}z_{2tt}\left(x\frac{\dddot{x}_s}{x_0}\right)z_{2x}\frac{x_0}{L-x_s}\right)dx\\
&\quad\leq C|\mathbf{z}(t,\cdot)|_{C^1([0,x_0];\mathbb{R}^2)}(z_{1tt}(t,x_0)+z_{2tt}(t,x_0))^2\\
&\qquad+O(|\mathbf{z}(t,\cdot)|_{C^1([0,x_0];\mathbb{R}^2)}|\mathbf{z}|_{H^2}^2),
\end{aligned}
\tag{4.45}
$$

where C denotes a constant, independent of $\mathbf{z}$, x_s, T and $t\in[0,T]$. Note that the first term on the right is now proportional to $\mathbf{z}_{tt}^2(t,x_0)$ with a proportionality coefficient $C|\mathbf{z}(t,\cdot)|_{C^1([0,x_0];\mathbb{R}^2)}$ that, by Sobolev inequality, can be made sufficiently small provided that $|\mathbf{z}|_{H^2}$ is sufficiently small and thus can be dominated by the boundary terms. More precisely, from (4.44) and (4.45) we have

$$
\begin{aligned}
\frac{dV_3}{dt}&\leq-\mu V_3-\left[p_1e^{\frac{-\mu x}{\eta_1}}\frac{x_0}{x_s}(z_{1tt}^2)\right]_0^{x_0}-\left[p_2e^{\frac{-\mu x}{\eta_2}}\frac{x_0}{L-x_s}z_{2tt}^2\right]_0^{x_0}\\
&\quad+O(|\mathbf{z}|_{H^2})(z_{1tt}^2(t,x_0)+z_{2tt}^2(t,x_0))+O((|\mathbf{z}|_{H^2}+|x_s-x_0|)^3).
\end{aligned}
\tag{4.46}
$$

Let us now deal with the term V_4 that takes into account the position of the jump. In the following, we use notations $\mathbf{z}(0)$ and $\mathbf{z}(x_0)$ instead of $\mathbf{z}(t,0)$ and $\mathbf{z}(t,x_0)$ for simplicity. We have

$$
\begin{aligned}
\frac{dV_4}{dt}&=-\int_0^{x_0}\bar{p}_1e^{\frac{-\mu x}{\eta_1}}\left(1+z_1-x\frac{\dot{x}_s}{x_0}\right)z_{1x}(x_s-x_0)\frac{x_0}{x_s}\,dx+\int_0^{x_0}\bar{p}_1e^{\frac{-\mu x}{\eta_1}}z_1\dot{x}_s\,dx\\
&\quad-\int_0^{x_0}\bar{p}_2e^{\frac{-\mu x}{\eta_2}}\left(1-z_2+x\frac{\dot{x}_s}{x_0}\right)z_{2x}(x_s-x_0)\frac{x_0}{L-x_s}\,dx
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^{x_0} \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} z_2 \dot{x}_s \, dx + 2\kappa \dot{x}_s (x_s - x_0) \\
&= -(x_s - x_0)\left[\bar{p}_1 e^{\frac{-\mu x}{\eta_1}} \frac{x_0}{x_s} z_1 + \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} \frac{x_0}{L - x_s} z_2\right]_0^{x_0} - \mu(V_4 - \kappa(x_s - x_0)^2) \\
&\quad + \frac{z_1(x_0) + z_2(x_0)}{2}\left(\int_0^{x_0} \bar{p}_1 e^{-\frac{\mu x}{\eta_1}} z_1 \, dx\right) + \frac{z_1(x_0) + z_2(x_0)}{2} \\
&\quad \times \left(\int_0^{x_0} \bar{p}_2 e^{-\frac{\mu x}{\eta_2}} z_2 \, dx\right) + \kappa(z_1(x_0) + z_2(x_0))(x_s - x_0) \\
&\quad + O((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3).
\end{aligned} \tag{4.47}
$$

According to Young's inequality, for any positive ε_1 and ε_2, we have

$$
\begin{aligned}
&\frac{z_1(x_0) + z_2(x_0)}{2}\left(\int_0^{x_0} \bar{p}_1 e^{-\frac{\mu x}{\eta_1}} z_1 \, dx\right) \\
&\qquad \leq \frac{\varepsilon_1}{4}\left(\frac{z_1(x_0) + z_2(x_0)}{2}\right)^2 + \frac{1}{\varepsilon_1}\left(\int_0^{x_0} \bar{p}_1 e^{-\frac{\mu x}{\eta_1}} z_1 \, dx\right)^2, \\
&\frac{z_1(x_0) + z_2(x_0)}{2}\left(\int_0^{x_0} \bar{p}_2 e^{-\frac{\mu x}{\eta_2}} z_2 \, dx\right) \\
&\qquad \leq \frac{\varepsilon_2}{4}\left(\frac{z_1(x_0) + z_2(x_0)}{2}\right)^2 + \frac{1}{\varepsilon_2}\left(\int_0^{x_0} \bar{p}_2 e^{-\frac{\mu x}{\eta_2}} z_2 \, dx\right)^2.
\end{aligned}
$$

Then using the boundary condition (3.4) and Cauchy–Schwarz inequality, (4.47) becomes

$$
\begin{aligned}
\frac{dV_4}{dt} &\leq -\mu V_4 - \bar{p}_1(x_s - x_0)\frac{x_0}{x_s}\left(\left(e^{-\frac{\mu x_0}{\eta_1}} - k_1\right) z_1(x_0) + b_1(x_s - x_0)\right) \\
&\quad - \bar{p}_2(x_s - x_0)\frac{x_0}{L - x_s}\left(\left(e^{-\frac{\mu x_0}{\eta_2}} - k_2\right) z_2(x_0) + b_2(x_s - x_0)\right) \\
&\quad + (\varepsilon_1 + \varepsilon_2)\frac{z_1^2(x_0) + z_2^2(x_0)}{8} + \max\left\{\frac{\Theta_1}{\varepsilon_1}, \frac{\Theta_2}{\varepsilon_2}\right\} V_1 + \kappa(x_s - x_0)(z_1(x_0) \\
&\quad + z_2(x_0)) + \mu\kappa(x_s - x_0)^2 + O((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3).
\end{aligned} \tag{4.48}
$$

Let us now consider V_5. From (4.8) and (4.41), one has similarly

$$
\begin{aligned}
\frac{dV_5}{dt} &= -\int_0^{x_0} \bar{p}_1 e^{\frac{-\mu x}{\eta_1}} z_{1tx} \dot{x}_s \frac{x_0}{x_s} \, dx + \int_0^{x_0} \bar{p}_1 e^{\frac{-\mu x}{\eta_1}} z_{1t} \ddot{x}_s \, dx \\
&\quad - \int_0^{x_0} \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} z_{2tx} \dot{x}_s \frac{x_0}{L - x_s} \, dx + \int_0^{x_0} \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} z_{2t} \ddot{x}_s \, dx + 2\kappa \ddot{x}_s \dot{x}_s \\
&\quad + O\left((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3\right)
\end{aligned}
$$

$$= -\dot{x}_s \left[\bar{p}_1 e^{\frac{-\mu x}{\eta_1}} \frac{x_0}{x_s} z_{1t} + \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} \frac{x_0}{L - x_s} z_{2t} \right]_0^{x_0} - \mu(V_5 - \kappa(\dot{x}_s)^2)$$
$$+ \frac{z_{1t}(x_0) + z_{2t}(x_0)}{2} \left(\int_0^{x_0} \bar{p}_1 e^{-\frac{\mu x}{\eta_1}} z_{1t}\, dx \right) + \frac{z_{1t}(x_0) + z_{2t}(x_0)}{2}$$
$$\times \left(\int_0^{x_0} \bar{p}_2 e^{-\frac{\mu x}{\eta_2}} z_{2t}\, dx \right) + \kappa(z_{1t}(x_0) + z_{2t}(x_0))\dot{x}_s + O((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3).$$

By differentiating (3.4) with respect to time, we have

$$\begin{aligned} z_{1t}(0) &= k_1 z_{1t}(x_0) - b_1 \dot{x}_s, \\ z_{2t}(0) &= k_2 z_{2t}(x_0) - b_2 \dot{x}_s, \end{aligned} \tag{4.49}$$

and therefore using Cauchy–Schwarz and Young's inequalities, we get

$$\begin{aligned} \frac{dV_5}{dt} \le{}& -\mu V_5 - \bar{p}_1 \dot{x}_s \frac{x_0}{x_s} \big(\big(e^{-\frac{\mu x_0}{\eta_1}} - k_1\big) z_{1t}(x_0) + b_1 \dot{x}_s \big) \\ &- \bar{p}_2 \dot{x}_s \frac{x_0}{L - x_s} \big(\big(e^{-\frac{\mu x_0}{\eta_2}} - k_2\big) z_{2t}(x_0) + b_2 \dot{x}_s \big) \\ &+ (\varepsilon_1 + \varepsilon_2) \frac{z_{1t}^2(x_0) + z_{2t}^2(x_0)}{8} + \max\left\{ \frac{\Theta_1}{\varepsilon_1}, \frac{\Theta_2}{\varepsilon_2} \right\} V_2 \\ &+ \kappa \dot{x}_s (z_{1t}(x_0) + z_{2t}(x_0)) + \mu\kappa(\dot{x}_s)^2 + O((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3). \end{aligned} \tag{4.50}$$

Furthermore, by differentiating (4.49) with respect to time, we have

$$\begin{aligned} z_{1tt}(0) &= k_1 z_{1tt}(x_0) - b_1 \ddot{x}_s, \\ z_{2tt}(0) &= k_2 z_{2tt}(x_0) - b_2 \ddot{x}_s, \end{aligned} \tag{4.51}$$

and therefore using also (4.43), one has

$$\begin{aligned} \frac{dV_6}{dt} ={}& -\int_0^{x_0} \bar{p}_1 e^{\frac{-\mu x}{\eta_1}} z_{1ttx} \ddot{x}_s \frac{x_0}{x_s}\, dx + \int_0^{x_0} \bar{p}_1 e^{\frac{-\mu x}{\eta_1}} z_{1tt} \dddot{x}_s\, dx - \int_0^{x_0} \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} z_{2ttx} \ddot{x}_s \\ &\times \frac{x_0}{L - x_s}\, dx + \int_0^{x_0} \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} z_{2tt} \dddot{x}_s\, dx + 2\kappa \dddot{x}_s \ddot{x}_s + \int_0^{x_0} \bar{p}_1 e^{\frac{-\mu x}{\eta_1}} \ddot{x}_s \\ &\times \left(x \frac{\dddot{x}_s}{x_0} \right) z_{1x} \frac{x_0}{x_s} dx - \int_0^{x_0} \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} \ddot{x}_s \left(x \frac{\dddot{x}_s}{x_0} \right) z_{2x} \frac{x_0}{L - x_s}\, dx \\ &+ O((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3) \\ ={}& -\ddot{x}_s \left[\bar{p}_1 e^{\frac{-\mu x}{\eta_1}} \frac{x_0}{x_s} z_{1tt} + \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} \frac{x_0}{L - x_s} z_{2tt} \right]_0^{x_0} - \mu(V_6 - \kappa(\ddot{x}_s)^2) \\ &+ \frac{z_{1tt}(x_0) + z_{2tt}(x_0)}{2} \left(\int_0^{x_0} \bar{p}_1 e^{-\frac{\mu x}{\eta_1}} z_{1tt}\, dx \right) \\ &+ \frac{z_{1tt}(x_0) + z_{2tt}(x_0)}{2} \left(\int_0^{x_0} \bar{p}_2 e^{-\frac{\mu x}{\eta_2}} z_{2tt}\, dx \right) \end{aligned}$$

$$+ \kappa(z_{1tt}(x_0) + z_{2tt}(x_0))\ddot{x}_s + \int_0^{x_0} \bar{p}_1 e^{\frac{-\mu x}{\eta_1}} \ddot{x}_s \left(x \frac{\dddot{x}_s}{x_0} \right) z_{1x} \frac{x_0}{x_s} dx$$

$$- \int_0^{x_0} \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} \ddot{x}_s \left(x \frac{\dddot{x}_s}{x_0} \right) z_{2x} \frac{x_0}{L - x_s} dx + O((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3).$$

Note that, as above for V_3, here appears again $\dddot{x}_s$ which is proportional to $\mathbf{z}_{tt}(t, x_0)$ and cannot be bounded by $|\mathbf{z}|_{H^2}$. We therefore use Cauchy–Schwarz and Young's inequalities as previously and the boundary condition (4.51) to get

$$\begin{aligned}
\frac{dV_6}{dt} \le & -\mu V_6 - \bar{p}_1 \ddot{x}_s \frac{x_0}{x_s} \big((e^{-\frac{\mu x_0}{\eta_1}} - k_1) z_{1tt}(x_0) + b_1 \ddot{x}_s \big) \\
& - \bar{p}_2 \ddot{x}_s \frac{x_0}{L - x_s} \big((e^{-\frac{\mu x_0}{\eta_2}} - k_2) z_{2tt}(x_0) + b_2 \ddot{x}_s \big) \\
& + (\varepsilon_1 + \varepsilon_2) \frac{z_{1tt}^2(x_0) + z_{2tt}^2(x_0)}{8} + \max\left\{ \frac{\Theta_1}{\varepsilon_1}, \frac{\Theta_2}{\varepsilon_2} \right\} V_2 \\
& + \kappa \ddot{x}_s (z_{1tt}(x_0) + z_{2tt}(x_0)) + \mu\kappa(\ddot{x}_s)^2 + O(|\mathbf{z}|_{H^2})(z_{1tt}^2(x_0) + z_{2tt}^2(x_0)) \\
& + O((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3).
\end{aligned} \tag{4.52}$$

Hence, from (4.40), (4.48) and the boundary conditions (3.4), we have

$$\frac{dV_1}{dt} + \frac{dV_4}{dt} \le -\mu(V_1 + V_4) \tag{4.53}$$

$$\begin{aligned}
& + \max\left\{ \frac{\Theta_1}{\varepsilon_1}, \frac{\Theta_2}{\varepsilon_2} \right\} V_1 \\
& + \left[\frac{x_0}{x_s} p_1 \big(k_1^2 - e^{-\frac{\mu x_0}{\eta_1}} \big) + \frac{\varepsilon_1 + \varepsilon_2}{8} \right] z_1^2(x_0) \\
& + \left[\frac{x_0}{L - x_s} p_2 \big(k_2^2 - e^{-\frac{\mu x_0}{\eta_2}} \big) + \frac{\varepsilon_1 + \varepsilon_2}{8} \right] z_2^2(x_0) \\
& + \left[-2 \frac{x_0}{x_s} p_1 b_1 k_1 - \frac{x_0}{x_s} \bar{p}_1 \big(e^{-\frac{\mu x_0}{\eta_1}} - k_1 \big) + \kappa \right] z_1(x_0)(x_s - x_0) \\
& + \left[-2 \frac{x_0}{L - x_s} p_2 b_2 k_2 - \frac{x_0}{L - x_s} \bar{p}_2 \big(e^{-\frac{\mu x_0}{\eta_2}} - k_2 \big) + \kappa \right] z_2(x_0)(x_s - x_0) \\
& + \left[\frac{x_0}{x_s} p_1 b_1^2 + \frac{x_0}{L - x_s} p_2 b_2^2 - \frac{x_0}{x_s} \bar{p}_1 b_1 - \frac{x_0}{L - x_s} \bar{p}_2 b_2 + \mu\kappa \right] (x_s - x_0)^2 \\
& + O((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3).
\end{aligned} \tag{4.54}$$

Let us now select ε_1 and ε_2 as follows:

$$\varepsilon_1 = 2 \frac{\Theta_1}{\mu}, \quad \varepsilon_2 = 2 \frac{\Theta_2}{\mu}, \tag{4.55}$$

where Θ_1 and Θ_2 are defined in (4.17). Then (4.54) can be rewritten in the following compact form:

$$\frac{dV_1}{dt} + \frac{dV_4}{dt} \leq -\frac{\mu}{2} V_1 - \mu V_4 - \mathbf{Z}^T \mathbf{A}_0 \, \mathbf{Z} + O((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3). \tag{4.56}$$

This expression involves the quadratic form $\mathbf{Z}^T \mathbf{A}_0 \, \mathbf{Z}$ with the vector $\mathbf{Z}$ defined as

$$\mathbf{Z} = (z_1(x_0) \ z_2(x_0) \ (x_s - x_0))^T \tag{4.57}$$

and the matrix $\mathbf{A}_0$ satisfies

$$\mathbf{A}_0 = \mathbf{A} + O(|x_s - x_0|), \tag{4.58}$$

where $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{4.59}$$

with

$$a_{11} = p_1 \big(e^{-\frac{\mu x_0}{\eta_1}} - k_1^2\big) - \frac{\varepsilon_1 + \varepsilon_2}{8}, \tag{4.60}$$

$$a_{13} = a_{31} = p_1 b_1 k_1 + \frac{\bar{p}_1}{2} \big(e^{-\frac{\mu x_0}{\eta_1}} - k_1\big) - \frac{\kappa}{2}, \tag{4.61}$$

$$a_{22} = \frac{x_0}{L - x_0} p_2 \big(e^{-\frac{\mu x_0}{\eta_2}} - k_2^2\big) - \frac{\varepsilon_1 + \varepsilon_2}{8}, \tag{4.62}$$

$$a_{23} = a_{32} = \frac{x_0}{L - x_0} p_2 b_2 k_2 + \frac{x_0}{L - x_0} \frac{\bar{p}_2}{2} \big(e^{-\frac{\mu x_0}{\eta_2}} - k_2\big) - \frac{\kappa}{2}, \tag{4.63}$$

$$a_{33} = -p_1 b_1^2 - \frac{x_0}{L - x_0} p_2 b_2^2 + \bar{p}_1 b_1 + \frac{x_0}{L - x_0} \bar{p}_2 b_2 - \mu\kappa. \tag{4.64}$$

Similarly, from (4.42) and (4.50), we get

$$\frac{dV_2}{dt} + \frac{dV_5}{dt} \leq -\frac{\mu}{2} V_2 - \mu V_5 - \mathbf{Z}_t^T \mathbf{A}_0 \, \mathbf{Z}_t + O((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3), \tag{4.65}$$

while from (4.46) and (4.52), we have

$$\frac{dV_3}{dt} + \frac{dV_6}{dt} \leq -\frac{\mu}{2} V_3 - \mu V_6 - \mathbf{Z}_{tt}^T \mathbf{A}_1 \, \mathbf{Z}_{tt} + O((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3) \tag{4.66}$$

with

$$\mathbf{A}_1 = \mathbf{A}_0 + \begin{pmatrix} O(|\mathbf{z}|_{H^2}) & 0 & 0 \\ 0 & O(|\mathbf{z}|_{H^2}) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.67}$$

If $\mathbf{A}$ is positive definite, from (4.58) and (4.67) and by continuity, $\mathbf{A}_0$ and $\mathbf{A}_1$ are also positive definite provided that $|\mathbf{z}|_{H^2}$ and $|x_s - x_0|$ are sufficiently small. Hence, from (4.56), (4.65), (4.66) and Lemma 3.1, there exists $\delta_1(T) > 0$ such that, if $|\mathbf{z}^0|_{H^2((0,x_0);\mathbb{R}^2)} \leq \delta_1(T)$ and $|x_{s0} - x_0| \leq \delta_1(T)$, one has

$$\frac{dV}{dt} \leq -\frac{\mu}{2}V + O((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3), \tag{4.68}$$

which ends the proof of Lemma 4.3. □

Let us now prove Theorem 4.1.

Proof of Theorem 4.1. From Lemmas 4.1 and 4.2, all it remains to do is to show that for any $\gamma > 0$, under conditions (2.16) there exist μ, p_1, p_2, $\bar{p}_1$ and $\bar{p}_2$ satisfying (4.16) and such that V given by (4.3)–(4.9) decreases exponentially with rate $\gamma/2$ along any C^3 solution of the system (3.3)–(3.5). Using Lemma 4.3 we first show that for any $\gamma > 0$ there exists $\mu > \gamma$, and positive parameters p_1, p_2, $\bar{p}_1$ and $\bar{p}_2$ satisfying (4.16) and such that the matrix $\mathbf{A}$ defined by (4.59)–(4.64) is positive definite, which implies that (4.39) holds. Then, we show that this implies the exponential decay of V with decay rate $\gamma/2$ along any C^3 solution of (3.3)–(3.5).

Let us start by selecting p_1 and p_2 as

$$p_1 = \frac{\bar{p}_1}{2b_1}, \quad p_2 = \frac{\bar{p}_2}{2b_2}. \tag{4.69}$$

Then the cross terms (4.61), (4.63) of the matrix $\mathbf{A}$ become

$$a_{13} = a_{31} = \frac{\bar{p}_1}{2}e^{-\frac{\mu x_0}{\eta_1}} - \frac{\kappa}{2}, \quad a_{23} = a_{32} = \frac{x_0}{L - x_0}\frac{\bar{p}_2}{2}e^{-\frac{\mu x_0}{\eta_2}} - \frac{\kappa}{2}. \tag{4.70}$$

Let $\bar{p}_1$ and $\bar{p}_2$ be selected as

$$\bar{p}_1 = \kappa e^{\frac{\mu x_0}{\eta_1}}, \quad \bar{p}_2 = \kappa\frac{L - x_0}{x_0}e^{\frac{\mu x_0}{\eta_2}}. \tag{4.71}$$

Then we have

$$a_{13} = a_{31} = 0, \quad a_{23} = a_{32} = 0 \tag{4.72}$$

such that $\mathbf{A}$ can now be rewritten as

$$\mathbf{A} = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}. \tag{4.73}$$

Moreover, from (4.69) and (4.71), we get

$$a_{33} = \frac{\bar{p}_1}{2}b_1 + \frac{x_0}{L - x_0}\frac{\bar{p}_2}{2}b_2 - \mu\kappa = \frac{\kappa}{2}b_1 e^{\frac{\mu x_0}{\eta_1}} + \frac{\kappa}{2}b_2 e^{\frac{\mu x_0}{\eta_2}} - \mu\kappa. \tag{4.74}$$

As conditions (2.16) are strict inequalities, by continuity it follows that we can select $\mu > \gamma$ such that these conditions (2.16) are still satisfied with μ instead of γ such that

$$\mu e^{-\frac{\mu x_0}{\eta_1}} < b_1 < \frac{\mu e^{-\frac{\mu x_0}{\eta_1}}}{1 - e^{-\frac{\mu x_0}{\eta_1}}}, \quad \mu e^{-\frac{\mu x_0}{\eta_2}} < b_2 < \frac{\mu e^{-\frac{\mu x_0}{\eta_2}}}{1 - e^{-\frac{\mu x_0}{\eta_2}}}, \tag{4.75}$$

this together with (4.74) gives

$$a_{33} > 0. \tag{4.76}$$

From (4.17), (4.55), (4.60), (4.62), (4.69) and (4.71), we have

$$a_{11} = \frac{\kappa}{2b_1}(1 - k_1^2 e^{\frac{\mu x_0}{\eta_1}}) - \frac{\kappa}{2\mu^2}\big[b_1\big(e^{\frac{\mu x_0}{\eta_1}} - 1\big) + b_2\big(e^{\frac{\mu x_0}{\eta_2}} - 1\big)\big], \tag{4.77}$$

$$a_{22} = \frac{\kappa}{2b_2}\big(1 - k_2^2 e^{\frac{\mu x_0}{\eta_2}}\big) - \frac{\kappa}{2\mu^2}\big[b_1\big(e^{\frac{\mu x_0}{\eta_1}} - 1\big) + b_2\big(e^{\frac{\mu x_0}{\eta_2}} - 1\big)\big]. \tag{4.78}$$

Then, under assumptions (2.16), it can be checked that

$$a_{11} > 0, \quad a_{22} > 0. \tag{4.79}$$

This implies that $\mathbf{A}$ is positive definite.

Thus from Lemma 4.3, for any $T > 0$, there exists $\delta_1(T) > 0$ such that for any $(\mathbf{z}, x_s) \in C^3([0,T]\times[0,x_0];\mathbb{R}^2)\times C^3([0,T];\mathbb{R})$ solution of (3.3)–(3.5) satisfying $|\mathbf{z}^0|_{H^2((0,L);\mathbb{R}^2)} \leq \delta_1(T)$ and $|x_{s0} - x_0| \leq \delta_1(T)$, one has

$$\frac{dV}{dt} \leq -\frac{\mu}{2}V + O((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3). \tag{4.80}$$

Now let us remark that from condition (4.75) we have

$$\max\left(2\frac{b_1\eta_1}{\mu}e^{\frac{\mu x_0}{\eta_1}}\big(1 - e^{-\frac{\mu x_0}{\eta_1}}\big),\ 2\frac{L - x_0}{x_0}\frac{b_2\eta_2}{\mu}e^{\frac{\mu x_0}{\eta_2}}\big(1 - e^{-\frac{\mu x_0}{\eta_2}}\big)\right) < 2. \tag{4.81}$$

Therefore, there exists $\kappa > 1$ such that

$$\max\left(2\kappa\frac{b_1\eta_1}{\mu}e^{\frac{\mu x_0}{\eta_1}}\big(1 - e^{-\frac{\mu x_0}{\eta_1}}\big),\ 2\kappa\frac{L - x_0}{x_0}\frac{b_2\eta_2}{\mu}e^{\frac{\mu x_0}{\eta_2}}\big(1 - e^{-\frac{\mu x_0}{\eta_2}}\big)\right) < 2, \tag{4.82}$$

which means from (4.69) and (4.71) that (4.16) is satisfied. Hence from (4.80) and Lemma 4.1, since $\mu > \gamma$, there exists $\delta_0(T) \leq \delta_1(T)$ such that, if $|\mathbf{z}^0|_{H^2((0,x_0);\mathbb{R}^2)} \leq \delta_0(T)$ and $|x_{s0} - x_0| \leq \delta_0(T)$, then

$$\frac{dV}{dt} \leq -\frac{\gamma}{2}V \tag{4.83}$$

along the C^3 solutions of the system (3.3)–(3.5). Thus from Lemma 4.2, (4.83) holds along the $C^0([0,T];H^2((0,x_0);\mathbb{R}^2))\times C^1([0,T];\mathbb{R})$ solutions of (3.3)–(3.5) in a distribution sense.

So far $\delta_0(T)$ may depend on T, while δ^* in Theorem 4.1 does not depend on T. The only thing left to check is that we can find δ^* independent of T such that if $|\mathbf{z}^0|_{H^2((0,x_0);\mathbb{R}^2)} \leq \delta^*$ and $|x_{s0} - x_0| \leq \delta^*$, then (4.83) holds on $(0,T)$ for any $T > 0$. As the constant β involved in Lemma 4.1 does not depend on T, there exists $T_1 > 0$ such that

$$\beta^{-2}e^{-\frac{\gamma}{2}T_1} < \frac{1}{2}. \tag{4.84}$$

As $T_1 \in (0, +\infty)$, from Lemma 3.1, we can choose $\delta_0(T_1) > 0$ satisfying $C(T_1)\delta_0(T_1) < \beta/2$, such that for every $x_{s0} \in (0, L)$ and $\mathbf{z}^0 \in H^2((0, x_0); \mathbb{R}^2)$ satisfying the compatibility conditions (3.7)–(3.8) and

$$|\mathbf{z}^0|_{H^2((0,x_0);\mathbb{R}^2)} \leq \delta_0(T_1), \quad |x_{s0} - x_0| \leq \delta_0(T_1),$$

there exists a unique solution $(\mathbf{z}, x_s) \in C^0([0, T_1]; H^2((0, x_0); \mathbb{R}^2)) \times C^1([0, T_1]; \mathbb{R})$ to the system (3.3)–(3.5) satisfying

$$|\mathbf{z}(t, \cdot)|_{H^2((0,x_0);\mathbb{R}^2)} + |x_s(t) - x_0| < \beta \tag{4.85}$$

and such that (4.83) holds on $(0, T_1)$ in a distribution sense. From (4.85), Lemma 4.1 and (4.84),

$$|\mathbf{z}(T_1, \cdot)|_{H^2((0,x_0);\mathbb{R}^2)} \leq \delta_0(T_1), \quad |x_s(T_1) - x_0| \leq \delta_0(T_1). \tag{4.86}$$

Moreover, the compatibility conditions hold now at time $t = T_1$ instead of $t = 0$. Thus, from Lemma 3.1 there exists a unique $(\mathbf{z}, x_s) \in C^0([T_1, 2T_1]; H^2((0, x_0); \mathbb{R}^2)) \times C^1([T_1, 2T_1]; \mathbb{R})$ solution of (3.3)–(3.5) on $[T_1, 2T_1]$ and (4.83) holds on $(T_1, 2T_1)$ in a distribution sense. One can repeat this analysis on $[jT_1, (j+1)T_1]$ where $j \in \mathbb{N}^* \backslash \{1\}$. Setting $\delta^* = \delta_0(T_1)$, we get that (4.83) holds on $(0, T)$ for any $T > 0$ in a distribution sense along the $C^0([0, T]; H^2((0, x_0); \mathbb{R}^2)) \times C^1([0, T]; \mathbb{R})$ solutions of the system (3.3)–(3.5). In fact, it also implies the global existence and uniqueness of $(\mathbf{z}, x_s) \in C^0([0, +\infty); H^2((0, x_0); \mathbb{R}^2)) \times C^1([0, +\infty); \mathbb{R})$ solution of (3.3)–(3.5) and the fact that (4.83) holds on $(0, +\infty)$. This concludes the proof of Theorem 4.1. ☐

5. Extension to a General Convex Flux

We can in fact extend this method to a more general convex flux. Let $f \in C^3(\mathbb{R})$ be a convex function, and consider the equation

$$\partial_t y + \partial_x (f(y)) = 0. \tag{5.1}$$

For this conservation law, the Rankine–Hugoniot condition becomes

$$\dot{x}_s = \frac{f(y(t, x_s(t)^+)) - f(y(t, x_s(t)^-))}{y(t, x_s(t)^+) - y(t, x_s(t)^-)} \tag{5.2}$$

and, let (y^*, x_0) be an entropic shock steady state of (5.1)–(5.2), without loss of generality we can assume that $y^*(x_0^+) = -1$ and $y^*(x_0^-) = 1$, thus $f(1) = f(-1)$. Then, for any $x_0 \in (0, L)$, we have the following result.

Theorem 5.1. *Let $f \in C^3(\mathbb{R})$ be a convex function such that $f(1) = f(-1)$ and assume in addition that*

$$f'(1) \geq 1 \quad \textit{and} \quad |f'(-1)| \geq 1. \tag{5.3}$$

Let $\gamma > 0$. If the following conditions hold:

$$b_1 \in \left(\frac{2\gamma e^{-\gamma x_0}}{f'(1)+|f'(-1)|}, \frac{\gamma e^{-\gamma x_0}}{1-e^{-\gamma x_0}}\right), \quad b_2 \in \left(\frac{2\gamma e^{-\gamma(L-x_0)}}{f'(1)+|f'(-1)|}, \frac{\gamma e^{-\gamma(L-x_0)}}{1-e^{-\gamma(L-x_0)}}\right), \tag{5.4a}$$

$$k_1^2 < e^{-\gamma x_0}\left(1 - f'(1)\frac{b_1}{\gamma}\left(b_1\frac{1-e^{-\gamma x_0}}{\gamma e^{-\gamma x_0}} + b_2\frac{1-e^{-\gamma(L-x_0)}}{\gamma e^{-\gamma(L-x_0)}}\right)\right), \tag{5.4b}$$

$$k_2^2 < e^{-\gamma(L-x_0)}\left(1 - |f'(-1)|\frac{b_2}{\gamma}\left(b_1\frac{1-e^{-\gamma x_0}}{\gamma e^{-\gamma x_0}} + b_2\frac{1-e^{-\gamma(L-x_0)}}{\gamma e^{-\gamma(L-x_0)}}\right)\right), \tag{5.4c}$$

then the steady state (y^*, x_0) of the system (5.1), (5.2), (2.3), (2.8) *is exponentially stable for the H^2-norm with decay rate $\gamma/4$.*

One can use exactly the same method as previously. We give in Appendix B a way to adapt the proof of Theorem 4.1.

Remark 5.1. One could wonder why we require condition (5.3). This condition ensures that there always exist parameters b_i and k_i satisfying (5.4).

6. Conclusion and Open Problems

The stabilization of shock-free regular solutions of quasilinear hyperbolic systems has been the subject of a large number of publications in the recent scientific literature. In contrast, there are no results concerning the Lyapunov stability of solutions with jump discontinuities, although they occur naturally in the form of shock waves or hydraulic jumps in many applications of fluid dynamics. For instance, the inviscid Burgers equation provides a simple scalar example of a hyperbolic system having natural solutions with jump discontinuities. The main contribution of this paper is precisely to address the issue of the boundary exponential feedback stabilization of an unstable shock steady state for the Burgers equation over a bounded interval. Our strategy to solve the problem relies on introducing a change of variables which allows to transform the scalar Burgers equation with shock wave solutions into an equivalent 2×2 quasilinear hyperbolic system having shock-free solutions over a bounded interval. Then, by a Lyapunov approach, we show that, for appropriately chosen boundary conditions, the exponential stability in H^2-norm of the steady state can be achieved with an arbitrary decay rate and with an exact exponential stabilization of the desired shock location. Compared with previous results in the literature for classical solutions of quasilinear hyperbolic systems, the selection of an appropriate Lyapunov function is challenging because the equivalent system is parameterized by the time-varying position of the jump discontinuity. In particular, the standard quadratic Lyapunov function used in the book Ref. 3 has to be augmented with suitable extra terms for the analysis of the stabilization of the jump position. Based on the result, some open questions could be addressed. Could these results be generalized to any convex flux, especially when (5.3) is not satisfied? As we show the rapid stabilization result, is it possible to obtain finite time stabilization? Could we replace the left/right state at the shock by measurements nearby or by averages close to the shock? If not, could the error on both the state and shock location be bounded?

Appendix A. Proof of Lemma 3.1

Proof of Lemma 3.1. We adapt the fixed point method used in Appendix B of Ref. 3 (see also Refs. 25 and 31). We first deal with the case where

$$T \in (0, \min(x_0,\, L - x_0)). \tag{A.1}$$

For any $\nu > 0$, $x_{s0} \in \mathbb{R}$ and $\mathbf{z}^0 \in H^2((0, x_0); \mathbb{R}^2)$, let $C_\nu(\mathbf{z}^0, x_{s0})$ be the set of

$$\mathbf{z} \in L^\infty((0,T); H^2((0,x_0);\mathbb{R}^2)) \cap W^{1,\infty}((0,T); H^1((0,x_0);\mathbb{R}^2)) \\ \cap W^{2,\infty}((0,T); L^2((0,x_0);\mathbb{R}^2))$$

such that

$$|\mathbf{z}|_{L^\infty((0,T);H^2((0,x_0);\mathbb{R}^2))} \le \nu, \tag{A.2}$$

$$|\mathbf{z}|_{W^{1,\infty}((0,T);H^1((0,x_0);\mathbb{R}^2))} \le \nu, \tag{A.3}$$

$$|\mathbf{z}|_{W^{2,\infty}((0,T);L^2((0,x_0);\mathbb{R}^2))} \le \nu, \tag{A.4}$$

$$\mathbf{z}(\cdot, x_0) \in H^2((0,T);\mathbb{R}^2), \quad |\mathbf{z}(\cdot, x_0)|_{H^2((0,T);\mathbb{R}^2)} \le \nu^2, \tag{A.5}$$

$$\mathbf{z}(0, \cdot) = \mathbf{z}^0, \tag{A.6}$$

$$\mathbf{z}_t(0, \cdot) = -A(\mathbf{z}^0, \cdot, x_s(\mathbf{z}(\cdot, x_0))(0))\mathbf{z}_x^0, \tag{A.7}$$

where we write $x_s(\mathbf{z}(\cdot, x_0))(t)$ in order to emphasize its dependence on $\mathbf{z}(\cdot, x_0)$ in the following proof and

$$x_s(\mathbf{z}(\cdot, x_0))(t) =: x_{s0} + \int_0^t \frac{z_1(s, x_0) + z_2(s, x_0)}{2}\, ds. \tag{A.8}$$

In (A.7),

$$A(\mathbf{z}, x, x_s(\mathbf{z}(\cdot, x_0))(t)) = \begin{pmatrix} a_1(\mathbf{z}, x, x_s(\mathbf{z}(\cdot, x_0))(t)) & 0 \\ 0 & a_2(\mathbf{z}, x, x_s(\mathbf{z}(\cdot, x_0))(t)) \end{pmatrix} \tag{A.9}$$

with

$$a_1(\mathbf{z}, x, x_s(\mathbf{z}(\cdot, x_0))(t)) \\ = \left(1 + z_1(t,x) - x\frac{z_1(t,x_0) + z_2(t,x_0)}{2x_0}\right) \frac{x_0}{x_s(\mathbf{z}(\cdot, x_0))(t)}, \tag{A.10}$$

$$a_2(\mathbf{z}, x, x_s(\mathbf{z}(\cdot, x_0))(t)) \\ = \left(1 - z_2(t,x) + x\frac{z_1(t,x_0) + z_2(t,x_0)}{2x_0}\right) \frac{x_0}{L - x_s(\mathbf{z}(\cdot, x_0))(t)}. \tag{A.11}$$

The set $C_\nu(\mathbf{z}^0, x_{s0})$ is not empty and is a closed subset of $L^\infty((0,T); L^2((0,L);\mathbb{R}^2))$ provided that $|\mathbf{z}^0|_{H^2((0,x_0);\mathbb{R}^2)} \le \delta$ and $|x_{s0} - x_0| \le \delta$, with δ sufficiently small (see for instance Appendix B of Ref. 3).

Let us define a mapping

$$\begin{aligned} \mathcal{F} : C_\nu(\mathbf{z}^0, x_{s0}) \to L^\infty((0,T); H^2((0,x_0);\mathbb{R}^2)) \cap W^{1,\infty}((0,T); H^1((0,x_0);\mathbb{R}^2)) \\ \cap W^{2,\infty}((0,T); L^2((0,x_0);\mathbb{R}^2)), \qquad \text{(A.12)} \end{aligned}$$

$$\mathbf{v} = (v_1, v_2)^T \mapsto \mathcal{F}(\mathbf{v}) = \mathbf{z} = (z_1, z_2)^T,$$

where $\mathbf{z}$ is the solution of the linear hyperbolic equation

$$\mathbf{z}_t + A(\mathbf{v}, x, x_s(\mathbf{v}(\cdot, x_0))(t))\mathbf{z}_x = 0, \tag{A.13}$$

$$\mathbf{z}(0,x) = \mathbf{z}^0(x), \tag{A.14}$$

with boundary conditions

$$z_1(t,0) = k_1 z_1(t,x_0) + b_1\psi(t), \tag{A.15}$$

$$z_2(t,0) = k_2 z_2(t,x_0) + b_2\psi(t), \tag{A.16}$$

where

$$\psi(t) = x_0 - x_s(\mathbf{v}(\cdot, x_0))(t). \tag{A.17}$$

In the following, we will treat z_1 in detail. For the sake of simplicity, we denote

$$f_1(t,x) := a_1(\mathbf{v}(t,x), x, x_s(\mathbf{v}(\cdot, x_0))(t)). \tag{A.18}$$

It is easy to check from (A.10) that if ν is sufficiently small, then $f_1(t,x)$ is strictly positive for any $(t,x) \in [0,T] \times [0,x_0]$. Let us now define the characteristic curve $\xi_1(s;t,x)$ passing through (t,x) as

$$\begin{aligned} \frac{d\xi_1(s;t,x)}{ds} &= f_1(s, \xi_1(s;t,x)), \\ \xi_1(t;t,x) &= x. \end{aligned} \tag{A.19}$$

One can see that for every $(t,x) \in [0,T] \times [0,x_0]$, $\xi_1(\cdot;t,x)$ is uniquely defined on some closed interval in $[0,T]$. From (A.1), only two cases can occur (see Fig. 2): If $\xi_1(t;0,0) < x \le x_0$, there exists $\beta_1 \in [0,x_0]$ depending on (t,x) such that

$$\beta_1 = \xi_1(0;t,x). \tag{A.20}$$

If $0 < x < \xi_1(t;0,0)$, there exists $\alpha_1 \in [0,t]$ depending on (t,x) such that

$$\xi_1(\alpha_1;t,x) = 0, \tag{A.21}$$

and in this case, there exists $\gamma_1 \in [0,x_0]$ depending on α_1 such that

$$\gamma_1 = \xi_1(0;\alpha_1,x_0). \tag{A.22}$$

Moreover, we have the following lemma which will be used in the estimations hereafter (the proof can be found at the end of this appendix).

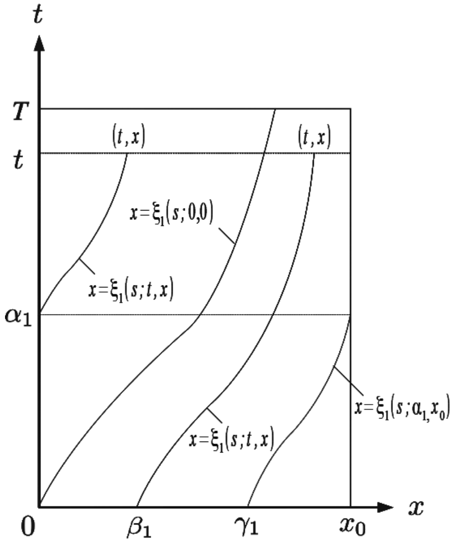

Fig. 2. Demonstration of the characteristics.

Lemma A.1. *There exist $\nu_0 > 0$ and $C > 0$ such that, for any T satisfying* (A.1), *for any $\nu \in (0, \nu_0]$ and for a.e. $t \in (0, T)$, we have*

$$|f_1(t,\cdot)|_0 \le C, \quad |f_{1x}(t,\cdot)|_0 \le C\nu, \quad |f_{1t}(t,\cdot)|_0 \le C\nu, \tag{A.23}$$

$$|\partial_x \xi_1(s;t,\cdot)|_0 \le C, \quad |\partial_t \xi_1(s;t,\cdot)|_0 \le C, \ \ s \in [0,t], \tag{A.24}$$

$$|\partial_x \beta_1(t,\cdot)|_0 \le C, \quad |(\partial_x \beta_1(t,\cdot))^{-1}|_0 \le C, \tag{A.25}$$

$$|\partial_t \beta_1(t,\cdot)|_0 \le C, \quad |(\partial_t \beta_1(t,\cdot))^{-1}|_0 \le C, \tag{A.26}$$

$$|\partial_x \alpha_1(t,\cdot)|_0 \le C, \quad |(\partial_x \alpha_1(t,\cdot))^{-1}|_0 \le C, \tag{A.27}$$

$$|\partial_x \gamma_1(t,\cdot)|_0 \le C, \quad |(\partial_x \gamma_1(t,\cdot))^{-1}|_0 \le C, \tag{A.28}$$

$$\int_0^T |\partial_{tt} \beta_1(t,x_0)|^2 \, dt \le C\nu, \tag{A.29}$$

$$\int_0^{x_0} |\partial_{xx} \alpha_1(t,x)|^2 dx \le C\nu, \tag{A.30}$$

$$\int_0^{x_0} |\partial_{xx} \beta_1(t,x)|^2 dx \le C\nu, \tag{A.31}$$

$$\int_0^{x_0} |\partial_{xx} \gamma_1(t,x)|^2 dx \le C\nu. \tag{A.32}$$

In these inequalities, and hereafter in this section, $|f|_0$ denotes the C^0-norm of a function f with respect to its variable and C may depend on x_0, x_{s0}, ν_0, k_1, k_2, b_1 and b_2, but is independent of ν, T, $\mathbf{v}$ and $\mathbf{z}$.

Our goal is now to use a fixed point argument to show the existence and uniqueness of the solution to (3.3)–(3.5). First, we show that for ν and δ sufficiently small, $\mathcal{F}$ maps $C_\nu(\mathbf{z}^0, x_{s0})$ into itself, i.e.

$$\mathcal{F}(C_\nu(\mathbf{z}^0, x_{s0})) \subset C_\nu(\mathbf{z}^0, x_{s0}).$$

Then, in a second step, we prove that $\mathcal{F}$ is a contraction mapping.

(1) $\mathcal{F}$ *maps* $C_\nu(\mathbf{z}^0, x_{s0})$ *into itself.*
For any $\mathbf{v} \in C_\nu(\mathbf{z}^0, x_{s0})$, let $\mathbf{z} = \mathcal{F}(\mathbf{v})$, we prove that $\mathbf{z} \in C_\nu(\mathbf{z}^0, x_{s0})$. By the definition of $\mathcal{F}$ in (A.12), using the method of characteristics, we can solve (A.13)–(A.16) for z_1 and obtain that

$$z_1(t,x) = \begin{cases} k_1 z_1^0(\gamma_1) + b_1\psi(\alpha_1), & 0 < x < \xi_1(t;0,0), \\ z_1^0(\beta_1), & \xi_1(t;0,0) < x < x_0. \end{cases} \tag{A.33}$$

Obviously $\mathbf{z}$ verifies the properties (A.6)–(A.7). Next, we prove that $\mathbf{z}$ verifies the property (A.5). Using the change of variables and from (A.26), we have

$$\int_0^T z_1(t,x_0)^2 dt = \int_0^T z_1^0(\beta_1(t,x_0))^2 dt \le C \int_0^{x_0} (z_1^0(x))^2 dx. \tag{A.34}$$

In (A.34) and hereafter, C denotes various constants that may depend on x_0, x_{s0}, ν_0, k_1, k_2, b_1 and b_2, but are independent of ν, T, $\mathbf{v}$ and $\mathbf{z}$. Similarly, by (A.26), we obtain

$$\int_0^T z_{1t}(t,x_0)^2 dt = \int_0^T (z_{1x}^0(\beta_1(t,x_0))\partial_t\beta_1(t,x_0))^2 dt \le C \int_0^T (z_{1x}^0(x))^2 dx. \tag{A.35}$$

From (A.29) and using Sobolev inequality, one has

$$\begin{aligned} \int_0^T z_{1tt}(t,x_0)^2 dt &= \int_0^T (z_{1xx}^0(\beta_1(t,x_0))(\partial_t\beta_1(t,x_0))^2 \\ &\qquad + z_{1x}^0(\beta_1(t,x_0))\partial_{tt}\beta_1(t,x_0))^2 dt \\ &\le C \int_0^{x_0} (z_{1xx}^0(x))^2 dx + 2|z_{1x}^0|_0^2 \int_0^T (\partial_{tt}\beta_1(t,x_0))^2\, dt \\ &\le C|z_1^0|^2_{H^2((0,x_0);\mathbb{R})}. \end{aligned} \tag{A.36}$$

Combining (A.34)–(A.36), we get

$$|z_1(\cdot,x_0)|_{H^2((0,T);\mathbb{R})} \le C|z_1^0|_{H^2((0,x_0);\mathbb{R})}. \tag{A.37}$$

Applying similar estimate to z_2 gives

$$|z_2(\cdot,x_0)|_{H^2((0,T);\mathbb{R})} \le C|z_2^0|_{H^2((0,x_0);\mathbb{R})}. \tag{A.38}$$

From (A.37) and (A.38), we can select δ sufficiently small such that

$$|\mathbf{z}(\cdot,x_0)|_{H^2((0,T);\mathbb{R}^2)} \le \nu^2, \tag{A.39}$$

which shows both the regularity and the boundedness property (A.5). We can again use the method of characteristics to prove properties (A.2)–(A.4). For a.e. $t \in (0, T)$,

$$z_{1x}(t,x) = \begin{cases} k_1 z^0_{1x}(\gamma_1)\partial_x\gamma_1 + b_1\dot{\psi}(\alpha_1)\partial_x\alpha_1, & 0 < x < \xi_1(t;0,0), \\ z^0_{1x}(\beta_1)\partial_x\beta_1, & \xi_1(t;0,0) < x < x_0. \end{cases} \tag{A.40}$$

$$z_{1xx}(t,x) = \begin{cases} k_1 z^0_{1x}(\gamma_1)\partial_{xx}\gamma_1 + k_1 z^0_{1xx}(\gamma_1)(\partial_x\gamma_1)^2 & \\ \qquad + b_1\ddot{\psi}(\alpha_1)(\partial_x\alpha_1)^2 + b_1\dot{\psi}(\alpha_1)\partial_{xx}\alpha_1, & \underbrace{0 < x < \xi_1(t;0,0)}, \\ z^0_{1x}(\beta_1)\partial_{xx}\beta_1 + z^0_{1xx}(\beta_1)(\partial_x\beta_1)^2, & \xi_1(t;0,0) < x < x_0. \end{cases} \tag{A.41}$$

Note that the last equation is true in distribution sense but shows that $z_1 \in L^\infty((0,T); H^2((0,x_0);\mathbb{R}))$. We first estimate $|\mathbf{z}|_{L^\infty((0,T);H^2((0,x_0);\mathbb{R}^2))}$. From (A.8) and (A.17), using Sobolev inequality, we get

$$|\psi|_0 \le |x_{s0} - x_0| + C|\mathbf{v}(\cdot, x_0)|_{H^2((0,T);\mathbb{R}^2)}, \tag{A.42}$$

$$|\dot{\psi}|_0 \le C|\mathbf{v}(\cdot, x_0)|_{H^2((0,T);\mathbb{R}^2)}, \tag{A.43}$$

$$|\ddot{\psi}|_0 \le C|\mathbf{v}(\cdot, x_0)|_{H^2((0,T);\mathbb{R}^2)}. \tag{A.44}$$

From (A.33), (A.40) and (A.41), we can compute directly using (A.25), (A.27)–(A.28) and (A.30)–(A.32) that

$$\begin{aligned} \int_0^{x_0} z_1^2 dx &\le (|(\partial_x\beta_1(t,\cdot))^{-1}|_0 + 2k_1^2|(\partial_x\gamma_1(t,\cdot))^{-1}|_0)\int_0^{x_0}(z_1^0(x))^2dx + 2b_1^2x_0|\psi|_0^2, \\ &\le C(|z_1^0|^2_{H^2((0,x_0);\mathbb{R})} + |x_{s0}-x_0|^2 + |\mathbf{v}(\cdot,x_0)|^2_{H^2((0,T);\mathbb{R}^2)}), \end{aligned} \tag{A.45}$$

$$\begin{aligned} \int_0^{x_0} z_{1x}^2 dx &\le (|\partial_x\beta_1(t,\cdot)|_0 + 2k_1^2|\partial_x\gamma_1(t,\cdot)|_0)\int_0^{x_0}(z^0_{1x}(x))^2dx \\ &\quad + 2x_0b_1^2|\dot{\psi}|_0^2|\partial_x\alpha_1(t,\cdot)|_0^2 \\ &\le C(|z_1^0|^2_{H^2((0,x_0);\mathbb{R})} + |\mathbf{v}(\cdot,x_0)|^2_{H^2((0,T);\mathbb{R}^2)}), \end{aligned} \tag{A.46}$$

$$\begin{aligned} \int_0^{x_0} z_{1xx}^2 dx &\le (2|\partial_x\beta_1(t,\cdot)|_0^3 + 4k_1^2|\partial_x\gamma_1(t,\cdot)|_0^3)\int_0^{x_0}(z^0_{1xx})^2dx \\ &\quad + 2|z^0_{1x}|_0^2\int_0^{x_0}|\partial_{xx}\beta_1(t,x)|^2dx + 4k_1^2|z^0_{1x}|_0^2\int_0^{x_0}|\partial_{xx}\gamma_1(t,x)|^2dx \\ &\quad + 4b_1^2|\partial_x\alpha_1(t,x)|_0^4\int_0^{x_0}|\ddot{\psi}(\alpha_1(t,x))|^2dx + 4b_1^2|\dot{\psi}|_0^2\int_0^{x_0}|\partial_{xx}\alpha_1(t,x)|^2dx \\ &\le C(|z_1^0|^2_{H^2((0,x_0);\mathbb{R})} + |\mathbf{v}(\cdot,x_0)|^2_{H^2((0,T);\mathbb{R}^2)}). \end{aligned} \tag{A.47}$$

Combining (A.45)–(A.47), we obtain

$$\begin{aligned} &|z_1(t,\cdot)|_{H^2((0,x_0);\mathbb{R})} \\ &\qquad \le C(|z_1^0|_{H^2((0,x_0);\mathbb{R})} + |x_{s0}-x_0| + |\mathbf{v}(\cdot,x_0)|_{H^2((0,T);\mathbb{R}^2)}). \end{aligned} \tag{A.48}$$

Similarly, one can get

$$
\begin{aligned}
&|z_2(t,\cdot)|_{H^2((0,x_0);\mathbb{R})}\\
&\qquad \le C(|z_2^0|_{H^2((0,x_0);\mathbb{R})} + |x_{s0}-x_0| + |\mathbf{v}(\cdot,x_0)|_{H^2((0,T);\mathbb{R}^2)}).
\end{aligned} \tag{A.49}
$$

Noticing from $\mathbf{v}\in C_\nu(\mathbf{z}^0,x_{s0})$ that

$$
|\mathbf{v}(\cdot,x_0)|_{H^2((0,T);\mathbb{R}^2)} \le \nu^2,
$$

thus by selecting δ and $\nu\in(0,\nu_0]$ sufficiently small, in addition to the previous hypothesis on δ, we have indeed

$$
|\mathbf{z}(t,\cdot)|_{H^2((0,x_0);\mathbb{R}^2)} \le \nu \quad \text{a.e. } t\in(0,T), \tag{A.50}
$$

which proves (A.2). The same method as to prove (A.50) enables us to show that z_1 verifies also (A.3) and (A.4). One only has to realize that

$$
z_{1t}(t,x) = \begin{cases} k_1 z_{1x}^0(\gamma_1)\partial_t\gamma_1 + b_1\dot{\psi}(\alpha_1)\partial_t\alpha_1, & 0<x<\xi_1(t;0,0),\\ z_{1x}^0(\beta_1)\partial_t\beta_1, & \xi_1(t;0,0)<x<x_0, \end{cases}
$$

$$
z_{1tt}(t,x) = \begin{cases} k_1 z_{1x}^0(\gamma_1)\partial_{tt}\gamma + k_1 z_{1xx}^0(\gamma_1)(\partial_t\gamma_1)^2 & \\ \qquad + b_1\ddot{\psi}(\alpha_1)(\partial_t\alpha_1)^2 + b_1\dot{\psi}(\alpha_1)\partial_{tt}\alpha_1, & \underbrace{0<x<\xi_1(t;0,0)},\\ z_{1x}^0(\beta_1)\partial_{tt}\beta_1 + z_{1xx}^0(\beta_1)(\partial_t\beta_1)^2, & \xi_1(t;0,0)<x<x_0, \end{cases}
$$

$$
z_{1tx}(t,x) = \begin{cases} k_1 z_{1x}^0(\gamma_1)\partial_x(\partial_t\gamma_1) + k_1 z_{1xx}^0(\gamma_1)(\partial_x\gamma_1\partial_t\gamma_1) & \\ \qquad + b_1\ddot{\psi}(\alpha_1)(\partial_t\alpha_1\partial_x\alpha_1) + b_1\dot{\psi}(\alpha_1)\partial_x(\partial_t\alpha_1), & \underbrace{0<x<\xi_1(t;0,0)},\\ z_{1x}^0(\beta_1)\partial_x(\partial_t\beta_1) + z_{1xx}^0(\beta_1)(\partial_x\beta_1\partial_t\beta_1), & \xi_1(t;0,0)<x<x_0, \end{cases}
$$

and to estimate $\int_0^{\xi_1(t;0,0)}|\partial_{tt}\alpha_1|^2dx$, $\int_{\xi_1(t;0,0)}^{x_0}|\partial_{tt}\beta_1|^2dx$, $\int_0^{\xi_1(t;0,0)}|\partial_{tt}\gamma_1|^2dx$, $\int_0^{\xi_1(t;0,0)}|\partial_x(\partial_t\alpha_1)|^2dx$, $\int_{\xi_1(t,0,0)}^{x_0}|\partial_x(\partial_t\beta_1)|^2dx$ and $\int_0^{\xi_1(t;0,0)}|\partial_x(\partial_t\gamma_1)|^2dx$ similarly as in (A.30)–(A.32) using the fact that $\mathbf{v}$ belongs to $L^\infty((0,T);H^2((0,x_0);\mathbb{R}^2))\cap W^{1,\infty}((0,T);H^1((0,x_0);\mathbb{R}^2))\cap W^{2,\infty}((0,T);L^2((0,x_0);\mathbb{R}^2))$ with bound ν in these norms.

We can clearly perform similar estimates for z_2. Consequently there exist δ and $\nu_1\in(0,\nu_0]$ sufficiently small depending only on C such that, for any $\nu\in(0,\nu_1]$, $\mathbf{z}=\mathcal{F}(\mathbf{v})$ verifies properties (A.2)–(A.7) and therefore $\mathcal{F}(C_\nu(\mathbf{z}^0,x_{s0}))\subset C_\nu(\mathbf{z}^0,x_{s0})$.

(2) $\mathcal{F}$ *is a contraction mapping.*
Next, we prove that $\mathcal{F}$ is a contraction mapping satisfying the following inequality:

$$
\begin{aligned}
&|\mathcal{F}(\mathbf{v})-\mathcal{F}(\bar{\mathbf{v}})|_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}^2))} + M|\mathcal{F}(\mathbf{v})(\cdot,x_0)-\mathcal{F}(\bar{\mathbf{v}})(\cdot,x_0)|_{L^2((0,T);\mathbb{R}^2)}\\
&\quad \le \frac{1}{2}|\mathbf{v}-\bar{\mathbf{v}}|_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}^2))} + \frac{M}{2}|\mathbf{v}(\cdot,x_0)-\bar{\mathbf{v}}(\cdot,x_0)|_{L^2((0,T);\mathbb{R}^2)},
\end{aligned} \tag{A.51}
$$

where $M > 0$ is a constant. We start with z_1, and with the estimate of $|z_1 - \bar{z}_1|_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}))}$. For any chosen $\mathbf{v}$ and $\bar{\mathbf{v}}$ from $C_\nu(\mathbf{z}^0, x_{s0})$, without loss of generality, we may assume that $\xi_1(t;0,0) < \bar{\xi}_1(t;0,0)$, where $\bar{\xi}_1$ is the characteristic defined in (A.19) associated to $\bar{\mathbf{v}}$. From (A.33), we have

$$\begin{aligned}
&\int_0^{x_0} |z_1(t,x) - \bar{z}_1(t,x)|^2\,dx \\
&\quad = \int_0^{\xi_1(t;0,0)} |k_1 z_1^0(\gamma_1) - k_1 z_1^0(\bar{\gamma}_1) + b_1\psi(\alpha_1) - b_1\bar{\psi}(\bar{\alpha}_1)|^2\,dx \\
&\qquad + \int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |z_1^0(\beta_1) - (k_1 z_1^0(\bar{\gamma}_1) + b_1\bar{\psi}(\bar{\alpha}_1))|^2\,dx \\
&\qquad + \int_{\bar{\xi}_1(t;0,0)}^{x_0} |z_1^0(\beta_1) - z_1^0(\bar{\beta}_1)|^2\,dx. \qquad (A.52)
\end{aligned}$$

From the definition of ψ in (A.17) and (A.8), using Sobolev and Cauchy–Schwarz inequalities, we have

$$\begin{aligned}
&\int_0^{\xi_1(t;0,0)} |b_1\psi(\alpha_1) - b_1\bar{\psi}(\bar{\alpha}_1)|^2\,dx \\
&\quad = \int_0^{\xi_1(t;0,0)} b_1^2 \left| \int_0^{\alpha_1} \frac{v_1(s,x_0) + v_2(s,x_0)}{2}\,ds - \int_0^{\bar{\alpha}_1} \frac{\bar{v}_1(s,x_0) + \bar{v}_2(s,x_0)}{2}\,ds \right|^2 dx \\
&\quad \le C|\mathbf{v}(\cdot,x_0) - \bar{\mathbf{v}}(\cdot,x_0)|^2_{L^2((0,T);\mathbb{R}^2)} + C|\bar{\mathbf{v}}(\cdot,x_0)|^2_{H^2((0,T);\mathbb{R}^2)} \\
&\qquad \times \int_0^{\xi_1(t;0,0)} |\alpha_1 - \bar{\alpha}_1|^2\,dx. \qquad (A.53)
\end{aligned}$$

By the definition of γ_1 in (A.22) and the corresponding definition of $\bar{\gamma}_1$ and using (A.24), we obtain

$$\int_0^{\xi_1(t;0,0)} |k_1 z_1^0(\gamma_1) - k_1 z_1^0(\bar{\gamma}_1)|^2\,dx \le C|z_1^0|^2_{H^2((0,x_0);\mathbb{R})} \int_0^{\xi_1(t;0,0)} |\alpha_1 - \bar{\alpha}_1|^2\,dx. \qquad (A.54)$$

Combining (A.52)–(A.54), we get

$$\begin{aligned}
&\int_0^{x_0} |z_1(t,x) - \bar{z}_1(t,x)|^2\,dx \\
&\quad \le C(|z_1^0|^2_{H^2((0,x_0);\mathbb{R})} + |\bar{\mathbf{v}}(\cdot,x_0)|^2_{H^2((0,T);\mathbb{R}^2)}) \int_0^{\xi_1(t;0,0)} |\alpha_1 - \bar{\alpha}_1|^2\,dx \\
&\qquad + |z_1^0|^2_{H^2((0,x_0);\mathbb{R})} \int_{\bar{\xi}_1(t;0,0)}^{x_0} |\beta_1 - \bar{\beta}_1|^2\,dx
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |z_1^0(\beta_1)-(k_1 z_1^0(\bar{\gamma}_1)+b_1\bar{\psi}(\bar{\alpha}_1))|^2\,dx \\
&+C|\mathbf{v}(\cdot,x_0)-\bar{\mathbf{v}}(\cdot,x_0)|^2_{L^2((0,T);\mathbb{R}^2)}.
\end{aligned}
\tag{A.55}
$$

We estimate each term in (A.55) separately. By the definition of β_1 in (A.20) and the corresponding definition of $\bar{\beta}_1$, we have

$$
\int_{\bar{\xi}_1(t;0,0)}^{x_0} |\beta_1-\bar{\beta}_1|^2\,dx=\int_{\bar{\xi}_1(t;0,0)}^{x_0} |\xi_1(0;t,x)-\bar{\xi}_1(0;t,x)|^2\,dx. \tag{A.56}
$$

Now, let us estimate $|\xi_1(0;t,x)-\bar{\xi}_1(0;t,x)|$. From the definition of x_s in (A.8) and the definitions of ξ_1 and $\bar{\xi}_1$, see (A.19), we get for any $s\in[0,t]$ that

$$
\begin{aligned}
&|\xi_1(s;t,x)-\bar{\xi}_1(s;t,x)| \\
&\quad=\left|\int_s^t f_1(\theta,\xi_1(\theta;t,x))\,d\theta-\int_s^t \bar{f}_1(\theta,\bar{\xi}_1(\theta;t,x))\,d\theta\right| \\
&\quad\le\int_s^t \left(\left|\left(1+v_1(\theta,\xi_1)-\xi_1\frac{v_1(\theta,x_0)+v_2(\theta,x_0)}{2x_0}\right)\right.\right. \\
&\qquad\left.\times\ \frac{x_0}{x_s(\mathbf{v}(\cdot,x_0))(\theta)x_s(\bar{\mathbf{v}}(\cdot,x_0))(\theta)}\right| \\
&\qquad\left.\cdot\int_0^\theta\left|\frac{v_1(\alpha,x_0)-\bar{v}_1(\alpha,x_0)+v_2(\alpha,x_0)-\bar{v}_2(\alpha,x_0)}{2}\right|\,d\alpha\right)\,d\theta \\
&\qquad+\int_s^t\left|\frac{x_0}{x_s(\bar{\mathbf{v}}(\cdot,x_0))(\theta)}\right|\cdot\left|v_1(\theta,\xi_1)-\bar{v}_1(\theta,\bar{\xi}_1)+\bar{\xi}_1\frac{\bar{v}_1(\theta,x_0)+\bar{v}_2(\theta,x_0)}{2x_0}\right. \\
&\qquad\left.-\xi_1\frac{v_1(\theta,x_0)+v_2(\theta,x_0)}{2x_0}\right|\,d\theta \\
&\quad\le C|\mathbf{v}(\cdot,x_0)-\bar{\mathbf{v}}(\cdot,x_0)|_{L^2((0,T);\mathbb{R}^2)}+C\nu\int_s^t|\xi_1(\theta;t,x)-\bar{\xi}_1(\theta;t,x)|\,d\theta \\
&\qquad+C\int_s^t|v_1(\theta,\bar{\xi}_1(\theta;t,x))-\bar{v}_1(\theta,\bar{\xi}_1(\theta;t,x))|d\theta.
\end{aligned}
\tag{A.57}
$$

From (A.57), we get for $\nu\in(0,\nu_0]$ sufficiently small and for $\bar{\xi}_1(t;0,0)<x\le x_0$ that

$$
\begin{aligned}
&|\xi_1(\cdot;t,x)-\bar{\xi}_1(\cdot;t,x)|_{C^0([0,t];\mathbb{R})} \\
&\qquad\le C|\mathbf{v}(\cdot,x_0)-\bar{\mathbf{v}}(\cdot,x_0)|_{L^2((0,T);\mathbb{R}^2)}+C\int_0^t|v_1(\theta,\bar{\xi}_1(\theta;t,x)) \\
&\qquad\quad-\bar{v}_1(\theta,\bar{\xi}_1(\theta;t,x))|\,d\theta.
\end{aligned}
\tag{A.58}
$$

Thus, from (A.56) and (A.58) we have

$$\begin{aligned}
&\int_{\bar{\xi}_1(t;0,0)}^{x_0} |\beta_1 - \bar{\beta}_1|^2 \, dx \\
&\quad \leq C|\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|^2_{L^2((0,T);\mathbb{R}^2)} + C \int_{\bar{\xi}_1(t;0,0)}^{x_0} \left(\int_0^t |v_1(\theta, \bar{\xi}_1(\theta; t, x)) \right. \\
&\qquad \left. - \bar{v}_1(\theta, \bar{\xi}_1(\theta; t, x))| \, d\theta \int_0^t \right)^2 dx \\
&\quad \leq C|\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|^2_{L^2((0,T);\mathbb{R}^2)} \\
&\qquad + C \int_0^t \int_{\bar{\xi}_1(t;0,0)}^{x_0} |v_1(\theta, \bar{\xi}_1(\theta; t, x)) - \bar{v}_1(\theta, \bar{\xi}_1(\theta; t, x))|^2 \, dx \, d\theta \\
&\quad \leq C|\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|^2_{L^2((0,T);\mathbb{R}^2)} + C|v_1 - \bar{v}_1|^2_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}))}.
\end{aligned} \tag{A.59}$$

The last inequality is obtained using the change of variable $y = \bar{\xi}_1(\theta; t, x)$, well-defined for $0 \leq \theta \leq t \leq T$ and $\bar{\xi}_1(t; 0, 0) < x \leq x_0$. Let us now estimate $|\alpha_1 - \bar{\alpha}_1|_{L^2((0,\xi_1(t;0,0));\mathbb{R})}$. Without loss of generality, we may assume that $\alpha_1 \leq \bar{\alpha}_1$. By definition of α_1 in (A.21) and the corresponding definition of $\bar{\alpha}_1$, we have

$$\int_{\alpha_1}^t f_1(s, \xi_1(s; t, x)) ds = x = \int_{\bar{\alpha}_1}^t \bar{f}_1(s, \bar{\xi}_1(s; t, x)) ds. \tag{A.60}$$

Hence, similarly to (A.57), we get

$$\begin{aligned}
|\alpha_1 - \bar{\alpha}_1| &\leq \frac{1}{\inf\limits_{(t,x) \in [0,T] \times [0,x_0]} |f_1(t,x)|} \int_{\bar{\alpha}_1}^t |f_1(s, \xi_1(s; t, x)) - \bar{f}_1(s, \bar{\xi}_1(s; t, x))| ds \\
&\leq C|\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|_{L^2((0,T);\mathbb{R}^2)} + C\nu \int_{\bar{\alpha}_1}^t |\xi_1(\theta; t, x) - \bar{\xi}_1(\theta; t, x)| \, d\theta \\
&\quad + C \int_{\bar{\alpha}_1}^t |v_1(\theta, \bar{\xi}_1(\theta; t, x)) - \bar{v}_1(\theta, \bar{\xi}_1(\theta; t, x))| d\theta.
\end{aligned} \tag{A.61}$$

Similarly to the proof of (A.58), for $\nu \in (0, \nu_0]$ sufficiently small, we can obtain that (note that $\xi_1(s; t, x)$ and $\bar{\xi}_1(s; t, x)$ for any $s \in [\bar{\alpha}_1, t]$ are well defined as we assume that $\alpha_1 \leq \bar{\alpha}_1$)

$$\begin{aligned}
|\xi_1(\cdot; t, x) - \bar{\xi}_1(\cdot; t, x)|_{C^0([\bar{\alpha}_1,t];\mathbb{R})} &\leq C|\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|_{L^2((0,T);\mathbb{R}^2)} \\
&\quad + C \int_{\bar{\alpha}}^t |v_1(\theta, \bar{\xi}_1(\theta; t, x)) - \bar{v}_1(\theta, \bar{\xi}_1(\theta; t, x))| \, d\theta.
\end{aligned} \tag{A.62}$$

Using this inequality in (A.61) and performing similarly as in (A.59), we can obtain

$$\int_0^{\xi_1(t;0,0)} |\alpha_1 - \bar{\alpha}_1|^2\, dx \leq C|\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|^2_{L^2((0,T);\mathbb{R}^2)} + C|v_1 - \bar{v}_1|^2_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}))}. \tag{A.63}$$

Let us now focus on the estimation of the term $\int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |z_1^0(\beta_1) - (k_1 z_1^0(\bar{\gamma}_1) + b_1\bar{\psi}(\bar{\alpha}_1))|^2\, dx$ in (A.55). Using the compatibility condition (3.7), we have

$$\begin{aligned}
&\int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |z_1^0(\beta_1) - (k_1 z_1^0(\bar{\gamma}_1) + b_1\bar{\psi}(\bar{\alpha}_1))|^2\, dx \\
&= \int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |z_1^0(\beta_1) - z_1^0(0) + z_1^0(0) - (k_1 z_1^0(\bar{\gamma}_1) + b_1\bar{\psi}(\bar{\alpha}_1))|^2\, dx \\
&= \int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |z_1^0(\beta_1) - z_1^0(0) + k_1 z_1^0(x_0) + b_1(x_0 - x_{s0}) \\
&\quad - (k_1 z_1^0(\bar{\gamma}_1) + b_1\bar{\psi}(\bar{\alpha}_1))|^2\, dx \\
&\leq C|z_1^0|^2_{H^2((0,x_0);\mathbb{R})} \int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |\beta_1|^2\, dx + C|z_1^0|^2_{H^2((0,x_0);\mathbb{R})} \int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |x_0 - \bar{\gamma}_1|^2\, dx \\
&\quad + C\int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} \left|\int_0^{\bar{\alpha}_1} \frac{\bar{v}_1(s,x_0) + \bar{v}_2(s,x_0)}{2}\, ds\right|^2 dx.
\end{aligned} \tag{A.64}$$

We first estimate $\int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |\beta_1|^2\, dx$. As $\xi_1(s;t,x)$ is increasing with respect to $s \in [0,t]$, we have

$$\begin{aligned}
|\beta_1| < |\xi_1(\bar{\alpha}_1;t,x)| &= |\xi_1(\bar{\alpha}_1;t,x) - \bar{\xi}_1(\bar{\alpha}_1;t,x)| \\
&\leq |\xi_1(\cdot;t,x) - \bar{\xi}_1(\cdot;t,x)|_{C^0([\bar{\alpha}_1,t];\mathbb{R})},
\end{aligned} \tag{A.65}$$

then by (A.62) and performing the same proof as in (A.59), we get

$$\int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |\beta_1|^2\, dx \leq C|\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|^2_{L^2((0,T);\mathbb{R}^2)} + C|v_1 - \bar{v}_1|^2_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}))}. \tag{A.66}$$

Let us now look at the second term in (A.64), from (A.24) and the definition of $\bar{\gamma}_1$, we have

$$\begin{aligned}
\int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |x_0 - \bar{\gamma}_1|^2\, dx &= \int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |\bar{\xi}_1(0;0,x_0) - \bar{\xi}_1(0;\bar{\alpha}_1,x_0)|^2\, dx \\
&\leq |\partial_t\bar{\xi}_1|_0^2 \int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |\bar{\alpha}_1|^2\, dx \leq C\int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |\bar{\alpha}_1|^2\, dx.
\end{aligned} \tag{A.67}$$

It is easy to deal with the last term in (A.64), one has

$$\int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} \left| \int_0^{\bar{\alpha}_1} \frac{\bar{v}_1(s,x_0)+\bar{v}_2(s,x_0)}{2}\, ds \right|^2 dx \\ \leq C|\bar{\mathbf{v}}(\cdot,x_0)|^2_{H^2((0,T);\mathbb{R}^2)} \int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |\bar{\alpha}_1|^2\, dx. \tag{A.68}$$

Thus, we only have to estimate $\int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |\bar{\alpha}_1|^2\, dx$. Noticing that for any fixed (t,x), the characteristic $\bar{\xi}_1(s;t,x)$ is increasing with respect to $s \in [\bar{\alpha}_1, t]$ and that $\xi_1^{-1}(\cdot;t,x)(\beta_1) = 0$, we obtain

$$\bar{\alpha}_1 < \bar{\xi}_1^{-1}(\cdot;t,x)(\beta_1) - \xi_1^{-1}(\cdot;t,x)(\beta_1).$$

Moreover,

$$\beta_1 = x + \int_t^{\xi_1^{-1}(\cdot;t,x)(\beta_1)} f_1(s;\xi_1(s;t,x))\, d\theta,$$
$$\beta_1 = x + \int_t^{\bar{\xi}_1^{-1}(\cdot;t,x)(\beta_1)} \bar{f}_1(s;\bar{\xi}_1(s;t,x))\, d\theta.$$

Then similarly as for (A.61), we can prove that

$$|\bar{\xi}_1^{-1}(\cdot;t,x)(\beta_1) - \xi_1^{-1}(\cdot;t,x)(\beta_1)| \\ \leq C|\mathbf{v}(\cdot,x_0) - \bar{\mathbf{v}}(\cdot,x_0)|_{L^2(0,T)} + C\nu \int_{\bar{\xi}_1^{-1}(\cdot;t,x)(\beta_1)}^t |\xi_1(\theta;t,x) - \bar{\xi}_1(\theta;t,x)|\, d\theta \\ + C\int_{\bar{\xi}_1^{-1}(\cdot;t,x)(\beta_1)}^t |v_1(\theta,\bar{\xi}_1(\theta;t,x)) - \bar{v}_1(\theta,\bar{\xi}_1(\theta;t,x))| d\theta.$$

Thus, similarly as in the proof for (A.63), we get

$$\int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |\bar{\alpha}_1|^2\, dx \leq C|\mathbf{v}(\cdot,x_0) - \bar{\mathbf{v}}(\cdot,x_0)|^2_{L^2((0,T);\mathbb{R}^2)} \\ + C|v_1 - \bar{v}_1|^2_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}))}. \tag{A.69}$$

Finally, using estimations (A.66) and (A.67)–(A.69), (A.64) becomes

$$\int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |z_1^0(\beta_1) - (k_1 z_1^0(\bar{\gamma}_1) + b_1\psi(\bar{\alpha}_1))|^2\, dx \\ \leq C(|z_1^0|^2_{H^2((0,x_0);\mathbb{R})} + |\bar{\mathbf{v}}(\cdot,x_0)|^2_{H^2((0,T);\mathbb{R}^2)})(|\mathbf{v}(\cdot,x_0) - \bar{\mathbf{v}}(\cdot,x_0)|^2_{L^2((0,T);\mathbb{R}^2)} \\ + |v_1 - \bar{v}_1|^2_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}))}). \tag{A.70}$$

Combining (A.55), (A.59), (A.63) and (A.70), we get

$$\begin{aligned}
&|z_1 - \bar{z}_1|^2_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}))} \\
&\quad \le C(|z_1^0|^2_{H^2((0,x_0);\mathbb{R})} + |\bar{\mathbf{v}}(\cdot,x_0)|^2_{H^2((0,T);\mathbb{R}^2)}) \\
&\qquad \times (|\mathbf{v}(\cdot,x_0) - \bar{\mathbf{v}}(\cdot,x_0)|^2_{L^2((0,T);\mathbb{R}^2)} + |v_1 - \bar{v}_1|^2_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}))}) \\
&\qquad + C|\mathbf{v}(\cdot,x_0) - \bar{\mathbf{v}}(\cdot,x_0)|^2_{L^2((0,T);\mathbb{R}^2)}.
\end{aligned} \tag{A.71}$$

We are left with estimating $|\mathbf{z}(\cdot,x_0) - \bar{\mathbf{z}}(\cdot,x_0)|_{L^2((0,T);\mathbb{R}^2)}$ in order to obtain (A.51). Here we give the estimation for z_1. Using (A.58), we get

$$\begin{aligned}
&\int_0^T |z_1(t,x_0) - \bar{z}_1(t,x_0)|^2\, dt \\
&\quad = \int_0^T |z_1^0(\xi_1(0;t,x_0)) - z_1^0(\bar{\xi}_1(0;t,x_0))|^2 \\
&\quad \le |z_{1x}^0|_0^2 \int_0^T |\xi_1(0;t,x_0) - \bar{\xi}_1(0;t,x_0)|^2\, dt \\
&\quad \le C|z_1^0|^2_{H^2((0,x_0);\mathbb{R})}|\mathbf{v}(\cdot,x_0) - \bar{\mathbf{v}}(\cdot,x_0)|^2_{L^2((0,T);\mathbb{R}^2)} \\
&\qquad + C|z_1^0|^2_{H^2((0,x_0);\mathbb{R})} \int_0^T \int_0^t |v_1(\theta,\bar{\xi}_1(\theta;t,x_0)) - \bar{v}_1(\theta,\bar{\xi}_1(\theta;t,x_0))|^2\, d\theta\, dt \\
&\quad \le C|z_1^0|^2_{H^2((0,x_0);\mathbb{R})}|\mathbf{v}(\cdot,x_0) - \bar{\mathbf{v}}(\cdot,x_0)|^2_{L^2((0,T);\mathbb{R}^2)} \\
&\qquad + C|z_1^0|^2_{H^2((0,x_0);\mathbb{R})} \int_0^T \int_\theta^T |v_1(\theta,\bar{\xi}_1(\theta;t,x_0)) - \bar{v}_1(\theta,\bar{\xi}_1(\theta;t,x_0))|^2\, dt\, d\theta \\
&\quad \le C|z_1^0|^2_{H^2((0,x_0);\mathbb{R})}(|\mathbf{v}(\cdot,x_0) - \bar{\mathbf{v}}(\cdot,x_0)|^2_{L^2((0,T);\mathbb{R}^2)} \\
&\qquad + |v_1 - \bar{v}_1|^2_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}))}).
\end{aligned} \tag{A.72}$$

The last inequality is obtained by changing the variable $y = \bar{\xi}_1(\theta;t,x_0)$. Similar estimates can be done for z_2. Hence, from (A.71) and (A.72), there exists $M > 0$ such that for δ sufficiently small and $\nu \in (0,\nu_2]$, where $\nu_2 \in (0,\nu_1]$ is sufficiently small and depends only on C, we have

$$\begin{aligned}
&|\mathbf{z} - \bar{\mathbf{z}}|_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}^2))} + M|\mathbf{z}(\cdot,x_0) - \bar{\mathbf{z}}(\cdot,x_0)|_{L^2((0,T);\mathbb{R}^2)} \\
&\quad \le \frac{1}{2}|\mathbf{v} - \bar{\mathbf{v}}|_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}^2))} + \frac{M}{2}|\mathbf{v}(\cdot,x_0) - \bar{\mathbf{v}}(\cdot,x_0)|_{L^2((0,T);\mathbb{R}^2)}.
\end{aligned} \tag{A.73}$$

Hence $\mathcal{F}$ is a contraction mapping and has a fixed point $\mathbf{z} \in C_\nu(\mathbf{z}^0, x_{s0})$, i.e. there exists a unique solution $\mathbf{z} \in C_\nu(\mathbf{z}^0, x_{s0})$ to the system (3.3)–(3.5). Noticing (A.8), we get that $x_s \in C^1([0,T];\mathbb{R})$. To get the extra regularity $\mathbf{z} \in C^0([0,T]; H^2((0,x_0);\mathbb{R}^2))$, we adapt the proof given by Majda (see pp. 44–46 of Ref. 33). There, the author used energy estimates method for an initial value problem. Using this method for our boundary value problem, we have to be careful with the boundary terms when integrating by parts. Substituting $\mathbf{v}$ by $\mathbf{z}$ in $\psi(t)$ and $f_1(t,x)$ in the expression of z_{1x}, z_{1xx} in (A.40) and (A.41), noticing (A.2)–(A.4) and computing similar estimates as in (A.46) and (A.47), we can obtain the "hidden" regularity $\mathbf{z}_x(\cdot,x_0) \in L^2((0,T);\mathbb{R}^2)$ and $\mathbf{z}_{xx}(\cdot,x_0) \in L^2((0,T);\mathbb{R}^2)$ together with estimates on $|\mathbf{z}_x(\cdot,x_0)|_{L^2((0,T);\mathbb{R}^2)}$ and $|\mathbf{z}_{xx}(\cdot,x_0)|_{L^2((0,T);\mathbb{R}^2)}$, which are sufficient to take care of the boundary terms when integrating by parts. This concludes the proof of the existence and uniqueness of a classical solution $x_s(t) \in C^1([0,T];\mathbb{R})$ and $\mathbf{z} \in C^0([0,T];H^2((0,x_0);\mathbb{R}^2))$ in $C_\nu(\mathbf{z}^0,x_{s0})$ to the system (3.3)–(3.5) for T satisfying (A.1).

The estimate (3.10) for $|\mathbf{z}(t,\cdot)|_{H^2((0,x_0);\mathbb{R}^2)}$ part can be obtained from estimates (A.48)–(A.49) by first replacing $\mathbf{v}$ with $\mathbf{z}$ and then applying (A.37)–(A.38). Noticing the definition of x_s in (A.8) and applying (A.37)–(A.38) again, the estimate for the $|x_s(t)-x_0|$ part follows.

Next, we show the uniqueness of the solution in $C^0([0,T];H^2((0,x_0);\mathbb{R}^2))$. Suppose that there is another solution $\tilde{\mathbf{z}} \in C^0([0,T];H^2((0,x_0);\mathbb{R}^2))$, we prove that $\tilde{\mathbf{z}} \in C_\nu(\mathbf{z}^0,x_{s0})$, for δ sufficiently small. To that end, assume that $\mathbf{z}(t,\cdot)=\tilde{\mathbf{z}}(t,\cdot)$ for any $t\in[0,\tau]$ with $\tau\in[0,T]$. If $\tau\neq T$, by (3.10), for δ sufficiently small and as $\tilde{\mathbf{z}} \in C^0([0,T];H^2((0,x_0);\mathbb{R}^2))$, one can choose $\tau'\in(\tau,T)$ small enough such that $\tilde{\mathbf{z}} \in C_\nu(\mathbf{z}(\tau),x_s(\tau))$ with T is replaced by $\tau'-\tau$ and by considering τ as the new initial time. Thus, $\mathbf{z}(t,\cdot)=\tilde{\mathbf{z}}(t,\cdot)$ for any $t\in[0,\tau']$. As $|\tilde{\mathbf{z}}(t,\cdot)|_{H^2((0,x_0);\mathbb{R}^2)}$ is uniformly continuous on $[0,T]$, and as, moreover C and ν do not depend on T, we can repeat this process and finally get $\mathbf{z}(t,\cdot)=\tilde{\mathbf{z}}(t,\cdot)$ on $[0,T]$.

For general $T>0$, one just needs to take T_1 satisfying (A.1) and, noticing that C and ν do not depend on T_1, one can apply the above procedure at most $[T/T_1]+1$ times. This concludes the proof of Lemma 3.1. □

Proof of Lemma A.1. From (A.19), we have

$$\begin{cases} \dfrac{\partial^2\xi_1(s;t,x)}{\partial s\partial x} = f_{1x}\dfrac{\partial\xi_1(s;t,x)}{\partial x}, \\ \dfrac{\partial\xi_1(t;t,x)}{\partial x} = 1, \end{cases} \tag{A.74}$$

and

$$\begin{cases} \dfrac{\partial^2\xi_1(s;t,x)}{\partial s\partial t} = f_{1x}\dfrac{\partial\xi_1(s;t,x)}{\partial t}, \\ \dfrac{\partial\xi_1(s;s,x)}{\partial s} + \dfrac{\partial\xi_1(s;s,x)}{\partial t} = 0. \end{cases} \tag{A.75}$$

Thus,

$$\partial_x\xi_1(s;t,x) = e^{-\int_s^t f_{1x}(\theta,\xi_1(\theta;t,x))d\theta}, \tag{A.76}$$

$$\partial_t\xi_1(s;t,x) = -f_1(t,x)e^{-\int_s^t f_{1x}(\theta,\xi_1(\theta;t,x))d\theta}. \tag{A.77}$$

From (A.76)–(A.77) and noticing $\beta_1 = \xi_1(0; t, x)$, we have

$$\frac{\partial \beta_1}{\partial t} = -f_1(t,x) e^{-\int_0^t f_{1x}(\theta,\xi_1(\theta;t,x)) d\theta}, \quad \frac{\partial \beta_1}{\partial x} = e^{-\int_0^t f_{1x}(\theta,\xi_1(\theta;t,x))\, d\theta}. \tag{A.78}$$

From (A.76), noticing $\xi_1(\alpha_1; t, x) = 0$ and by chain rules, we have

$$\frac{\partial \alpha_1}{\partial x} = -\frac{1}{f_1(\alpha_1, 0)} e^{-\int_{\alpha_1}^t f_{1x}(s,\xi_1(s;t,x))\, ds}, \tag{A.79}$$

and as $\gamma_1 = \xi_1(0; \alpha_1, x_0)$, we obtain from (A.77) that

$$\begin{aligned}\frac{\partial \gamma_1}{\partial x} &= \frac{d\gamma_1}{d\alpha_1}\frac{\partial \alpha_1}{\partial x} \\ &= \frac{f_1(\alpha_1, x_0)}{f_1(\alpha_1, 0)} e^{-\int_0^{\alpha_1} f_{1x}(s,\xi_1(s;\alpha_1,x_0))\, ds - \int_{\alpha_1}^t f_{1x}(s,\xi_1(s;t,x)) ds}.\end{aligned} \tag{A.80}$$

Observe that for a.e. $s \in (0, T)$ and $x \in [0, x_0]$,

$$|v_1(s,x)| \le \left| \int_\theta^x v_{1x}(s,l) dl \right| + |v_1(s,\theta)|, \quad \forall\, \theta \in [0, x_0] \tag{A.81}$$

and as v_1 is H^1 in x and its L^2-norm is bounded by ν, there exists θ such that $|v_1(s, \theta)| \le \nu/\sqrt{x_0}$, therefore

$$|v_1(s,x)| \le C\nu, \tag{A.82}$$

and similarly as v_1 is H^2 in x with the same bound and v_{1t} is in $L^\infty((0,T); H^1((0,x_0);\mathbb{R}))$ with bound ν from (A.3)

$$\begin{aligned} &x \in [0, x_0], \quad |v_{1x}(s,x)| \le C\nu \quad \text{for a.e. } s \in (0,T), \\ &x \in [0, x_0], \quad |v_{1t}(s,x)| \le C\nu \quad \text{for a.e. } s \in (0,T). \end{aligned} \tag{A.83}$$

From the expression of f_1 defined in (A.18) and using (A.76)–(A.80) and (A.82)–(A.83), after some direct computations, estimates (A.23)–(A.28) can be obtained. We now demonstrate the estimate (A.29) in detail, while (A.30)–(A.32) can be treated in a similar way, thus we omit them. From (A.78), we have

$$\begin{aligned}\partial_{tt}\beta_1 = &\left(-f_{1t}(t,x) + f_1(t,x) \left(f_{1x}(t,x) + \int_0^t f_{1xx}(\theta, \xi_1(\theta;t,x)) \partial_t \xi_1(\theta;t,x)\, d\theta \right) \right) \\ &\times e^{-\int_0^t f_{1x}(\theta,\xi_1(\theta;t,x))\, d\theta}.\end{aligned}$$

Looking at (A.18), as $\mathbf{v}$ is only in $L^\infty((0,T); H^2((0,x_0);\mathbb{R}^2)) \cap W^{1,\infty}((0,T); H^1((0,x_0);\mathbb{R}^2)) \cap W^{2,\infty}((0,T); L^2((0,x_0);\mathbb{R}^2))$, this equation is expressed *a priori* formally in the distribution sense. Thus, we have to be careful when we estimate (A.29). By (A.18) and using estimates (A.23), (A.24), we get by Cauchy–Schwarz inequality together with the change of variable $y = \xi_1(\theta; t, x_0)$ that

$$\int_0^T |\partial_{tt}\beta_1(t,x_0)|^2\, dt \le C\nu + C \int_0^T \left| \int_0^t f_{1xx}(\theta, \xi_1(\theta;t,x_0)) \partial_t \xi_1(\theta;t,x_0)\, d\theta \right|^2 dt$$

$$
\begin{aligned}
&\le C\nu + C\int_0^T \int_0^t v_{1xx}^2(\theta,\xi_1(\theta;t,x_0))\partial_t^2\xi_1(\theta;t,x_0)\,d\theta dt \\
&= C\nu + C\int_0^T \int_\theta^T v_{1xx}^2(\theta,\xi_1(\theta;t,x_0))\partial_t^2\xi_1(\theta;t,x_0)\,dt d\theta \\
&\le C\nu + C\int_0^T \int_0^{x_0} v_{1xx}^2(\theta,y)\,dy d\theta \\
&\le C\nu. \qquad \text{(A.84)}
\end{aligned}
$$

□

Appendix B. Proof of Theorem 5.1

First observe that, after the change of variables (3.1), (3.2), the new equations are

$$
\begin{aligned}
&z_{1t} + \left(f'(1) + (f'(z_1+1) - f'(1)) - x\frac{\dot{x}_s}{x_0}\right) z_{1x}\frac{x_0}{x_s} = 0,\\
&z_{2t} + \left(-f'(-1) + (f'(-1) - f'(z_2-1)) + x\frac{\dot{x}_s}{x_0}\right) z_{2x}\frac{x_0}{L-x_s} = 0,\\
&\dot{x}_s(t) = \frac{f'(1)z_1(t,x_0) - f'(-1)z_2(t,x_0)}{2 + (z_1(t,x_0) - z_2(t,x_0))}\\
&\qquad + \frac{(f(z_1(t,x_0)+1) - f'(1)z_1(t,x_0) - f(1)) - (f(z_2(t,x_0)-1) - f'(-1)z_2(t,x_0) - f(-1))}{2 + (z_1(t,x_0) - z_2(t,x_0))}
\end{aligned}
\tag{B.1}
$$

and the boundary conditions remain given by (3.4). Note that in (B.1) the expression of $\dot{x}_s$ can actually be written as

$$
\dot{x}_s(t) = \frac{f'(1)z_1(t,x_0) - f'(-1)z_2(t,x_0)}{2} + O(|\mathbf{z}(t,x_0)|^2). \tag{B.2}
$$

Thus, to prove Theorem 5.1, it suffices to show Theorem 4.1 with (B.1) instead of (3.3). We still define the Lyapunov function candidate as previously by (4.3)–(4.9). Then Lemmas 4.1 and 4.2 remain unchanged. To adapt Lemma 4.3, one can check that, when differentiating V_1, V_2 and V_3 along the C^3 solutions of (B.1), (3.4) with associated initial conditions and noticing that under assumption $f(-1) = f(1)$, one has $f'(-1) \le 0, f'(1) \ge 0$ from the property of convex function, we obtain as previously (4.40), (4.42) and (4.46) but with $f'(1)p_1$ instead of p_1 and $|f'(-1)|p_2$ instead of p_2 in the boundary terms and μV_i being replaced by $\mu\min(f'(1),|f'(-1)|)V_i$. Then, from (5.3) and dealing with V_4, we finally get

$$
\begin{aligned}
&\frac{dV_1}{dt} + \frac{dV_4}{dt}\\
&\quad \le -\mu(V_1+V_4) + \max\left\{\frac{\Theta_1}{\varepsilon_1}, \frac{\Theta_2}{\varepsilon_2}\right\} V_1
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\frac{x_0}{x_s}p_1(k_1^2-e^{-\frac{\mu x_0}{\eta_1}})f'(1)+\frac{\varepsilon_1+\varepsilon_2}{8}f'(1)^2\right]z_1^2(x_0)\\
&+\left[\frac{x_0}{L-x_s}p_2(k_2^2-e^{-\frac{\mu x_0}{\eta_2}})|f'(-1)|+\frac{\varepsilon_1+\varepsilon_2}{8}|f'(-1)|^2\right]z_2^2(x_0)\\
&+f'(1)\left[-2\frac{x_0}{x_s}p_1b_1k_1-\frac{x_0}{x_s}\bar{p}_1(e^{-\frac{\mu x_0}{\eta_1}}-k_1)+\kappa\right]z_1(x_0)(x_s-x_0)\\
&+|f'(-1)|\left[-2\frac{x_0}{L-x_s}p_2b_2k_2-\frac{x_0}{L-x_s}\bar{p}_2(e^{-\frac{\mu x_0}{\eta_2}}-k_2)+\kappa\right]z_2(x_0)(x_s-x_0)\\
&+\left[\frac{x_0}{x_s}p_1b_1^2f'(1)+\frac{x_0}{L-x_s}p_2b_2^2|f'(-1)|-\frac{x_0}{x_s}\bar{p}_1b_1f'(1)\right.\\
&\left.-\frac{x_0}{L-x_s}\bar{p}_2b_2|f'(-1)|+\mu\kappa\right](x_s-x_0)^2+O((|\mathbf{z}|_{H^2}+|x_s-x_0|)^3) \qquad \text{(B.3)}
\end{aligned}
$$

and a similar expression for V_2+V_5 and V_3+V_6 as previously. Thus Lemma 4.3 still holds but with $\mathbf{A}$ now defined by

$$a_{11}=p_1\big(e^{-\frac{\mu x_0}{\eta_1}}-k_1^2\big)f'(1)-\frac{\varepsilon_1+\varepsilon_2}{8}f'(1)^2, \tag{B.4}$$

$$a_{13}=a_{31}=f'(1)p_1b_1k_1+f'(1)\frac{\bar{p}_1}{2}\big(e^{-\frac{\mu x_0}{\eta_1}}-k_1\big)-f'(1)\frac{\kappa}{2}, \tag{B.5}$$

$$a_{22}=\frac{x_0}{L-x_0}p_2\big(e^{-\frac{\mu x_0}{\eta_2}}-k_2^2\big)|f'(-1)|-\frac{\varepsilon_1+\varepsilon_2}{8}|f'(-1)|^2, \tag{B.6}$$

$$
\begin{aligned}
a_{23}=a_{32}=&\,|f'(-1)|\frac{x_0}{L-x_0}p_2b_2k_2\\
&+|f'(-1)|\frac{x_0}{L-x_0}\frac{\bar{p}_2}{2}\big(e^{-\frac{\mu x_0}{\eta_2}}-k_2\big)-|f'(-1)|\frac{\kappa}{2},
\end{aligned} \tag{B.7}
$$

$$
\begin{aligned}
a_{33}=&-p_1b_1^2f'(1)-\frac{x_0}{L-x_0}p_2b_2^2|f'(-1)|+\bar{p}_1b_1f'(1)\\
&+\frac{x_0}{L-x_0}\bar{p}_2b_2|f'(-1)|-\mu\kappa
\end{aligned} \tag{B.8}
$$

instead of (4.60)–(4.64). We can then choose p_1, p_2, $\bar{p}_1$, $\bar{p}_2$ as previously by (4.69)–(4.71) and $\mathbf{A}$ becomes again diagonal with the expression of its elements given by

$$a_{33}=\frac{\kappa}{2}f'(1)b_1e^{\frac{\mu x_0}{\eta_1}}+\frac{\kappa}{2}|f'(-1)|b_2e^{\frac{\mu x_0}{\eta_2}}-\mu\kappa, \tag{B.9}$$

$$a_{11}=\frac{\kappa f'(1)}{2b_1}\big(1-k_1^2e^{\frac{\mu x_0}{\eta_1}}\big)-\frac{\kappa f'(1)^2}{2\mu^2}\big[b_1\big(e^{\frac{\mu x_0}{\eta_1}}-1\big)+b_2\big(e^{\frac{\mu x_0}{\eta_2}}-1\big)\big], \tag{B.10}$$

$$a_{22}=\frac{\kappa|f'(-1)|}{2b_2}\big(1-k_2^2e^{\frac{\mu x_0}{\eta_2}}\big)-\frac{\kappa|f'(-1)|^2}{2\mu^2}\big[b_1\big(e^{\frac{\mu x_0}{\eta_1}}-1\big)+b_2\big(e^{\frac{\mu x_0}{\eta_2}}-1\big)\big], \tag{B.11}$$

instead of (4.74), (4.77) and (4.78), respectively. Then to prove Theorem 4.1 with (B.1) instead of (3.3), we only need to show now that under assumption (5.4) there exist $\mu > \gamma$ and $\kappa > 1$ such that $a_{ii} > 0, i = 1, 2, 3$ and such that (4.16) holds where $\Theta_i, i = 1, 2$ are still defined by (4.17). But this can be checked exactly as in the proof of Theorem 4.1. With condition (5.3), one can now check as in Remark 2.4 that there always exist parameters b_i and k_i such that conditions (5.4) are satisfied.

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