METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 26, No. 2, pp. 101–112, June 2019 001

EXPONENTIAL STABILITY OF PI CONTROL FOR SAINT-VENANT EQUATIONS WITH A FRICTION TERM*

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Dedicated to Roland Glowinski, a master and a friend, on the occasion of his 80th birthday

Abstract. We consider open channels represented by Saint-Venant equations that are monitored and controlled at the downstream boundary and subject to unmeasured flow disturbances at the upstream boundary. We address the issue of feedback stabilization and disturbance rejection under Proportional-Integral (PI) boundary control. For channels with uniform steady states, the analysis has been carried out previously in the literature with spectral methods as well as with Lyapunov functions in Riemann coordinates. In this article, our main contribution is to show how the analysis can be extended to channels with non-uniform steady states with a Lyapunov function in physical coordinates.

Key words. Hyperbolic systems, Boundary control, Lyapunov Stability, Saint-Venant equations.

Mathematics Subject Classification. 35F61, 93D15, 93D30.

Introduction. The hyperbolic Saint-Venant equations are commonly used for the description of water flow dynamics in open channels and for the design of management and control systems in irrigation networks and navigable rivers. In particular, the exponential stabilization of Saint-Venant equations by boundary feedback control has been a recurring research topic in the literature for more than twenty years.

The earlier results dealt with *static proportional control*. In the simplest case of horizontal channels with negligible friction, the stability analysis was carried out in [8] with an entropy Lyapunov function, in [17, 12] with the method of characteristics, and in [9, Section VI] with a Lyapunov function in Riemann coordinates. The stability analysis was then extended to channels with slope and friction. In the special case of a uniform steady state, the stability analysis was carried out with a spectral method for linearized equations in [18, Section 6]. However the linearized system stability does not directly imply the stability of the steady state for the nonlinear Saint-Venant equations (see e.g. [10]). For this nonlinear case, the stability analysis is done in [4, 14] with a Lyapunov function in Riemann coordinates. More recently, the case of channels with friction and slope and non-uniform steady state was considered in [3] and [16] with dedicated Lyapunov functions expressed in physical coordinates.

The boundary feedback stabilization of Saint-Venant equations by *Proportional-Integral (PI) control* has received much less attention in the literature. It has been analyzed for channels with uniform steady states in [6] with a spectral method and in [15, Section 4], [2, Section 5.5] with Lyapunov functions in Riemann coordinates. *In the present article, our main contribution is to show how the analysis of* [3] *can be extended to channels with non-uniform steady states under PI control, using a Lyapunov function in physical coordinates.*

Obviously, in principle, stabilization is also possible with more sophisticated control laws. In particular, the recent backstepping method for 2×2 hyperbolic systems,

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see e.g. [11, 1, 13], allows to design stabilizing boundary output feedbacks in *observer-controller form* for Saint-Venant equations. However, it is clear that such advanced solutions are far from being used in practice and that PI controllers are the only regulators that are really implemented in the vast majority of field applications. The reason is obviously that PI regulators, besides their great ease of implementation, are the simplest solution to cancel off-set errors and attenuate load disturbances. In a PI regulator, the parameter k_i is a measure of the disturbance attenuation efficiency, but too large values may produce instability. The analysis of the stability of a closed-loop system under PI control, as we present in this article, is therefore an important and relevant issue.

Saint-Venant equations. We consider a pool of a prismatic horizontal open channel with a rectangular cross section and a unit width, as shown in Fig.1. The dynamics of the system are described by the Saint-Venant equations

$$H_t + (HV)_x = 0, \tag{1a}$$

$$V_t + \left(gH + \frac{1}{2}V^2\right)_x + S_f(H,V) = 0, \tag{1b}$$

with the state variables $H(t,x)$ = water depth and $V(t,x)$ = horizontal water velocity at the time instant t and the location x along the channel. L is the length of the pool and g is the gravity acceleration. $S_f(H,V)$ is the friction term for which various empirical models are available in the engineering literature. In this article, we adopt the simple model

$$S_f(H,V) \triangleq C\frac{V^2}{H} \tag{2}$$

with C a constant friction coefficient. This model is an approximation of the classical Manning-Strickler formula (see e.g. [7, Section 5.8], [19, Section 2.1]) which is valid when the channel is much wider than deep and the friction on the bottom is dominant. In that case, indeed, it is quite natural to assume a viscous friction (proportional to V^2) decreasing proportionally to the water depth H.

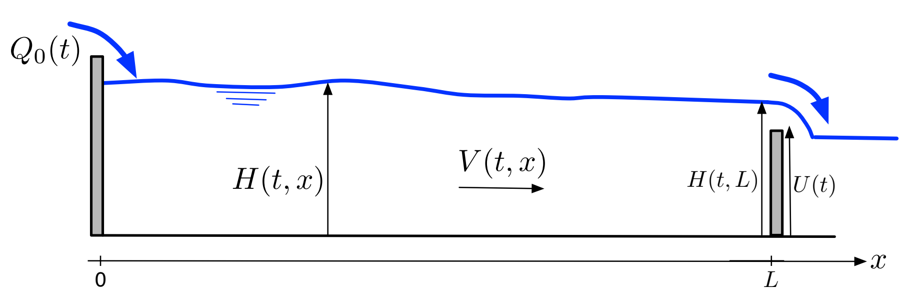

Fig. 1. *Pool of an open channel with an overflow gate at the downstream side.*

The system is subject to the following boundary conditions:

$$H(t,0)V(t,0) = Q_0(t), \tag{3a}$$

$$H(t,L)V(t,L) = v_G\big(H(t,L) - U(t)\big). \tag{3b}$$

The first boundary condition (3a) imposes the value of the canal inflow rate which is an unknown disturbance denoted $Q_0(t)$. The second boundary condition (3b) is a simple linear model of an overshot gate with $U(t)$ the gate elevation used as control input and υ_G a constant discharge coefficient.

Proportional-Integral control. In this article we are concerned with the case where the outflow gate is provided with a Proportional-Integral (PI) control law

$$U(t) \triangleq U_r + k_p(H_{sp} - H(t,L)) + k_i \int_0^t (H_{sp} - H(\tau, L))d\tau \tag{4}$$

where H_{sp} denotes the set-point for the downstream level $H(t,L)$ which is assumed to be measured on line. The first term U_r is an arbitrary constant value for the gate elevation. The second term is the proportional correction action with the tuning parameter k_p. The last term is the integral action with the tuning parameter k_i.

With this control law, defining $Z(t) \triangleq U(t) + k_p H(t,L)$, the boundary conditions are written in differential form as follows:

$$H(t,0)V(t,0) = Q_0(t), \tag{5a}$$

$$H(t,L)V(t,L) = \upsilon_G\big[(1+k_p)H(t,L) - Z(t)\big], \tag{5b}$$

$$\frac{dZ}{dt} = k_i(H_{sp} - H(t,L)), \quad Z(0) = U_r + k_p H_{sp}. \tag{5c}$$

When $U(t)$ is the feedback command signal (4), the system (1), (5) is a closed loop boundary control system.

In this article, our main purpose is to analyze the exponential stability of this closed loop control system.

Fluvial steady state. In case of a constant positive disturbance $Q_0 > 0$ and a constant positive set point $H_{sp} > 0$, a steady state of the closed loop control system is a time-invariant solution $H^*(x), V^*(x), Z^*$, $x \in [0,L]$, given by:

$$H^*(x) \text{ solution of } (gH^{*3} - Q_0^2)H^*_x + CQ_0^2 = 0, \quad H^*(L) = H_{sp}, \tag{6a}$$

$$V^*(x) = \frac{Q_0}{H^*(x)}, \tag{6b}$$

$$Z^* = (1+k_p)H_{sp} - \frac{Q_0}{\upsilon_G}. \tag{6c}$$

The existence of a solution to (6a) requires that $gH_{sp}^3 \neq Q_0^2$. If $gH_{sp}^3 > Q_0^2$, then (6a) has a solution (note that H^* is then decreasing) and the steady state flow is subcritical (or fluvial). In such case, from (6a) and (6b), according to the physical evidence, the state (H^*, V^*) is positive :

$$H^*(x) > 0, \quad V^*(x) > 0, \quad \text{for all } x \in [0,L], \tag{7}$$

and satisfies the following inequality:

$$0 < gH^*(x) - V^{*2}(x), \quad \forall x \in [0,L]. \tag{8}$$

The fluvial flow is the natural regime of open channels in irrigation networks and navigable rivers for which the feedback control addressed in this paper is a relevant and delicate task.

In the case where $gH_{sp}^3 < Q_0^2$, the steady state, if it exists, is supercritical (or torrential). We do not consider that case in the present article because the control of open channels under torrential flow, generally with hydraulic jumps, cannot be achieved with a single controller located at the downstream side. It is a more complex problem which is beyond the scope of this article and is not yet very studied in the scientific literature. The interested reader is referred to the recent paper [5] and the references therein.

Linearization. In order to linearize the model, we define the deviations of the states $H(t,x)$, $V(t,x)$ and $Z(t)$ with respect to the steady states $H^*(x)$, $V^*(x)$ and Z^*:

$$h(t,x) \triangleq H(t,x) - H^*(x), \qquad v(t,x) \triangleq V(t,x) - V^*(x), \qquad z = Z(t) - Z^*. \tag{9}$$

Then the linearized Saint-Venant equations around the steady-state are

$$\begin{pmatrix} h_t \\ v_t \end{pmatrix} + \begin{pmatrix} V^* & H^* \\ g & V^* \end{pmatrix} \begin{pmatrix} h_x \\ v_x \end{pmatrix} + \begin{pmatrix} V_x^* & H_x^* \\ -C\dfrac{V^{*2}}{H^{*2}} & V_x^* + 2C\dfrac{V^*}{H^*} \end{pmatrix} \begin{pmatrix} h \\ v \end{pmatrix} = 0, \tag{10}$$

and the linearized boundary conditions are

$$v(t,0) = -b_0 h(t,0) \quad \text{with} \quad b_0 = \frac{V^*(0)}{H^*(0)}, \tag{11a}$$

$$v(t,L) = b_L h(t,L) - b_z z(t), \quad b_L = \frac{\upsilon_G(1+k_p) - V^*(L)}{H^*(L)}, \quad b_z = \frac{\upsilon_G}{H^*(L)}, \tag{11b}$$

$$z_t = -k_i h(t,L). \tag{11c}$$

Exponential stability of the linearized system. Let us consider the linearized system (10), (11) under an initial condition

$$h(0,x) = h^o(x), \;\; v(0,x) = v^o(x), \;\; z(0) = z^o, \tag{12}$$

such that

$$(h^o, v^o) \in L^2((0,L);\mathbb{R}^2), \;\; z^o \in \mathbb{R}. \tag{13}$$

The Cauchy problem (10)-(11)-(12) is well-posed (see [2, Appendix A]).

Our concern is to analyze the exponential stability of the system (10)-(11) according to the following definition.

DEFINITION 1. *The system* (10)-(11) *is exponentially stable (for the L^2-norm) if there exist $\nu > 0$ and $C_o > 0$ such that, for every initial condition $(h^o, v^o) \in L^2((0,L);\mathbb{R}^2)$, $z^o \in \mathbb{R}$, the solution to the Cauchy problem* (10), (11), (12) *satisfies*

$$\|(h(t,\cdot), v(t,\cdot))\|_{L^2} + |z(t)| \leqslant C_o e^{-\nu t}\Big[\|(h^o, v^o)\|_{L^2} + |z^o|\Big]. \tag{14}$$

We now prove that the linearized control system (10)-(11) is exponentially stable if the steady state is subcritical and the control tuning parameters are positive: $k_p > 0$

and $k_i > 0$. For this stability analysis, the following candidate Lyapunov function is considered:

$$\mathbf{V}(h,v,z) = \int_0^L (gh^2 + H^* v^2)dx + qz^2. \tag{15}$$

Note that there exists $c > 0$ such that

$$\begin{gathered}\frac{1}{c}\left(\|(h,v)\|_{L^2}^2 + |z|^2\right) \leq \mathbf{V}(h,v,z) \leq c\left(\|(h,v)\|_{L^2}^2 + |z|^2\right) \\ \forall (h,v,z) \in L^2(0,L) \times L^2(0,L) \times \mathbb{R}.\end{gathered} \tag{16}$$

The time derivative of this function $\mathbf{V}$ along the C^1 solutions of the Cauchy problem (10), (11), (12) is

$$\frac{d\mathbf{V}}{dt} = 2\int_0^L (ghh_t + H^* vv_t)dx + 2qzz_t. \tag{17}$$

Using the system equation (10) and the boundary condition (11c),we have

$$\begin{aligned}\frac{d\mathbf{V}}{dt} = -2\int_0^L \Big(& gh(V^* h_x + H^* v_x + V_x^* h + H_x^* v) \\ & + H^* v\big(gh_x + V^* v_x - C\frac{V^{*2}}{H^{*2}}h + (V_x^* + 2C\frac{V^*}{H^*}v)\big)\Big)dx - 2qk_i zh(t,L).\end{aligned} \tag{18}$$

Then, using integration by parts together with (6a), we have

$$\frac{d\mathbf{V}}{dt} = -\left[\begin{pmatrix} h & v \end{pmatrix} M(x) \begin{pmatrix} h \\ v \end{pmatrix}\right]_0^L - \int_0^L \begin{pmatrix} h & v \end{pmatrix} N(x) \begin{pmatrix} h \\ v \end{pmatrix} dx - 2qk_i zh(t,L), \tag{19}$$

with

$$M(x) = \begin{pmatrix} gV^*(x) & gH^*(x) \\ gH^*(x) & H^*(x)V^*(x) \end{pmatrix}, \tag{20}$$

and

$$N(x) = \begin{pmatrix} \dfrac{gCV^{*3}}{H^*(gH^* - V^{*2})} & -\dfrac{CV^{*2}}{H^*} \\ -\dfrac{CV^{*2}}{H^*} & \dfrac{2CV^{*3}}{(gH^* - V^{*2})} + 4CV^* \end{pmatrix}. \tag{21}$$

We introduce the notations

$$h_0 = h(t,0), \quad h_L = h(t,L), \quad v_0 = v(t,0), \quad v_L = v(t,L), \tag{22}$$

$$H_0 = H^*(0), \quad V_0 = V^*(0), \quad H_L = H^*(L), \quad V_L = V^*(L). \tag{23}$$

Then, using the boundary conditions (11a), (11b), we have

$$
\begin{aligned}
&\left[\begin{pmatrix} h & v \end{pmatrix} M(x) \begin{pmatrix} h \\ v \end{pmatrix}\right]_0^L \\
&= gV_L h_L^2 + 2gH_L h_L v_L + Q_0 v_L^2 - gV_0 h_0^2 - 2gH_0 h_0 v_0 - Q_0 v_0^2 &\quad (24)\\
&= gV_L h_L^2 + 2gH_L h_L (b_L h_L - b_z z) + Q_0 (b_L h_L - b_z z)^2 - gV_0 h_0^2 \\
&\quad + 2gH_0 h_0 (b_0 h_0) - Q_0 (b_0 h_0)^2 &\quad (25)\\
&= (gV_L + 2gb_L H_L + Q_0 b_L^2) h_L^2 + (-gV_0 + 2gb_0 H_0 - Q_0 b_0^2) h_0^2 + Q_0 b_z^2 z^2 \\
&\quad + (-2gb_z H_L - 2Q_0 b_L b_z) h_L z. &\quad (26)
\end{aligned}
$$

Consequently

$$
\frac{d\mathbf{V}}{dt} = -m_0 h_0^2 - \begin{pmatrix} h_L & z \end{pmatrix} \mathcal{M} \begin{pmatrix} h_L \\ z \end{pmatrix} - \int_0^L \begin{pmatrix} h & v \end{pmatrix} N(x) \begin{pmatrix} h \\ v \end{pmatrix} dx - 2aqk_i z^2, \tag{27}
$$

with

$$
m_0 = -gV_0 + 2gb_0 H_0 - Q_0 b_0^2, \tag{28}
$$

$$
\mathcal{M} = \begin{pmatrix} gV_L + 2gb_L H_L + Q_0 b_L^2 & -gb_z H_L - Q_0 b_L b_z + qk_i \\ -gb_z H_L - Q_0 b_L b_z + qk_i & Q_0 b_z^2 - 2aqk_i \end{pmatrix}, \tag{29}
$$

and a is a real positive constant to be determined.

Under the subcritical flow condition (8), using the definition of b_0 (11a), we have that

$$
m_0 = -gV_0 + 2gb_0 H_0 - H_0 V_0 b_0^2 = b_0 (gH_0 - V_0^2) > 0 \tag{30}
$$

and that the matrix $N(x)$ is positive definite for all $x \in [0, L]$ since

$$
\det[N(x)] = \left(\frac{CV^{*2}}{H^*}\right)^2 \left(\frac{2gH^* V^{*2}}{(gH^* - V^{*2})^2} + \frac{4gH^*}{(gH^* - V^{*2})} - 1\right) > 0. \tag{31}
$$

On the other hand, $\mathcal{M}$ is positive definite if

$$
\text{(a)}\quad gV_L + 2gb_L H_L + Q_0 b_L^2 > 0, \tag{32}
$$

$$
\begin{aligned}
\text{(b)}\quad \det(\mathcal{M}) = {} & (gV_L + 2gb_L H_L + Q_0 b_L^2)(Q_0 b_z^2 - 2aqk_i) \\
& - (gb_z H_L + Q_0 b_L b_z - qk_i)^2 > 0.
\end{aligned} \tag{33}
$$

It follows from (3b) that $v_G > V_L$. Hence, since $k_p > 0$, we have from (11b)

$$
b_L = \frac{(v_G - V_L) + v_G k_p}{H^*(L)} > 0 \tag{34}
$$

and Condition (a) is satisfied.

Regarding condition (b), using the definition of b_L, we have

$$
\det(\mathcal{M}) = -\alpha + 2\beta k_i q - k_i^2 q^2 = \mathcal{P}(q), \tag{35}
$$

with

$$\alpha = g b_z^2 H_L (g H_L - V_L^2) \tag{36}$$

and

$$\beta = g b_z H_L + Q_0 b_L b_z - a(g V_L + 2 g b_L H_L + Q_0 b_L^2). \tag{37}$$

$\mathcal{P}(q)$ is a degree-2 polynomial in q with discriminant

$$\Delta = 4 k_i^2 (\beta^2 - \alpha). \tag{38}$$

We observe that $\alpha > 0$ under the subcritical flow condition (8). Moreover, it is easy to check that the positive parameter a can be selected sufficiently small so that $\beta > 0$ and $\beta^2 - \alpha > 0$. Hence, if $k_i > 0$, $\mathcal{P}(q)$ has two positive real roots and there exists a positive value of q (depending on k_i) such that $\det(\mathcal{M}) > 0$ and condition (b) is satisfied. Then, it follows directly from the definition (15) of $\mathbf{V}$ and from (27) that there exists a positive real constant μ such that

$$\frac{d\mathbf{V}}{dt} \leqslant -\mu \mathbf{V} \tag{39}$$

along the C^1-solutions of the system. However, since the C^1-solutions are dense in the set of L^2-solutions, inequality (39) is also satisfied in the sense of distributions for L^2-solutions (see [2] for details). Consequently, $\mathbf{V}$ is an exponentially decaying Lyapunov function for the L^2-norm and the system (10)-(11) is exponentially stable in the sense of Definition 1.

Exponential stability of the steady state of the Saint-Venant equations. In the previous section, we have shown that the PI controller (4) stabilizes the **linearized** Saint-Venant equations if the steady state is subcritical and the control tuning parameters are positive: $k_p > 0$ and $k_i > 0$. In this section, we briefly explain how it can be shown that the same PI controller is also sufficient to guarantee the local exponential stability for the H^2-norm of the steady state $H^*(x)$, $V^*(x)$ of the **nonlinear** system of Saint-Venant equations (1), (2) under the nonlinear boundary conditions (5).

Let us rewrite the Saint-Venant equations in the (h, v) coordinates (see (9)),

$$\begin{pmatrix} h_t \\ v_t \end{pmatrix} + \begin{pmatrix} V^*(x) + v & H^*(x) + h \\ g & V^*(x) + v \end{pmatrix} \begin{pmatrix} h_x \\ v_x \end{pmatrix} + \begin{pmatrix} V_x^*(x) & H_x^*(x) \\ -C \dfrac{V^{*2}(x)}{H^*(x)(H^*(x)+h)} & V_x^*(x) + C \dfrac{2V^*(x) + v}{H^*(x) + h} \end{pmatrix} \begin{pmatrix} h \\ v \end{pmatrix} = 0, \tag{40}$$

with the boundary conditions (using the notations (22) and (23))

$$v_0 = -b_0 h_0 + \frac{V_0}{H_0(H_0 + h_0)} h_0^2, \tag{41a}$$

$$v_L = b_L h_L - b_z z + \frac{(V_L - v_G(1 + k_P)) h_L^2 + v_G h_L z}{H_L(H_L + h_L)}, \tag{41b}$$

$$z_t = -k_i h_L. \tag{41c}$$

Then, we transform the system into Riemann coordinates which are defined as follows:

$$\mathbf{R} = \begin{pmatrix} R^+ \\ R^- \end{pmatrix} = \begin{pmatrix} v + 2\eta(h) \\ v - 2\eta(h) \end{pmatrix} \text{ with } \eta(h) = \sqrt{g(H^* + h)} - \sqrt{gH^*}. \tag{42}$$

With these coordinates, the system (40) is written in the following characteristic form:

$$\mathbf{R}_t + \Lambda(\mathbf{R}, x)\mathbf{R}_x + \mathbf{B}(\mathbf{R}, x) = \mathbf{0}, \tag{43}$$

with the diagonal matrix

$$\Lambda(\mathbf{R}, x) = \begin{pmatrix} \lambda^+(\mathbf{R}, x) & 0 \\ 0 & \lambda^-(\mathbf{R}, x) \end{pmatrix} \text{ with } \lambda^{\pm}(\mathbf{R}, x) = V^* \pm \sqrt{gH^*} + v \pm \eta(h), \tag{44}$$

and an appropriate definition of $\mathbf{B}(\mathbf{R}, x)$.

The goal is to prove the H^2 exponential stability of the zero steady state for the system (44) under the boundary conditions (41) and under an initial condition

$$\mathbf{R}(0, x) = \mathbf{R}_{\mathrm{o}}(x), \quad z(0) = z^o. \tag{45}$$

according to the following definition.

Definition 2. *The steady state $\mathbf{R}(t, x) \equiv 0$ of the system* (41), (43), *and* (45) *is exponentially stable (for the H^2-norm) if there exist $\delta > 0$, $\nu > 0$ and $C_0 > 0$ such that, for every initial condition $\mathbf{R}_o \in H^2((0, L); \mathbb{R}^n)$ satisfying $\|\mathbf{R}_o\|_{H^2((0,L);\mathbb{R}^n)} \leqslant \delta$ and compatibility*[1] *conditions of order 1 , the solution $\mathbf{R}$ of the Cauchy problem* (44), (41), (45) *is defined on $[0, +\infty) \times [0, L]$ and satisfies*

$$\|\mathbf{R}(t, .)\|_{H^2((0,L);\mathbb{R}^n)} + |z(t)| \leq C_0 e^{-\nu t}\Big[\|\mathbf{R}_o\|_{H^2((0,L);\mathbb{R}^n)} + |z^o|\Big]. \tag{46}$$

The proof can be build in a way very similar to the proof given in [2, Chapter 6] for a general class of quasi-linear hyperbolic systems with static boundary conditions. Here we limit ourselves to the key points of the proof and we refer the reader to [2, Section 6.2] for a comprehensive development.

First, we consider an augmented system with state $(\mathbf{R}, \mathbf{R}_t, \mathbf{R}_{tt})$ where the dynamics of $\mathbf{R}_t$ and $\mathbf{R}_{tt}$ are simply obtained by taking partial time derivatives of the system equation (44) and the boundary conditions (41).

Then the candidate Lyapunov function is defined as

$$\mathbf{V}_{\mathrm{NL}} = \mathbf{V_1}(\mathbf{R}, z) + \mathbf{V_2}(\mathbf{R}_t, z_t) + \mathbf{V_3}(\mathbf{R}_{tt}, z_{tt}), \tag{47}$$

with

$$\mathbf{V_1}(\mathbf{R}, z) = \int_0^L \tfrac{1}{2} H^* \mathbf{R}^T \mathbf{R} dx + q z^2, \tag{48}$$

$$\mathbf{V_2}(\mathbf{R}_t, z_t) = \int_0^L \tfrac{1}{2} H^* \mathbf{R}_t^T \mathbf{R}_t dx + q z_t^2, \tag{49}$$

$$\mathbf{V_3}(\mathbf{R}_{tt}, z_{tt}) = \int_0^L \tfrac{1}{2} H^* \mathbf{R}_{tt}^T \mathbf{R}_{tt} dx + q z_{tt}^2. \tag{50}$$

[1] For an explanation of the concept of compatibility of initial conditions, see [2, Section 4.5.2]

For a vector $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n)^T \in \mathbb{R}^n$, we denote $|\boldsymbol{\xi}|_\infty = \max\{|\xi_j|;\ j \in \{1, \dots, n\}\}$. For a map $f \in C^0([0, L]; \mathbb{R}^n)$, we denote $|f|_0 = \max\{|f(x)|_\infty;\ x \in [0, L]\}$. We remark that, for small $|h|_0$, the function $\mathbf{V_1}(\mathbf{R}, z)$ can be viewed as a perturbation of the Lyapunov function $\mathbf{V}(h, v, z)$ of the linearized system (see equation (15)). More precisely, for $|h|_0$ small enough,

$$\begin{aligned}
\mathbf{V_1}(\mathbf{R}, z) &= \int_0^L \tfrac{1}{2} H^* \mathbf{R}^T \mathbf{R} dx + qz^2 && (51)\\
&= \int_0^L (4H^* \eta^2(h) + H^* v^2) dx + qz^2 \\
&= \int_0^L (gh^2 + H^* v^2 + O(h^3)) dx + qz^2 \\
&= \mathbf{V}(h, v, z) + \int_0^L O(|h|^3) dx. && (52)
\end{aligned}$$

Similar expressions of $\mathbf{V_2}$ and $\mathbf{V_3}$ are obtained as follows: for $|h|_0$ small enough

$$\mathbf{V_2}(\mathbf{R}_t, z_t) = \mathbf{V}(h_t, v_t, z_t) + \int_0^L O(|h h_t^2|) dx, \tag{53}$$

$$\mathbf{V_3}(\mathbf{R}_{tt}, z_{tt}) = \mathbf{V}(h_{tt}, v_{tt}, z_{tt}) + \int_0^L O(|h_t^2 h_{tt}| + |h h_{tt}^2|) dx. \tag{54}$$

Let us now introduce a notation to deal with "higher order terms" in the time derivative of the Lyapunov function. We denote by $\mathcal{O}(X_1; X_2)$, with $X_1 \geqslant 0$ and $X_2 \geqslant 0$, quantities for which there exist $C_0 > 0$ and $\varepsilon > 0$ independent of $\mathbf{R}$, $\mathbf{R}_t$ and $\mathbf{R}_{tt}$, such that

$$(X_2 \leqslant \varepsilon) \Longrightarrow (|\mathcal{O}(X_1; X_2)| \leqslant C_0 X_1).$$

It follows that the time derivatives of $\mathbf{V_1}$, $\mathbf{V_2}$ and $\mathbf{V_3}$ along the system solutions can be expressed in the following form

$$\begin{aligned}
\frac{d\mathbf{V_1}}{dt} = &- \left[\begin{pmatrix} h & v \end{pmatrix} M(x) \begin{pmatrix} h \\ v \end{pmatrix} \right]_0^L - \int_0^L \begin{pmatrix} h & v \end{pmatrix} N(x) \begin{pmatrix} h \\ v \end{pmatrix} dx - 2qzz_t \\
&+ \mathcal{O}\big(|\mathbf{R}(t,0)|^3 + |\mathbf{R}(t,L)|^3; |\mathbf{R}(t,0)| + |\mathbf{R}(t,L)|\big) \\
&+ \mathcal{O}\Big(\int_0^L (|\mathbf{R}|^3 + |\mathbf{R}_t||\mathbf{R}|^2) dx; |\mathbf{R}(t,.)|_0\Big),
\end{aligned} \tag{55}$$

$$\begin{aligned}
\frac{d\mathbf{V_2}}{dt} = &- \left[\begin{pmatrix} h_t & v_t \end{pmatrix} M(x) \begin{pmatrix} h_t \\ v_t \end{pmatrix} \right]_0^L - \int_0^L \begin{pmatrix} h_t & v_t \end{pmatrix} N(x) \begin{pmatrix} h_t \\ v_t \end{pmatrix} dx - 2qz_t z_{tt} \\
&+ \mathcal{O}\big(|\mathbf{R}_t(t,0)|^2 |\mathbf{R}(t,0)| + |\mathbf{R}_t(t,L)|^2 |\mathbf{R}(t,L)|; |\mathbf{R}(t,0)| + |\mathbf{R}(t,L)|\big) \\
&+ \mathcal{O}\Big(\int_0^L |\mathbf{R}_t|^2 (|\mathbf{R}_t| + |\mathbf{R}|) dx; |\mathbf{R}(t,.)|_0\Big),
\end{aligned} \tag{56}$$

$$
\begin{aligned}
\frac{d\mathbf{V_3}}{dt} = &-\left[\begin{pmatrix} h_t & v_t \end{pmatrix} M(x) \begin{pmatrix} h_t \\ v_t \end{pmatrix}\right]_0^L - \int_0^L \begin{pmatrix} h_t & v_t \end{pmatrix} N(x) \begin{pmatrix} h_t \\ v_t \end{pmatrix} dx - 2q z_{tt} z_{ttt} \\
&+ \mathcal{O}\big(|\mathbf{R}_{tt}(t,0)|^2 |\mathbf{R}_t(t,0)| + |\mathbf{R}_{tt}(t,0)||\mathbf{R}_t(t,0)|^2 + |\mathbf{R}_t(t,0)|^4 \\
&+ |\mathbf{R}_{tt}(t,L)|^2 |\mathbf{R}_t(t,L)| + |\mathbf{R}_{tt}(t,L)||\mathbf{R}_t(t,L)|^2 + |\mathbf{R}_t(t,L)|^4; |\mathbf{R}(t,0)| + |\mathbf{R}(t,L)|\big) \\
&+ \mathcal{O}\Big(\int_0^L \big(|\mathbf{R}_{tt}|^2 (|\mathbf{R}_t| + |\mathbf{R}|) + |\mathbf{R}_{tt}||\mathbf{R}_t|^2\big) dx; |\mathbf{R}(t,.)|_0 + |\mathbf{R}_t(t,.)|_0\Big). \qquad (57)
\end{aligned}
$$

We observe that, in each case, we recover the quadratic formula of the linear case augmented with (at least) cubic terms that are negligible for small $|\mathbf{R}(t,.)|_0 + |\mathbf{R}_t(t,.)|_0$. It is therefore not surprising that the local H^2 stability of the nonlinear steady state can be deduced from the global L^2 stability of the linear system. By proceeding analogously to [2, Chapter 6], it can be shown that there exist positive constants α and δ such that, for every $\mathbf{R}$ such that $|\mathbf{R}(t,.)|_0 + |\mathbf{R}_t(t,.)|_0 < \delta$, we have

$$
\frac{d\mathbf{V}_{\mathrm{NL}}}{dt} \leqslant -\alpha \mathbf{V}_{\mathrm{NL}} \qquad (58)
$$

along the system solutions (compared to [2, Chapter 6], note that, by (41b), v_L can be expressed in terms of h_L and z). It follows that the system steady-state is locally exponentially stable for the H^2-norm in the sense of Definition 2.

Numerical simulation. We consider a pool with the following parameters:

length: $L = 1000$ (meters),

friction coefficient: $C = 0.03$,

discharge coefficient: $\upsilon_g = 3$ m/sec

constant input flow rate: $Q_0(t) = 2\ \mathrm{m}^3/\mathrm{sec}\ \forall t$,

reference gate elevation: $U_r = 4$ m,

initial state: $H(0,x) = 5.05$ m, $V(0,x) = 0.4$ m/sec, $\forall x \in [0, L]$,

control set point: $H_{sp} = 5$ m,

control tuning: $k_p = 2,\ k_i = 0.02\ \mathrm{sec}^{-1}$.

The simulation is done with the 'hpde' solver [20]. The simulation results are illustrated with Figures 2 and 3. The convergence of the water level $H(t, L)$ from the initial state towards the set point H_{sp} is shown in Figure 2. The exponential decay of the Lyapunov function is shown in Figure 3.

Conclusion. In this article, our main contribution was to exhibit a Lyapunov function which allows to study the exponential stability of Saint-Venant equations with nonuniform steady-states under boundary feedback PI control.

Acknowledgments. The two authors thank Amaury Hayat for useful comments. They also thank ETH-ITS and ETH-FIM for their hospitality. This work was partly completed during their stay in these two institutions. Both authors were supported by ANR Project Finite4SoS (ANR 15-CE23-0007).

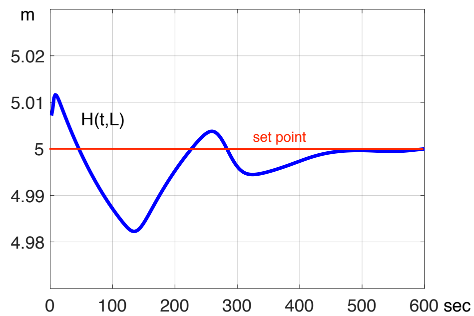

FIG. 2. *Convergence of the water level towards the set point.*

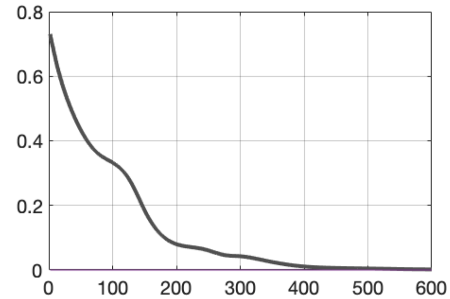

FIG. 3. *Exponential decay of the Lyapunov function.*

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