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Boundary feedback stabilization of hydraulic jumps

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ABSTRACT

In an open channel, a hydraulic jump is an abrupt transition between a torrential (supercritical) flow and a fluvial (subcritical) flow. In this article hydraulic jumps are represented by discontinuous shock solutions of hyperbolic Saint-Venant equations. Using a Lyapunov approach, we prove that we can stabilize the state of the system in H^2-norm as well as the hydraulic jump location, with simple feedback boundary controls and an arbitrary decay rate, by appropriately choosing the gains of the feedback boundary controls.

1. Introduction and main result

Nonlinear hyperbolic equations are well-known to give rise to discontinuities in finite time that are physically meaningful. Hydraulic jump is one of the most known example. A hydraulic jump is a phenomenon that frequently occurs in open channel flow, such as rivers and spillways. It describes a transition between a torrential (or supercritical) regime and a fluvial (or subcritical) regime, i.e., an abrupt transition between a fast flow and a slow flow with a higher height. As a consequence, a part of the initial kinetic energy of the flow is converted into an increase in potential energy, while some energy is irreversibly lost through turbulence and heat. This phenomenon can be seen not only in rivers and spillways but also in air flows of the atmosphere. This is for instance believed to explain the phenomenon of "Morning Glory cloud" (Clarke, 1972) and may be at the origin of some gliders' crashes (Kuettner & Hertenstein, 2002). Hydraulic jumps are important not only because they occur naturally but also because they are sometimes engineered on purpose and are very useful in hydraulic applications to dissipate energy in water and prevent in this way the erosion of the streambed or damages on hydraulic installations (Hager, 1992). However, when studying the flow equations, the stabilization of hydraulic jumps is seldom considered and almost all the studies focus on the stabilization of the dynamics of the fluvial regime (Bastin & Coron, 2011, 2016, 2017; Bastin, Coron, & d'Andréa Novel, 2009; Coron, d'Andréa Novel, & Bastin, 1999; Gugat, Leugering, Schittkowski, & Schmidt, 2001; de Halleux, Prieur, Coron, d'Andréa Novel, & Bastin, 2003; Leugering & Schmidt, 2002). In this paper, we explicitly address the issue of the stabilization of a hydraulic jump represented by a discontinuous shock solution of the flow equations, switching from the torrential regime to the fluvial regime. In other words, the two eigenvalues of the hyperbolic system modeling the shallow water are both positive in the torrential regime and one of them changes sign and switches to a negative value in the fluvial regime. Our goal is to achieve the stability of the channel with a general class of local feedback controls at the boundary. Fundamentally, the stabilization of shock steady states for hyperbolic systems, while being very interesting, has rarely been studied. One can refer to Bastin, Coron, Hayat, and Shang (2017) and Perrollaz (2018) for the scalar case and to our knowledge, no such result exists for systems. By a Lyapunov approach we prove the exponential H^2-stability of the steady state, with an arbitrary decay rate and with an exact exponential stabilization of the desired location of the hydraulic jump.

We consider a channel with a rectangular cross section with constant width, which is taken to be 1 without loss of generality. We denote by $Q(t,x)$ the flux and $H(t,x)$ the water depth, where t and x are, respectively, the time and space independent variables as usual. As the channel has a finite length $L>0$, the spatial domain is bounded and noted $[0,L]$. The Saint-Venant model which, neglecting friction, consists in a continuity equation and an equilibrium of

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forces, is written as

$$\begin{aligned} &\partial_t H + \partial_x Q = 0, \\ &\partial_t Q + \partial_x \left(\frac{gH^2}{2} + \frac{Q^2}{H} \right) = 0. \end{aligned} \tag{1}$$

We are interested with solution trajectories $(H(t,x), Q(t,x))^\intercal$ that may have a jump discontinuity at some point $x_s(t) \in (0, L)$ and are classical otherwise. Thus, in order to close the system, we need a relationship between Q and H before and after this jump. From the Rankine–Hugoniot condition applied to (1), two quantities are conserved through the jump in the jump's referential: the flux Q and the momentum $gH^2/2 + Q^2/H$. This gives the following relationships at the jump $x_s(t)$:

$$\begin{aligned} &[Q]_-^+ = \dot{x}_s [H]_-^+, \\ &[Q]_-^+ \dot{x}_s = \left[\frac{Q^2}{H} + \frac{1}{2} g H^2 \right]_-^+, \end{aligned} \tag{2}$$

where, as usual, $\dot{x}_s$ denotes the time derivative of x_s, i.e., the speed of the jump. These relationships can be reformulated as:

$$\dot{x}_s = \frac{[Q]_-^+}{[H]_-^+}, \tag{3}$$

$$\text{and} \quad ([Q]_-^+)^2 = [H]_-^+ \left[\frac{Q^2}{H} + \frac{1}{2} g H^2 \right]_-^+, \tag{4}$$

where we define for any bounded function f in a neighborhood of x_s: $[f]_-^+ = f(x_s^+(t)) - f(x_s^-(t))$. This relation (4) can be regarded as the generalization for non-stationary states of the well-known Bélanger equation (8) below.

Our goal is to stabilize the steady states of the system (1), (3) and (4) where a (single) hydraulic jump occurs, meaning that the flow switches from the torrential regime to the fluvial regime with a discontinuity in height. Therefore, such steady states $((H^*, Q^*)^\intercal, x_s^*)$ satisfy the following conditions:

1. Q^* is constant and positive, $x_s^* \in (0, L)$ and

$$H^* = \begin{cases} H_1^*, & x \in [0, x_s^*), \\ H_2^*, & x \in (x_s^*, L], \end{cases} \tag{5}$$

   where H_1^*, H_2^* are positive constants.
2. The steady state flow is in the torrential regime before the jump and in the fluvial regime after the jump. This means that in the torrential regime the two system eigenvalues are positive,

$$\lambda_1 = \frac{Q^*}{H_1^*} - \sqrt{gH_1^*} > 0, \quad \lambda_2 = \frac{Q^*}{H_1^*} + \sqrt{gH_1^*} > 0, \quad \text{for } x \in [0, x_s^*), \tag{6}$$

   while there is one positive and one negative eigenvalue in the fluvial regime (Bastin & Coron, 2016),

$$-\lambda_3 = \frac{Q^*}{H_2^*} - \sqrt{gH_2^*} < 0, \quad \lambda_4 = \frac{Q^*}{H_2^*} + \sqrt{gH_2^*} > 0, \quad \text{for } x \in (x_s^*, L]. \tag{7}$$

   In particular this implies that $H_1^* < H_2^*$.
3. Furthermore, the Rankine–Hugoniot conditions applied to (1) in the stationary case are equivalent to the following well-known Bélanger equation (Chanson, 2009)

$$\frac{H_2^*}{H_1^*} = \frac{-1 + \sqrt{1 + 8\frac{(Q^*)^2}{g(H_1^*)^3}}}{2}. \tag{8}$$

Physical remarks.

- The switch from the torrential regime to the fluvial regime corresponds to a transition (shock) between a state where the system (1) has two positive eigenvalues and a state where the system has one positive and one negative eigenvalue. As we will see later (from Theorem 1.1 together with (6) and (7)), this transition (shock) induces a discontinuity not only for the eigenvalue that changes sign but also for the eigenvalue that keeps the same sign. More precisely, if we denote by λ_s the eigenvalue that changes sign, then $\lambda_s(x_s^-(t)) > 0 > \lambda_s(x_s^+(t))$ for all $t > 0$. And if we denote by λ_c the eigenvalue that does not change sign, then $\lambda_c(x_s^-(t)) \neq \lambda_c(x_s^+(t))$ for all $t > 0$. We point out that smooth transitions could happen around critical equilibria or when source terms are considered (see Coron, Glass, & Wang, 2009/10; Gugat & Ulbrich, 2017). Such smooth transitions are also related to coupling conditions for networks for the transition from supersonic to subsonic fluid states, such as natural gas pipeline transportation systems that have been analyzed in Gugat, Herty, and Müller (2017).
- Note that when the solutions are classical, the formulation (1) of the Saint-Venant equations with the level H and the flux Q as state variables is equivalent to the alternative formulation with the level H and the velocity Q/H that is obtained by replacing the equilibrium of forces by an energy equation and is used for instance in Bastin and Coron (2016, 2017) and Hayat and Shang (2017). When the solutions are not classical however, the two formulations are not equivalent anymore and this can be seen by looking at the stationary states: the formulation (1) in level and flux is compatible with shock and discontinuity of $H^*(x)$ while the version with the energy equation is not. This is logical as there is a pointwise loss of energy in the hydraulic jump, which implies that the energy conservation does not hold anymore.
- From (3), the location of the shock x_s may be moving around its initial location and potentially all along the channel. This can be seen in practical phenomena such as tidal bores. The main challenge of this work is to also stabilize this location when stabilizing the state of the system. This is not obvious as one can see that for given heights and flux (H_1^*, H_2^*, Q^*) satisfying (6)–(8), any shock location $x_s^* \in [0, L]$ induces an admissible steady state $((H^*, Q^*)^\intercal, x_s^*)$, where H^* is given by (5). Thus the steady states are not isolated and therefore not asymptotically stable in open loop. Indeed, any small perturbation on x_s^* corresponds to another steady state with the same heights and flux at the two ends.

As illustrated in Fig. 1, let us consider a channel which is equipped with devices allowing a feedback control on $H(t, 0) = H_0(t)$, $Q(t, L) = Q_L(t)$ and $Q(t, 0) \approx Q_0(t)$ (quasi-steady state approximation). Let the set point for the control be a steady state $((H^*, Q^*)^\intercal, x_s^*)$ defined as previously by (5)–(8). We assume that static boundary feedback control laws are selected so that the boundary conditions can be written in the following general form:

$$\begin{pmatrix} H(t,0) - H_1^* \\ Q(t,0) - Q^* \\ Q(t,L) - Q^* \end{pmatrix} = G \begin{pmatrix} Q(t, x_s^-) - Q^* \\ Q(t, x_s^+) - Q^* \\ H(t, x_s^-) - H_1^* \\ x_s - x_s^* \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ G_4(H(t,L) - H_2^*) \end{pmatrix} \tag{9}$$

where $G = (G_1, G_2, G_3)^\intercal : \mathbb{R}^4 \to \mathbb{R}^3$ and $G_4 : \mathbb{R} \to \mathbb{R}$ are of class C^2 and satisfy

$$G(\mathbf{0}) = \mathbf{0}, \quad G_4(0) = 0, \quad G_4'(0) = -\lambda_4. \tag{10}$$

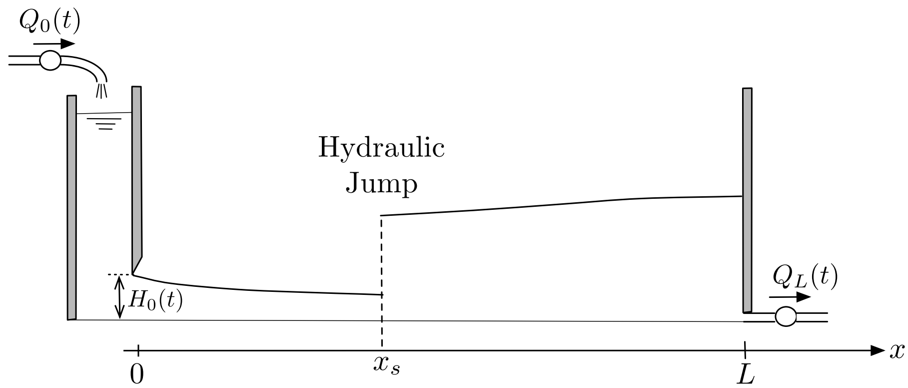

Fig. 1. Open channel with a hydraulic jump and three control devices: the gate opening $H_0(t)$ and the inflow $Q_0(t)$ and outflow $Q_L(t)$ which are driven by two pumps.

Obviously, by (5), the steady state $((H^*, Q^*)^\mathsf{T}, x_s^*)$ satisfies the boundary conditions (9), as $H^*(0) = H_1^*$ and $H^*(L) = H_2^*$. Note that this boundary feedback is quite simple to implement as it only requires a pointwise measure of $H(t, L), x_s(t), H(t, x_s^-), Q(t, x_s^+)$ and $Q(t, x_s^-)$.

In order to state the main stability result of this article, we first introduce the following notations:

$$D(x, \gamma) = \operatorname{diag}\left(\frac{s_i(1 - s_i\frac{\lambda_i}{\lambda_4})}{b_i} e^{\frac{\gamma}{x_i\lambda_i}(x_s^* - x)}, i \in \{1, 2, 3\}\right),$$

$$\widetilde{D}(\gamma) = \operatorname{diag}\left(\left(\sum_{j=1}^{3} e^{\frac{\gamma x_s^*}{x_j\lambda_j}} - \frac{\gamma x_s^*}{x_j\lambda_j}\right)\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right)^2, \quad i \in \{1, 2, 3\}\right),$$

$$K = \begin{pmatrix} \frac{\lambda_2\lambda_1}{\lambda_2 - \lambda_1} & -\frac{\lambda_1}{\lambda_2 - \lambda_1} & 0 \\ \frac{\lambda_2\lambda_1}{\lambda_1 - \lambda_2} & -\frac{\lambda_2}{\lambda_1 - \lambda_2} & 0 \\ 0 & 0 & \frac{\lambda_3}{\lambda_3 + \lambda_4} \end{pmatrix} \times G'(\mathbf{0}) \begin{pmatrix} 1 & 1 & 0 \\ \frac{\lambda_1}{\lambda_4} & \frac{\lambda_2}{\lambda_4} & 1 + \frac{\lambda_3}{\lambda_4} \\ \frac{1}{\lambda_1} & \frac{1}{\lambda_2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{11}$$

$$d = \frac{1}{H_1^* - H_2^*},$$

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} \frac{\lambda_2\lambda_1}{\lambda_2 - \lambda_1} & -\frac{\lambda_1}{\lambda_2 - \lambda_1} & 0 \\ \frac{\lambda_2\lambda_1}{\lambda_1 - \lambda_2} & -\frac{\lambda_2}{\lambda_1 - \lambda_2} & 0 \\ 0 & 0 & \frac{\lambda_3}{\lambda_3 + \lambda_4} \end{pmatrix} G'(\mathbf{0}) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

with $s_1 = s_2 = 1, s_3 = -1, x_1 = x_2 = 1, x_3 = x_s^*/(L - x_s^*)$ and $x_4 = x_s^*/(x_s^* - L)$.

We consider the following initial condition

$$H(0, x) = H_0(x), \quad Q(0, x) = Q_0(x), \quad x_s(0) = x_{s,0} \tag{12}$$

where $x_{s,0} \in (0, L)$ and $(H_0(x), Q_0(x))^\mathsf{T} \in H^2((0, x_{s,0}); \mathbb{R}^2) \cap H^2((x_{s,0}, L); \mathbb{R}^2)$. We assume that the initial condition satisfies the first order compatibility conditions derived from (9), (see Bastin & Coron, 2016 for a proper definition of the first order compatibility condition which is omitted here for the sake of simplicity).

Now, we give the following definition:

Definition 1.1. The steady state $((H^*, Q^*)^\mathsf{T}, x_s^*)$ is locally exponentially stable for the H^2-norm with decay rate γ, if there exist $\delta^* > 0$ and $C^* > 0$ such that for any initial data $(H_0(x), Q_0(x))^\mathsf{T} \in H^2((0, x_{s,0}); \mathbb{R}^2) \cap H^2((x_{s,0}, L); \mathbb{R}^2)$ and $x_{s,0} \in (0, L)$ satisfying

$$|(H_0 - H_1^*, Q_0 - Q^*)^\mathsf{T}|_{H^2((0,x_{s,0});\mathbb{R}^2)} + |(H_0 - H_2^*, Q_0 - Q^*)^\mathsf{T}|_{H^2((x_{s,0},L);\mathbb{R}^2)} \leq \delta^*, \tag{13}$$

$$|x_{s,0} - x_s^*| \leq \delta^*, \tag{14}$$

and the corresponding first order compatibility conditions derived from (9), and for any $T > 0$, the system (1), (3), (4), (9) and (12) has a unique solution $(H, Q)^\mathsf{T} \in C^0([0, T]; H^2((0, x_s(t)); \mathbb{R}^2) \cap H^2((x_s(t), L); \mathbb{R}^2))$ and $x_s \in C^1([0, T])$ and

$$\begin{aligned} &|(H(t, \cdot) - H_1^*, Q(t, \cdot) - Q^*)^\mathsf{T}|_{H^2((0,x_s(t));\mathbb{R}^2)} \\ &\qquad + |(H(t, \cdot) - H_2^*, Q(t, \cdot) - Q^*)^\mathsf{T}|_{H^2((x_s(t),L);\mathbb{R}^2)} \\ &\qquad + |x_s(t) - x_s^*| \\ &\leq C^* e^{-\gamma t}\Big(|(H_0 - H_1^*, Q_0 - Q^*)^\mathsf{T}|_{H^2((0,x_{s,0});\mathbb{R}^2)} \\ &\qquad + |(H_0 - H_2^*, Q_0 - Q^*)^\mathsf{T}|_{H^2((x_{s,0},L);\mathbb{R}^2)} \\ &\qquad + |x_{s,0} - x_s^*|\Big), \quad \forall t \in [0, T). \end{aligned} \tag{15}$$

Remark 1. A function f in $C^0([0, T]; H^2((0, x_s(t)); \mathbb{R}^2) \cap H^2((x_s(t), L); \mathbb{R}^2))$ is a function f in $C^0([0, T]; L^2((0, L); \mathbb{R}^2))$ such that, if one defines

$$f_1(t, x) := f(t, x_s(t)x), \ t \in (0, T), \ x \in (0, 1), \tag{16}$$

$$f_2(t, x) := f(t, L + (x_s(t) - L)x), \ t \in (0, T), \ x \in (0, 1), \tag{17}$$

then f_1 and f_2 are both in $C^0([0, T]; H^2((0, 1); \mathbb{R}^2))$. The transformation $f \to (f_1, f_2)$ enables us to reduce the problem to a time-invariant domain and to define the stability of a function $f \in C^0([0, T]; H^2((0, x_s(t)); \mathbb{R}^2) \cap H^2((x_s(t), L); \mathbb{R}^2))$, a function that is piecewise H^2 with a discontinuity that is potentially moving. This transformation will also be used later on in the analysis of the problem (see (23) below).

Based on Definition 1.1, we have the following theorem.

Theorem 1.1. *For any given steady state $((H^*, Q^*)^\intercal, x_s^*)$ of the system (1) satisfying (5)–(8) and the boundary conditions (9), for any $\gamma > 0$,*

if for $i = 1, 2, 3$

$$
\begin{aligned}
& b_i \in \left(\frac{-\gamma e^{-\frac{\gamma}{x_i\lambda_i}x_s^*}}{3ds_i\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right)(1 - e^{-\frac{\gamma}{x_i\lambda_i}x_s^*})}, \frac{-\gamma e^{-\frac{\gamma}{x_i\lambda_i}x_s^*}}{3ds_i\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right)} \right), \\
& \quad \text{if} \quad s_i\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right) < 0, \\
& b_i \in \left(\frac{-\gamma e^{-\frac{\gamma}{x_i\lambda_i}x_s^*}}{3ds_i\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right)}, \frac{-\gamma x_i e^{-\frac{\gamma}{x_i\lambda_i}x_s^*}}{3ds_i\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right)(1 - e^{-\frac{\gamma}{x_i\lambda_i}x_s^*})} \right), \\
& \quad \text{if} \quad s_i\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right) > 0,
\end{aligned}
\tag{18}
$$

and if the matrix

$$
D(x_s^*, \gamma) - K^\intercal D(0, \gamma)K - \left(\sum_{k=1}^{3} \frac{2d^2}{\gamma^2} b_k s_k (1 - s_k\frac{\lambda_k}{\lambda_4})(e^{\frac{\gamma x_s^*}{x_k\lambda_k}} - 1) \right) \widetilde{D}(\gamma)
\tag{19}
$$

is positive definite, with $(b_1, b_2, b_3)^\intercal$, D, $\widetilde{D}$ and K defined in (11), then the steady state $((H^*, Q^*)^\intercal, x_s^*)$ is locally exponentially stable for the H^2-norm with decay rate $\gamma/4$.

Remark 2. Note that it is not obvious that there always exists G such that K and $(b_1, b_2, b_3)^\intercal$ defined in (11) satisfy (18)–(19). We will prove in details that such G indeed exists in Appendix.

2. Well-posedness of the system

In this section, we prove the well-posedness of the Saint-Venant equations (1) with the hydraulic jump conditions (3) and (4), the boundary feedback control conditions (9) and initial condition (12). We have the following well-posedness theorem.

Theorem 2.1. *For any $T > 0$, there exists $\delta(T) > 0$ such that, for any given initial condition (12) satisfying the first order compatibility conditions and*

$$
\begin{aligned}
& |(H_0 - H_1^*, Q_0 - Q^*)^\intercal|_{H^2((0,x_{s,0});\mathbb{R}^2)} \\
& \quad + |(H_0 - H_2^*, Q_0 - Q^*)^\intercal|_{H^2((x_{s,0},L);\mathbb{R}^2)} \leq \delta(T),
\end{aligned}
\tag{20}
$$

$$
|x_{s,0} - x_s^*| \leq \delta(T),
\tag{21}
$$

the system (1), (3), (4), (9) and *(12) has a unique solution $(H, Q)^\intercal \in C^0([0, T]; H^2((0, x_{s,0}); \mathbb{R}^2) \cap H^2((x_{s,0}, L); \mathbb{R}^2))$ and $x_s \in C^1([0, T])$. Moreover, the following estimate holds for any $t \in [0, T]$*

$$
\begin{aligned}
& |(H(t,\cdot) - H_1^*, Q(t,\cdot) - Q^*)^\intercal|_{H^2((0,x_s(t));\mathbb{R}^2)} \\
& \quad + |(H(t,\cdot) - H_2^*, Q(t,\cdot) - Q^*)^\intercal|_{H^2((x_s(t),L);\mathbb{R}^2)} \\
& \quad + |x_s(t) - x_s^*| \\
& \leq C(T)\Big(|(H_0 - H_1^*, Q_0 - Q^*)^\intercal|_{H^2((0,x_{s,0});\mathbb{R}^2)} \\
& \quad + |(H_0 - H_2^*, Q_0 - Q^*)^\intercal|_{H^2((x_{s,0},L);\mathbb{R}^2)} + |x_{s,0} - x_s^*| \Big).
\end{aligned}
\tag{22}
$$

Proof. One can see that the shock location x_s depends on t in general. In order to avoid the time-varying domains $[0, x_s(t)]$ and $[x_s(t), L]$, under the assumption that $x_s \in C^0([0, T])$, we perform, as in Diagne, Shang, and Wang (2016) and Li and Yu (1985), a transformation of the space coordinate x which allows to define new state variables on the fixed domain $[0, x_s^*]$ as follows:

$$
\begin{aligned}
H_1(t, x) &= H(t, x\frac{x_s}{x_s^*}), \\
Q_1(t, x) &= Q(t, x\frac{x_s}{x_s^*}), \\
H_2(t, x) &= H(t, L + x\frac{x_s - L}{x_s^*}), \\
Q_2(t, x) &= Q(t, L + x\frac{x_s - L}{x_s^*}).
\end{aligned}
\tag{23}
$$

Let us denote by h_i and q_i the deviations

$$
h_i = H_i - H_i^*, \quad q_i = Q_i - Q^*, \quad i = 1, 2.
\tag{24}
$$

Then, the system (1), (3) and (4) is equivalent to the following 4×4 system, which is diagonalizable by blocks and defined on $\mathbb{R}^+ \times [0, x_s^*]$:

$$
\begin{aligned}
& \partial_t h_1 - \left(x\frac{\dot{x}_s}{x_s^*} \right) \frac{x_s^*}{x_s} \partial_x h_1 + \frac{x_s^*}{x_s} \partial_x q_1 = 0, \\
& \partial_t q_1 + \left(\frac{2(q_1 + Q^*)}{h_1 + H_1^*} - x\frac{\dot{x}_s}{x_s^*} \right) \frac{x_s^*}{x_s} \partial_x q_1 \\
& \quad + \left(g(h_1 + H_1^*) - \frac{(q_1 + Q^*)^2}{(h_1 + H_1^*)^2} \right) \frac{x_s^*}{x_s} \partial_x h_1 = 0, \\
& \partial_t h_2 + \left(x\frac{\dot{x}_s}{x_s^*} \right) \frac{x_s^*}{L - x_s} \partial_x h_2 - \frac{x_s^*}{L - x_s} \partial_x q_2 = 0, \\
& \partial_t q_2 - \left(\frac{2(q_2 + Q^*)}{h_2 + H_2^*} - x\frac{\dot{x}_s}{x_s^*} \right) \frac{x_s^*}{L - x_s} \partial_x q_2 \\
& \quad - \left(g(h_2 + H_2^*) - \frac{(q_2 + Q^*)^2}{(h_2 + H_2^*)^2} \right) \frac{x_s^*}{L - x_s} \partial_x h_2 = 0,
\end{aligned}
\tag{25}
$$

where

$$
\dot{x}_s = \frac{q_2(t, x_s^*) - q_1(t, x_s^*)}{h_2(t, x_s^*) - h_1(t, x_s^*) + H_2^* - H_1^*}
\tag{26}
$$

and with, from the jump condition (4), the following boundary condition at $x = x_s^*$:

$$
\begin{aligned}
(q_2 - q_1)^2 = (h_2 - h_1 + H_2^* - H_1^*) \bigg(& \frac{(q_2 + Q^*)^2}{h_2 + H_2^*} + \frac{g}{2}(h_2 + H_2^*)^2 \\
& - \frac{(q_1 + Q^*)^2}{h_1 + H_1^*} - \frac{g}{2}(h_1 + H_1^*)^2 \bigg).
\end{aligned}
\tag{27}
$$

Now, we introduce the following Riemann coordinates

$$
\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} \begin{pmatrix} h_1 \\ q_1 \\ h_2 \\ q_2 \end{pmatrix}
\tag{28}
$$

with

$$
S_1 = \begin{pmatrix} \frac{1}{\lambda_1} & \frac{1}{\lambda_2} \\ 1 & 1 \end{pmatrix}^{-1}, \quad S_2 = \begin{pmatrix} -\frac{1}{\lambda_3} & \frac{1}{\lambda_4} \\ 1 & 1 \end{pmatrix}^{-1}
\tag{29}
$$

and λ_i defined in (6), (7). Then the system (25) can be rewritten as

$$
\mathbf{u}_t + (\Lambda(x_s) + A(\mathbf{u}, x_s) + x\dot{x}_s B(x_s))\mathbf{u}_x = 0,
\tag{30}
$$

where

$$\Lambda = \begin{pmatrix} \frac{x_s^*}{x_s}\lambda_1 & 0 & 0 & 0 \\ 0 & \frac{x_s^*}{x_s}\lambda_2 & 0 & 0 \\ 0 & 0 & \frac{x_s^*}{L-x_s}\lambda_3 & 0 \\ 0 & 0 & 0 & -\frac{x_s^*}{L-x_s}\lambda_4 \end{pmatrix} \tag{31}$$

and where A, B are two matrices of class C^2 that can be obtained by direct computations (omitted here for simplicity) and such that A satisfies $A(\mathbf{0}, x_s) = 0$. Using the change of coordinates (28), Eq. (26) becomes:

$$\dot{x}_s = \frac{u_1(t,x_s^*) + u_2(t,x_s^*) - u_3(t,x_s^*) - u_4(t,x_s^*)}{\sum_{i=1}^{3}\frac{u_i(t,x_s^*)}{\lambda_i} - \frac{u_4(t,x_s^*)}{\lambda_4} + (H_1^* - H_2^*)} \tag{32}$$

and the boundary condition (27) becomes:

$$\frac{2Q^*}{H_2^*}(u_3+u_4) - \frac{2Q^*}{H_1^*}(u_1+u_2) + (gH_2^* - \frac{Q^{*2}}{H_2^{*2}})(\frac{u_4}{\lambda_4} - \frac{u_3}{\lambda_3})$$
$$- (gH_1^* - \frac{Q^{*2}}{H_1^{*2}})(\frac{u_1}{\lambda_1} + \frac{u_2}{\lambda_2}) = O\left(|\mathbf{u}(t,x_s^*)|^2\right). \tag{33}$$

Here and hereafter, $O(s)$ (with $s \geq 0$) means that for any $\varepsilon > 0$, there exists $C_1 > 0$ such that

$(s \leq \varepsilon) \Longrightarrow (|O(s)| \leq C_1 s)$.

With the expression of the eigenvalues given by (6) and (7), (33) becomes

$$\lambda_4 u_4(t,x_s^*) = \lambda_1 u_1(t,x_s^*) + \lambda_2 u_2(t,x_s^*) + \lambda_3 u_3(t,x_s^*) + O\left(|\mathbf{u}(t,x_s^*)|^2\right). \tag{34}$$

Using (23), (24), (28) and (34), the boundary conditions (9) now become

$$\begin{pmatrix} u_1(t,0) \\ u_2(t,0) \\ u_3(t,0) \end{pmatrix} = \mathcal{B}\left(\begin{pmatrix} u_1(t,x_s^*) \\ u_2(t,x_s^*) \\ u_3(t,x_s^*) \end{pmatrix}, u_4(t,0), x_s - x_s^*\right), \tag{35}$$

where $\mathcal{B} = (B_1, B_2, B_3)^\intercal : \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3$ is of class C^2 and where B_1 and B_2 are defined by

$$B_1 = \left(\lambda_2 G_1(\mathbf{u}(t,x_s^*), x_s) - G_2(\mathbf{u}(t,x_s^*), x_s)\right)\frac{\lambda_1}{\lambda_2 - \lambda_1}, \tag{36}$$

$$B_2 = \left(\lambda_1 G_1(\mathbf{u}(t,x_s^*), x_s) - G_2(\mathbf{u}(t,x_s^*), x_s)\right)\frac{\lambda_2}{\lambda_1 - \lambda_2}. \tag{37}$$

To define B_3, from the boundary conditions (9) and the change of variables (24), (28), we have

$$\begin{aligned} u_3(t,0) = &\frac{-\lambda_4\lambda_3}{\lambda_3+\lambda_4}\left(\frac{u_4(t,0)}{\lambda_4} - \frac{u_3(t,0)}{\lambda_3}\right) \\ &+ \frac{\lambda_3}{\lambda_3+\lambda_4}G_3(\mathbf{u}(t,x_s^*), x_s) \\ &- \frac{\lambda_3}{\lambda_3+\lambda_4}G_4\left(\frac{u_4(t,0)}{\lambda_4} - \frac{u_3(t,0)}{\lambda_3}\right). \end{aligned} \tag{38}$$

From condition (10), applying the implicit function theorem, one obtains

$$B_3 = \mathcal{F}(u_4(t,0), G_3(\mathbf{u}(t,x_s^*), x_s)) \tag{39}$$

in a neighborhood of $\mathbf{u} = 0$ with

$$\mathcal{F}(0,0) = 0, \quad \partial_1\mathcal{F}(0,0) = 0, \quad \partial_2\mathcal{F}(0,0) = \frac{\lambda_3}{\lambda_3+\lambda_4}, \tag{40}$$

where $\partial_i\mathcal{F},\ i = 1, 2$, denote the partial derivative of $\mathcal{F}$ with respect to its ith variable.

Remark 3. For simplicity, in (36)–(39), we have used the following slight abuse of notation adapted from (9):

$$G_i(\mathbf{u}(t,x_s^*), x_s) = G_i\begin{pmatrix} u_1(t,x_s^*) + u_2(t,x_s^*) \\ u_3(t,x_s^*) + u_4(t,x_s^*) \\ \frac{u_1(t,x_s^*)}{\lambda_1} + \frac{u_2(t,x_s^*)}{\lambda_2} \\ x_s - x_s^* \end{pmatrix}, \quad i = 1, 2, 3. \tag{41}$$

From (23), (24), (28) and (34), one can see that, as expressed in (35), $\mathcal{B}$ only depends on $u_i(t,x_s^*)$, $i = 1, 2, 3$, $u_4(t,0)$ and $x_s - x_s^*$ because from (34) $u_4(t,x_s^*)$ can be considered as a function of $u_i(t,x_s^*)$, $i = 1, 2, 3$.

The initial condition (12) becomes

$$\begin{aligned} &\mathbf{u}(0,x) = \mathbf{u}_0(x) = (u_{10}(x), u_{20}(x), u_{30}(x), u_{40}(x))^\intercal, \\ &x_s(0) = x_{s,0} \end{aligned} \tag{42}$$

that satisfies the first order compatibility conditions corresponding to (35). Thus, to study the well-posedness of (1), (3), (4), (9) and (12) is equivalent to study the well-posedness of (30), (32), (34), (35) and (42). We have the following lemma from which one can easily obtain Theorem 2.1.

Lemma 2.1. *For any $T > 0$, there exists $\delta(T) > 0$ such that, for any $x_{s,0} \in (0, L)$ and $\mathbf{u}_0 \in H^2((0,x_s^*); \mathbb{R}^4)$ satisfying the first order compatibility conditions and*

$$|\mathbf{u}_0|_{H^2((0,x_s^*);\mathbb{R}^4)} \leq \delta(T) \text{ and } |x_{s,0} - x_s^*| \leq \delta(T), \tag{43}$$

*the system (30), (32), (34), (35) and (42) has a unique solution $\mathbf{u} \in C^0([0,T];$
$H^2((0,x_s^*);\mathbb{R}^4))$ and $x_s \in C^1([0,T])$. Moreover, the following estimate holds for any $t \in [0,T]$*

$$|\mathbf{u}(t,\cdot)|_{H^2((0,x_s^*);\mathbb{R}^4)} + |x_s(t) - x_s^*| \leq C(T)(|\mathbf{u}_0|_{H^2((0,x_s^*);\mathbb{R}^4)} + |x_{s,0} - x_s^*|). \tag{44}$$

Remark 4. For the proof of Lemma 2.1, we refer to Bastin et al. (2017, Appendix), where the well-posedness of a 2×2 nonlinear hyperbolic system coupled with an ODE was studied. But the proof there can be easily adapted to the 4×4 nonlinear hyperbolic system coupled with an ODE. Noticing that $A(\mathbf{0}, x_s) = 0$ and that, from (32), $\dot{x}_s = 0$ when $\mathbf{u} = \mathbf{0}$, one has

$$\Lambda(x_s) + A(\mathbf{u}, x_s) + x\dot{x}_s B(x_s) = \Lambda(x_s)$$

when $\mathbf{u} = \mathbf{0}$. Thus, (30) is indeed strictly hyperbolic provided that $|\mathbf{u}|_{C^0([0,T];H^2((0,x_s^*);\mathbb{R}^4))}$ is small enough and can be diagonalized in a neighborhood of $\mathbf{u} = \mathbf{0}$. Then we can perform similar fixed point argument as in Bastin et al. (2017, Appendix) by carefully estimating the related norms of the solution along the characteristic curves. The C^1 regularity of x_s is then obtained directly from (32). We omit the details.

This completes the proof of Theorem 2.1. □

3. Exponential stability of the steady state for the H^2-norm

In this section we prove Theorem 1.1.

Proof of Theorem 1.1. It is worth noticing that due to the equivalence of the system (1), (3), (4), (9) and the system (30), (32), (34) and (35), one only needs to prove the exponential stability of the null-steady state of the system (30), (32), (34) and (35) for the H^2-norm.

Motivated by Coron, Bastin, and d'Andréa Novel (2008), see also Bastin and Coron (2016, Section 4.4), and by Bastin et al. (2017), we introduce the following Lyapunov function:

$$V(\mathbf{u}, x_s) = V_1(\mathbf{u})+V_2(\mathbf{u})+V_3(\mathbf{u})+V_4(\mathbf{u}, x_s)+V_5(\mathbf{u}, x_s)+V_6(\mathbf{u}, x_s), \tag{45}$$

where:

$$V_1(\mathbf{u}) = \int_0^{x_s^*} \sum_{i=1}^4 p_i e^{-\frac{\mu}{x_i\lambda_i}x} u_i^2 dx, \tag{46}$$

$$V_2(\mathbf{u}) = \int_0^{x_s^*} \sum_{i=1}^4 p_i e^{-\frac{\mu}{x_i\lambda_i}x} u_{it}^2 dx, \tag{47}$$

$$V_3(\mathbf{u}) = \int_0^{x_s^*} \sum_{i=1}^4 p_i e^{-\frac{\mu}{x_i\lambda_i}x} u_{itt}^2 dx, \tag{48}$$

$$V_4(\mathbf{u}, x_s) = \int_0^{x_s^*} \sum_{i=1}^3 \frac{p_i'}{\lambda_i} e^{-\frac{\mu}{x_i\lambda_i}x} u_i(t,x)(x_s - x_s^*)dx + C_0(x_s - x_s^*)^2, \tag{49}$$

$$V_5(\mathbf{u}, x_s) = \int_0^{x_s^*} \sum_{i=1}^3 \frac{p_i'}{\lambda_i} e^{-\frac{\mu}{x_i\lambda_i}x} u_{it}(t,x)\dot{x}_s dx + C_0(\dot{x}_s)^2, \tag{50}$$

$$V_6(\mathbf{u}, x_s) = \int_0^{x_s^*} \sum_{i=1}^3 \frac{p_i'}{\lambda_i} e^{-\frac{\mu}{x_i\lambda_i}x} u_{itt}(t,x)\ddot{x}_s dx + C_0(\ddot{x}_s)^2, \tag{51}$$

where p_i and C_0 are positive constants that shall be determined later on, while p_i' are constants, not necessarily positive, which will also be determined later on. Besides we impose $C_0 > 3/2$ and we recall that $x_1 = x_2 = 1$, $x_3 = x_s^*/(L - x_s^*)$ and $x_4 = x_s^*/(x_s^* - L)$. In the following we may denote for simplicity $V_i := V_i(\mathbf{u}, x_s)$ and $|\mathbf{u}|_{H^2} := |\mathbf{u}(t,\cdot)|_{H^2((0,x_s^*);\mathbb{R}^4)}$ in the computations. Similarly to what is done in Bastin et al. (2017), from the Cauchy–Schwarz inequality and as $C_0 > 3/2$, it can be shown that the Lyapunov function V considered here is equivalent to $(|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^2$ provided that $|\mathbf{u}|_{H^2} + |x_s - x_s^*|$ is small enough and that

$$\max_i \left(\frac{p_i'^2 x_i}{\mu\lambda_i p_i}(1 - e^{-\frac{\mu}{x_i\lambda_i}x_s^*}) \right) < 2. \tag{52}$$

This means that, under condition (52), there exists $\bar{\rho} > 0$ and $\bar{C}$ such that, for every $T > 0$ and $\mathbf{u} \in C^0([0,T]; H^2((0,x_s^*);\mathbb{R}^4))$ and for every $x_s \in C^1([0,T])$ solution of the system (30), (32), (34) and (35), if $|\mathbf{u}|_{H^2} + |x_s - x_s^*| \leq \bar{\rho}$

$$\frac{1}{\bar{C}}(|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^2 \leq V(\mathbf{u}, x_s) \leq \bar{C}(|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^2. \tag{53}$$

This can be proved by direct estimations (see Bastin et al., 2017 for more details).

From the boundary condition (35), as $\mathcal{B}$ is of class C^2, we have

$$\begin{aligned}\mathbf{v}(t,0) = &\partial_1\mathcal{B}(\mathbf{0},0,0)\mathbf{v}(t,x_s^*) + \partial_2\mathcal{B}(\mathbf{0},0,0)u_4(t,0)\\ &+ \partial_3\mathcal{B}(\mathbf{0},0,0)(x_s - x_s^*) + O((|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^2),\end{aligned} \tag{54}$$

where $\mathbf{v} = (u_1, u_2, u_3)^{\mathsf{T}}$ is the vector of the components of $\mathbf{u}$ on which the feedback (35) applies. This notation is practical as it isolates u_1, u_2 and u_3 from u_4 on which we have no control and whose boundary condition is imposed by the condition (34). In (54), the notation $\partial_1\mathcal{B}$ is the 3×3 Jacobian matrix of the vector-valued function $\mathcal{B}$ with respect to its first variable which is a 3-D vector (see the expression of $\mathcal{B}$ in (35)). From (36)–(40), one can check that $\partial_2\mathcal{B}(\mathbf{0},0,0) \equiv \mathbf{0}$. Moreover, from (36)–(39), noticing (40), it can be verified that the matrix K and the vector $(b_1, b_2, b_3)^{\mathsf{T}}$ defined in (11) satisfy

$$K = (k_{ij})_{(i,j)\in\{1,2,3\}^2} = \partial_1\mathcal{B}(\mathbf{0},0,0), \quad \partial_3\mathcal{B}(\mathbf{0},0,0) = (b_1,b_2,b_3)^{\mathsf{T}}. \tag{55}$$

Let $\bar{T} > 0$ be given and let $x_{s,0} \in (0,L)$ and $\mathbf{u}_0 \in H^2((0,x_s^*);\mathbb{R}^4)$ satisfying the first order compatibility conditions and (43). Let $\mathbf{u} \in C^0([0,\bar{T}]; H^2((0,x_s^*);\mathbb{R}^4))$ and $x_s \in C^1([0,\bar{T}])$ be the solution of the system (30), (32), (34), (35) and (42). Let us start with the case where $\mathbf{u}$ is of class C^3. Taking the time derivative of V_1 along this solution and integrating by parts, we obtain

$$\begin{aligned}\frac{dV_1}{dt} &= -\mu V_1 - \left[\sum_{i=1}^4 p_i x_i \lambda_i e^{-\frac{\mu}{x_i\lambda_i}x} u_i^2\right]_0^{x_s^*}\\ &\quad + O\left((|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^3\right).\end{aligned} \tag{56}$$

By differentiating (30), similarly as (56), we can obtain

$$\begin{aligned}\frac{dV_2}{dt} &= -\mu V_2 - \left[\sum_{i=1}^4 p_i x_i \lambda_i e^{-\frac{\mu}{x_i\lambda_i}x} u_{it}^2\right]_0^{x_s^*}\\ &\quad + O\left((|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^3\right).\end{aligned} \tag{57}$$

Now, let us deal with the V_3 term. To that end, we derive from (30) that

$$\begin{aligned}&\mathbf{u}_{ttt} + \Lambda(x_s)\mathbf{u}_{ttx} + 2\dot{x}_s\Lambda'(x_s)\mathbf{u}_{tx} + (\Lambda''(x_s)(\dot{x}_s)^2 + \Lambda'(x_s)\ddot{x}_s)\mathbf{u}_x\\ &+ (A(\mathbf{u},x_s)\mathbf{u}_x)_{tt} + x\dddot{x}_s B(x_s)\mathbf{u}_x + 2x\ddot{x}_s(B(x_s)\mathbf{u}_x)_t\\ &+ x\dot{x}_s(B(x_s)\mathbf{u}_x)_{tt} = 0.\end{aligned} \tag{58}$$

Thus,

$$\begin{aligned}\frac{dV_3}{dt} &= -\mu V_3 - \left[\sum_{i=1}^4 p_i x_i \lambda_i e^{-\frac{\mu}{x_i\lambda_i}x} u_{itt}^2\right]_0^{x_s^*}\\ &\quad - \int_0^{x_s^*} \sum_{i=1}^4 2p_i e^{-\frac{\mu}{x_i\lambda_i}x} x\dddot{x}_s u_{itt}\left(\sum_{j=1}^4 B_{ij}u_{jx}\right) dx\\ &\quad + O\left((|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^3\right).\end{aligned} \tag{59}$$

We observe that now $\dddot{x}_s$ appears. As $\dddot{x}_s$ is proportional to $\mathbf{u}_{tt}(x_s^*)$, it cannot be bounded by $|\mathbf{u}|_{H^2}$. However, we can use Young's inequality to compensate it with the boundary terms. Using (54), one has

$$\begin{aligned}\frac{dV_3}{dt} &\leq -\mu V_3 - \sum_{i=1}^4 \left(\left(p_i x_i \lambda_i e^{-\frac{\mu x_s^*}{x_i\lambda_i}} + O(|\mathbf{u}|_{H^2})\right)\right.\\ &\quad \left.\times\, u_{itt}^2(x_s^*) - u_{itt}^2(0)\right)\\ &\quad + O\left((|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^3\right).\end{aligned} \tag{60}$$

Differentiating (49), from (30), one has

$$\begin{aligned}\frac{dV_4}{dt} =&(x_s - x_s^*)\int_0^{x_s^*} \sum_{i=1}^3 \frac{p_i'}{\lambda_i} e^{-\frac{\mu}{x_i\lambda_i}x} u_{it}(t,x)dx\\ &+ \dot{x}_s \int_0^{x_s^*} \sum_{i=1}^3 \frac{p_i'}{\lambda_i} e^{-\frac{\mu}{x_i\lambda_i}x} u_i(t,x)dx + 2C_0\dot{x}_s(x_s - x_s^*)\\ =& -(x_s - x_s^*)\int_0^{x_s^*} \sum_{i=1}^3 x_i p_i' e^{-\frac{\mu}{x_i\lambda_i}x} u_{ix}(t,x)dx\\ &+ d\left(u_1(x_s^*) + u_2(x_s^*) - u_3(x_s^*) - u_4(x_s^*)\right)\\ &\times \int_0^{x_s^*} \sum_{i=1}^3 \frac{p_i'}{\lambda_i} e^{-\frac{\mu}{x_i\lambda_i}x} u_i(t,x)dx\\ &+ 2dC_0(x_s - x_s^*)\left(u_1(x_s^*) + u_2(x_s^*) - u_3(x_s^*) - u_4(x_s^*)\right)\\ &+ O\left((|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^3\right),\end{aligned} \tag{61}$$

where we recall that $d = (H_1^* - H_2^*)^{-1} < 0$ is defined in (11). Thus, integrating by parts and using (34),

$$\begin{aligned}
\frac{dV_4}{dt} &= -(x_s - x_s^*)\left[\sum_{i=1}^{3} x_i p_i' e^{-\frac{\mu}{x_i\lambda_i}x} u_i(t,x)\right]_0^{x_s^*} \\
&\quad -\mu(V_4 - C_0(x_s - x_s^*)^2) \\
&\quad +d\left(u_1(x_s^*)\left(1-\frac{\lambda_1}{\lambda_4}\right) + u_2(x_s^*)\left(1-\frac{\lambda_2}{\lambda_4}\right)\right. \\
&\quad \left. -u_3(x_s^*)\left(1+\frac{\lambda_3}{\lambda_4}\right)\right) \\
&\quad \times\left(2C_0(x_s - x_s^*) + \int_0^{x_s^*}\sum_{i=1}^{3}\frac{p_i'}{\lambda_i}e^{-\frac{\mu}{x_i\lambda_i}x}u_i(t,x)dx\right) \\
&\quad +O\left((|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^3\right).
\end{aligned} \tag{62}$$

Similarly for V_5, from (30), one has

$$\begin{aligned}
\frac{dV_5}{dt} &= -\dot{x}_s\left[\sum_{i=1}^{3} x_i p_i' e^{-\frac{\mu}{x_i\lambda_i}x} u_{it}(t,x)\right]_0^{x_s^*} - \mu(V_5 - C_0(\dot{x}_s)^2) \\
&\quad + d\left(\sum_{i=1}^{3}\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right)s_i u_{it}(x_s^*)\right) \\
&\quad \times\left(2C_0\dot{x}_s + \int_0^{x_s^*}\sum_{i=1}^{3}\frac{p_i'}{\lambda_i}e^{-\frac{\mu}{x_i\lambda_i}x}u_{it}(t,x)dx\right) \\
&\quad + O\left((|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^3\right).
\end{aligned} \tag{63}$$

By (58), for V_6, one has

$$\begin{aligned}
\frac{dV_6}{dt} &= -\ddot{x}_s\left[\sum_{i=1}^{3} x_i p_i' e^{-\frac{\mu}{x_i\lambda_i}x} u_{itt}(t,x)\right]_0^{x_s^*} - \mu(V_6 - C_0(\ddot{x}_s)^2) \\
&\quad + d\left(\sum_{i=1}^{3}\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right)s_i u_{itt}(x_s^*)\right) \\
&\quad \times\left(2C_0\ddot{x}_s + \int_0^{x_s^*}\sum_{i=1}^{3}\frac{p_i'}{\lambda_i}e^{-\frac{\mu}{x_i\lambda_i}x}u_{itt}(t,x)dx\right) \\
&\quad - \int_0^{x_s^*}\sum_{i=1}^{3}\frac{p_i'}{\lambda_i}e^{-\frac{\mu}{x_i\lambda_i}x}x\dddot{x}_s\left(\sum_{j=1}^{4}B_{ij}u_{jx}\right)(x_s - x_s^*)\,dx \\
&\quad + O\left((|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^3\right).
\end{aligned} \tag{64}$$

Dealing with the $\dddot{x}_s$ term in (64) similarly as for V_3, we have

$$\begin{aligned}
\frac{dV_6}{dt} &= -\ddot{x}_s\sum_{i=1}^{3}\left(\left(x_i p_i' e^{-\frac{\mu}{x_i\lambda_i}x_s^*} + O\left(|\mathbf{u}|_{H^2}\right)\right)u_{itt}(x_s^*) - x_i p_i' u_{itt}(0)\right) \\
&\quad - \mu(V_6 - C_0(\ddot{x}_s)^2) \\
&\quad + d\left(\sum_{i=1}^{3}\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right)s_i u_{itt}(x_s^*)\right) \\
&\quad \times\left(2C_0\ddot{x}_s + \int_0^{x_s^*}\sum_{i=1}^{3}\frac{p_i'}{\lambda_i}e^{-\frac{\mu}{x_i\lambda_i}x}u_{itt}(t,x)dx\right) \\
&\quad + O\left((|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^3\right).
\end{aligned} \tag{65}$$

Note that $V_2 + V_5$ has the same structure as $V_1 + V_4$ with u_i and $x_s - x_s^*$ being replaced by u_{it} and $\dot{x}_s$ respectively. The same applies for $V_3 + V_6$ by replacing u_i and $x_s - x_s^*$ in $V_1 + V_4$ with u_{itt} and $\ddot{x}_s$ respectively. Hence, we only need to analyze $V_1 + V_4$. From (56) and (62), recalling that $s_i = 1$ if $i \in \{1, 2\}$ and $s_3 = -1$, one has

$$\begin{aligned}
\frac{d(V_1+V_4)}{dt} &= -\left[\sum_{i=1}^{4} p_i x_i \lambda_i e^{-\frac{\mu}{x_i\lambda_i}x}u_i^2\right]_0^{x_s^*} - \mu(V_1 + V_4) \\
&\quad - (x_s - x_s^*)\left[\sum_{i=1}^{3} x_i p_i' e^{-\frac{\mu}{x_i\lambda_i}x}u_i\right]_0^{x_s^*} + \mu C_0(x_s - x_s^*)^2 \\
&\quad + d\left(\sum_{i=1}^{3} u_i(x_s^*)s_i\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right)\right) \\
&\quad \times\left(2C_0(x_s - x_s^*) + \int_0^{x_s^*}\sum_{i=1}^{3}\frac{p_i'}{\lambda_i}e^{-\frac{\mu}{x_i\lambda_i}x}u_i(t,x)dx\right) \\
&\quad + O\left((|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^3\right).
\end{aligned} \tag{66}$$

Using now the boundary conditions (34), (54) and noticing (55), (66) becomes

$$\begin{aligned}
\frac{d(V_1+V_4)}{dt} &= -\mu(V_1 + V_4) \\
&\quad - \mathbf{v}(x_s^*)^{\mathsf{T}}\left(F(x_s^*,\mu) - K^{\mathsf{T}}F(0,\mu)K\right)\mathbf{v}(x_s^*) \\
&\quad - \frac{x_4 p_4}{\lambda_4}e^{-\frac{\mu}{x_4\lambda_4}x_s^*}(\lambda_1 u_1(x_s^*) + \lambda_2 u_2(x_s^*) + \lambda_3 u_3(x_s^*))^2 \\
&\quad - \lambda_4|x_4|p_4 u_4^2(0) + \sum_{i=1}^{3} x_i p_i \lambda_i b_i^2(x_s - x_s^*)^2 \\
&\quad + 2\sum_{i=1}^{3} x_i p_i \lambda_i b_i\left(\sum_{j=1}^{3} k_{ij}u_j(x_s^*)(x_s - x_s^*)\right) \\
&\quad - \left(\sum_{i=1}^{3} u_i(x_s^*)(x_s - x_s^*)\left(x_i p_i' e^{-\frac{\mu}{x_i\lambda_i}x_s^*}\right.\right. \\
&\quad \left.\left. -2dC_0 s_i\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right)\right) - \sum_{j=1}^{3} k_{ij}u_j(x_s^*)(x_s - x_s^*)x_i p_i'\right) \\
&\quad + \sum_{i=1}^{3} x_i p_i' b_i(x_s - x_s^*)^2 + \mu C_0(x_s - x_s^*)^2 \\
&\quad + d\left(\sum_{i=1}^{3} u_i(x_s^*)s_i\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right)\right) \\
&\quad \times\left(\int_0^{x_s^*}\sum_{j=1}^{3}\frac{p_j'}{\lambda_j}e^{-\frac{\mu}{x_j\lambda_j}x}u_j(t,x)dx\right) \\
&\quad + O\left((|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^3\right),
\end{aligned} \tag{67}$$

where

$$F(x,\mu) = \operatorname{diag}\left(\lambda_i p_i x_i e^{-\frac{\mu}{x_i\lambda_i}x}, i \in \{1,2,3\}\right). \tag{68}$$

We observe that, except from the last product proportional to d, a quadratic form in $(\mathbf{v}(x_s^*)^{\mathsf{T}}, u_4(0), x_s - x_s^*)$ appears. Using successively the Young and Cauchy–Schwarz inequalities to deal with the last product, and noticing that

$$\int_0^{x_s^*} e^{-\frac{\mu}{x_i\lambda_i}x}dx = \frac{\lambda_i x_i}{\mu}(1 - e^{-\frac{\mu}{x_i\lambda_i}x_s^*}), \tag{69}$$

we get that, for any $j \in \{1, 2, 3\}$,

$$d\left(\sum_{i=1}^{3} u_i(x_s^*)s_i\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right)\right)\left(\int_0^{x_s^*}\frac{p_j'}{\lambda_j}e^{-\frac{\mu}{x_j\lambda_j}x}u_j(t,x)dx\right) \leq$$
$$\frac{\varepsilon_j}{\mu}\left(\frac{p_j'^2 x_j(1-e^{-\frac{\mu}{x_j\lambda_j}x_s^*})}{\lambda_j p_j}\right)\left(\int_0^{x_s^*} p_j e^{-\frac{\mu}{x_j\lambda_j}x}u_j^2(t,x)dx\right)$$
$$+\frac{d^2}{4\varepsilon_j}\left(\sum_{i=1}^{3}u_i(x_s^*)\left(1-s_i\frac{\lambda_i}{\lambda_4}\right)s_i\right)^2.$$

Using again the Cauchy–Schwarz inequality, we get that

$$\begin{aligned}&\frac{d^2}{4\varepsilon_j}\left(\sum_{i=1}^{3}u_i(x_s^*)\left(1-s_i\frac{\lambda_i}{\lambda_4}\right)s_i\right)^2\\ &\leq \frac{d^2}{4\varepsilon_j}\left(\sum_{i=1}^{3}u_i^2(x_s^*)\left(1-s_i\frac{\lambda_i}{\lambda_4}\right)^2\left(\sum_{j=1}^{3}e^{\frac{\mu x_s^*}{x_i\lambda_i}-\frac{\mu x_s^*}{x_j\lambda_j}}\right)\right).\end{aligned} \tag{70}$$

Therefore, combining (67)–(70), one has

$$\begin{aligned}\frac{d(V_1+V_4)}{dt} \leq &-\mu(V_1+V_4) - \mathbf{v}(x_s^*)^{\mathsf{T}}\Bigg(F(x_s^*,\mu) - K^{\mathsf{T}}F(0,\mu)K\\ &-\frac{d^2}{4}\left(\sum_{k=1}^{3}\frac{1}{\varepsilon_k}\right)\operatorname{diag}\left(\left(\sum_{j=1}^{3}e^{\frac{\mu x_s^*}{x_i\lambda_i}-\frac{\mu x_s^*}{x_j\lambda_j}}\right)\right.\\ &\times\left.\left(1-s_i\frac{\lambda_i}{\lambda_4}\right)^2\right)_{i\in\{1,2,3\}}\Bigg)\mathbf{v}(x_s^*)\\ &-\frac{x_4p_4}{\lambda_4}e^{-\frac{\mu}{x_4\lambda_4}x_s^*}(\lambda_1u_1(x_s^*)+\lambda_2u_2(x_s^*)\\ &+\lambda_3u_3(x_s^*))^2 - \lambda_4|x_4|p_4u_4^2(0)\\ &+\left(\mu C_0+\sum_{i=1}^{3}(x_ip_i\lambda_ib_i^2+x_ip_i'b_i)\right)(x_s-x_s^*)^2\\ &+\sum_{i=1}^{3}\Bigg(\frac{\varepsilon_i}{\mu}\left(\frac{p_i'^2x_i(1-e^{-\frac{\mu}{\lambda_ix_i}x_s^*})}{\lambda_ip_i}\right)\\ &\times\left(\int_0^{x_s^*}p_ie^{-\frac{\mu}{x_i\lambda_i}x}u_i^2(t,x)dx\right)\Bigg)\\ &+\sum_{j=1}^{3}\Bigg(2dC_0s_j\left(1-s_j\frac{\lambda_j}{\lambda_4}\right)-x_jp_j'e^{-\frac{\mu}{x_j\lambda_j}x_s^*}\\ &+\sum_{i=1}^{3}(2x_ip_i\lambda_ib_ik_{ij}+x_ip_i'k_{ij})\Bigg)u_j(x_s^*)(x_s-x_s^*)\\ &+O\left((|\mathbf{u}|_{H^2}+|x_s-x_s^*|)^3\right).\end{aligned} \tag{71}$$

In order to obtain an exponential decay, we first choose ε_i such that

$$\frac{1}{\varepsilon_i} = \frac{2p_i'^2x_i(1-e^{-\frac{\mu}{x_i\lambda_i}x_s^*})}{\mu^2\lambda_ip_i}, \quad i=1,2,3. \tag{72}$$

Therefore, (71) becomes

$$\begin{aligned}\frac{d(V_1+V_4)}{dt} \leq &-\frac{\mu}{2}V_1 - \mu V_4 - \mathbf{v}(x_s^*)^{\mathsf{T}}\Bigg(F(x_s^*,\mu)-K^{\mathsf{T}}F(0,\mu)K\\ &-\frac{d^2}{4}\left(\sum_{k=1}^{3}\frac{1}{\varepsilon_k}\right)\widetilde{D}(\mu)\Bigg)\mathbf{v}(x_s^*)\\ &-\frac{x_4p_4}{\lambda_4}e^{-\frac{\mu}{x_4\lambda_4}x_s^*}(\lambda_1u_1(x_s^*)+\lambda_2u_2(x_s^*)+\lambda_3u_3(x_s^*))^2\\ &-\lambda_4|x_4|p_4u_4^2(0)\\ &+\left(\mu C_0+\sum_{i=1}^{3}(x_ip_i\lambda_ib_i^2+x_ip_i'b_i)\right)(x_s-x_s^*)^2\\ &+\sum_{j=1}^{3}\Bigg(2dC_0s_j\left(1-s_j\frac{\lambda_j}{\lambda_4}\right)-x_jp_j'e^{-\frac{\mu}{x_j\lambda_j}x_s^*}\\ &+\sum_{i=1}^{3}(2x_ip_i\lambda_ib_ik_{ij}+x_ip_i'k_{ij})\Bigg)u_j(x_s^*)(x_s-x_s^*)\\ &+O\left((|\mathbf{u}|_{H^2}+|x_s-x_s^*|)^3\right).\end{aligned} \tag{73}$$

We clearly see now two terms proportional to V_1 and V_4 respectively that will bring the exponential decay, and a quadratic form in $(\mathbf{v}(x_s^*)^{\mathsf{T}}, u_4(0), x_s - x_s^*)$ appears. In order to simplify the quadratic form by canceling the cross terms, we choose

$$p_i' = 2\frac{dC_0s_i\left(1-s_i\frac{\lambda_i}{\lambda_4}\right)e^{\frac{\mu}{x_i\lambda_i}x_s^*}}{x_i}, \quad i=1,2,3. \tag{74}$$

Observe that from (18), one always has $b_ix_ip_i' < 0$ for $i = 1, 2, 3$, thus we can choose

$$p_i = -\frac{p_i'}{2b_i\lambda_i} > 0, \quad i=1,2,3. \tag{75}$$

Therefore we have, using (74), (75) and Young's inequality

$$\begin{aligned}\frac{d(V_1+V_4)}{dt} \leq &-\frac{\mu}{2}V_1 - \mu V_4 - \mathbf{v}(x_s^*)^{\mathsf{T}}\\ &\times\Bigg(F(x_s^*,\mu)-K^{\mathsf{T}}F(0,\mu)K-\frac{d^2}{4}\left(\sum_{k=1}^{3}\frac{1}{\varepsilon_k}\right)\widetilde{D}(\mu)\\ &-\operatorname{diag}\left(\frac{3|x_4|p_4\lambda_i^2}{\lambda_4}e^{-\frac{\mu}{x_4\lambda_4}x_s^*}\right)_{i\in\{1,2,3\}}\Bigg)\mathbf{v}(x_s^*)\\ &-\lambda_4|x_4|p_4u_4^2(0)+\left(\mu C_0-\frac{1}{2}\sum_{i=1}^{3}|x_ip_i'b_i|\right)\\ &\times(x_s-x_s^*)^2\\ &+O\left((|\mathbf{u}|_{H^2}+|x_s-x_s^*|)^3\right).\end{aligned} \tag{76}$$

Observe that the conditions (18) and (19) are satisfied for $\gamma > 0$, but as the inequalities are strict, there exists $\mu > \gamma$ such that (18) and (19) are also verified with μ instead of γ. We choose such μ and using (74), one can see that

$$\mu C_0 - \frac{1}{2}\sum_{i=1}^{3}|x_ip_i'b_i| < 0, \tag{77}$$

and one can also check from (72), (74), (75), (11) and condition (19) and using that $-dC_0 > 0$, the matrix defined by

$$F(x_s^*,\mu)-K^{\mathsf{T}}F(0,\mu)K-\frac{d^2}{4}\left(\sum_{k=1}^{3}\frac{1}{\varepsilon_k}\right)\widetilde{D}(\mu) \tag{78}$$

is positive definite. This implies that there exists $p_4 > 0$ such that the quadratic form in $\mathbf{v}(x_s^*)$ in (76) is non-positive. Therefore

$$\frac{d(V_1+V_4)}{dt} \leq -\frac{\mu}{2}V_1 - \mu V_4 + O\left((|\mathbf{u}|_{H^2}+|x_s-x_s^*|)^3\right). \tag{79}$$

As $\mu > \gamma$, at least if $|\mathbf{u}|_{H^2} + |x_s - x_s^*|$ is small enough which can be guaranteed from Lemma 2.1 by requiring $\delta(\bar{T})$ small enough

$$\frac{d(V_1+V_4)}{dt} \leq -\frac{\gamma}{2}(V_1+V_4), \tag{80}$$

thus,

$$\frac{dV}{dt} \leq -\frac{\gamma}{2}V. \tag{81}$$

We have derived (81) under the assumption that the trajectories of (30), (32), (34) and (35) are of class C^3, but one can use a density argument to generalize the result for trajectories in $C^0([0, \bar{T}]; H^2((0, x_s^*); \mathbb{R}^4))$ by noticing that γ does not depend on any C^2 or C^3-norm of $\mathbf{u}$. The inequality (81) is then understood in the distribution sense. One can refer to Bastin et al. (2017) or Bastin and Coron (2016, Comment 4.6) for more details.

By the equivalence between the Lyapunov function V and $(|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^2$ if this last quantity is small, we get immediately the exponential stability of the null steady state of the system (30), (32), (34) and (35) for the H^2-norm with decay rate $\gamma/4$. It remains to check that under assumption (19), (52) holds with p_i' and p_i defined as (74) and (75). Indeed,

$$\max_i \left(\frac{p_i'^2 x_i}{\mu \lambda_i p_i} (1 - e^{-\frac{\mu}{x_i \lambda_i} x_s^*}) \right) < \frac{4C_0}{3}, \tag{82}$$

therefore there exists $C_0 > 3/2$ such that the condition (52) is satisfied.

So far $\delta(\bar{T})$ depends on $\bar{T}$, we next prove that for any given $T > 0$, we can choose δ^* independent of T such that (81) holds on $(0, T)$ as required in Definition 1.1.

Let us now assume that $x_{s,0} \in (0, L)$ and $\mathbf{u}_0 \in H^2((0, x_s^*); \mathbb{R}^4)$ satisfying the first order compatibility conditions and

$$|\mathbf{u}_0|_{H^2((0,x_s^*);\mathbb{R}^4)} + |x_{s,0} - x_s^*| < \bar{\rho} \text{ and } V(\mathbf{u}_0, x_{s,0}) \leq \nu, \tag{83}$$

where $\nu > 0$ is going to be chosen small enough. Then, for any $t \in [0, \bar{T}]$, at least if $\nu > 0$ is small enough, from (44), (53) and (81),

$$|\mathbf{u}(t)|_{H^2((0,x_s^*);\mathbb{R}^4)} + |x_s(t) - x_s^*| < \bar{\rho} \text{ and } V(\mathbf{u}(t), x_s(t)) \leq \nu. \tag{84}$$

Using (84) for $t = \bar{T}$ one can keep going on $[\bar{T}, 2\bar{T}]$ and then on $[2\bar{T}, 3\bar{T}]$, etc. So we get that, for every $j = 1, 2, 3, \ldots,$

$$V(\mathbf{u}(t), x_s(t)) \leq \nu, \ t \in [(j-1)\bar{T}, j\bar{T}], \tag{85}$$

$$(|\mathbf{u}(t)|_{H^2((0,x_s^*);\mathbb{R}^4)} + |x_s(t) - x_s^*|) < \bar{\rho}, \ t \in [(j-1)\bar{T}, j\bar{T}], \tag{86}$$

$$\frac{dV}{dt} \leq -\frac{\gamma}{2}V \text{ in the distribution sense on } (0, j\bar{T}). \tag{87}$$

Noticing (28), there exists a δ^* such that if (13)–(14) hold, one has (83). Thus, noticing also that for any $T > 0$ there exists $j \in \mathbb{N}$ such that $(0, T) \subset (0, j\bar{T})$, one gets that the steady state $((H^*, Q^*)^\intercal, x_s^*)$ is locally exponentially stable for the H^2-norm with decay rate $\gamma/4$. The proof of Theorem 1.1 is thus complete. □

Remark 5. Given the assumptions of Theorem 1.1, it is obvious that this stability result is robust with respect to small variations of G in the feedback control. However, it is actually also robust with respect to small variations of G_4. Indeed, if $|G_4'(0) + \lambda_4|$ is sufficiently small but with a bound independent of the state $(H, Q)^\intercal$ and x_s, we can still define $\mathcal{B}$ as in (36)–(39) using the implicit function theorem. Then looking at (54), $\partial_2\mathcal{B}(\mathbf{0}, 0, 0) \neq \mathbf{0}$, but for any $\delta > 0$, $|\partial_2\mathcal{B}(\mathbf{0}, 0, 0)| < \delta$ provided $|G_4'(0) + \lambda_4|$ is sufficiently small. Then all the additional terms about $u_4^2(0)$ and $u_i^2(x_s^*)$, $i = 1, 2, 3$ will be compensated by the fact that $p_4 > 0$ in (73) and that $|G_4'(0) + \lambda_4|$ is sufficiently small. The rest of the proof is the same as in the case where $G_4'(0) = -\lambda_4$.

4. Conclusion

In this article, we have considered the problem of the boundary feedback stabilization of an open channel with a hydraulic jump. We focused on the case where the channel has a rectangular cross section without friction or slope. The channel dynamics are modeled by a version of the homogeneous Saint-Venant equations with the water level H and the flow rate Q as state variables. The hydraulic jump is represented by a discontinuous shock solution of the system. The main contribution of this paper is to analyze the boundary feedback stabilization of the system with a general class of static feedback controls that require pointwise measurements of the level and the flux at the boundary and in the immediate vicinity of the hydraulic jump. In order to prove the well-posedness of the system, we first introduce a change of variables which allows to transform the Saint-Venant equations with shock wave solutions into an equivalent 4×4 quasilinear hyperbolic system which is parametrized by the jump position but has shock-free solutions. Then, by a Lyapunov approach, we show that, for the considered class of boundary feedback controls, the exponential stability in H^2-norm of the steady state can be achieved with an arbitrary decay rate and with an exponential stabilization of the desired location of the hydraulic jump. Compared with previous results in the literature for classical solutions of quasilinear hyperbolic systems, the H^2-Lyapunov function introduced in Coron et al. (2008) (see also Bastin & Coron, 2016, Section 4.4) has to be augmented with suitable extra terms for the analysis of the stabilization of the jump position. In the case where the cross section is irregular and with friction or slope, the jump stabilization issue is much more challenging and remains an open problem.

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Appendix

In this appendix we prove that there always exists G such that K and $(b_1, b_2, b_3)^\intercal$ defined in (11) satisfy (18)–(19). Let us first point out that, for every $K \in \mathbb{R}^{3\times 3}$, there exists a linear map $G : \mathbb{R}^4 \to \mathbb{R}^3$ such that the third equation of (11) holds. Hence it remains only to show that there always exist K and $(b_1, b_2, b_3)^\intercal$ satisfying (18) and (19). In the special case where $K = \text{diag}(k_i, i \in \{1, 2, 3\})$, the condition that the matrix defined in (19) is positive definite becomes

$$k_i^2 < e^{-\frac{\gamma}{x_i \lambda_i} x_s^*} D_i, \quad \forall i \in \{1, 2, 3\}, \tag{88}$$

with

$$D_i := 1 - \frac{2d^2 b_i}{\gamma^2 s_i (1 - s_i \frac{\lambda_i}{\lambda_4})} \left(\sum_{k=1}^{3} b_k s_k (1 - s_k \frac{\lambda_k}{\lambda_4})(e^{\frac{\gamma x_s^*}{x_k \lambda_k}} - 1) \right) \times (\sum_{j=1}^{3} e^{\frac{\gamma x_s^*}{x_i \lambda_i} - \frac{\gamma x_s^*}{x_j \lambda_j}})(1 - s_i \frac{\lambda_i}{\lambda_4})^2. \tag{89}$$

Let us look at a limiting case in (18) and take $b_i = -\gamma e^{-\gamma x_s^*/(x_i \lambda_i)} / 3ds_i \left(1 - s_i \frac{\lambda_i}{\lambda_4}\right)$. Then we have

$$D_i = 1 - \frac{2}{9} \left(\sum_{k=1}^{3} (1 - e^{-\frac{\gamma x_s^*}{x_k \lambda_k}}) \right) (\sum_{j=1}^{3} e^{-\frac{\gamma x_s^*}{x_j \lambda_j}}). \tag{90}$$

We denote $y = \left(\sum_{k=1}^{3} e^{-\frac{\gamma x_s^*}{x_k \lambda_k}} \right)$. Thus we get

$$D_i := 1 - \frac{2}{3} y + \frac{2}{9} y^2. \tag{91}$$

This is a second order polynomial with negative discriminant, thus D_i is always strictly positive. As D_i depends continuously on b_i, this implies that there exist $K = \mathrm{diag}(k_i, i \in \{1, 2, 3\})$ and $(b_1, b_2, b_3)^{\mathsf{T}}$, satisfying (18) and (19).

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