A necessary and sufficient robustness condition for boundary observer-controllers that stabilize 2×2 hyperbolic systems

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Abstract

We investigate the robustness of the stabilization of a class of 2×2 hyperbolic systems with boundary control subject to uncertainties in characteristic velocities. We focus on the single-input-single-output setting, examining separately the cases of co-located and anti-located sensing and actuation. In both cases, stabilizing controls are designed via the backstepping technique.

For the co-located case, we recall that output feedback stabilization is unconditionally robust with respect to arbitrarily small perturbations in the characteristic velocities.

The main contribution addresses the anti-located case. We provide a sharp necessary and sufficient dissipative boundary condition ensuring robustness against small perturbations in the characteristic velocities. The originality of our contribution lies essentially in the proof strategy for robustness in L^2-norm. The analysis proceeds in three stages: (i) robustness is first established in H^1-norm using a basic quadratic Lyapunov function; (ii) robustness in H^1 is shown to imply robustness in L^2 because there is an isomorphism between the respective solution spaces, revealing in addition a new, less intuitive Lyapunov function; and (iii) using degree theory, we prove that the robustness boundary condition is also necessary, by showing that there exist arbitrarily small perturbations for which the associated eigenvalue problem has unstable solutions whenever the condition is not satisfied.

Keywords: Linear hyperbolic system; Boundary control; Feedback stabilization; Observer-Controller; Robustness; Dissipative condition.

1 Introduction and motivation

Hyperbolic systems of conservation and balance laws (e.g. [1], [2], [3], [4], [5],) are an important class of partial differential equations (PDEs). They are ubiquitous in many fields of natural and human sciences, including for instance physics, fluid mechanics, biology or economics. Their solutions are known to potentially exhibit non-trivial features. A classical example is the spontaneous formation of discontinuities (shock waves), even starting from smooth initial data, when the system is genuinely nonlinear. Other pathological behaviours can also occur in nonlinear hyperbolic systems, often associated with a lack of robustness of the solutions with respect to perturbations of the wave-propagation speeds.

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In control theory, the stabilization of hyperbolic systems has attracted much interest over the past three decades, both for its usefulness in applications and for the underlying mathematical challenge. Despite these efforts, the stabilization of PDE systems by means of boundary control remains an open question and a challenging problem in general. For one-dimensional spatial domains, however, a systematic method known as backstepping was developed by Krstić and collaborators (e.g.[6, 7, 8]), initially for parabolic systems and subsequently extended to hyperbolic systems [9, 10] (see also [11, 12, 13, 14, 15] for recent generalizations).

Over the past two decades, the backstepping method has become a standard tool for stabilization of linear hyperbolic systems, owing to its ability to deliver strong results under relatively mild assumptions. During this period, more than a thousand papers have studied or applied this method. However, a notable limitation, and probably the main one, is that the backstepping method generally produces complex observer-based control laws, for which the issue of robustness is seldom addressed, particularly with respect to wave propagation speeds. This question is essential for extending the control laws to quasi-linear systems, which encompasses most physical hyperbolic systems of interest. Although the question has been known for nearly two decades, it remains largely open so far. The purpose of this article is to provide a sharp answer for 2×2 systems.

Specifically, we consider the application of observer-controllers to the following 2×2 linear hyperbolic (open-loop) control system defined on the domain $\{(t,x) | t \geqslant 0, x \in [0,1]\}$:

$$\partial_t y_1(t,x) + \lambda_1 \partial_x y_1(t,x) + c_1 y_2(t,x) = 0, \tag{1.1a}$$

$$\partial_t y_2(t,x) - \lambda_2 \partial_x y_2(t,x) + c_2 y_1(t,x) = 0, \tag{1.1b}$$

$$y_1(t,0) = k_1 y_2(t,0) + U(t), \tag{1.1c}$$

$$y_2(t,1) = k_2 y_1(t,1), \tag{1.1d}$$

where (y_1, y_2) is the state, $U(t)$ is the control input, $\lambda_1 > 0$ and $\lambda_2 > 0$ are the transport velocity parameters, $c_1 \neq 0$ and $c_2 \neq 0$ are the coupling (or source terms) parameters, k_1 and $k_2 \neq 0$ are the boundary parameters.

The input-to-state stability (ISS) of the open-loop control system (1.1) depends on the values of the system parameters λ_i, k_i and c_i $(i = 1, 2)$. It has been analyzed in several recent publications, e.g. [16, Section 3], [17, Section 5], [18], [19, Chapter 9]. In particular, we know that the system (1.1) is ISS if the following two conditions hold:

(a) Boundary condition: damping $\mathcal{D}_b = |k_1 k_2| < 1$,

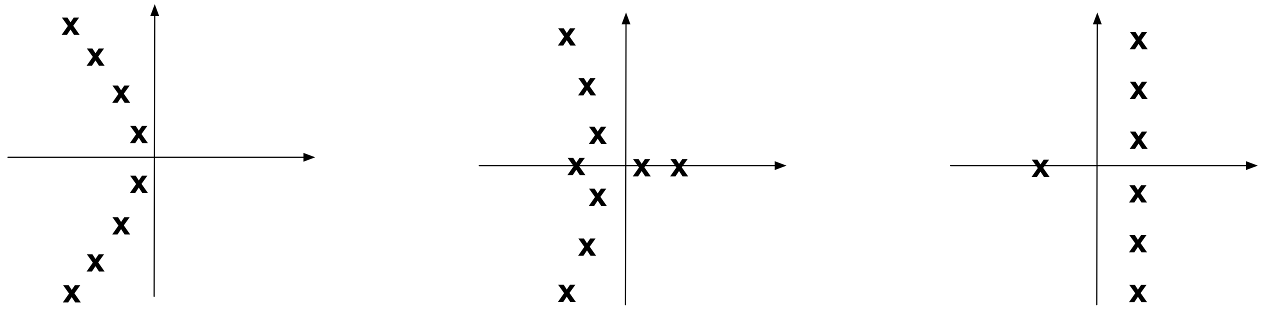
(b) Internal condition: $\max(|c_1|, |c_2|)$ is sufficiently small.

Condition **(a)** is a **necessary** boundary dissipation condition (see [20]) ensuring that system solutions are exponentially damped on average at the boundaries. The factor $\mathcal{D}_b$ quantifies the effect of boundary reflections: the smaller $\mathcal{D}_b$, the stronger the damping efficiency.

Condition **(b)** constrains the internal coupling between the two states of the system. If the coupling is too strong, it may amplify the solutions and induce instability.

Figure 1 illustrates these stability conditions in the special case of system (1.1) with $\lambda_1 = \lambda_2 = 1$ and $c_1 = c_2$. In Figure 1a, with $k_1 = 0$, $k_2 = 1$ (i.e. $\mathcal{D}_b = |k_1 k_2| = 0$) and $c_1 = c_2 = 1$, all eigenvalues are strictly located in the left complex half-plane and the system is therefore ISS because the two conditions **(a)** and **(b)** are satisfied.

In contrast, when condition **(a)** or condition **(b)** is not satisfied, the system can be unstable. This is illustrated in Figure 1b where the spectrum is shown for $k_1 = 0$, $k_2 = -1$ and $c_1 = c_2 = 4$. In that case, condition **(a)** is still satisfied (since $\mathcal{D}_b = |k_1 k_2| = 0$) but condition **(b)** is violated and the system has two unstable eigenvalues.



(a) $\mathcal{D}_b=0,\ c_1=c_2=1.$ (b) $\mathcal{D}_b=0,\ c_1=c_2=4.$ (c) $\mathcal{D}_b=2,\ c_1=c_2=1.$

Figure 1 – Shape of the system spectrum.

Another example of instability is shown in Figure 1c with the spectrum obtained for $k_1 = 2$, $k_2 = 1$ and $c_1 = c_2 = 1$. In that case, condition **(a)** fails ($\mathcal{D}_b = |k_1k_2| = 2$) and the system has an infinity of unstable eigenvalues on a vertical line in the right half-plane.

Regardless of the parameter values λ_i, c_i and k_i, it is well established (see e.g. [9, Section 3], [10, Section 3], [21, Section 8.2]) that the system (1.1) can be stabilized by a full state feedback designed with the backstepping technique *even if the two conditions* **(a)** *and* **(b)** *are not met at all.* The control law takes a dynamical form which is recalled in Section 2. However, the practical implementation of this control law is not possible because it requires the feedback of the full state function $y(t,x)$ on $[0,1]$ that cannot realistically be assumed to be entirely measured on-line. This limitation is usually avoided by designing a state observer which provides an on-line full-state estimate denoted $\hat{y}(t,x)$ from available pointwise measurements.

In this article, we focus on the single-input-single-output case where both actuation and sensing are located at the boundaries. From (1.1c), the single control $U(t)$ is assumed to be located at the left boundary ($x = 0$). Concerning the sensing location, we consider the case when the single output $Y(t)$ is measured at the same boundary (*co-located input/output*):

$$Y(t) = y_2(t,0), \tag{1.2}$$

and the case when the single output measurement $Y(t)$ is located at the opposite boundary (*anti-located input/output*):

$$Y(t) = y_1(t,1). \tag{1.3}$$

In both cases, regardless of the parameter values λ_i, c_i and k_i, the system can be stabilized by an output feedback control designed with the backstepping technique. The control laws take the form of observer-controllers, the presentation of which are given in Sections 3.1 and 4.1.

In this framework of output feedback stabilization of system (1.1) with backstepping observer-controllers, it is however well known (see e.g. [22, Section 3]) that the stability and performance of the closed-loop system can be highly sensitive to various perturbations, namely static perturbations (such as parametric uncertainties or external disturbances) or dynamic perturbations (such as neglected dynamics or parasitic delays in the loop).

Literature review. In the following table, we present a brief chronological overview of representative papers that have addressed some of these robustness aspects for 2×2 hyperbolic systems similar to system (1.1).

- Aamo 2013 [23]
  1. System with spatially-varying parameters and boundary damping $\mathcal{D}_b = 0$.
  2. Robustness to external boundary time-varying disturbances (ODE signal model).
  3. Control law: co-located output feedback by a backstepping Observer-Controller.
  4. Stability analysis in the time domain: L^2-stability with a quadratic Lyapunov function.

- Lamare-Di Meglio 2016 [24]
  1. System with spatially-varying parameters and an arbitrary open-loop boundary damping $\mathcal{D}_b$.
  2. Robustness to external in-domain and boundary time-invariant disturbances.
  3. Control law: anti-located output feedback by a backstepping Observer-Controller with an additional integral action.
  4. Stability analysis in the time domain: L^∞-stability via the method of characteristics.

- Deutscher 2017 [25]
  1. System with spatially-varying parameters and boundary damping $\mathcal{D}_b = 0$.
  2. Robustness to external in-domain and boundary time-varying disturbances (ODE signal model) and robustness of the output regulation to non-destabilizing parametric uncertainties.
  3. Control law: backstepping full-state feedback including an internal disturbance model.
  4. Stability analysis in the time domain: L^2-stability of the nominal system with a quadratic Lyapunov function.

- Anfinsen-Aamo 2017 [26]
  1. System with spatially-varying parameters and boundary damping $\mathcal{D}_b = 0$.
  2. Robustness to unknown coupling parameters and one unknown boundary parameter.
  3. Control law: anti-located output feedback by a backstepping adaptive Observer-Controller.
  4. Stability analysis in the time domain: L^2-stability with a quadratic Lyapunov function.

- Anfinsen-Aamo 2018 [27]
  1. System with constant parameters and boundary damping $\mathcal{D}_b = 0$.
  2. Robustness to unknown coupling parameters and one unknown boundary parameter.
  3. Control law: backstepping full-state feedback with additional parameter adaptation.
  4. Stability analysis in the time domain: L^2-stability with a quadratic Lyapunov function.

- Lamare-Auriol-Di Meglio-Aarsnes 2018 [28]
  1. System with spatially-varying parameters and boundary damping $\mathcal{D}_b < 1$.
  2. Robustness to external in-domain and boundary time-varying disturbances.
  3. Control law: co-located output feedback by a backstepping adaptive Observer-Controller with an additional integral action.
  4. Stability analysis in the time domain: L^∞-stability with the method of characteristics.

- Auriol-Bribriesca Argomedo-Bou Saba-Di Loreto-Di Meglio 2018 [29]

  1. System with constant velocity and boundary parameters, spatially-varying coupling parameters and boundary damping $\mathcal{D}_b < 1$.
  2. Robustness to a small parasitic transmission delay between the output measurement and the control input.
  3. Control law: backstepping full state feedback.
  4. Stability analysis: Spectral stability in the frequency domain.

- Auriol-Aarsnes-Martin-Di Meglio 2018 [30]

  1. System with constant velocity and boundary parameters, spatially-varying coupling parameters and boundary damping $\mathcal{D}_b < 1$.
  2. Robustness to a small parasitic transmission delay between the output measurement and the control input.
  3. Control law: backstepping full state feedback.
  4. Stability analysis: Spectral stability in the frequency domain.

- Auriol-Di Meglio 2020 [31]

  1. System with constant velocity and boundary parameters, spatially-varying coupling parameters and boundary damping $\mathcal{D}_b < 1$.
  2. Robustness to external disturbances, parametric uncertainties and a parasitic delay in the feedback loop.
  3. Control law: co-located output feedback by a backstepping adaptive Observer-Controller with an additional integral action.
  4. Stability analysis: Spectral stability in the frequency domain.

Beyond these contributions, many other papers extend robustness analysis to larger hyperbolic systems (see e.g. [21], [32], [33], [34],), to cascades of interconnected PDE and ODE systems (see e.g. [29], [35], [36]) or to hyperbolic systems with non-local source terms (see e.g. [37], [38]). From this brief review of the literature, it can be noted that the robustness of backstepping control for hyperbolic systems has been approached in very diverse and varied ways, leaving a large number of open questions.

Contribution. We establish sufficient and necessary conditions for the robustness of backstepping control with respect to uncertainties in the characteristic velocity parameters λ_i. The central question is whether arbitrarily small perturbations of the parameters λ_i can destabilize a closed-loop system that is theoretically exponentially stable in the absence of perturbations. The results are sharp and are established for solutions in H^1 and L^2 spaces. They may be summarized as follows:

a) Full state feedback: the backstepping full-state feedback stabilization is robust to small perturbations of the parameters λ_i (see Theorem 1, Section 2.2).

b) Output feedback, **co-located** input/output: the backstepping output feedback stabilization with an observer-controller is robust to small perturbations of the parameters λ_i, without restrictions on coupling parameters c_i or on boundary damping factor $\mathcal{D}_b = |k_1 k_2|$ (see Theorem 3, Section 3.2).

c) Output feedback, **anti-located** input/output: the backstepping output feedback stabilization with an observer-controller can be robust to small perturbations of the parameters λ_i **if and only if** the boundary damping $\mathcal{D}_b = |k_1 k_2| < 1$, i.e. when the natural (physical) boundary conditions of the system are sufficiently dissipative (see Theorem 5, Section 4.2 for the "sufficient" condition and Theorem 7, Section 7 for the "necessary" condition). In other words, when the open-loop boundary damping factor satisfies $\mathcal{D}_b = |k_1 k_2| \geqslant 1$, output feedback stabilization, while theoretically feasible in the absence of perturbations, fails to be robust.

To the best of our knowledge, the approach followed in the article for this robustness analysis has not yet been considered in the literature. From a technical standpoint, the main originality of our contribution lies in the way in which the proof of robustness in L^2-norm is developed. Indeed, it appears that the structure of the perturbed system does not allow us to directly prove robustness in L^2-norm using a classical basic quadratic Lyapunov function (i.e. a Lyapunov function which is the square of a L^2-norm, see e.g. in [39] and [40]). This difficulty leads us to organize the demonstration in two distinct stages. First in Section 5, robustness is established in H^1-norm using a basic quadratic Lyapunov function. Then, in Section 6, it is shown that robustness in H^1 implies robustness in L^2, owing to an isomorphism between the two solution spaces. This, furthermore, leads to define a new Lyapunov function which is less intuitive and more involved than a simple quadratic L^2-norm (see Remark 4 at the end of Section 6).

Finally in Section 7, we prove that the robustness condition is necessary : there exist arbitrarily small non-zero perturbations $(\varepsilon_1, \varepsilon_2)$ for which, when the robustness condition is not verified, the associated eigenvalue problem has solutions with strictly positive real parts or with negative real parts that are not bounded away from zero. The proof relies on degree theory and a careful analysis of the stability conditions.

Some final concluding remarks are provided in Section 8.

2 Stabilizing full state feedback control

2.1 Backstepping design

Using the vector notation

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{2.1}$$

it is convenient for our purposes in this paper to rewrite system (1.1) in the following matrix form:

$$y_t(t,x) + \mathbf{\Lambda} y_x(t,x) + \mathbf{C} y(t,x) = 0, \tag{2.2a}$$

$$\begin{pmatrix} y_1(t,0) \\ y_2(t,1) \end{pmatrix} = \mathbf{K} \begin{pmatrix} y_1(t,1) \\ y_2(t,0) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} U(t), \tag{2.2b}$$

with

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 & c_1 \\ c_2 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{K} = \begin{pmatrix} 0 & k_1 \\ k_2 & 0 \end{pmatrix}. \tag{2.3}$$

The backstepping design relies on the direct/inverse Volterra transformation pair

$$z(t,x) = y(t,x) - \int_x^1 P(x,\xi) y(t,\xi) d\xi, \tag{2.4a}$$

$$y(t,x) = z(t,x) - \int_x^1 Q(x,\xi) z(t,\xi) d\xi, \tag{2.4b}$$

where $P(x,\xi)$ and $Q(x,\xi)$ are, respectively, direct and inverse kernels that are solutions of the kernel partial differential equations given in Appendix A.1.

Using the new coordinates[1] $z = (z_1, z_2)^\mathsf{T}$ defined by (2.4), the dynamics of the system (1.1) are equivalent to

$$\partial_t z(t,x) + \mathbf{\Lambda}\partial_x z(t,x) = 0, \tag{2.5a}$$

$$z_1(t,0) = k_1 z_2(t,0) - \int_0^1 \Big[P_{(1)}(0,\xi) - k_1 P_{(2)}(0,\xi)\Big] y(t,\xi)d\xi + U(t), \tag{2.5b}$$

$$z_2(t,1) = k_2 z_1(t,1), \tag{2.5c}$$

where $P_{(i)}$ ($i \in \{1,2\}$) denote the lines of the matrix P (see (A.1) in Appendix A).

In order to stabilize the system (2.5) at the origin, the following dynamical backstepping state feedback is defined:

$$U(t) = -k_c y_2(t,0) + \int_0^1 \Big[P_{(1)}(0,\xi) - (k_1 - k_c)P_{(2)}(0,\xi)\Big] y(t,\xi)d\xi \tag{2.6}$$

where k_c is a controller tuning parameter.

With this control law, the closed-loop system (2.5), (2.6) expressed in the z coordinates is

$$\partial_t z(t,x) + \mathbf{\Lambda}\partial_x z(t,x) = 0, \tag{2.7a}$$

$$z_1(t,0) = (k_1 - k_c) z_2(t,0), \tag{2.7b}$$

$$z_2(t,1) = k_2 z_1(t,1). \tag{2.7c}$$

This closed-loop system (2.7) is usually called *target system* in the literature. It has a single uniform equilibrium state

$$z_1 \equiv 0, \quad z_2 \equiv 0. \tag{2.8}$$

This system has been widely studied in the literature and is well-known to be exponentially stable (see for instance [41, Theorem 2.4]) if and only if the tuning parameter k_c is selected such that

$$|(k_1 - k_c)k_2| < 1. \tag{2.9}$$

Using the Volterra transformation (2.4), we know that the dynamics of the closed-loop system in y coordinates are equivalent to the dynamics in z coordinates and, therefore, that the closed-loop system (2.2)–(2.6) has a single uniform equilibrium state

$$y_1 \equiv 0, \quad y_2 \equiv 0, \tag{2.10}$$

which is exponentially stable under the same condition (2.9).

2.2 Robustness

The question addressed in this article is whether arbitrarily small perturbations of the parameters λ_i can destabilize the closed-loop system. The first step of this robustness analysis is to rewrite the equations of the closed-loop system with such perturbations.

Therefore we introduce perturbations ε_i that are supposed to modify the characteristic transport velocities λ_i without changing their sign (i.e. $|\varepsilon_i| < \lambda_i$). The open-loop system (1.1) is then rewritten

[1]Throughout this paper, the transpose of a vector or matrix is denoted by the superscript $^\mathsf{T}$.

as follows:

$$\partial_t y(t,x) + (\boldsymbol{\Lambda} + \boldsymbol{\mathcal{E}})\partial_x y(t,x) + \mathbf{C}y(t,x) = 0, \quad \boldsymbol{\mathcal{E}} = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & -\varepsilon_2 \end{pmatrix}, \tag{2.11a}$$

$$y_1(t,0) = k_1 y_2(t,0) + U(t), \tag{2.11b}$$

$$y_2(t,1) = k_2 y_1(t,1). \tag{2.11c}$$

Using the Volterra transformations (2.4), this system is equivalent to

$$\begin{aligned}\partial_t z(t,x) + (\boldsymbol{\Lambda} + \boldsymbol{\mathcal{E}})\partial_x z(t,x) = &\int_x^1 \Big[P_\xi(x,\xi)\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}}P_x(x,\xi)\Big]\Big(\int_\xi^1 Q(\xi,\zeta)z(t,\zeta)d\zeta - z(t,\xi)\Big)d\xi \\ &+ \Big[P(x,x)\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{E}}P(x,x)\Big]\Big(\int_x^1 Q(x,\xi)z(t,\xi)d\xi - z(t,x)\Big) \\ &+ P(x,1)\boldsymbol{\mathcal{E}}z(t,1),\end{aligned} \tag{2.12a}$$

$$z_1(t,0) = k_1 z_2(t,0) - \int_0^1 P_{(1)}(0,\xi)y(t,\xi)d\xi + k_1\int_0^1 P_{(2)}(0,\xi)y(t,\xi)d\xi + U(t), \tag{2.12b}$$

$$z_2(t,1) = k_2 z_1(t,1). \tag{2.12c}$$

Notation. To simplify the notations, for any function $\varphi \in L^2((0,1);\mathbb{R}^2)$, we introduce the operator

$$\mathbf{h}_{1\varepsilon} : L^2((0,1);\mathbb{R}^2) \to L^2((0,1);\mathbb{R}^2) : \varphi \mapsto \mathbf{h}_{1\varepsilon}(\varphi) \tag{2.13}$$

which is defined by the statement that the value of the function $\mathbf{h}_{1\varepsilon}(\varphi) \in L^2((0,1);\mathbb{R}^2)$ at $x \in (0,1)$ is given by

$$\begin{aligned}\mathbf{h}_{1\varepsilon}(\varphi)(x) = &\int_x^1 \Big[P_\xi(x,\xi)\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}}P_x(x,\xi)\Big]\Big(\int_\xi^1 Q(\xi,\zeta)\varphi(\zeta)d\zeta - \varphi(\xi)\Big)d\xi \\ &+ \Big[P(x,x)\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{E}}P(x,x)\Big]\Big(\int_x^1 Q(x,\xi)\varphi(\xi)d\xi - \varphi(x)\Big) \\ &+ P(x,1)\boldsymbol{\mathcal{E}}\varphi(1).\end{aligned} \tag{2.14}$$

An important point for our robustness analysis in L^2 is that an upper bound of the following form can be established :

$$\|\mathbf{h}_{1\varepsilon}(\varphi)\|_{L^2} \leqslant C\max(|\varepsilon_1|,|\varepsilon_2|)\Big(\|\varphi\|_{L^2} + |\varphi(1)|\Big) \tag{2.15}$$

where C is a positive constant which depends only on the system parameters λ_i, c_i, k_i.

We now assume that the control law (2.6) is applied to the system (2.12). Using the definition (2.14), the closed-loop target system (2.7) is perturbed and becomes

$$\partial_t z(t,x) + (\boldsymbol{\Lambda} + \boldsymbol{\mathcal{E}})\partial_x z(t,x) = \mathbf{h}_{1\varepsilon}(z(t,\cdot))(x), \tag{2.16a}$$

$$z_1(t,0) = (k_1 - k_c)z_2(t,0), \tag{2.16b}$$

$$z_2(t,1) = k_2 z_1(t,1). \tag{2.16c}$$

Note that this perturbed closed-loop system has a uniform zero steady-state independent of $\boldsymbol{\mathcal{E}}$.

Then we have the following robustness property.

Theorem 1. If the tuning parameter k_c is selected such that $|(k_1 - k_c)k_2| < 1$, there exists $\varepsilon_0 > 0$ such that, if $\max(|\varepsilon_1|,|\varepsilon_2|) \leqslant \varepsilon_0$, then the closed-loop system (2.11) and (2.6) is exponentially stable for the L^2-norm. In other words, there exist $C_0 > 0$ and $\gamma > 0$ such that for any $T > 0$, and any initial condition $y_0 \in L^2((0,1);\mathbb{R}^2)$ the system (2.16) has a unique solution $y \in C^0([0,T];L^2((0,1);\mathbb{R}^2))$ and

$$\|y(t,\cdot)\|_{L^2} \leqslant C_0 e^{-\gamma t}\|y_0\|_{L^2}, \ \forall\, t \in [0,T]. \tag{2.17}$$

Proof. The proof is based on a standard basic quadratic Lyapunov function of the form

$$\mathbf{V}(t) = \int_0^1 \Big(a_1 z_1^2(t,x)e^{-b_1 x} + a_2 z_2^2(t,x)e^{b_2 x}\Big)dx. \tag{2.18}$$

and is a direct extension of Theorem 3.2 of [41]. It relies on the well-known robustness of Lyapunov methods with respect to small perturbations of the system dynamics. □

3 Observer-Controller with co-located input/output

3.1 Backstepping design

In this section, we consider the single-input-single-output case with a co-located boundary output measurement

$$Y(t) = y_2(t,0). \tag{3.1}$$

The state observer is a copy of system (1.1) driven by additional so-called *output injection* terms, as follows:

$$\partial_t \hat{y}(t,x) + \mathbf{\Lambda}\partial_x \hat{y}(t,x) + \mathbf{C}\hat{y}(t,x) + \begin{pmatrix} \upsilon_1(x) \\ \upsilon_2(x) \end{pmatrix} (Y(t) - \hat{y}_2(t,0)) = 0, \tag{3.2a}$$

$$\hat{y}_1(t,0) = k_1 \hat{y}_2(t,0) + U(t) + k_0(Y(t) - \hat{y}_2(t,0)), \tag{3.2b}$$

$$\hat{y}_2(t,1) = k_2 \hat{y}_1(t,1), \tag{3.2c}$$

where $\upsilon_1(x)$, $\upsilon_2(x)$ and k_0 are output injection gains to be selected to guarantee the convergence $\hat{y}(t,x) \to y(t,x)$.

Defining the observation error

$$\tilde{y} = y - \hat{y}, \tag{3.3}$$

we can derive from (1.1) and (3.2) the following dynamics of the observation error system:

$$\partial_t \tilde{y}(t,x) + \mathbf{\Lambda}\partial_x \tilde{y}(t,x) + \mathbf{C}\tilde{y}(t,x) - \begin{pmatrix} \upsilon_1(x) \\ \upsilon_2(x) \end{pmatrix} \tilde{y}_2(t,0) = 0, \tag{3.4a}$$

$$\tilde{y}_1(t,0) = (k_1 - k_0)\tilde{y}_2(t,0), \tag{3.4b}$$

$$\tilde{y}_2(t,1) = k_2 \tilde{y}_1(t,1). \tag{3.4c}$$

To determine the stability conditions of this system, we again use direct/inverse Volterra transformations denoted as follows:

$$\tilde{z}(t,x) = \tilde{y}(t,x) - \int_0^x \widetilde{P}(x,\xi)\tilde{y}(t,\xi)d\xi, \tag{3.5a}$$

$$\tilde{y}(t,x) = \tilde{z}(t,x) - \int_0^x \widetilde{Q}(x,\xi)\tilde{z}(t,\xi)d\xi, \tag{3.5b}$$

where $\widetilde{P}(x,\xi)$ and $\widetilde{Q}(x,\xi)$ are, respectively, the direct and inverse kernels that are solutions of the kernel partial differential equations given in Appendix A.2.

Then, by selecting the internal output injection gains

$$\begin{pmatrix} \upsilon_1(x) \\ \upsilon_2(x) \end{pmatrix} = \widetilde{Q}(x,0) \begin{pmatrix} -\lambda_1(k_1 - k_0) \\ \lambda_2 \end{pmatrix}, \tag{3.6}$$

it can be shown that the observer error system in $\tilde{z} = (\tilde{z}_1, \tilde{z}_2)^\mathsf{T}$ coordinates is written

$$\partial_t \tilde{z}(t,x) + \mathbf{\Lambda} \partial_x \tilde{z}(t,x) = 0, \tag{3.7a}$$

$$\tilde{z}_1(t,0) = (k_1 - k_0)\tilde{z}_2(t,0), \tag{3.7b}$$

$$\tilde{z}_2(t,1) = k_2 \tilde{z}_1(t,1). \tag{3.7c}$$

This system is exponentially stable if and only if k_0 is selected such that

$$|(k_1 - k_0)k_2| < 1, \tag{3.8}$$

which guarantees the convergence of the estimate $\hat{y}$ of the system state to its actual value y.

Combining the full state feedback control law with the observer (3.2), an output feedback control law is obtained by replacing the state y by its estimate $\hat{y}$ in equation (2.6) as follows:

$$U(t) = -k_c Y(t) + \int_0^1 \Big[P_{(1)}(0,\xi) - (k_1 - k_c) P_{(2)}(0,\xi) \Big] \hat{y}(t,\xi) d\xi. \tag{3.9}$$

With this control law, the dynamics of the closed-loop system expressed in $z, \tilde{z}$ coordinates are written

$$\partial_t z(t,x) + \mathbf{\Lambda} \partial_x z(t,x) = 0, \tag{3.10a}$$

$$\partial_t \tilde{z}(t,x) + \mathbf{\Lambda} \partial_x \tilde{z}(t,x) = 0, \tag{3.10b}$$

$$\begin{pmatrix} z_1(t,0) \\ z_2(t,1) \\ \tilde{z}_1(t,0) \\ \tilde{z}_2(t,1) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & k_1 - k_c & 0 & 0 \\ k_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_1 - k_0 \\ 0 & 0 & k_2 & 0 \end{pmatrix}}_{\widetilde{\mathbf{K}}} \begin{pmatrix} z_1(t,1) \\ z_2(t,0) \\ \tilde{z}_1(t,1) \\ \tilde{z}_2(t,0) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \int_0^1 f(\zeta) \tilde{z}(t,\zeta) d\zeta, \tag{3.10c}$$

where, using Fubini's theorem, the function $f(\zeta)$ is defined as

$$f(\zeta) = \Big(- P_{(1)}(0,\zeta) + (k_1 - k_c) P_{(2)}(0,\zeta) \Big) + \int_\zeta^1 \Big(P_{(1)}(0,\xi) - (k_1 - k_c) P_{(2)}(0,\xi) \Big) \widetilde{Q}(\xi,\zeta) d\xi. \tag{3.11}$$

This closed-loop system (3.10) has a single equilibrium $z \equiv 0$, $\tilde{z} \equiv 0$ with the following stability property.

Theorem 2. If the tuning parameters k_c and k_0 are selected such that

$$|(k_1 - k_c)k_2| < 1 \quad \text{and} \quad |(k_1 - k_0)k_2| < 1, \tag{3.12}$$

then the equilibrium of the closed-loop system (3.10) is globally exponentially stable for the L^2-norm. In other words, there exist $C_0 > 0$ and $\gamma > 0$ such that for any $T > 0$, and any initial condition $(z_0, \tilde{z}_0) \in L^2((0,1); \mathbb{R}^4)$ the system (3.10) has a unique solution $(z, \tilde{z}) \in C^0([0,T]; L^2((0,1); \mathbb{R}^4))$ and

$$\|(z(t,\cdot), \tilde{z}(t,\cdot))\|_{L^2} \leqslant C_0 e^{-\gamma t} \|(z_0, \tilde{z}_0)\|_{L^2}, \ \forall\, t \in [0,T]. \tag{3.13}$$

Proof. If $|(k_1 - k_0)k_2| < 1$, the $\tilde{z}$ sub-system is exponentially stable for the L^2-norm ([41, Theorem 2.4]). Then the theorem follows because, if $|(k_1 - k_c)k_2| < 1$, the z subsystem with input $\tilde{z}$ is ISS for the L^2-norm ([16, Section 3]). $\square$

Note that, here in the statement of this theorem and also in the rest of the article, we use for brevity the simplified notation $(z, \tilde{z})$ to represent, indifferently, either the row vector $(z^{\mathsf{T}}, \tilde{z}^{\mathsf{T}}) = (z_1, z_2, \tilde{z}_1, \tilde{z}_2)$ or the column vector

$$\begin{pmatrix} z \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix}. \tag{3.14}$$

3.2 Robustness

We consider the situation where the observer-controller (3.2)–(3.9) is applied to the perturbed open-loop system (2.11). In that case, the observation error system (3.4) must be modified as follows:

$$\partial_t \tilde{y}(t,x) + \mathbf{\Lambda}\partial_x \tilde{y}(t,x) + \boldsymbol{\mathcal{E}}\partial_x y(t,x) + \mathbf{C}\tilde{y}(t,x) - \begin{pmatrix} \upsilon_1(x) \\ \upsilon_2(x) \end{pmatrix} \tilde{y}_2(t,0) = 0, \tag{3.15a}$$

$$\tilde{y}_1(t,0) = (k_1 - k_0)\tilde{y}_2(t,0), \tag{3.15b}$$

$$\tilde{y}_2(t,1) = k_2 \tilde{y}_1(t,1). \tag{3.15c}$$

Let us emphasize that, in the above equation, the term $\boldsymbol{\mathcal{E}}\partial_x y$ appears since the error $\boldsymbol{\mathcal{E}}$ is unknown and cannot therefore be used when constructing $\hat{y}$.

Using the Volterra transformations (3.5), this system is equivalent to

$$\begin{aligned} \partial_t \tilde{z}(t,x) + \mathbf{\Lambda}\partial_x \tilde{z}(t,x) + \boldsymbol{\mathcal{E}}\partial_x z(t,x) &= -\int_0^x \widetilde{P}_\xi(x,\xi)\boldsymbol{\mathcal{E}}\left(z(t,\xi) - \int_\xi^1 Q(\xi,\zeta)z(t,\zeta)d\zeta\right)d\xi \\ &+ \widetilde{P}(x,x)\boldsymbol{\mathcal{E}}\left(z(t,x) - \int_x^1 Q(x,\xi)z(t,\xi)d\xi\right) + \widetilde{P}(x,0)\boldsymbol{\mathcal{E}}\left(z(t,0) - \int_0^1 Q(0,\xi)z(t,\xi)d\xi\right) \\ &+ \int_x^1 \boldsymbol{\mathcal{E}}Q_x(x,\xi)z(t,\xi)d\xi - \boldsymbol{\mathcal{E}}Q(x,x)z(t,x), \end{aligned} \tag{3.16a}$$

$$\tilde{z}_1(t,0) = (k_1 - k_0)\tilde{z}_2(t,0), \tag{3.16b}$$

$$\tilde{z}_2(t,1) = k_2 \tilde{z}_1(t,1). \tag{3.16c}$$

Then the perturbed closed-loop system is written as follows in the $z, \tilde{z}$ coordinates:

$$\partial_t z(t,x) + (\mathbf{\Lambda} + \boldsymbol{\mathcal{E}})\partial_x z(t,x) = \mathbf{h}_{1\varepsilon}(z(t,\cdot)(x), \tag{3.17a}$$

$$\partial_t \tilde{z}(t,x) + \mathbf{\Lambda}\partial_x \tilde{z}(t,x) + \boldsymbol{\mathcal{E}}\partial_x z(t,x) = \widetilde{\mathbf{h}}_{2\varepsilon}(z(t,\cdot))(x), \tag{3.17b}$$

$$\begin{pmatrix} z_1(t,0) \\ z_2(t,1) \\ \tilde{z}_1(t,0) \\ \tilde{z}_2(t,1) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & k_1 - k_c & 0 & 0 \\ k_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_1 - k_0 \\ 0 & 0 & k_2 & 0 \end{pmatrix}}_{\widetilde{\mathbf{K}}} \begin{pmatrix} z_1(t,1) \\ z_2(t,0) \\ \tilde{z}_1(t,1) \\ \tilde{z}_2(t,0) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \int_0^1 f(\zeta)\tilde{z}(t,\zeta)d\zeta, \tag{3.17c}$$

where the operator $\mathbf{h}_{1\varepsilon}$ has been defined in (2.14) and, for any $\varphi \in L^2((0,1);\mathbb{R}^2)$, the operator $\widetilde{\mathbf{h}}_{2\varepsilon}$

is similarly defined such that

$$
\begin{aligned}
\widetilde{\mathbf{h}}_{2\varepsilon}(\varphi)(x) = &-\int_0^x \widetilde{P}_\xi(x,\xi)\boldsymbol{\mathcal{E}}\left(\varphi(\xi) - \int_\xi^1 Q(\xi,\zeta)\varphi(\zeta)d\zeta\right)d\xi \\
&+ \widetilde{P}(x,x)\boldsymbol{\mathcal{E}}\left(\varphi(x) - \int_x^1 Q(x,\xi)\varphi(\xi)d\xi\right) + \widetilde{P}(x,0)\boldsymbol{\mathcal{E}}\left(\varphi(0) - \int_0^1 Q(0,\xi)\varphi(\xi)d\xi\right) \\
&+ \int_x^1 \boldsymbol{\mathcal{E}}Q_x(x,\xi)\varphi(\xi)d\xi - \boldsymbol{\mathcal{E}}Q(x,x)\varphi(x).
\end{aligned}
\tag{3.18}
$$

The system (3.17) can be transformed into a diagonal characteristic form by defining the auxiliary state variable

$$\hat{z}(t,x) = z(t,x) - \tilde{z}(t,x) \tag{3.19}$$

such that the perturbed closed-loop system may now be written as follows in Riemann coordinates $z, \hat{z}$:

$$\partial_t z(t,x) + (\boldsymbol{\Lambda} + \boldsymbol{\mathcal{E}})\partial_x z(t,x) = \mathbf{h}_{1\varepsilon}(z(t,\cdot))(x), \tag{3.20a}$$

$$\partial_t \hat{z}(t,x) + \boldsymbol{\Lambda}\partial_x \hat{z}(t,x) = \mathbf{h}_{1\varepsilon}(z(t,\cdot))(x) - \widetilde{\mathbf{h}}_{2\varepsilon}(z(t,\cdot))(x), \tag{3.20b}$$

$$
\begin{pmatrix} z_1(t,0) \\ z_2(t,1) \\ \hat{z}_1(t,0) \\ \hat{z}_2(t,1) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & k_1 - k_c & 0 & 0 \\ k_2 & 0 & 0 & 0 \\ 0 & k_0 - k_c & 0 & k_1 - k_0 \\ 0 & 0 & k_2 & 0 \end{pmatrix}}_{\widehat{\mathbf{K}}} \begin{pmatrix} z_1(t,1) \\ z_2(t,0) \\ \hat{z}_1(t,1) \\ \hat{z}_2(t,0) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \int_0^1 f(\zeta)\Big(z(t,\zeta) - \hat{z}(t,\zeta)\Big)d\zeta. \tag{3.20c}
$$

For this system, we have the following robustness theorem.

Theorem 3 (Exponential stability in L^2)**.** Assume that the tuning parameters k_c and k_0 are selected such that (3.12) holds. Then there exists $\varepsilon_0 > 0$ such that the system (3.20) is exponentially stable for the L^2 norm if $\max(|\varepsilon_1|, |\varepsilon_2|) < \varepsilon_0$. In other words, there exist $C_0 > 0$ and $\gamma > 0$ such that for any $T > 0$, and any initial condition $(z_0, \hat{z}_0) \in L^2((0,1);\mathbb{R}^4)$ the system (3.20) has a unique solution $(z, \hat{z}) \in C^0([0,T]; L^2((0,1);\mathbb{R}^4))$ and

$$\|(z(t,\cdot), \hat{z}(t,\cdot))\|_{L^2} \leqslant C_0 e^{-\gamma t} \|(z_0, \hat{z}_0)\|_{L^2}, \ \forall\, t \in [0,T]. \tag{3.21}$$

Proof. The proof of this theorem is omitted because it can be carried on in a similar way to the proof of Theorem 5. □

In summary, Theorem 3 shows that output feedback stabilization of the system (1.1)–(1.2) by the observer-controller (3.2)–(3.9) is robust with respect to small perturbations in characteristic velocities λ_i.

4 Observer-Controller with anti-located input/output

4.1 Backstepping design

In this section, we consider the single-input-single-output case with an anti-located boundary output measurement

$$Y(t) = y_1(t,1). \tag{4.1}$$

The state observer is a copy of the system (1.1) driven by additional output injection terms as follows:

$$\partial_t \hat{y}(t,x) + \mathbf{\Lambda}\partial_x \hat{y}(t,x) + \mathbf{C}\hat{y}(t,x) + \begin{pmatrix} \upsilon_1(x) \\ \upsilon_2(x) \end{pmatrix} (Y(t) - \hat{y}_1(t,1)) = 0, \tag{4.2a}$$

$$\hat{y}_1(t,0) = k_1 \hat{y}_2(t,0) + U(t), \tag{4.2b}$$

$$\hat{y}_2(t,1) = k_2 \hat{y}_1(t,1) + k_0(Y(t) - \hat{y}_1(t,1)), \tag{4.2c}$$

where $\upsilon_1(x)$, $\upsilon_2(x)$ and k_0 are output injection gains to be selected to guarantee the convergence $\hat{y}(t,x) \to y(t,x)$.

Then, defining the observation error

$$\tilde{y} = y - \hat{y}, \tag{4.3}$$

we can derive from (1.1) and (4.2) the following dynamics of the observation error system:

$$\partial_t \tilde{y}(t,x) + \mathbf{\Lambda}\partial_x \tilde{y}(t,x) + \mathbf{C}\tilde{y}(t,x) - \begin{pmatrix} \upsilon_1(x) \\ \upsilon_2(x) \end{pmatrix} \tilde{y}_1(t,1) = 0, \tag{4.4a}$$

$$\tilde{y}_1(t,0) = k_1 \tilde{y}_2(t,0), \tag{4.4b}$$

$$\tilde{y}_2(t,1) = (k_2 - k_0)\tilde{y}_1(t,1). \tag{4.4c}$$

To determine the stability conditions of this system, we again use direct/inverse Volterra transformations denoted as follows:

$$\tilde{z}(t,x) = \tilde{y}(t,x) - \int_x^1 \widetilde{P}(x,\xi)\tilde{y}(t,\xi)d\xi, \tag{4.5a}$$

$$\tilde{y}(t,x) = \tilde{z}(t,x) - \int_x^1 \widetilde{Q}(x,\xi)\tilde{z}(t,\xi)d\xi, \tag{4.5b}$$

where $\widetilde{P}(x,\xi)$ and $\widetilde{Q}(x,\xi)$ are, respectively, the direct and inverse kernels that are solutions of the kernel partial differential equations given in Appendix A.3.

Then, by selecting the internal output injection gains

$$\begin{pmatrix} \upsilon_1(x) \\ \upsilon_2(x) \end{pmatrix} = \widetilde{Q}(x,1) \begin{pmatrix} \lambda_1 \\ \lambda_2(k_2 - k_0) \end{pmatrix}, \tag{4.6}$$

it can be shown that the observer error system in $\tilde{z}$ coordinates is written

$$\partial_t \tilde{z}(t,x) + \mathbf{\Lambda}\partial_x \tilde{z}(t,x) = 0, \tag{4.7a}$$

$$\tilde{z}_1(t,0) = k_1 \tilde{z}_2(t,0), \tag{4.7b}$$

$$\tilde{z}_2(t,1) = (k_2 - k_0)\tilde{z}_1(t,1). \tag{4.7c}$$

This system is exponentially stable if and only if k_0 is selected such that

$$|k_1(k_2 - k_0)| < 1, \tag{4.8}$$

which guarantees the convergence of the estimate $\hat{y}$ of the system state to its actual value y.

Combining the full state feedback control law with the observer (4.2), an output feedback control law is obtained by replacing the state y by its estimate $\hat{y}$ in equation (2.6) as follows:

$$U(t) = -k_c \hat{y}_2(t,0) + \int_0^1 \Big[P_{(1)}(0,\xi) - (k_1 - k_c)P_{(2)}(0,\xi)\Big]\hat{y}(t,\xi)d\xi. \tag{4.9}$$

With this control law, the dynamics of the closed-loop system expressed in $z, \tilde{z}$ coordinates are written

$$\partial_t z(t,x) + \mathbf{\Lambda}\partial_x z(t,x) = 0, \tag{4.10a}$$

$$\partial_t \tilde{z}(t,x) + \mathbf{\Lambda}\partial_x \tilde{z}(t,x) = 0, \tag{4.10b}$$

$$\begin{pmatrix} z_1(t,0) \\ z_2(t,1) \\ \tilde{z}_1(t,0) \\ \tilde{z}_2(t,1) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & k_1 - k_c & 0 & k_c \\ k_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 - k_0 & 0 \end{pmatrix}}_{\widetilde{\mathbf{K}}} \begin{pmatrix} z_1(t,1) \\ z_2(t,0) \\ \tilde{z}_1(t,1) \\ \tilde{z}_2(t,0) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \int_0^1 f(\zeta)\tilde{z}(t,\zeta)d\zeta, \tag{4.10c}$$

where, using Fubini's theorem, the function $f(\zeta)$ is defined as

$$f(\zeta) = (f_1(\zeta), f_2(\zeta)) = \Big(- P_{(1)}(0,\zeta) + (k_1 - k_c)P_{(2)}(0,\zeta) - k_c\widetilde{Q}_{(2)}(0,\zeta)\Big) \\ + \int_0^\zeta \Big(P_{(1)}(0,\xi) - (k_1 - k_c)P_{(2)}(0,\xi)\Big)\widetilde{Q}(\xi,\zeta)d\xi. \tag{4.11}$$

Theorem 4. The closed-loop system (4.10) has a single equilibrium $z \equiv 0$, $\tilde{z} \equiv 0$ which is exponentially stable for the L^2-norm if the tuning parameters k_c and k_0 are selected such that

$$|(k_1 - k_c)k_2| < 1 \quad \text{and} \quad |k_1(k_2 - k_0)| < 1. \tag{4.12}$$

Proof. The proof is similar to the proof of Theorem 2. □

4.2 Robustness

We consider the situation where the observer-controller (4.2)–(4.9) is applied to the perturbed open-loop system (2.11). In that case, the observation error system (4.4) must be modified as follows:

$$\partial_t \tilde{y}(t,x) + \mathbf{\Lambda}\partial_x \tilde{y}(t,x) + \boldsymbol{\mathcal{E}}\partial_x y(t,x) + \mathbf{C}\tilde{y}(t,x) - \begin{pmatrix} \upsilon_1(x) \\ \upsilon_2(x) \end{pmatrix} \tilde{y}_1(t,1) = 0, \tag{4.13a}$$

$$\tilde{y}_1(t,0) = k_1\tilde{y}_2(t,0), \tag{4.13b}$$

$$\tilde{y}_2(t,1) = (k_2 - k_0)\tilde{y}_1(t,1). \tag{4.13c}$$

Now, using the Volterra transformations (4.5), this system is equivalent to

$$\partial_t \tilde{z}(t,x) + \mathbf{\Lambda}\partial_x \tilde{z}(t,x) + \boldsymbol{\mathcal{E}}\partial_x z(t,x) = -\int_x^1 \widetilde{P}_\xi(x,\xi)\boldsymbol{\mathcal{E}}\left(z(t,\xi) - \int_\xi^1 Q(\xi,\zeta)z(t,\zeta)d\zeta\right)d\xi \\ - \widetilde{P}(x,x)\boldsymbol{\mathcal{E}}\left(z(t,x) - \int_x^1 Q(x,\xi)z(t,\xi)d\xi\right) + \widetilde{P}(x,1)\boldsymbol{\mathcal{E}}z(t,1) \\ + \int_x^1 \boldsymbol{\mathcal{E}}Q_x(x,\xi)z(t,\xi)d\xi - \boldsymbol{\mathcal{E}}Q(x,x)z(t,x), \tag{4.14a}$$

$$\tilde{z}_1(t,0) = k_1\tilde{z}_2(t,0), \tag{4.14b}$$

$$\tilde{z}_2(t,1) = (k_2 - k_0)\tilde{z}_1(t,1). \tag{4.14c}$$

The perturbed closed-loop system can then be expressed in the $z, \tilde{z}$ coordinates as

$$\partial_t z(t,x) + (\mathbf{\Lambda} + \mathbf{\mathcal{E}})\partial_x z(t,x) = \mathbf{h}_{1\varepsilon}(z(t,\cdot))(x), \tag{4.15a}$$

$$\partial_t \tilde{z}(t,x) + \mathbf{\Lambda}\partial_x \tilde{z}(t,x) + \mathbf{\mathcal{E}}\partial_x z(t,x) = \widetilde{\mathbf{h}}_{2\varepsilon}(z(t,\cdot))(x), \tag{4.15b}$$

$$\begin{pmatrix} z_1(t,0) \\ z_2(t,1) \\ \tilde{z}_1(t,0) \\ \tilde{z}_2(t,1) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & k_1 - k_c & 0 & k_c \\ k_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 - k_0 & 0 \end{pmatrix}}_{\widetilde{\mathbf{K}}} \begin{pmatrix} z_1(t,1) \\ z_2(t,0) \\ \tilde{z}_1(t,1) \\ \tilde{z}_2(t,0) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \int_0^1 f(\zeta)\tilde{z}(t,\zeta)d\zeta. \tag{4.15c}$$

Here the operator $\mathbf{h}_{1\varepsilon}$ has been defined in (2.14) and, for any $\varphi \in L^2((0,1);\mathbb{R}^2)$, the operator $\widetilde{\mathbf{h}}_{2\varepsilon}$ is now redefined by

$$\begin{aligned} \widetilde{\mathbf{h}}_{2\varepsilon}(\varphi)(x) = &- \int_x^1 \widetilde{P}_\xi(x,\xi)\mathbf{\mathcal{E}}\left(\varphi(\xi) - \int_\xi^1 Q(\xi,\zeta)\varphi(\zeta)d\zeta\right)d\xi \\ &- \widetilde{P}(x,x)\mathbf{\mathcal{E}}\left(\varphi(x) - \int_x^1 Q(x,\xi)\varphi(\xi)d\xi\right) + \widetilde{P}(x,1)\mathbf{\mathcal{E}}\varphi(1) \\ &+ \int_x^1 \mathbf{\mathcal{E}}Q_x(x,\xi)\varphi(\xi)d\xi - \mathbf{\mathcal{E}}Q(x,x)\varphi(x). \end{aligned} \tag{4.16}$$

As in (2.15), we note that an upper bound of the following form can be established in L^2:

$$\|\widetilde{\mathbf{h}}_{2\varepsilon}(\varphi)\|_{L^2} \leqslant C\max(|\varepsilon_1|,|\varepsilon_2|)\Big(\|\varphi\|_{L^2} + |\varphi(1)|\Big), \tag{4.17}$$

where C denotes a positive constant which depends only on the system parameters λ_i, c_i, k_i.

As in the previous section, to diagonalize the system, we introduce the auxiliary state variable

$$\hat{z}(t,x) = z(t,x) - \tilde{z}(t,x) \tag{4.18}$$

so that the perturbed closed-loop system is written in the Riemann coordinates $z, \hat{z}$ as

$$\Sigma_{cl} \quad \begin{cases} \partial_t z(t,x) + (\mathbf{\Lambda} + \mathbf{\mathcal{E}})\partial_x z(t,x) = \mathbf{h}_{1\varepsilon}(z(t,\cdot))(x), & (4.19a) \\ \partial_t \hat{z}(t,x) + \mathbf{\Lambda}\partial_x \hat{z}(t,x) = \mathbf{h}_{1\varepsilon}(z(t,\cdot))(x) - \widetilde{\mathbf{h}}_{2\varepsilon}(z(t,\cdot))(x), & (4.19b) \\ \begin{pmatrix} z_1(t,0) \\ z_2(t,1) \\ \hat{z}_1(t,0) \\ \hat{z}_2(t,1) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & k_1 & 0 & -k_c \\ k_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_1 - k_c \\ k_0 & 0 & k_2 - k_0 & 0 \end{pmatrix}}_{\widehat{\mathbf{K}}} \begin{pmatrix} z_1(t,1) \\ z_2(t,0) \\ \hat{z}_1(t,1) \\ \hat{z}_2(t,0) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \int_0^1 f(\zeta)\Big(z(t,\zeta) - \hat{z}(t,\zeta)\Big)d\zeta. & (4.19c) \end{cases}$$

For this system, we have the following robustness property.

Theorem 5 (Exponential stability in L^2)**.** Assume that

$$|(k_1 - k_c)k_2| < 1, \quad |k_1(k_2 - k_0)| < 1 \quad \text{and} \quad \rho_2(\widehat{\mathbf{K}}) < 1 \tag{4.20}$$

with

$$\rho_2(\widehat{\mathbf{K}}) = \inf_D \Big\{\|D\widehat{\mathbf{K}}D^{-1}\|_2, \;\; D \text{ diagonal positive matrix}\Big\}. \tag{4.21}$$

Then there exists $\varepsilon_0 > 0$ such that the system Σ_{cl} is exponentially stable for the L^2 norm if $\max(|\varepsilon_1|, |\varepsilon_2|) < \varepsilon_0$. In other words, there exist $C_1 > 0$ and $\gamma > 0$ such that for any $T > 0$, and any initial condition $(z_0, \hat{z}_0) \in L^2((0,1), \mathbb{R}^4)$ the system Σ_{cl} has a unique solution $(z, \hat{z}) \in C^0([0,T]; L^2((0,1); \mathbb{R}^4))$ and

$$\left\|(z(t,\cdot), \hat{z}(t,\cdot))\right\|_{L^2} \leqslant C_1 e^{-\gamma t} \left\|(z_0, \hat{z}_0)\right\|_{L^2}, \ \forall\, t \in [0,T]. \tag{4.22}$$

Proof. The proof is given in Section 6. □

An explicit expression of the function $\rho_2(\widehat{\mathbf{K}})$ is given in the following proposition.

Proposition 1. For the matrix $\widehat{\mathbf{K}}$ defined in equation (6.1c), we have

$$\rho_2(\widehat{\mathbf{K}}) = \sqrt{\begin{array}{l} \frac{1}{2}\Big[|k_1k_2| + |k_0k_c| + |(k_2-k_0)(k_1-k_c)|\Big] \\ \quad +\frac{1}{2}\sqrt{\Big(|k_1k_2| + |k_0k_c| + |(k_2-k_0)(k_1-k_c)|\Big)^2 - 4|k_1k_2||(k_2-k_0)(k_1-k_c)|} \end{array}}. \tag{4.23}$$

Proof. See Appendix B.

Remark 1. From a practical point of view, it is natural to ask whether the control tuning parameters k_c and k_0 can be chosen to satisfy assumption (4.20) in the statement of Theorem 5. From expression (4.23), it can be checked that

$$\min_{k_c, k_0} \rho_2(\widehat{\mathbf{K}}) = \sqrt{|k_1k_2|}, \tag{4.24}$$

which implies that the condition

$$\mathcal{D}_b = |k_1k_2| < 1 \tag{4.25}$$

is necessary (and clearly sufficient) to allow the selection of tuning parameters k_c and k_0 satisfying conditions (4.20). Let us emphasize again that k_1 and k_2 are system parameters that are a priori given and thus (4.25) is an intrinsic limit on the system. It is important to note that this limitation does not exist when seeking only exponential stability and not robustness.

The next two sections of the article are now devoted to proving Theorem 5. The proof is organized in two successive stages. Since the system Σ_{cl} is expressed in characteristic form, one might expect that a classical quadratic Lyapunov function, equivalent to the square of a L^2-norm, would suffice. In fact, this is not possible because there is not enough freedom to determine an upper bound on the time derivative of the Lyapunov function in the presence of the integral term $\int_0^1 f(\zeta)\Big(z(t,\zeta) - \hat{z}(t,\zeta)\Big)d\zeta$ in the boundary condition (4.19c). However, as we will see in the next section (Theorem 6), it is actually possible to show that, under the same assumptions as Theorem 5, the system Σ_{cl} is exponentially stable for the H^1-norm using a Lyapunov approach.

In the second stage (Section 6), we will then show that stability in H^1 implies stability in L^2 because there is an isomorphism between the spaces of solutions of Σ_{cl} in $C^0([0,T]; L^2((0,1); \mathbb{R}^4))$ and in $C^0([0,T]; H^1((0,1); \mathbb{R}^4))$.

5 Lyapunov stability analysis in H^1

In this section, we prove the following theorem concerning the stability of the closed-loop system Σ_{cl} in the H^1-norm.

Theorem 6 (Exponential stability in H^1)**.** Assume that

$$|(k_1 - k_c)k_2| < 1, \quad |k_1(k_2 - k_0)| < 1 \quad \text{and} \quad \rho_2(\widehat{\mathbf{K}}) < 1. \tag{5.1}$$

Then there exists $\varepsilon_0^* > 0$ such that, if $\max(|\varepsilon_1|, |\varepsilon_2|) < \varepsilon_0^*$, the system (4.19) is exponentially stable for the H^1-norm. In other words, there exist $C_0 > 0$ and $\gamma > 0$ such that for any $T > 0$, and any initial condition $(z_0, \hat{z}_0) \in H^1((0,1);\mathbb{R}^4)$ satisfying the boundary conditions (4.19c), the system Σ_{cl} has a unique solution $(z, \hat{z}) \in C^0([0,T]; H^1((0,1);\mathbb{R}^4))$ and

$$\|(z(t,\cdot), \hat{z}(t,\cdot))\|_{H^1} \leqslant C_0 e^{-\gamma t} \|(z_0, \hat{z}_0)\|_{H^1}, \ \forall\, t \in [0,T]. \tag{5.2}$$

Proof. Let $(z_0, \hat{z}_0) \in H^1((0,1);\mathbb{R}^4)$ satisfying the boundary conditions (4.19c). There exists a unique solution $(z, \hat{z}) \in C^0([0,T]; H^1((0,1);\mathbb{R}^4))$ to the system Σ_{cl} (see Appendix. C). We introduce the auxiliary state variables:

$$s(t,x) = \partial_t z(t,x) \quad \text{and} \quad \hat{s}(t,x) = \partial_t \hat{z}(t,x) \quad \text{and} \quad \tilde{z}(t,x) = z(t,x) - \hat{z}(t,x). \tag{5.3}$$

Using (4.15a) together with (5.3), we have (in $L^2((0,1);\mathbb{R}^2)$)

$$\partial_x z(t,x) = -(\mathbf{\Lambda} + \boldsymbol{\mathcal{E}})^{-1} s(t,x) + (\mathbf{\Lambda} + \boldsymbol{\mathcal{E}})^{-1} \mathbf{h}_{1\varepsilon}(z(t,\cdot))(x). \tag{5.4}$$

Then, by differentiating (4.15) with respect to time, we can derive the dynamics of the state variables s and $\hat{s}$ and write the following extended form for the closed-loop system dynamics:

$$\partial_t z(t,x) + (\mathbf{\Lambda} + \boldsymbol{\mathcal{E}})\partial_x z(t,x) = \mathbf{h}_{1\varepsilon}(z(t,\cdot))(x), \tag{5.5a}$$

$$\partial_t \tilde{z}(t,x) + \mathbf{\Lambda}\partial_x \tilde{z}(t,x) - \boldsymbol{\mathcal{E}}(\mathbf{\Lambda} + \boldsymbol{\mathcal{E}})^{-1} s(t,x) = \mathbf{h}_{2\varepsilon}(z(t,\cdot))(x), \tag{5.5b}$$

$$\begin{pmatrix} z_1(t,0) \\ z_2(t,1) \\ \tilde{z}_1(t,0) \\ \tilde{z}_2(t,1) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & k_1 - k_c & 0 & k_c \\ k_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 - k_0 & 0 \end{pmatrix}}_{\widetilde{\mathbf{K}}} \begin{pmatrix} z_1(t,1) \\ z_2(t,0) \\ \tilde{z}_1(t,1) \\ \tilde{z}_2(t,0) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \int_0^1 f(\zeta)\tilde{z}(t,\zeta)d\zeta, \tag{5.5c}$$

$$\partial_t s(t,x) + (\mathbf{\Lambda} + \boldsymbol{\mathcal{E}})\partial_x s(t,x) = \mathbf{h}_{1\varepsilon}(s(t,\cdot))(x), \tag{5.6a}$$

$$\partial_t \hat{s}(t,x) + \mathbf{\Lambda}\partial_x \hat{s}(t,x) = \mathbf{h}_{1\varepsilon}(s(t,\cdot))(x) - \widetilde{\mathbf{h}}_{2\varepsilon}(s(t,\cdot))(x), \tag{5.6b}$$

$$\begin{pmatrix} s_1(t,0) \\ s_2(t,1) \\ \hat{s}_1(t,0) \\ \hat{s}_2(t,1) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & k_1 & 0 & -k_c \\ k_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_1 - k_c \\ k_0 & 0 & k_2 - k_0 & 0 \end{pmatrix}}_{\widehat{\mathbf{K}}} \begin{pmatrix} s_1(t,1) \\ s_2(t,0) \\ \hat{s}_1(t,1) \\ \hat{s}_2(t,0) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \int_0^1 f(\zeta)\Big(s(t,\zeta) - \hat{s}(t,\zeta)\Big)d\zeta. \tag{5.6c}$$

Here the operator $\mathbf{h}_{2\varepsilon}$ is defined by

$$\mathbf{h}_{2\varepsilon} := \widetilde{\mathbf{h}}_{2\varepsilon} - \boldsymbol{\mathcal{E}}(\mathbf{\Lambda} + \boldsymbol{\mathcal{E}})^{-1}\mathbf{h}_{1\varepsilon}. \tag{5.7}$$

Notice that the extended closed-loop system (5.5)–(5.6) is in diagonal characteristic form with Riemann coordinates $(z, \tilde{z}, s, \hat{s})$.

For any function $\varphi = (\varphi_1, \varphi_2)^\mathsf{T} : [0,1] \to \mathbb{R}^2$, we define the functionals

$$\boldsymbol{\theta}(\varphi) = \int_0^1 f(\zeta)\varphi(\zeta)d\zeta, \tag{5.8}$$

$$\mathbf{W}_i(\varphi) = \int_0^1 \Big(a_i(x)\varphi_1^2(x) + b_i(x)\varphi_2^2(x)\Big)dx, \tag{5.9}$$

$$a_i(x) = \bar{a}_i e^{-\bar{\mu}_i x}, \quad b_i(x) = \bar{b}_i e^{\bar{\mu}_i x}, \tag{5.10}$$

where $\bar{a}_i, \bar{b}_i, \bar{\mu}_i$ are three positive real numbers to be determined.

The stability of the system (5.5)–(5.6) will be analysed using the candidate Lyapunov function

$$\mathbf{V} = \mathbf{V}_0 + \mathbf{V}_1, \tag{5.11}$$

where

$$\mathbf{V}_0 = \mathbf{W}_1(z(t,\cdot)) + M\mathbf{W}_2(\tilde{z}(t,\cdot)), \tag{5.12}$$

$$\mathbf{V}_1 = \mathbf{W}_3(s(t,\cdot)) + \mathbf{W}_4(\hat{s}(t,\cdot)), \tag{5.13}$$

with $M > 0$ a constant to be determined.

To carry out the stability analysis, we compute the time derivative of the function $\mathbf{V}$ along C^1-solutions of the system (5.5)–(5.6).

<u>Time derivative of the functions $\mathbf{W}_i$.</u>

The first step is to explicit the time derivatives of the functions $\mathbf{W}_i$ which have the following structure:

$$\frac{d\mathbf{W}_i}{dt} = \mathcal{B}_i(t) + \mathcal{I}_i(t), \quad i = 1,2,3,4, \tag{5.14}$$

where $\mathcal{B}_i(t)$ are boundary terms and $\mathcal{I}_i(t)$ are internal integral terms. Using the notation

$$A_i(x) = \operatorname{diag}(a_i(x), b_i(x)), \tag{5.15}$$

these terms are

$$\begin{aligned}\mathcal{B}_1(t) = &-(\lambda_1+\varepsilon_1)\Big[a_1(1)z_1^2(t,1) - a_1(0)\Big((k_1-k_c)z_2(t,0) + k_c\tilde{z}_2(t,0) + \boldsymbol{\theta}(\tilde{z}(t,\cdot))\Big)^2\Big] \\ &+ (\lambda_2+\varepsilon_2)\Big[b_1(1)k_2^2 z_1^2(t,1) - b_1(0)z_2^2(t,0)\Big],\end{aligned} \tag{5.16}$$

$$\mathcal{I}_1(t) = \int_0^1 \Big((\lambda_1+\varepsilon_1)a_1'(x)z_1^2(t,x) - (\lambda_2+\varepsilon_2)b_1'(x)z_2^2(t,x) + 2\mathbf{h}_{1\varepsilon}^\mathsf{T}(z(t,\cdot))(x)A_1(x)z(t,x)\Big)dx, \tag{5.17}$$

$$\mathcal{B}_2(t) = -\lambda_1\Big[a_2(1)\tilde{z}_1^2(t,1) - a_2(0)k_1^2\tilde{z}_2^2(t,0)\Big] + \lambda_2\Big[b_2(1)(k_2-k_0)^2\tilde{z}_1^2(t,1) - b_2(0)\tilde{z}_2^2(t,0)\Big], \tag{5.18}$$

$$\begin{aligned}\mathcal{I}_2(t) = \int_0^1 \Big(&\lambda_1 a_2'(x)\tilde{z}_1^2(t,x) - \lambda_2 b_2'(x)\tilde{z}_2^2(t,x) + \frac{2\varepsilon_1 a_2(x)}{\lambda_1+\varepsilon_1}\tilde{z}_1(t,x)s_1(t,x) \\ &+ \frac{2\varepsilon_2 b_2(x)}{\lambda_2+\varepsilon_2}\tilde{z}_2(t,x)s_2(t,x) + 2\mathbf{h}_{2\varepsilon}^\mathsf{T}(z(t,\cdot))(x)A_2(x)\tilde{z}(t,x)\Big)dx,\end{aligned} \tag{5.19}$$

$$
\begin{aligned}
\mathcal{B}_3(t) =& -(\lambda_1+\varepsilon_1)\Big[a_3(1)s_1^2(t,1) - a_3(0)\Big(k_1 s_2(t,0) - k_c\hat{s}_2(t,0) + \boldsymbol{\theta}(s(t,\cdot)-\hat{s}(t,\cdot))\Big)^2\Big] \\
&+(\lambda_2+\varepsilon_2)\Big[b_3(1)k_2^2 s_1^2(t,1) - b_3(0)s_2^2(t,0)\Big],
\end{aligned}
\tag{5.20}
$$

$$
\mathcal{I}_3(t) = \int_0^1 \Big((\lambda_1+\varepsilon_1)a_3'(x)s_1^2(t,x) - (\lambda_2+\varepsilon_2)b_3'(x)s_2^2(t,x) + 2\mathbf{h}_{1\varepsilon}^{\mathsf{T}}(s(t,\cdot))(x)A_3(x)s(t,x)\Big)dx, \tag{5.21}
$$

$$
\begin{aligned}
\mathcal{B}_4(t) =& -\lambda_1\Big[a_4(1)\hat{s}_1^2(t,1) - a_4(0)\Big((k_1-k_c)\hat{s}_2(t,0) + \boldsymbol{\theta}(s(t,\cdot)-\hat{s}(t,\cdot))\Big)^2\Big] \\
&+\lambda_2\Big[b_4(1)\Big(k_0 s_1(t,1) + (k_2-k_0)\hat{s}_1(t,1)\Big)^2 - b_4(0)\hat{s}_2^2(t,0)\Big],
\end{aligned}
\tag{5.22}
$$

$$
\mathcal{I}_4(t) = \int_0^1 \Big(\lambda_1 a_4'(x)\hat{s}_1^2(t,x) - \lambda_2 b_4'(x)\hat{s}_2^2(t,x) + 2\mathbf{h}_{2\varepsilon}^{\mathsf{T}}(s(t,\cdot))(x)A_4(x)\hat{s}(t,x)\Big)dx. \tag{5.23}
$$

Time derivative of the function $\mathbf{V}_0$.

We now examine the time derivative of the function $\mathbf{V}_0$:

$$
\frac{d\mathbf{V}_0}{dt} = \frac{d\mathbf{W}_1}{dt} + M\frac{d\mathbf{W}_2}{dt} = \mathcal{B}_1(t) + M\mathcal{B}_2(t) + \mathcal{I}_1(t) + M\mathcal{I}_2(t). \tag{5.24}
$$

The boundary term is

$$
\begin{aligned}
\mathcal{B}_1(t) + M\mathcal{B}_2(t) =& -\mathbf{X}_0^{\mathsf{T}}(t)F_0\mathbf{X}_0(t) \\
&+(\lambda_1+\varepsilon_1)\bar{a}_1\Big[\boldsymbol{\theta}^2(\tilde{z}(t,\cdot)) + \Big(2(k_1-k_c)z_2(t,0) + 2k_c\tilde{z}_2(t,0)\Big)\boldsymbol{\theta}(\tilde{z}(t,\cdot))\Big]
\end{aligned}
\tag{5.25}
$$

with the vector

$$
\mathbf{X}_0^{\mathsf{T}}(t) = \Big(z_1(t,1), z_2(t,0), \tilde{z}_1(t,1), \tilde{z}_2(t,0)\Big), \tag{5.26}
$$

and the symmetric matrix

$$
F_0 = \begin{pmatrix} f_{01} & 0 & 0 & 0 \\ 0 & f_{02} & 0 & f_{06} \\ 0 & 0 & Mf_{03} & 0 \\ 0 & f_{06} & 0 & f_{04}+Mf_{05} \end{pmatrix} \tag{5.27}
$$

where

$$
f_{01} = (\lambda_1+\varepsilon_1)\bar{a}_1 e^{-\bar{\mu}_1} - k_2^2(\lambda_2+\varepsilon_2)\bar{b}_1 e^{\bar{\mu}_1}, \tag{5.28a}
$$

$$
f_{02} = -(k_1-k_c)^2(\lambda_1+\varepsilon_1)\bar{a}_1 + (\lambda_2+\varepsilon_2)\bar{b}_1, \tag{5.28b}
$$

$$
f_{03} = \lambda_1\bar{a}_2 e^{-\bar{\mu}_2} - (k_2-k_0)^2\lambda_2\bar{b}_2 e^{\bar{\mu}_2}, \tag{5.28c}
$$

$$
f_{04} = -k_c^2(\lambda_1+\varepsilon_1)\bar{a}_1, \tag{5.28d}
$$

$$
f_{05} = -k_1^2\lambda_1\bar{a}_2 + \lambda_2\bar{b}_2, \tag{5.28e}
$$

$$
f_{06} = -k_c(k_1-k_c)(\lambda_1+\varepsilon_1)\bar{a}_1. \tag{5.28f}
$$

The integral term is

$$
\begin{aligned}
\mathcal{I}_1(t) + M\mathcal{I}_2(t) = &\int_0^1 \Big((\lambda_1+\varepsilon_1)a_1'(x)z_1^2(t,x) - (\lambda_2+\varepsilon_2)b_1'(x)z_2^2(t,x)\Big)dx \\
&+ M\int_0^1 \Big(\lambda_1 a_2'(x)\tilde{z}_1^2(t,x) - \lambda_2 b_2'(x)\tilde{z}_2^2(t,x)\Big)dx \\
&+ M\int_0^1 \Big(\frac{2\varepsilon_1 a_2(x)}{\lambda_1+\varepsilon_1}\tilde{z}_1(t,x)s_1(t,x) + \frac{2\varepsilon_2 b_2(x)}{\lambda_2+\varepsilon_2}\tilde{z}_2(t,x)s_2(t,x)\Big)dx \\
&+ \int_0^1 2\mathbf{h}_{1\varepsilon}^{\mathsf{T}}(z(t,\cdot))(x)A_1(x)z(t,x)dx + M\int_0^1 2\mathbf{h}_{2\varepsilon}^{\mathsf{T}}(z(t,\cdot))(x)A_2(x)\tilde{z}(t,x)dx.
\end{aligned}
\tag{5.29}
$$

Time derivative of the function $\mathbf{V}_1$.

We now examine the time derivative of the function $\mathbf{V}_1$:

$$
\frac{d\mathbf{V}_1}{dt} = \frac{d\mathbf{W}_3}{dt} + \frac{d\mathbf{W}_4}{dt} = \mathcal{B}_3(t) + \mathcal{B}_4(t) + \mathcal{I}_3(t) + \mathcal{I}_4(t). \tag{5.30}
$$

The boundary term is

$$
\begin{aligned}
\mathcal{B}_3(t) + \mathcal{B}_4(t) = &-\mathbf{X}_1^{\mathsf{T}}(t)F_1\mathbf{X}_1(t) + \Big((\lambda_1+\varepsilon_1)\bar{a}_3 + \lambda_1\bar{a}_4\Big)\boldsymbol{\theta}^2(s(t,\cdot)-\hat{s}(t,\cdot)) \\
&+ 2\Big((\lambda_1+\varepsilon_1)\bar{a}_3(k_1 s_2(t,0) - k_c\hat{s}_2(t,0)) + (k_1-k_c)\lambda_1\bar{a}_4\hat{s}_2(t,0)\Big)\boldsymbol{\theta}(s(t,\cdot)-\hat{s}(t,\cdot)),
\end{aligned}
\tag{5.31}
$$

with the vector

$$
\mathbf{X}_1^{\mathsf{T}}(t) := \Big(s_1(t,1), s_2(t,0), \hat{s}_1(t,1), \hat{s}_2(t,0)\Big), \tag{5.32}
$$

and the symmetric matrix

$$
F_1 = \begin{pmatrix} f_{11} & 0 & f_{15} & 0 \\ 0 & f_{12} & 0 & f_{16} \\ f_{15} & 0 & f_{13} & 0 \\ 0 & f_{16} & 0 & f_{14} \end{pmatrix} \tag{5.33}
$$

where

$$
f_{11} = (\lambda_1+\varepsilon_1)\bar{a}_3 e^{-\bar{\mu}_3} - k_2^2(\lambda_2+\varepsilon_2)\bar{b}_3 e^{\bar{\mu}_3} - k_0^2\lambda_2\bar{b}_4 e^{\bar{\mu}_4}, \tag{5.34a}
$$

$$
f_{12} = -k_1^2(\lambda_1+\varepsilon_1)\bar{a}_3 + (\lambda_2+\varepsilon_2)\bar{b}_3, \tag{5.34b}
$$

$$
f_{13} = \lambda_1\bar{a}_4 e^{-\bar{\mu}_4} - (k_2-k_0)^2\lambda_2\bar{b}_4 e^{\bar{\mu}_4}, \tag{5.34c}
$$

$$
f_{14} = -k_c^2(\lambda_1+\varepsilon_1)\bar{a}_3 - (k_1-k_c)^2\lambda_1\bar{a}_4 + \lambda_2\bar{b}_4, \tag{5.34d}
$$

$$
f_{15} = -k_0(k_2-k_0)\lambda_2\bar{b}_4 e^{\bar{\mu}_4}, \tag{5.34e}
$$

$$
f_{16} = k_1 k_c(\lambda_1+\varepsilon_1)\bar{a}_3. \tag{5.34f}
$$

The integral term is

$$
\begin{aligned}
\mathcal{I}_3(t) + \mathcal{I}_4(t) = &\int_0^1 \Big((\lambda_1+\varepsilon_1)a_3'(x)s_1^2(t,x) - (\lambda_2+\varepsilon_2)b_3'(x)s_2^2(t,x)\Big)dx \\
&+ \int_0^1 \Big(\lambda_1 a_4'(x)\hat{s}_1^2(t,x) - \lambda_2 b_4'(x)\hat{s}_2^2(t,x)\Big)dx \\
&+ \int_0^1 2\mathbf{h}_{1\varepsilon}^{\mathsf{T}}(s(t,\cdot))(x)A_3(x)s(t,x)dx + \int_0^1 2\mathbf{h}_{2\varepsilon}^{\mathsf{T}}(s(t,\cdot))(x)A_4(x)\hat{s}(t,x)dx
\end{aligned}
\tag{5.35}
$$

Determination of an upper bound for $d\mathbf{V}_1/dt$.

Since $\rho_2(\widehat{\mathbf{K}}) < 1$ by assumption (see (5.1)), following e.g. [41, Chapter 3], we can select $\bar{a}_3$, $\bar{b}_3$, $\bar{a}_4$, $\bar{b}_4$, $\bar{\mu}_3$, $\bar{\mu}_4$ such that there are two real positive numbers ν_1 and $\bar{\nu}_1$ for which the following inequalities hold:

$$-\mathbf{X}_1^{\mathsf{T}}(t)F_1\mathbf{X}_1(t) \leqslant -2\nu_1\|\mathbf{X}_1(t)\|^2, \tag{5.36}$$

$$\begin{aligned}\int_0^1 &\Big((\lambda_1+\varepsilon_1)a_3'(x)s_1^2(t,x)-(\lambda_2+\varepsilon_2)b_3'(x)s_2^2(t,x)\Big)dx\\ &+\int_0^1\Big(\lambda_1a_4'(x)\hat{s}_1^2(t,x)-\lambda_2b_4'(x)\hat{s}_2^2(t,x)\Big)dx \leqslant -\bar{\nu}_1\Big(\|s(t,\cdot)\|_{L^2}^2+\|\hat{s}(t,\cdot)\|_{L^2}^2\Big).\end{aligned}\tag{5.37}$$

This implies from (5.31) and (5.35) that

$$\begin{aligned}\frac{d\mathbf{V}_1}{dt} &\leqslant -2\nu_1\|\mathbf{X}_1(t)\|^2-\bar{\nu}_1\Big(\|s(t,\cdot)\|_{L^2}^2+\|\hat{s}(t,\cdot)\|_{L^2}^2\Big)\\ &+\Big((\lambda_1+\varepsilon_1)\bar{a}_3+\lambda_1\bar{a}_4\Big)\boldsymbol{\theta}^2(s(t,\cdot)-\hat{s}(t,\cdot))\\ &+2\Big((\lambda_1+\varepsilon_1)\bar{a}_3k_1(s_2(t,0)-\hat{s}_2(t,0))+\lambda_1\bar{a}_4(k_1-k_c)\hat{s}_2(t,0)\Big)\boldsymbol{\theta}(s(t,\cdot)-\hat{s}(t,\cdot))\\ &+\int_0^1 2\mathbf{h}_{1\varepsilon}^{\mathsf{T}}(s(t,\cdot))(x)A_3(x)s(t,x)dx+\int_0^1 2\mathbf{h}_{2\varepsilon}^{\mathsf{T}}(s(t,\cdot))(x)A_4(x)\hat{s}(t,x)dx.\end{aligned}\tag{5.38}$$

By using Young's inequality we can write, for any $\eta_1>0$,

$$\begin{aligned}\frac{d\mathbf{V}_1}{dt} &\leqslant -2\nu_1\|\mathbf{X}_1(t)\|^2-\bar{\nu}_1\Big(\|s(t,\cdot)\|_{L^2}^2+\|\hat{s}(t,\cdot)\|_{L^2}^2\Big)\\ &+(\lambda_1+\varepsilon_1)\bar{a}_3k_1^2\eta_1s_2^2(t,0)\\ &+\Big((\lambda_1+\varepsilon_1)\bar{a}_3k_c^2+\lambda_1\bar{a}_4(k_1-k_c)^2\Big)\eta_1\hat{s}_2^2(t,0)\\ &+\Big((\lambda_1+\varepsilon_1)\bar{a}_3(1+2\eta_1^{-1})+\lambda_1\bar{a}_4(1+\eta_1^{-1})\Big)\boldsymbol{\theta}^2(s(t,\cdot)-\hat{s}(t,\cdot))\\ &+C\|s(t,\cdot)\|_{L^2}\|\mathbf{h}_{1\varepsilon}(s(t,\cdot))\|_{L^2}+C\|\hat{s}(t,\cdot)\|_{L^2}\|\mathbf{h}_{2\varepsilon}(s(t,\cdot))\|_{L^2}dx.\end{aligned}\tag{5.39}$$

As above, and everywhere in the article, the generic notation C represents a positive constant which is used in several upper bounds and may vary from one place to another, but which only depends on the system parameters λ_i, c_i, k_i and the control parameters k_c, k_0.

Let us now choose η_1 such that

$$\eta_1=\frac{\nu_1}{\max\{[(\lambda_1+\varepsilon_1)\bar{a}_3k_1^2],[(\lambda_1+\varepsilon_1)\bar{a}_3k_c^2+\lambda_1\bar{a}_4(k_1-k_c)^2]\}}\tag{5.40}$$

and therefore such that

$$\begin{aligned}\frac{d\mathbf{V}_1}{dt} &\leqslant -\nu_1\|\mathbf{X}_1(t)\|^2-\bar{\nu}_1\Big(\|s(t,\cdot)\|_{L^2}^2+\|\hat{s}(t,\cdot)\|_{L^2}^2\Big)\\ &+\Big((\lambda_1+\varepsilon_1)\bar{a}_3(1+2\eta_1^{-1})+\lambda_1\bar{a}_4(1+\eta_1^{-1})\Big)\boldsymbol{\theta}^2(s(t,\cdot)-\hat{s}(t,\cdot))\\ &+C\|s(t,\cdot)\|_{L^2}\|\mathbf{h}_{1\varepsilon}(s(t,\cdot))\|_{L^2}+C\|\hat{s}(t,\cdot)\|_{L^2}\|\mathbf{h}_{2\varepsilon}(s(t,\cdot))\|_{L^2}.\end{aligned}\tag{5.41}$$

Using (2.15), (4.17) and (5.7), we have the following upper bounds on the L^2-norms of the functions $\mathbf{h}_{1\varepsilon}(s(t,\cdot))$ and $\mathbf{h}_{2\varepsilon}(s(t,\cdot))$:

$$\begin{aligned}\|\mathbf{h}_{1\varepsilon}(s(t,\cdot))\|_{L^2} &\leqslant C\max(|\varepsilon_1|,|\varepsilon_2|)\Big(\|s(t,\cdot)\|_{L^2}+|(s_1(t,1))|\Big),\\ \|\mathbf{h}_{2\varepsilon}(s(t,\cdot))\|_{L^2} &\leqslant C\max(|\varepsilon_1|,|\varepsilon_2|)\Big(\|s(t,\cdot)\|_{L^2}+|(s_1(t,1))|\Big),\end{aligned}\tag{5.42}$$

which imply the inequalities

$$
\begin{aligned}
&\|s(t,\cdot)\|_{L^2}\|\mathbf{h}_{1\varepsilon}(s(t,\cdot))\|_{L^2} \leqslant C\max(|\varepsilon_1|,|\varepsilon_2|)\Big(s_1^2(t,1)+\|s(t,\cdot)\|_{L^2}^2\Big),\\
&\|\hat{s}(t,\cdot)\|_{L^2}\|\mathbf{h}_{2\varepsilon}(s(t,\cdot))\|_{L^2} \leqslant C\max(|\varepsilon_1|,|\varepsilon_2|)\Big(s_1^2(t,1)+\|s(t,\cdot)\|_{L^2}^2+\|\hat{s}(t,\cdot)\|_{L^2}^2\Big),
\end{aligned}
\tag{5.43}
$$

and consequently

$$
\begin{aligned}
\|s(t,\cdot)\|_{L^2}&\|\mathbf{h}_{1\varepsilon}(s(t,\cdot))\|_{L^2}+\|\hat{s}(t,\cdot)\|_{L^2}\|\mathbf{h}_{2\varepsilon}(s(t,\cdot))\|_{L^2}\\
&\leqslant C\max(|\varepsilon_1|,|\varepsilon_2|)\Big(\|\mathbf{X}_1(t)\|^2+\|s(t,\cdot)\|_{L^2}^2+\|\hat{s}(t,\cdot)\|_{L^2}^2\Big).
\end{aligned}
\tag{5.44}
$$

Moreover, from (4.15b), (5.3) and the definition (5.8) of $\boldsymbol{\theta}$, we have

$$
\begin{aligned}
\boldsymbol{\theta}(s(t,\cdot)-\hat{s}(t,\cdot)) &= \int_0^1 f(x)(s(t,x)-\hat{s}(t,x))dx\\
&= -\int_0^1 f(x)\Lambda\partial_x\tilde{z}(t,x)dx-\int_0^1 f(x)\boldsymbol{\mathcal{E}}\Big[\partial_x z(t,x)\Big]dx+\int_0^1 f(x)\widetilde{\mathbf{h}}_{2\varepsilon}(z(t,\cdot))(x)dx\\
&= [f(0)\Lambda\tilde{z}(t,0)-f(1)\Lambda\tilde{z}(t,1)]+[f(0)\boldsymbol{\mathcal{E}}z(t,0)-f(1)\boldsymbol{\mathcal{E}}z(t,1)]\\
&\quad+\int_0^1 f'(x)\Lambda\tilde{z}(t,x)dx+\int_0^1 f'(x)\mathcal{E}z(t,x))dx+\int_0^1 f(x)\widetilde{\mathbf{h}}_{2\varepsilon}(z(t,\cdot))(x)dx.
\end{aligned}
\tag{5.45}
$$

It follows that $\boldsymbol{\theta}^2$ is upper bounded as

$$
\begin{aligned}
\boldsymbol{\theta}^2(s(t,\cdot)-\hat{s}(t,\cdot)) \leqslant C\bigg(&\Big[\tilde{z}_1^2(t,1)+\tilde{z}_2^2(t,0)+\|\tilde{z}(t,\cdot)\|_{L^2}^2\Big]\\
&+\max(|\varepsilon_1|,|\varepsilon_2|)\Big[z_1^2(t,1)+z_2^2(t,0)+\|z(t,\cdot)\|_{L^2}^2\Big]\bigg).
\end{aligned}
\tag{5.46}
$$

Finally, combining (5.41), (5.44) and (5.46), we get

$$
\begin{aligned}
\frac{d\mathbf{V}_1}{dt} \leqslant &-\Big(\nu_1-C\max(|\varepsilon_1|,|\varepsilon_2|)\Big)\|\mathbf{X}_1(t)\|^2\\
&-\bar{\nu}_1\Big(\|s(t,\cdot)\|_{L^2}^2+\|\hat{s}(t,\cdot)\|_{L^2}^2\Big)\\
&+C\max(|\varepsilon_1|,|\varepsilon_2|)\Big(\|s(t,\cdot)\|_{L^2}^2+\|\hat{s}(t,\cdot)\|_{L^2}^2\Big)\\
&+C\Big(\tilde{z}_1^2(t,1)+\tilde{z}_2^2(t,0)+\|\tilde{z}(t,\cdot)\|_{L^2}^2\Big)\\
&+C\max(|\varepsilon_1|,|\varepsilon_2|)\Big(z_1^2(t,1)+z_2^2(t,0)+\|z(t,\cdot)\|_{L^2}^2\Big).
\end{aligned}
\tag{5.47}
$$

Remark 2. As can be seen in (5.45)–(5.46), we make use of the fact that f belongs to H^1.

<u>Determination of an upper bound for $d\mathbf{V}_0/dt$.</u>

Since $|(k_1-k_c)k_2|<1$ and $|k_1(k_2-k_0)|<1$ by assumption (see (5.1)), following e.g. [41, Chapter 3], we can select $\bar{a}_1$, $\bar{b}_1$, $\bar{a}_2$, $\bar{b}_2$, $\bar{\mu}_1$, $\bar{\mu}_2$ such that there are two real positive numbers ν_0 and $\bar{\nu}_0$ for which the following inequalities hold:

$$
-\begin{pmatrix} z_1(t,1)\\ z_2(t,0)\end{pmatrix}^{\mathsf{T}}\begin{pmatrix} f_{01} & 0\\ 0 & f_{02}\end{pmatrix}\begin{pmatrix} z_1(t,1)\\ z_2(t,0)\end{pmatrix} \leqslant -2\nu_0\Big(z_1^2(t,1)+z_2^2(t,0)\Big),
\tag{5.48a}
$$

$$
-\begin{pmatrix} \tilde{z}_1(t,1)\\ \tilde{z}_2(t,0)\end{pmatrix}^{\mathsf{T}}\begin{pmatrix} f_{03} & 0\\ 0 & f_{05}\end{pmatrix}\begin{pmatrix} \tilde{z}_1(t,1)\\ \tilde{z}_2(t,0)\end{pmatrix} \leqslant -2\nu_0\Big(\tilde{z}_1^2(t,1)+\tilde{z}_2^2(t,0)\Big),
\tag{5.48b}
$$

and

$$\int_0^1 \Big((\lambda_1+\varepsilon_1)a_1'(x)z_1^2(t,x)-(\lambda_2+\varepsilon_2)b_1'(x)z_2^2(t,x)\Big)dx \leqslant -2\bar{\nu}_0\|z(t,\cdot)\|_{L^2}^2, \tag{5.49a}$$

$$\int_0^1 \Big(\lambda_1 a_2'(x)\tilde{z}_1^2(t,x)-\lambda_2 b_2'(x)\tilde{z}_2^2(t,x)\Big)dx \leqslant -2\bar{\nu}_0\|\tilde{z}(t,\cdot)\|_{L^2}^2. \tag{5.49b}$$

Furthermore, using a Young's inequality we have, with any $\eta_2>0$,

$$\begin{aligned}\int_0^1 \Big(\frac{2\varepsilon_1 a_2(x)}{\lambda_1+\varepsilon_1}\tilde{z}_1(t,x)s_1(t,x)+\frac{2\varepsilon_2 b_2(x)}{\lambda_2+\varepsilon_2}\tilde{z}_2(t,x)s_2(t,x)\Big)dx&\\ \leqslant C\max(|\varepsilon_1|,|\varepsilon_2|)\Big(\eta_2\|\tilde{z}(t,\cdot)\|_{L^2}^2+\eta_2^{-1}\|s(t,\cdot)\|_{L^2}^2\Big).&\end{aligned} \tag{5.50}$$

Then, combining (5.48), (5.49), (5.50) with (5.25) and (5.29), we get

$$\begin{aligned}\frac{d\mathbf{V}_0}{dt}=&\frac{d\mathbf{W}_1(z(t,\cdot))}{dt}+M\frac{d\mathbf{W}_2(\tilde{z}(t,\cdot))}{dt}\\ \leqslant&-2\nu_0\Big(z_1^2(t,1)+z_2^2(t,0)\Big)-2M\nu_0\Big(\tilde{z}_1^2(t,1)+\tilde{z}_2^2(t,0)\Big)\\ &-2\bar{\nu}_0\|z(t,\cdot)\|_{L^2}^2-2M\bar{\nu}_0\|\tilde{z}(t,\cdot)\|_{L^2}^2\\ &+MC\max(|\varepsilon_1|,|\varepsilon_2|)\Big(\eta_2\|\tilde{z}(t,\cdot)\|_{L^2}^2+\eta_2^{-1}\|s(t,\cdot)\|_{L^2}^2\Big)\\ &+(\lambda_1+\varepsilon_1)\bar{a}_1k_c^2\tilde{z}_2^2(t,0)\\ &+2(\lambda_1+\varepsilon_1)\bar{a}_1k_c(k_1-k_c)z_2(t,0)\tilde{z}_2(t,0)\\ &+(\lambda_1+\varepsilon_1)\bar{a}_1\Big[\boldsymbol{\theta}^2(\tilde{z}(t,\cdot))+\Big(2(k_1-k_c)z_2(t,0)+2k_c\tilde{z}_2(t,0)\Big)\boldsymbol{\theta}(\tilde{z}(t,\cdot))\Big]\\ &+C\|z(t,\cdot)\|_{L^2}\|\mathbf{h}_{1\varepsilon}(z(t,\cdot))\|_{L^2}+MC\|\tilde{z}(t,\cdot)\|_{L^2}\|\mathbf{h}_{2\varepsilon}(z(t,\cdot))\|_{L^2}.\end{aligned} \tag{5.51}$$

We introduce the notation

$$\bar{\alpha}_1=(\lambda_1+\varepsilon_1)\bar{a}_1>0. \tag{5.52}$$

Then, using Young's inequalities with any $\eta_3>0$, (5.51) becomes

$$\begin{aligned}\frac{d\mathbf{V}_0}{dt}\leqslant&-2\nu_0\Big(z_1^2(t,1)+z_2^2(t,0)\Big)-2M\nu_0\Big(\tilde{z}_1^2(t,1)+\tilde{z}_2^2(t,0)\Big)\\ &-2\bar{\nu}_0\|z(t,\cdot)\|_{L^2}^2-2M\bar{\nu}_0\|\tilde{z}(t,\cdot)\|_{L^2}^2\\ &+MC\max(|\varepsilon_1|,|\varepsilon_2|)\Big(\eta_2\|\tilde{z}(t,\cdot)\|_{L^2}^2+\eta_2^{-1}\|s(t,\cdot)\|_{L^2}^2\Big)\\ &+\bar{\alpha}_1\Big(k_c^2\tilde{z}_2^2(t,0)+\boldsymbol{\theta}^2(\tilde{z}(t,\cdot))\Big)\\ &+\bar{\alpha}_1\Big(\eta_3z_2^2(t,0)+\eta_3^{-1}k_c^2(k_1-k_c)^2\tilde{z}_2^2(t,0)\Big)\\ &+\bar{\alpha}_1\Big(\eta_3(k_1-k_c)^2z_2^2(t,0)+\eta_3^{-1}\boldsymbol{\theta}^2(\tilde{z}(t,\cdot)\Big)\\ &+\bar{\alpha}_1\Big(k_c^2\tilde{z}_2^2(t,0)+\boldsymbol{\theta}^2(\tilde{z}(t,\cdot)\Big)\\ &+C\|z(t,\cdot)\|_{L^2}\|\mathbf{h}_{1\varepsilon}(z(t,\cdot))\|_{L^2}+MC\|\tilde{z}(t,\cdot)\|_{L^2}\|\mathbf{h}_{2\varepsilon}(z(t,\cdot))\|_{L^2}.\end{aligned} \tag{5.53}$$

Since

$$\boldsymbol{\theta}^2(\tilde{z}(t,\cdot))\leqslant\|f\|_{L^2}^2\|\tilde{z}(t,\cdot)\|_{L^2}^2 \tag{5.54}$$

and

$$\begin{aligned}\|z(t,\cdot)\|_{L^2}\|\mathbf{h}_{1\varepsilon}(z(t,\cdot))\|_{L^2}&\leqslant C\max(|\varepsilon_1|,|\varepsilon_2|)\Big(z_1^2(t,1)+\|z(t,\cdot)\|_{L^2}^2\Big),\\ \|\tilde{z}(t,\cdot)\|_{L^2}\|\mathbf{h}_{2\varepsilon}(z(t,\cdot))\|_{L^2}&\leqslant C\max(|\varepsilon_1|,|\varepsilon_2|)\Big(z_1^2(t,1)+\|z(t,\cdot)\|_{L^2}^2+\|\tilde{z}(t,\cdot)\|_{L^2}^2\Big),\end{aligned} \tag{5.55}$$

combining (5.53), (5.54), (5.55), we finally get

$$
\begin{aligned}
\frac{d\mathbf{V}_0}{dt} \leqslant &-\Big(2\nu_0 - C\max(|\varepsilon_1|,|\varepsilon_2|)(1+M)\Big)z_1^2(t,1)\\
&-\Big(2\nu_0 - \bar{\alpha}_1\eta_3(1+(k_1-k_c)^2)\Big)z_2^2(t,0)\\
&-2M\nu_0\tilde{z}_1^2(t,1)\\
&-\Big(2M\nu_0 - \bar{\alpha}_1k_c^2(2+\eta_3^{-1}(k_1-k_c)^2)\Big)\tilde{z}_2^2(t,0)\\
&-\Big(2\bar{\nu}_0 - C\max(|\varepsilon_1|,|\varepsilon_2|)(1+M)\Big)\|z(t,\cdot)\|_{L^2}^2\\
&-\Big(2M\bar{\nu}_0 - \bar{\alpha}_1\|f\|_{L^2}^2(2+\eta_3^{-1}) - MC\max(|\varepsilon_1|,|\varepsilon_2|)(1+\eta_2)\Big)\|\tilde{z}(t,\cdot)\|_{L^2}^2\\
&+MC\max(|\varepsilon_1|,|\varepsilon_2|)\eta_2^{-1}\|s(t,\cdot)\|_{L^2}^2.
\end{aligned}
\tag{5.56}
$$

<ins>Determination of an upper bound for $d\mathbf{V}/dt = d\mathbf{V}_0/dt + d\mathbf{V}_1/dt$.</ins>

Combining (5.47) and (5.56), we can write

$$
\begin{aligned}
\frac{d\mathbf{V}}{dt} = \frac{d\mathbf{V}_0}{dt} + \frac{d\mathbf{V}_1}{dt}&\\
\leqslant &-\Big(2\nu_0 - C\max(|\varepsilon_1|,|\varepsilon_2|)(2+M)\Big)z_1^2(t,1)\\
&-\Big(2\nu_0 - \bar{\alpha}_1\eta_3(1+(k_1-k_c)^2) - C\max(|\varepsilon_1|,|\varepsilon_2|)\Big)z_2^2(t,0)\\
&-\Big(2M\nu_0 - C)\tilde{z}_1^2(t,1)\\
&-\Big(2M\nu_0 - \bar{\alpha}_1k_c^2(2+\eta_3^{-1}(k_1-k_c)^2) - C\Big)\tilde{z}_2^2(t,0)\\
&-\Big(\nu_1 - C\max(|\varepsilon_1|,|\varepsilon_2|)\Big)\|\mathbf{X}_1(t)\|^2\\
&-\Big(2\bar{\nu}_0 - C\max(|\varepsilon_1|,|\varepsilon_2|)(2+M)\Big)\|z(t,\cdot)\|_{L^2}^2\\
&-\Big(2M\bar{\nu}_0 - \bar{\alpha}_1\|f\|_{L^2}^2(1+\eta_3^{-1}) - MC\max(|\varepsilon_1|,|\varepsilon_2|)(1+\eta_2) - C\Big)\|\tilde{z}(t,\cdot)\|_{L^2}^2\\
&-\bar{\nu}_1\Big(\|s(t,\cdot)\|_{L^2}^2 + \|\hat{s}(t,\cdot)\|_{L^2}^2\Big)\\
&+C\max(|\varepsilon_1|,|\varepsilon_2|)\Big((1+M\eta_2^{-1})\|s(t,\cdot)\|_{L^2}^2 + \|\hat{s}(t,\cdot)\|_{L^2}^2\Big).
\end{aligned}
\tag{5.57}
$$

Let us now choose successively η_3, η_2 and M such that

$$
\eta_3 = \frac{\nu_0}{\bar{\alpha}_1(1+(k_1-k_c)^2)},
\tag{5.58a}
$$

$$
\eta_2 = M = \max\left(\frac{C+\bar{\alpha}_1k_c^2(2+\eta_3^{-1}(k_1-k_c)^2)}{\nu_0}\,,\;\frac{C+\bar{\alpha}_1\|f\|_{L^2}^2\|(1+\eta_3^{-1})}{\bar{\nu}_0}\right).
\tag{5.58b}
$$

Then, inequality (5.57) becomes equivalent to

$$
\begin{aligned}
\frac{d\mathbf{V}}{dt} = \frac{d\mathbf{V}_0}{dt} + \frac{d\mathbf{V}_1}{dt} \leqslant &-\Big(\nu_0 - C(1+M)\max(|\varepsilon_1|,|\varepsilon_2|)\Big)\Big(2z_1^2(t,1) + z_2^2(t,0)\Big)\\
&-M\nu_0\Big(\tilde{z}_1^2(t,1) + \tilde{z}_2^2(t,0)\Big)\\
&-\Big(\nu_1 - C\max(|\varepsilon_1|,|\varepsilon_2|)\Big)\|\mathbf{X}_1(t)\|^2\\
&-2\Big(\bar{\nu}_0 - C(1+M)\max(|\varepsilon_1|,|\varepsilon_2|)\Big)\|z(t,\cdot)\|_{L^2}^2\\
&-M\Big(\bar{\nu}_0 - C(1+M)\max(|\varepsilon_1|,|\varepsilon_2|)\Big)\|\tilde{z}(t,\cdot)\|_{L^2}^2\\
&-\Big(\bar{\nu}_1 - 2C\max(|\varepsilon_1|,|\varepsilon_2|)\Big)\Big(\|s(t,\cdot)\|_{L^2}^2 + \|\hat{s}(t,\cdot)\|_{L^2}^2\Big).
\end{aligned}
\tag{5.59}
$$

We know that, by the definition of $\mathbf{V}$, there exists $\beta_0 > 0$ such that

$$\frac{1}{\beta_0}\mathbf{V} \leqslant \Big(\|z(t,\cdot)\|_{L^2}^2 + \|\tilde{z}(t,\cdot)\|_{L^2}^2 + \|s(t,\cdot)\|_{L^2}^2 + \|\hat{s}(t,\cdot)\|_{L^2}^2\Big) \leqslant \beta_0 \mathbf{V}. \tag{5.60}$$

It follows that there exists $\nu > 0$ and $\varepsilon_0^* > 0$ such that if $\max(|\varepsilon_1|, |\varepsilon_2|) < \varepsilon_0^*$ then

$$\frac{d\mathbf{V}}{dt} \leqslant -\nu \mathbf{V}. \tag{5.61}$$

Therefore $\mathbf{V}$ is a strict Lyapunov function and the solutions of the extended system (5.5)–(5.6) exponentially converge to zero for the L^2-norm.

This implies that there exists $C, \gamma > 0$ such that, using (5.3),

$$\|(z(t,\cdot), z(t,\cdot) - \hat{z}(t,\cdot), s(t,\cdot), \hat{s}(t,\cdot))\|_{L^2} \leq Ce^{-\gamma t}\|(z_0, z_0 - \hat{z}_0, s_0, \hat{s}_0)\|_{L^2}. \tag{5.62}$$

Since there exists $\bar{\beta}_0 > 0$ such that

$$\begin{aligned}\bar{\beta}_0^{-1}\|(z(t,\cdot), z(t,\cdot) - \hat{z}(t,\cdot), s(t,\cdot), \hat{s}(t,\cdot))\|_{L^2} &\leq \|(z(t,\cdot), \hat{z}(t,\cdot))\|_{H^1} \\ &\leq \bar{\beta}_0\|(z(t,\cdot), z(t,\cdot) - \hat{z}(t,\cdot), s(t,\cdot), \hat{s}(t,\cdot))\|_{L^2},\end{aligned} \tag{5.63}$$

this implies that the solutions of the perturbed closed-loop system Σ_{cl} exponentially converge to zero for the H^1-norm. This concludes the proof of Theorem 6. □

6 Stability analysis in L^2

In this section, we give the proof of Theorem 5. Let us recall the closed-loop system (4.19) in $z, \hat{z}$ coordinates

$$\Sigma_{cl} \begin{cases} \partial_t z(t,x) + (\mathbf{\Lambda} + \mathbf{\mathcal{E}})\partial_x z(t,x) = \mathbf{h}_{1\varepsilon}(z(t,\cdot))(x), & \text{(6.1a)} \\ \partial_t \hat{z}(t,x) + \mathbf{\Lambda}\partial_x \hat{z}(t,x) = \mathbf{h}_{1\varepsilon}(z(t,\cdot))(x) - \widetilde{\mathbf{h}}_{2\varepsilon}(z(t,\cdot))(x), & \text{(6.1b)} \\ \begin{pmatrix} z_1(t,0) \\ z_2(t,1) \\ \hat{z}_1(t,0) \\ \hat{z}_2(t,1) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & k_1 & 0 & -k_c \\ k_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_1 - k_c \\ k_0 & 0 & k_2 - k_0 & 0 \end{pmatrix}}_{\widehat{\mathbf{K}}} \begin{pmatrix} z_1(t,1) \\ z_2(t,0) \\ \hat{z}_1(t,1) \\ \hat{z}_2(t,0) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \int_0^1 f(\zeta)\big(z(t,\zeta) - \hat{z}(t,\zeta)\big)d\zeta. & \text{(6.1c)} \end{cases}$$

where $\mathbf{h}_{1\varepsilon}$ and $\widetilde{\mathbf{h}}_{2\varepsilon}$ are defined in (2.14) and (4.16), respectively.

Our strategy is to show that, under the stability conditions (4.20) in the statement of Theorem 5 and if $\max(|\varepsilon_1|, |\varepsilon_2|)$ is sufficiently small, there exists, for any $T > 0$, a linear isomorphism $\mathcal{T}$ between the solution spaces of Σ_{cl} in $C^0([0,T]; L^2((0,1) : \mathbb{R}^4))$ and in $C^0([0,T]; H^1((0,1); \mathbb{R}^4))$ so that the exponential stability in H^1 which follows from Theorem 6 may be transferred to exponential stability in L^2. The proof proceeds in four steps:

Step 1: We show that for any $(z, \hat{z}) \in C^0([0,T]; H^1((0,1); \mathbb{R}^4))$ solution to Σ_{cl} there exists a (unique) solution $(Z, \hat{Z}) \in C^0([0,T]; H^2((0,1); \mathbb{R}^4))$ to the same system Σ_{cl} such that

$$(z, \hat{z}) = (\mathbf{h}_{1\varepsilon}(Z) - (\mathbf{\Lambda} + \mathbf{\mathcal{E}})Z_x,\ \mathbf{h}_{1\varepsilon}(Z) - \widetilde{\mathbf{h}}_{2\varepsilon}(Z) - \mathbf{\Lambda}\hat{Z}_x). \tag{6.2}$$

Step 2: We consider the linear transform $\mathcal{T}$ which maps the space of $C^0([0,T];H^1((0,1);\mathbb{R}^4))$ solutions of Σ_{cl} to the space of $C^0([0,T];H^2((0,1);\mathbb{R}^4))$ solutions of Σ_{cl}. We show that this mapping is an isomorphism between $L^2((0,1);\mathbb{R}^4)$ and $\mathcal{T}(L^2((0,1);\mathbb{R}^4)) \subset H^1((0,1);\mathbb{R}^4)$. In particular there exists a constant $C>0$ such that, for any $(z,\hat{z}) \in L^2((0,1);\mathbb{R}^4)$,

$$\frac{1}{C}\|(z,\hat{z})\|_{L^2} \leqslant \|\mathcal{T}(z,\hat{z})\|_{H^1} \leqslant C\|(z,\hat{z})\|_{L^2}. \tag{6.3}$$

Step 3: Using a density argument, we show that for any $(z,\hat{z}) \in C^0([0,T];L^2((0,1);\mathbb{R}^4))$ solution to Σ_{cl}, $(Z,\hat{Z}) = \mathcal{T}(z,\hat{z}) \in C^0([0,T];H^1((0,1);\mathbb{R}^4))$ is a solution to Σ_{cl}.

Step 4: Combining the previous steps, we show that for any $(z,\hat{z}) \in C^0([0,T];L^2((0,1);\mathbb{R}^4))$ solution of Σ_{cl}, $\mathcal{T}(z,\hat{z})$ exponentially converges to zero in H^1-norm under the assumptions of Theorem 6. Using (6.3), we then deduce that $(z,\hat{z})$ itself converges exponentially to zero in L^2-norm and therefore that the system Σ_{cl} is exponentially stable in L^2-norm.

Step 1: We introduce the following notation to represent the components of the operators $\mathbf{h}_{1\varepsilon}$ and $\widetilde{\mathbf{h}}_{2\varepsilon}$:

$$\mathbf{h}_{1\varepsilon} = \begin{pmatrix} h_{11\varepsilon} \\ h_{12\varepsilon} \end{pmatrix}, \quad \widetilde{\mathbf{h}}_{2\varepsilon} = \begin{pmatrix} \widetilde{h}_{21\varepsilon} \\ \widetilde{h}_{22\varepsilon} \end{pmatrix}. \tag{6.4}$$

Let $(z,\hat{z}) \in C^0([0,T];H^1((0,1);\mathbb{R}^4))$ be a solution of Σ_{cl}. For any C^1 functions $d = (d_1,d_2)^\mathsf{T}$ and $\hat{d} = (\hat{d}_1,\hat{d}_2)^\mathsf{T}$, we consider the solutions $Z = (Z_1,Z_2)^\mathsf{T}$ and $\hat{Z} = (\hat{Z}_1,\hat{Z}_2)^\mathsf{T}$ of the following equations:

$$Z_1(t,x) = (\lambda_1+\varepsilon_1)^{-1}\int_0^x \Big(h_{11\varepsilon}(Z(t,\cdot))(\xi) - z_1(t,\xi)\Big)d\xi + d_1(t), \tag{6.5a}$$

$$Z_2(t,x) = (\lambda_2+\varepsilon_2)^{-1}\int_x^1 \Big(h_{12\varepsilon}(Z(t,\cdot))(\xi) - z_2(t,\xi)\Big)d\xi + d_2(t), \tag{6.5b}$$

$$\hat{Z}_1(t,x) = \lambda_1^{-1}\int_0^x \Big(h_{11\varepsilon}(Z(t,\cdot))(\xi) - \widetilde{h}_{21\varepsilon}(Z(t,\cdot))(\xi) - \hat{z}_1(t,\xi)\Big)d\xi + \hat{d}_1(t), \tag{6.5c}$$

$$\hat{Z}_2(t,x) = \lambda_2^{-1}\int_x^1 \Big(h_{12\varepsilon}(Z(t,\cdot))(\xi) - \widetilde{h}_{22\varepsilon}(Z(t,\cdot))(\xi) - \hat{z}_2(t,\xi)\Big)d\xi + \hat{d}_2(t). \tag{6.5d}$$

Let us first remark that these equations are selected such that (6.2) trivially holds, i.e.

$$-(\mathbf{\Lambda}+\boldsymbol{\mathcal{E}})\partial_x Z + \mathbf{h}_{1\varepsilon}(Z) = z, \quad -\mathbf{\Lambda}\partial_x \hat{Z} + \mathbf{h}_{1\varepsilon}(Z) - \widetilde{\mathbf{h}}_{2\varepsilon}(Z) = \hat{z}. \tag{6.6}$$

Our objective is, in a first stage, to show that for a given $(z,\hat{z}) \in C^0([0,T];H^1((0,1);\mathbb{R}^4))$ solution of Σ_{cl} and for any C^1 function $(d,\hat{d})$, (6.5) has a unique solution $(Z,\hat{Z})$, and, in a second stage, that $(d(t),\hat{d}(t))$ can be selected such that the solution $(Z,\hat{Z}) \in C^0([0,T];H^2((0,1);\mathbb{R}^4))$ is exactly also a solution of the system Σ_{cl}. This result will be formally stated in Lemma 2 at the end of the section.

Equations (6.5) can be rewritten in a compact form as

$$(\mathbf{I_d} - \mathcal{A}_0)(Z(t,.),\hat{Z}(t,.)) = \mathcal{B}_0(z(t,.),\hat{z}(t,.)) + \mathbf{d}(t) \tag{6.7}$$

where $\mathbf{I_d}$ denotes the identity operator and $\mathcal{A}_0$, $\mathcal{B}_0$, $\mathbf{d}$ are linear operators defined by

$$\mathcal{A}_0(Z(t,\cdot),\hat{Z}(t,\cdot))(x) = \begin{pmatrix} (\lambda_1+\varepsilon_1)^{-1}\displaystyle\int_0^x h_{11\varepsilon}(Z(t,\cdot))(\xi)d\xi \\ (\lambda_2+\varepsilon_2)^{-1}\displaystyle\int_x^1 h_{12\varepsilon}(Z(t,\cdot))(\xi)d\xi \\ \lambda_1^{-1}\displaystyle\int_0^x \Big(h_{11\varepsilon}(Z(t,\cdot))(\xi) - \widetilde{h}_{21\varepsilon}(Z(t,\cdot))(\xi)\Big)d\xi \\ \lambda_2^{-1}\displaystyle\int_x^1 \Big(h_{12\varepsilon}(Z(t,\cdot))(\xi) - \widetilde{h}_{22\varepsilon}(Z(t,\cdot))(\xi)\Big)d\xi \end{pmatrix}, \tag{6.8}$$

$$\mathcal{B}_0(z(t,\cdot),\hat{z}(t,\cdot))(x) = \begin{pmatrix} -(\lambda_1+\varepsilon_1)^{-1}\displaystyle\int_0^x z_1(t,\xi)d\xi \\ -(\lambda_2+\varepsilon_2)^{-1}\displaystyle\int_x^1 z_2(t,\xi)d\xi \\ -\lambda_1^{-1}\displaystyle\int_0^x \hat{z}_1(t,\xi)d\xi \\ -\lambda_2^{-1}\displaystyle\int_x^1 \hat{z}_2(t,\xi)d\xi \end{pmatrix}, \qquad \mathbf{d}(t) = \begin{pmatrix} d(t) \\ \hat{d}(t) \end{pmatrix} = \begin{pmatrix} d_1(t) \\ d_2(t) \\ \hat{d}_1(t) \\ \hat{d}_2(t) \end{pmatrix}. \tag{6.9}$$

Note that, from the definitions of $\mathbf{h}_{1\varepsilon}$ and $\widetilde{\mathbf{h}}_{2\varepsilon}$ given by (2.14) and (4.16), we have for any $\varphi \in H^1$ (resp. $\varphi \in H^2$)

$$\begin{aligned} \|\mathbf{h}_{1\varepsilon}(\varphi)\|_{H^1} + \|\widetilde{\mathbf{h}}_{2\varepsilon}(\varphi)\|_{H^1} \le C\max(|\varepsilon_1|,|\varepsilon_2|)\|\varphi\|_{H^1}, \\ \|\mathbf{h}_{1\varepsilon}(\varphi)\|_{H^2} + \|\widetilde{\mathbf{h}}_{2\varepsilon}(\varphi)\|_{H^2} \le C\max(|\varepsilon_1|,|\varepsilon_2|)\|\varphi\|_{H^2}, \end{aligned} \tag{6.10}$$

and consequently for the operator norms

$$\begin{aligned} \|\mathcal{A}_0\|_{\mathcal{L}(\mathcal{H}^1)} \le C\max(|\varepsilon_1|,|\varepsilon_2|), \\ \|\mathcal{A}_0\|_{\mathcal{L}(\mathcal{H}^2)} \le C\max(|\varepsilon_1|,|\varepsilon_2|), \end{aligned} \tag{6.11}$$

where, as above, C denotes positive constants which may be different from one equation to another and depend only on the system and control parameters but are independent of the perturbations ε_1 and ε_2. Note that we used here that the kernels P, $\widetilde{P}$, Q, $\widetilde{Q}$ defined in Appendix A are C^2 (see [10, Theorem A.2]). In particular, it follows that there exists $\varepsilon_1^* > 0$ such that if $\max(|\varepsilon_1|,|\varepsilon_2|) < \varepsilon_1^*$, then $\mathbf{I_d} - \mathcal{A}_0$ is an isomorphism of $H^1(0,1)$ into itself and an isomorphism of $H^2(0,1)$ into itself.

Moreover, we observe that since $(z,\hat{z}) \in C^0([0,T];H^1((0,1);\mathbb{R}^4))$,

$$\mathcal{B}_0(z(t,.),\hat{z}(t,.)) + \mathbf{d}(t) \in C^0([0,T];H^2((0,1);\mathbb{R}^4)). \tag{6.12}$$

Therefore, if $\max(|\varepsilon_1|,|\varepsilon_2|) < \varepsilon_1^*$, equation (6.5) is equivalent to

$$(Z(t,.),\hat{Z}(t,.)) = (\mathbf{I_d} - \mathcal{A}_0)^{-1}\mathcal{B}_0(z(t,.),\hat{z}(t,.)) + (\mathbf{I_d} - \mathcal{A}_0)^{-1}\mathbf{d}(t) \tag{6.13}$$

and defines a unique $(Z,\hat{Z})$ in $C^0([0,T];H^2((0,1);\mathbb{R}^4))$.

This completes the first stage of Step 1. Now, to complete Step 1, we still have to determine $d(t)$ and $\hat{d}(t)$ so that $(Z,\hat{Z})$ satisfies the boundary conditions (6.1c). Using (6.5) we first observe that

$$\begin{pmatrix} Z_1(t,0) \\ Z_2(t,1) \\ \hat{Z}_1(t,0) \\ \hat{Z}_2(t,1) \end{pmatrix} = \begin{pmatrix} d_1(t) \\ d_2(t) \\ \hat{d}_1(t) \\ \hat{d}_2(t) \end{pmatrix} = \begin{pmatrix} d(t) \\ \hat{d}(t) \end{pmatrix}. \tag{6.14}$$

We introduce the notations $w = (w_1,w_2)^\mathsf{T}$ and $\hat{w} = (\hat{w}_1,\hat{w}_2)^\mathsf{T}$ given by

$$(w(t,\cdot),\hat{w}(t,\cdot)) := (\mathbf{I_d} - \mathcal{A}_0)^{-1}\mathcal{B}_0(z(t,.),\hat{z}(t,.)) \tag{6.15}$$

and which are so that equation (6.13) is rewritten as

$$(Z(t,.),\hat{Z}(t,.)) = (w(t,\cdot),\hat{w}(t,\cdot)) + (\mathbf{I_d} - \mathcal{A}_0)^{-1}\mathbf{d}(t). \tag{6.16}$$

Using this expression and motivated by the form of the boundary condition (6.1c), we write

$$\begin{aligned}
&\widehat{\mathbf{K}}\begin{pmatrix} Z_1(t,1)\\ Z_2(t,0)\\ \hat{Z}_1(t,1)\\ \hat{Z}_2(t,0)\end{pmatrix} + \begin{pmatrix}1\\0\\1\\0\end{pmatrix}\int_0^1 f(\zeta)(Z(t,\zeta)-\hat{Z}(t,\zeta))d\zeta \\
&= \widehat{\mathbf{K}}\begin{pmatrix} w_1(t,1)\\ w_2(t,0)\\ \hat{w}_1(t,1)\\ \hat{w}_2(t,0)\end{pmatrix} + \widehat{\mathbf{K}}\mathcal{F}_1\left((\mathbf{I_d}-\mathcal{A}_0)^{-1}\begin{pmatrix} d(t)\\ \hat{d}(t)\end{pmatrix}\right) \\
&+ \begin{pmatrix}1\\0\\1\\0\end{pmatrix}\int_0^1 f(\zeta)\mathcal{F}\begin{pmatrix} w(t,\zeta)\\ \hat{w}(t,\zeta)\end{pmatrix}d\zeta + \begin{pmatrix}1\\0\\1\\0\end{pmatrix}\int_0^1 f(\zeta)\mathcal{F}(\mathbf{I_d}-\mathcal{A}_0)^{-1}\begin{pmatrix} d(t)\\ \hat{d}(t)\end{pmatrix}d\zeta.
\end{aligned} \tag{6.17}$$

where $\mathcal{F}_1$ and $\mathcal{F}$ are linear operators defined respectively on $H^1((0,1);\mathbb{R}^4)$ and $\mathbb{R}^4$ by

$$\mathcal{F}_1\begin{pmatrix} g_1\\ g_2\\ g_3\\ g_4\end{pmatrix} = \begin{pmatrix} g_1(1)\\ g_2(0)\\ g_3(1)\\ g_4(0)\end{pmatrix}, \quad \mathcal{F}\begin{pmatrix} g_1\\ g_2\\ g_3\\ g_4\end{pmatrix} = \begin{pmatrix} g_1-g_3\\ g_2-g_4\end{pmatrix}. \tag{6.18}$$

Combining with (6.14), $(Z,\hat{Z})$ satisfies the boundary conditions (6.1c) if and only if

$$\begin{aligned}
&(\mathbf{I_d}-\widehat{\mathbf{K}}\mathcal{F}_1(\mathbf{I_d}-\mathcal{A}_0)^{-1})\begin{pmatrix} d(t)\\ \hat{d}(t)\end{pmatrix} - \begin{pmatrix}1\\0\\1\\0\end{pmatrix}\int_0^1 f(\zeta)\mathcal{F}(\mathbf{I_d}-\mathcal{A}_0)^{-1}\begin{pmatrix} d(t)\\ \hat{d}(t)\end{pmatrix}d\zeta \\
&= \widehat{\mathbf{K}}\begin{pmatrix} w_1(t,1)\\ w_2(t,0)\\ \hat{w}_1(t,1)\\ \hat{w}_2(t,0)\end{pmatrix} + \begin{pmatrix}1\\0\\1\\0\end{pmatrix}\int_0^1 f(\zeta)\mathcal{F}\begin{pmatrix} w(t,\xi)\\ \hat{w}(t,\xi)\end{pmatrix}d\zeta.
\end{aligned} \tag{6.19}$$

This (linear) equation in $d(t), \hat{d}(t)$ has a (unique) solution if and only if the operator

$$\mathcal{D}_{\boldsymbol{\varepsilon}} : \begin{pmatrix} d\\ \hat{d}\end{pmatrix} \mapsto (\mathbf{I_d}-\widehat{\mathbf{K}}\mathcal{F}_1(\mathbf{I_d}-\mathcal{A}_0)^{-1})\begin{pmatrix} d\\ \hat{d}\end{pmatrix} - \begin{pmatrix}1\\0\\1\\0\end{pmatrix}\int_0^1 f(\zeta)\mathcal{F}(\mathbf{I_d}-\mathcal{A}_0)^{-1}\begin{pmatrix} d\\ \hat{d}\end{pmatrix}d\zeta \tag{6.20}$$

is invertible. Note that when $\varepsilon_1=\varepsilon_2=0$ then $(\mathbf{I_d}-\mathcal{A}_0)^{-1}=\mathbf{I_d}$ and

$$\int_0^1 f(\zeta)\mathcal{F}(\mathbf{I_d}-\mathcal{A}_0)^{-1}\begin{pmatrix} d(t)\\ \hat{d}(t)\end{pmatrix}d\zeta = \left(\int_0^1 f(\zeta)d\zeta\right)(d(t)-\hat{d}(t)). \tag{6.21}$$

Therefore the invertibility of $\mathcal{D}_{\boldsymbol{\varepsilon}}$ (as given by (6.20)) when $\boldsymbol{\varepsilon}=0$ is equivalent to the invertibility of the matrix

$$\mathcal{D}_0 := (\mathbf{I_d}-\widehat{\mathbf{K}}) - \begin{pmatrix} \int_0^1 f_1(\zeta)d\zeta & \int_0^1 f_2(\zeta)d\zeta & -\int_0^1 f_1(\zeta)d\zeta & -\int_0^1 f_2(\zeta)d\zeta \\ 0 & 0 & 0 & 0 \\ \int_0^1 f_1(\zeta)d\zeta & \int_0^1 f_2(\zeta)d\zeta & -\int_0^1 f_1(\zeta)d\zeta & -\int_0^1 f_2(\zeta)d\zeta \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6.22}$$

Using the expression of $\widehat{\mathbf{K}}$ given in (4.19c), we have

$$\mathcal{D}_0 = \begin{pmatrix} 1-\int_0^1 f_1(\zeta)d\zeta & -k_1-\int_0^1 f_2(\zeta)d\zeta & +\int_0^1 f_1(\zeta)d\zeta & k_c+\int_0^1 f_2(\zeta)d\zeta \\ -k_2 & 1 & 0 & 0 \\ -\int_0^1 f_1(\zeta)d\zeta & -\int_0^1 f_2(\zeta)d\zeta & 1+\int_0^1 f_1(\zeta)d\zeta & -k_1+k_c+\int_0^1 f_2(\zeta)d\zeta \\ -k_0 & 0 & -k_2+k_0 & 1 \end{pmatrix} \tag{6.23}$$

and

$$\begin{aligned} \det(\mathcal{D}_0) &= \begin{vmatrix} 1 & -k_1 & -1 & k_1 \\ -k_2 & 1 & 0 & 0 \\ -\int_0^1 f_1(\zeta)d\zeta & -\int_0^1 f_2(\zeta)d\zeta & 1+\int_0^1 f_1(\zeta)d\zeta & -k_1+k_c+\int_0^1 f_2(\zeta)d\zeta \\ -k_0 & 0 & -k_2+k_0 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -k_1 & 0 & 0 \\ -k_2 & 1 & -k_2 & 1 \\ -\int_0^1 f_1(\zeta)d\zeta & -\int_0^1 f_2(\zeta)d\zeta & 1 & -k_1+k_c \\ -k_0 & 0 & -k_2 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -k_1 & 0 & 0 \\ -(k_2-k_0) & 1 & 0 & 0 \\ -\int_0^1 f_1(\zeta)d\zeta & -\int_0^1 f_2(\zeta)d\zeta & 1 & -k_1+k_c \\ -k_0 & 0 & -k_2 & 1 \end{vmatrix} = (1-k_2(k_1-k_c))(1-k_1(k_2-k_0)). \end{aligned} \tag{6.24}$$

According to assumption (4.20) in the statement of Theorem 5, $(1-k_2(k_1-k_c))(1-k_1(k_2-k_0)) \neq 0$ and $\mathcal{D}_0$ is indeed invertible. Now, in the general case where ε_1 and ε_2 are not necessarily 0, since $\boldsymbol{\varepsilon} \mapsto \mathcal{A}_0$ is continuous with values in $\mathcal{L}(H^{-1})$ (and hence $\boldsymbol{\varepsilon} \mapsto (\mathbf{Id}-\mathcal{A}_0)^{-1}$ is too), $\boldsymbol{\varepsilon} \mapsto \mathcal{D}_{\boldsymbol{\varepsilon}}$ is also continuous with respect to ε_1 and ε_2 around 0, and there exists $\varepsilon_2^* > 0$ such that, if $\max(|\varepsilon_1|, |\varepsilon_2|) < \varepsilon_2^*$, the operator $\mathcal{D}_{\boldsymbol{\varepsilon}}$ is invertible (note that this is an operator from $\mathbb{R}^4$ to $\mathbb{R}^4$). Hence (6.19) has a unique solution (for each t) that is

$$\begin{pmatrix} d(t) \\ \hat{d}(t) \end{pmatrix} = \mathcal{D}_{\boldsymbol{\varepsilon}}^{-1} \mathcal{J}(w(t,\cdot), \hat{w}(t,\cdot)) \tag{6.25}$$

with

$$\mathcal{J}(w(t,\cdot),\hat{w}(t,\cdot)) := \widehat{\mathbf{K}} \begin{pmatrix} w_1(t,1) \\ w_2(t,0) \\ \hat{w}_1(t,1) \\ \hat{w}_2(t,0) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \int_0^1 f(\zeta)\mathcal{F}\begin{pmatrix} w(t,\zeta) \\ \hat{w}(t,\zeta) \end{pmatrix} d\zeta. \tag{6.26}$$

It remains to show that $(Z, \hat{Z})$ is a solution to Σ_{cl} with this choice of $(d, \hat{d})$. We first prove the following lemma.

Lemma 1.

$$\frac{d}{dt}\begin{pmatrix} d(t) \\ \hat{d}(t) \end{pmatrix} = \mathbf{X}_2(t) \quad \text{with} \quad \mathbf{X}_2(t) := \begin{pmatrix} z_1(t,0) \\ z_2(t,1) \\ \hat{z}_1(t,0) \\ \hat{z}_2(t,1) \end{pmatrix}. \tag{6.27}$$

Proof of Lemma 1. From (6.25), since $(z, \hat{z})$ is a solution of Σ_{cl}, we have

$$\frac{d}{dt}\begin{pmatrix} d(t) \\ \hat{d}(t) \end{pmatrix} = \mathcal{D}_{\mathcal{E}}^{-1}\left[\widehat{\mathbf{K}}\begin{pmatrix} \partial_t w_1(t,1) \\ \partial_t w_2(t,0) \\ \partial_t \hat{w}_1(t,1) \\ \partial_t \hat{w}_2(t,0) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}\int_0^1 f(\zeta)\mathcal{F}\begin{pmatrix} \partial_t w(t,\zeta) \\ \partial_t \hat{w}(t,\zeta) \end{pmatrix} d\zeta\right]. \tag{6.28}$$

Then, from the definition of $(w, \hat{w})$ given in (6.15), using the system definition (6.1a)–(6.1b) and the definition (6.8) of $\mathcal{A}_0$, we have

$$\begin{aligned}
\partial_t \begin{pmatrix} w(t,x) \\ \hat{w}(t,x) \end{pmatrix} &= (\mathbf{I_d} - \mathcal{A}_0)^{-1}\begin{pmatrix} -(\lambda_1+\varepsilon_1)^{-1}\displaystyle\int_0^x \partial_t z_1(t,\xi)d\xi \\ -(\lambda_2+\varepsilon_2)^{-1}\displaystyle\int_x^1 \partial_t z_2(t,\xi)d\xi \\ -\lambda_1^{-1}\displaystyle\int_0^x \partial_t \hat{z}_1(t,\xi)d\xi \\ -\lambda_2^{-1}\displaystyle\int_x^1 \partial_t \hat{z}_2(t,\xi)d\xi \end{pmatrix} \\
&= (\mathbf{I_d} - \mathcal{A}_0)^{-1}\begin{pmatrix} \displaystyle\int_0^x \partial_\xi z_1(t,\xi)d\xi - (\lambda_1+\varepsilon_1)^{-1}\int_0^x h_{11\varepsilon}(z(t,\cdot))(\xi)d\xi \\ -\displaystyle\int_x^1 \partial_\xi z_2(t,\xi)d\xi - (\lambda_2+\varepsilon_2)^{-1}\int_x^1 h_{12\varepsilon}(z(t,\cdot))(\xi)d\xi \\ \displaystyle\int_0^x \partial_\xi \hat{z}_1(t,\xi)d\xi - \lambda_1^{-1}\int_0^x \Big(h_{11\varepsilon}(z(t,\cdot))(\xi) - \widetilde{h}_{21\varepsilon}(z(t,\cdot))(\xi)\Big)d\xi \\ -\displaystyle\int_x^1 \partial_\xi \hat{z}_2(t,\xi)d\xi - \lambda_2^{-1}\int_x^1 \Big(h_{12\varepsilon}(z(t,\cdot))(\xi) - \widetilde{h}_{22\varepsilon}(z(t,\cdot))(\xi)\Big)d\xi \end{pmatrix} \\
&= (\mathbf{I_d} - \mathcal{A}_0)^{-1}\left[\begin{pmatrix} z(t,x) \\ \hat{z}(t,x) \end{pmatrix} - \mathbf{X}_2(t) - \mathcal{A}_0(z(t,x), \hat{z}(t,x))\right] \\
&= (\mathbf{I_d} - \mathcal{A}_0)^{-1}\left[(\mathbf{I_d} - \mathcal{A}_0)\begin{pmatrix} z(t,x) \\ \hat{z}(t,x) \end{pmatrix} - \mathbf{X}_2(t)\right] = \begin{pmatrix} z(t,x) \\ \hat{z}(t,x) \end{pmatrix} - (\mathbf{I_d} - \mathcal{A}_0)^{-1}\mathbf{X}_2(t).
\end{aligned} \tag{6.29}$$

Then, introducing (6.29) into (6.28), using the boundary condition (6.1c), the definitions (6.18) of $\mathcal{F}_1$ and $\mathcal{F}$ and the definition (6.20) of $\mathcal{D}_{\mathcal{E}}$, we get

$$\begin{aligned}
\frac{d}{dt}\begin{pmatrix} d(t) \\ \hat{d}(t) \end{pmatrix} &= \mathcal{D}_{\mathcal{E}}^{-1}\Big[\widehat{\mathbf{K}}\,\mathbf{X}_2(t) - \widehat{\mathbf{K}}\mathcal{F}_1\Big((\mathbf{I_d} - \mathcal{A}_0)^{-1}\mathbf{X}_2(t)\Big) \\
&\quad + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}\int_0^1 f(\zeta)(z(t,\zeta) - \hat{z}(t,\zeta))d\zeta - \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}\int_0^1 f(\zeta)\mathcal{F}(\mathbf{I_d} - \mathcal{A}_0)^{-1}\mathbf{X}_2(t)d\zeta\Bigg] \\
&= \mathcal{D}_{\mathcal{E}}^{-1}\left[\mathbf{X}_2(t) - \widehat{\mathbf{K}}\mathcal{F}_1\Big((\mathbf{I_d} - \mathcal{A}_0)^{-1}\mathbf{X}_2(t)\Big) - \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}\int_0^1 f(\zeta)\mathcal{F}(\mathbf{I_d} - \mathcal{A}_0)^{-1}\mathbf{X}_2(t)d\zeta\right] \\
&= \mathcal{D}_{\mathcal{E}}^{-1}\mathcal{D}_{\mathcal{E}}\mathbf{X}_2(t) = \mathbf{X}_2(t).
\end{aligned} \tag{6.30}$$

Hence (6.27) holds and this completes the proof of Lemma 1. $\square$

Using Lemma 1 and (6.5), we have

$$
\begin{aligned}
\partial_t Z_1(t,x) &= (\lambda_1+\varepsilon_1)^{-1}\int_0^x \Big(\partial_t h_{11\varepsilon}(Z(t,.))(\xi) - \partial_t z_1(t,\xi)\Big)d\xi + d_1'(t)\\
&= (\lambda_1+\varepsilon_1)^{-1}\int_0^x \Big(\partial_t h_{11\varepsilon}(Z(t,\cdot))(\xi) - h_{11\varepsilon}(z(t,\cdot)(\xi)\Big)d\xi + \int_0^x \partial_x z_1(t,\xi)d\xi + z_1(t,0)\\
&= (\lambda_1+\varepsilon_1)^{-1}\int_0^x h_{11\varepsilon}\Big(\partial_t Z(t,\cdot) - z(t,\cdot)\Big)(\xi)d\xi + z_1(t,x).
\end{aligned}
\tag{6.31}
$$

In a similar way, we can show that

$$
\begin{aligned}
\partial_t Z_2(t,x) &= (\lambda_2+\varepsilon_2)^{-1}\int_x^1 h_{12\varepsilon}\Big(\partial_t Z(t,\cdot) - z(t,\cdot)\Big)(\xi)d\xi + z_2(t,x),\\
\partial_t \hat{Z}_1(t,x) &= \lambda_1^{-1}\int_0^x \Big(h_{11\varepsilon} - \widetilde{h}_{21\varepsilon}\Big)\Big(\partial_t Z(t,\cdot) - z(t,\cdot)\Big)(\xi)d\xi + \hat{z}_1(t,x),\\
\partial_t \hat{Z}_2(t,x) &= \lambda_2^{-1}\int_x^1 \Big(h_{12\varepsilon} - \widetilde{h}_{22\varepsilon}\Big)\Big(\partial_t Z(t,\cdot) - z(t,\cdot)\Big)(\xi)d\xi + \hat{z}_2(t,x).
\end{aligned}
\tag{6.32}
$$

Note that, in the derivation of these equations, we use that $(z,\hat{z}) \in C^1([0,T];L^2((0,1);\mathbb{R}^4))$ since it is a solution of Σ_{cl} in $C^0([0,T];H^1((0,1);\mathbb{R}^4))$.

Then, using the definition (6.8) of the operator $\mathcal{A}_0$, it is readily verified that equations (6.31) and (6.32) are equivalent to the simple equation

$$
(\mathbf{I_d} - \mathcal{A}_0)\left(\partial_t\begin{pmatrix} Z\\ \hat{Z}\end{pmatrix} - \begin{pmatrix} z\\ \hat{z}\end{pmatrix}\right) = 0. \tag{6.33}
$$

Since $(\mathbf{I_d} - \mathcal{A}_0)$ is invertible, this implies that

$$
\partial_t Z = z \quad \text{and} \quad \partial_t \hat{Z} = \hat{z}. \tag{6.34}
$$

Finally, using (6.6) together with (6.34), it follows that $(Z,\hat{Z})$ defined by equations (6.5) satisfies the equations

$$
\partial_t Z(t,x) + (\mathbf{\Lambda}+\mathbf{\mathcal{E}})\partial_x Z(t,x) = \mathbf{h}_{1\varepsilon}(Z(t,\cdot))(x), \tag{6.35a}
$$

$$
\partial_t \hat{Z}(t,x) + \mathbf{\Lambda}\partial_x \hat{Z}(t,x) = \mathbf{h}_{1\varepsilon}(Z(t,\cdot))(x) - \widetilde{\mathbf{h}}_{2\varepsilon}(Z(t,\cdot))(x) \tag{6.35b}
$$

and is therefore a solution of Σ_{cl}, since $d(t),\hat{d}(t)$ in (6.5) have been selected to achieve the boundary condition (6.1c).

The result we have just shown is summarized in the following lemma.

Lemma 2. Assume that the stability conditions (4.20) in the statement of Theorem 5 are satisfied and let $T>0$. For any $(z,\hat{z}) \in C^0([0,T];H^1((0,1);\mathbb{R}^4))$ solution of Σ_{cl}, if $\max(|\varepsilon_1|,|\varepsilon_2|) < \min(\varepsilon_1^*,\varepsilon_2^*)$, there is a unique $(Z,\hat{Z}) \in C^0([0,T];H^2((0,1);\mathbb{R}^4))$ also solution of Σ_{cl} which is defined by equations (6.5) with $(d,\hat{d})$ given by (6.25), (6.26) and $(w,\hat{w})$ given by (6.15).

From now on, we assume that $\max(|\varepsilon_1|,|\varepsilon_2|) < \varepsilon_0 = \min(\varepsilon_0^*,\varepsilon_1^*,\varepsilon_2^*)$ whenever we mention the existence of a solution to Σ_{cl}.

Step 2: Let us now introduce the mapping

$$
\mathcal{T} : L^2((0,1);\mathbb{R}^4) \to H^1((0,1);\mathbb{R}^4) : (z,\hat{z}) \mapsto (Z,\hat{Z}) \tag{6.36}
$$

defined by

$$\mathcal{T} = (\mathbf{I_d} - \mathcal{A}_0)^{-1}(\mathbf{I_d} + \mathcal{D}_{\boldsymbol{\mathcal{E}}}^{-1}\mathcal{J}(\mathbf{I_d} - \mathcal{A}_0)^{-1})\mathcal{B}_0 \tag{6.37}$$

or, equivalently, by

$$(Z, \hat{Z}) = \mathcal{T}(z, \hat{z}) = (\mathbf{I_d} - \mathcal{A}_0)^{-1}(\mathbf{I_d} + \mathcal{D}_{\boldsymbol{\mathcal{E}}}^{-1}\mathcal{J}(\mathbf{I_d} - \mathcal{A}_0)^{-1})\mathcal{B}_0(z, \hat{z}) \tag{6.38}$$

where the operator $\mathcal{A}_0$ is defined by (6.8), the operator $\mathcal{B}_0$ is defined by (6.9), the operator $\mathcal{D}_{\boldsymbol{\mathcal{E}}}$ is defined by (6.20) and the operator $\mathcal{J}$ is defined by (6.26) (see (6.15), (6.16), (6.25)).

This bounded operator $\mathcal{T}$ is chosen because, according to Lemma 2, it is actually the injective mapping that we have built above (in Step 1) from the space of $C^0([0,T]; H^1((0,1);\mathbb{R}^4))$ solutions of Σ_{cl} to the space of $C^0([0,T]; H^2((0,1);\mathbb{R}^4))$ solutions of Σ_{cl}.

In this section, the domain of this operator $\mathcal{T}$ is extended to the space $L^2((0,1);\mathbb{R}^4)$ (that we abbreviate as L^2 hereafter) without imposing, for a while, that $(z, \hat{z})$ be a solution to Σ_{cl}.

Using (6.6), we introduce the operator $\mathcal{T}^{-1}$ defined by

$$\begin{pmatrix} z \\ \hat{z} \end{pmatrix} = \mathcal{T}^{-1}(Z, \hat{Z}) = \begin{pmatrix} \mathbf{h}_{1\varepsilon}(Z) - (\boldsymbol{\Lambda} + \boldsymbol{\mathcal{E}})\partial_x Z \\ \mathbf{h}_{1\varepsilon}(Z) - \widetilde{\mathbf{h}}_{2\varepsilon}(Z) - \boldsymbol{\Lambda}\partial_x \hat{Z} \end{pmatrix}. \tag{6.39}$$

Note that with this definition, $\mathcal{T}^{-1}$ is the inverse of $\mathcal{T}$ as, for any $(z, \hat{z}) \in L^2$, $\mathcal{T}^{-1} \circ \mathcal{T}(z, \hat{z}) = (z, \hat{z})$ and as a consequence $\mathcal{T}$ is bijective from L^2 to $\mathcal{T}(L^2)$ (i.e. with a co-domain restricted to the range of $\mathcal{T}$).

We are now going to show that $\mathcal{T}$ is an isomorphism between L^2 and $\mathcal{T}(L^2)$ or, in other words, that there exists $C > 0$ such that, for any $(Z, \hat{Z}) \in \mathcal{T}(L^2)$

$$\|\mathcal{T}^{-1}(Z, \hat{Z})\|_{L^2} \leqslant C\|(Z, \hat{Z})\|_{H^1}, \tag{6.40}$$

and for any $(z, \hat{z}) \in L^2$,

$$\|\mathcal{T}(z, \hat{z})\|_{H^1} \leqslant C\|(z, \hat{z})\|_{L^2}. \tag{6.41}$$

Estimate (6.40) follows directly from (6.10) and (6.39). On the other hand, from (6.38), together with (6.11), (6.20) and (6.26) we have

$$\begin{aligned}
\|\mathcal{T}\|_{H^1} &= \left\|(\mathbf{I_d} - \mathcal{A}_0)^{-1}(\mathbf{I_d} + \mathcal{D}_{\boldsymbol{\mathcal{E}}}^{-1}\mathcal{J}(\mathbf{I_d} - \mathcal{A}_0)^{-1})\mathcal{B}_0\right\|_{H^1} \\
&\leqslant C\left\|(\mathbf{I_d} + \mathcal{D}_{\boldsymbol{\mathcal{E}}}^{-1}\mathcal{J}(\mathbf{I_d} - \mathcal{A}_0)^{-1})\mathcal{B}_0\right\|_{H^1} \\
&\leqslant C\left\|\mathcal{B}_0\right\|_{H^1} + C\left|\mathcal{D}_{\boldsymbol{\mathcal{E}}}^{-1}\mathcal{J}(\mathbf{I_d} - \mathcal{A}_0)^{-1})\mathcal{B}_0\right| \\
&\leqslant C\left\|\mathcal{B}_0\right\|_{H^1} + C\left\|(\mathbf{I_d} - \mathcal{A}_0)^{-1})\mathcal{B}_0\right\|_{H^1} \\
&\leqslant C\left\|\mathcal{B}_0\right\|_{H^1}
\end{aligned} \tag{6.42}$$

where we used the fact that $\mathcal{D}_{\boldsymbol{\mathcal{E}}}^{-1}\mathcal{J}(\mathbf{I_d} - \mathcal{A}_0)^{-1})\mathcal{B}_0$ is a constant and that $\mathcal{J}$ is a continuous linear operator from H^1 into $\mathbb{R}^4$ (see (6.26)). In (6.42), C denotes constants that can change from one line to another and may depend on an upper bound of $\max(|\varepsilon_1|, |\varepsilon_2|)$ (but not on $\max(|\varepsilon_1|, |\varepsilon_2|)$ itself). Then estimate (6.41) follows from (6.42) together with the definition of $\mathcal{B}_0$ given by (6.9).

Step 3: In this step we establish the following lemma.

Lemma 3. If $(z, \hat{z}) \in C^0([0,T]; L^2((0,1);\mathbb{R}^4))$ is a solution of Σ_{cl}, then $\mathcal{T}(z, \hat{z}) \in C^0([0,T]; H^1((0,1);\mathbb{R}^4))$ is a solution of Σ_{cl}.

Proof. Let $(z, \hat{z}) \in C^0([0,T]; L^2((0,1);\mathbb{R}^4))$ be a solution of Σ_{cl}. By the definition of L^2-solutions, there exists a sequence $(z_n, \hat{z}_n)_{n\in\mathbb{N}}$ such that for any $n \in \mathbb{N}$, $(z_n, \hat{z}_n) \in C^0([0,T]; H^1((0,1);\mathbb{R}^4))$ is a solution of Σ_{cl} and

$$(z_n, \hat{z}_n) \xrightarrow[n\to+\infty]{} (z, \hat{z}) \text{ in } C^0([0,T]; L^2((0,1);\mathbb{R}^4)), \tag{6.43}$$

as it is shown for instance in [42]. From Lemma 2, $(Z_n, \hat{Z}_n) = \mathcal{T}(z_n, \hat{z}_n) \in C^0([0,T]; H^2((0,1);\mathbb{R}^4))$ is a solution of Σ_{cl}. Moreover, since $\mathcal{T}$ is continuous from $L^2(0,1)$ to $H^1(0,1)$, one has

$$(Z_n, \hat{Z}_n) \xrightarrow[n\to+\infty]{} \mathcal{T}(z, \hat{z}) =: (Z, \hat{Z}) \text{ in } C^0([0,T]; H^1((0,1);\mathbb{R}^4)). \tag{6.44}$$

It remains to show that $(Z, \hat{Z})$ satisfies Σ_{cl}. For any $n \in \mathbb{N}$, since $(Z_n, \hat{Z}_n)$ is a solution of Σ_{cl} we have

$$(Z_n, \hat{Z}_n) \in C^0([0,T]; H^2((0,1);\mathbb{R}^4)) \cap C^1([0,T]; H^1((0,1);\mathbb{R}^4)) \tag{6.45}$$

and therefore, for any $t_1, t_2 \in [0,T]$,

$$Z_n(t_2,\cdot) - Z_n(t_1,\cdot) = -\int_{t_1}^{t_2} \Big((\mathbf{\Lambda} + \boldsymbol{\mathcal{E}})\partial_x Z_n(s,\cdot) - \mathbf{h}_{1\varepsilon}(Z_n(s,\cdot))\Big)\, ds \quad \text{in } H^1. \tag{6.46}$$

Taking the limit as $n \to +\infty$, and using the continuity of ∂_x from H^1 to L^2, we obtain

$$Z(t_2,\cdot) - Z(t_1,\cdot) = -\int_{t_1}^{t_2} (\mathbf{\Lambda} + \boldsymbol{\mathcal{E}})\partial_x Z(s,\cdot) - \mathbf{h}_{1\varepsilon}(Z(s,\cdot))ds \quad \text{in } L^2. \tag{6.47}$$

In particular, $Z \in C^1([0,T]; L^2((0,1);\mathbb{R}^2))$ and

$$\partial_t Z + (\mathbf{\Lambda} + \boldsymbol{\mathcal{E}})\partial_x Z = \mathbf{h}_{1\varepsilon}(Z). \tag{6.48}$$

The same reasoning applies to $\hat{Z}$. Finally, by passing to the limit in the boundary conditions (6.1c) satisfied by $(Z_n, \hat{Z}_n)$, we deduce that $(Z, \hat{Z})$ also satisfies (6.1c). Hence $(Z, \hat{Z})$ is indeed a solution of Σ_{cl}. $\square$

Step 4: We can now give the proof of Theorem 5 on the basis of the two previous steps and Theorem 6.

Proof of Theorem 5. For any $T > 0$ and any initial condition $(z_0, \hat{z}_0) \in L^2((0,1);\mathbb{R}^4)$, we know that there exists a unique solution $(z, \hat{z}) \in C^0([0,T]; L^2((0,1);\mathbb{R}^4))$ to the system Σ_{cl} (see Appendix C). Then from Lemma 3, $(Z, \hat{Z}) = \mathcal{T}(z, \hat{z})$ is a solution of Σ_{cl} that belongs to $C^0([0,T]; H^1((0,1);\mathbb{R}^4))$. Consequently, by assumption (4.20) in the statement of Theorem 5, the stability conditions (5.1) hold and, from Theorem 6, there exists $C_0 > 0$, $\gamma > 0$ such that

$$\|(Z(t,\cdot), \hat{Z}(t,\cdot))\|_{H^1} \leqslant C_0 e^{-\gamma t} \|(Z(0,\cdot), \hat{Z}(0,\cdot))\|_{H^1}, \text{ for any } t \in [0,T]. \tag{6.49}$$

Using (6.3) (or (6.40) and (6.41)) on both sides of this estimate, this implies that

$$\|(z(t,\cdot), \hat{z}(t,\cdot))\|_{L^2} \leqslant C^2 C_0 e^{-\gamma t} \|(z_0, \hat{z}_0)\|_{L^2}, \text{ for any } t \in [0,T]. \tag{6.50}$$

Thus, it is clear that inequality (4.22) is satisfied with $C_1 = C^2 C_0$ and, therefore, that the system Σ_{cl} is exponentially stable for the L^2-norm. This completes the proof of Theorem 5. $\square$

Remark 3 (Lyapunov stability in L^2)**.** In Section 5, for the stability of the solution $(Z, \hat{Z})$ in H^1, we have defined the following quadratic Lyapunov function which is equivalent to the H^1-norm:

$$\mathbf{V}(Z, \hat{Z}) = \mathbf{W}_1(Z) + M\mathbf{W}_2(Z - \hat{Z}) + \mathbf{W}_3(\partial_t Z) + \mathbf{W}_4(\partial_t \hat{Z}). \tag{6.51}$$

Remarkably, using the operator $\mathcal{T}$, this function can be translated into a Lyapunov function of $(z, \hat{z})$ which is equivalent to the L^2-norm. For this we introduce the following notation to represent the components of $\mathcal{T}$:

$$\begin{pmatrix} Z \\ \hat{Z} \end{pmatrix} = \mathcal{T}(z, \hat{z}) = \begin{pmatrix} \mathcal{T}_{(1)}(z, \hat{z}) \\ \mathcal{T}_{(2)}(z, \hat{z}) \end{pmatrix}. \tag{6.52}$$

Then, using (6.34) and (6.52), we get

$$\mathbf{V}(z, \hat{z}) = \mathbf{W}_1(\mathcal{T}_{(1)}(z, \hat{z})) + M\mathbf{W}_2(\mathcal{T}_{(1)}(z, \hat{z}) - \mathcal{T}_{(2)}(z, \hat{z})) + \mathbf{W}_3(z) + \mathbf{W}_4(\hat{z}). \tag{6.53}$$

Hence, with our proof of Theorem 5 above, we have shown that the exponential stability is guaranteed in L^2 by this Lyapunov function (6.53) which is not the integral of a simple weighted sum of squares of state components and, thus, is not a so-called basic quadratic Lyapunov function as defined e.g. in [39] and [40].

Remark 4 (A Lyapunov function independent of the perturbations)**.** The Lyapunov function (6.53) depends on the perturbations $(\varepsilon_1, \varepsilon_2)$ which are unknown. Although the control does not depend on these quantities, it can be interesting and useful to have a Lyapunov function independent of $(\varepsilon_1, \varepsilon_2)$. In fact, by defining the mapping

$$\mathcal{T}^0(z, \hat{z}) = \begin{pmatrix} \mathcal{T}^0_{(1)}(z, \hat{z}) \\ \mathcal{T}^0_{(2)}(z, \hat{z}) \end{pmatrix} = \mathcal{T}|_{(\varepsilon_1=0, \varepsilon_2=0)}, \tag{6.54}$$

it turns out that

$$\mathcal{V}(z, \hat{z}) = \mathbf{W}_1(\mathcal{T}^0_{(1)}(z, \hat{z})) + M\mathbf{W}_2(\mathcal{T}^0_{(1)}(z, \hat{z}) - \mathcal{T}^0_{(2)}(z, \hat{z})) + \mathbf{W}_3(z) + \mathbf{W}_4(\hat{z}) \tag{6.55}$$

is also a valid Lyapunov function with respect to $(z, \hat{z})$ for the L^2 norm which is independent of $(\varepsilon_1, \varepsilon_2)$. This is shown in Proposition 6 in Appendix D. Introducing the notations

$$\mathcal{T}^0_{(1)}(z, \hat{z}) = \begin{pmatrix} \mathcal{T}^0_{(11)}(z, \hat{z}) \\ \mathcal{T}^0_{(12)}(z, \hat{z}) \end{pmatrix}, \quad \mathcal{T}^0_{(2)}(z, \hat{z}) = \begin{pmatrix} \mathcal{T}^0_{(21)}(z, \hat{z}) \\ \mathcal{T}^0_{(22)}(z, \hat{z}) \end{pmatrix} \tag{6.56}$$

we can also note that this Lyapunov function can be written in the following more explicit equivalent form:

$$\begin{aligned} \mathcal{V}(z, \hat{z}) &= \int_0^1 a_1(x)(\mathcal{T}^0_{(11)}(z, \hat{z}))^2 + b_1(x)(\mathcal{T}^0_{(12)}(z, \hat{z}))^2 dx \\ &+ M \int_0^1 a_2(x)((\mathcal{T}^0_{(11)}(z, \hat{z})) - (\mathcal{T}^0_{(21)}(z, \hat{z})))^2 + b_2(x)((\mathcal{T}^0_{(12)}(z, \hat{z})) - (\mathcal{T}^0_{(22)}(z, \hat{z})))^2 dx \\ &+ \int_0^1 a_3(x) z_1^2(x) + a_4(x) z_2^2(x) dx + \int_0^1 a_3(x) \hat{z}_1^2(x) + a_4(x) \hat{z}_2^2(x) dx, \end{aligned} \tag{6.57}$$

where $a_i(x)$ are the functions depending only on the system parameters which have been defined in Section 5 (see (5.10)).

7 The necessary condition

In the previous sections (Theorems 5 and 6), we have shown that the closed-loop system Σ_{cl} given by Equation (6.1) in $z, \hat{z}$ coordinates is exponentially stable under the sufficient condition

$$\rho_2(\widehat{\mathbf{K}}) < 1. \tag{7.1}$$

In this section, we shall now prove that this condition is also **necessary**. For that purpose, we consider the associated eigenvalue problem with the objective of proving that, if $\rho_2(\widehat{\mathbf{K}}) \geqslant 1$, there exist necessarily arbitrarily small non-zero perturbations $(\varepsilon_1, \varepsilon_2)$ for which there are eigenvalues with strictly positive real parts or with negative real parts that are not bounded away from zero.

The eigenvalue problem associated to Σ_{cl} is written

$$\mu z(x) + (\boldsymbol{\Lambda} + \boldsymbol{\mathcal{E}})\partial_x z(x) = \mathbf{h}_{3\varepsilon}(z)(x) + P(x,1)\boldsymbol{\mathcal{E}} z(1), \tag{7.2a}$$

$$\mu \hat{z}(x) + \boldsymbol{\Lambda}\partial_x \hat{z}(x) = \widehat{\mathbf{h}}_{3\varepsilon}(z)(x) + (P(x,1) - \widetilde{P}(x,1))\boldsymbol{\mathcal{E}} z(1), \tag{7.2b}$$

$$\begin{pmatrix} z_1(0) \\ z_2(1) \\ \hat{z}_1(0) \\ \hat{z}_2(1) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & k_1 & 0 & -k_c \\ k_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_1 - k_c \\ k_0 & 0 & k_2 - k_0 & 0 \end{pmatrix}}_{\widehat{\mathbf{K}}} \begin{pmatrix} z_1(1) \\ z_2(0) \\ \hat{z}_1(1) \\ \hat{z}_2(0) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \int_0^1 f(\zeta)\Big(z(\zeta) - \hat{z}(\zeta)\Big)d\zeta \tag{7.2c}$$

where, for any function $z \in L^2((0,1);\mathbb{R}^2)$, the linear integral operators $\mathbf{h}_{3\varepsilon}$ and $\widehat{\mathbf{h}}_{3\varepsilon}$ (from $L^2((0,1);\mathbb{R}^2)$ to $L^2((0,1);\mathbb{R}^2)$) are defined by:

$$\begin{aligned} \mathbf{h}_{3\varepsilon}(z)(x) =&\ \mathbf{h}_{1\varepsilon}(z)(x) - P(x,1)\mathbf{E}z(1) \\ =& - \int_x^1 \Big[P_\xi(x,\xi)\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}} P_x(x,\xi)\Big] z(\xi)d\xi \\ &+ \int_x^1 \Big[P_\xi(x,\xi)\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}} P_x(x,\xi)\Big] \left(\int_\xi^1 Q(\xi,\zeta)z(\zeta)d\zeta\right) d\xi \\ &+ \Big[P(x,x)\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{E}} P(x,x)\Big] \int_x^1 Q(x,\xi)z(\xi)d\xi \\ &- \Big[P(x,x)\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{E}} P(x,x)\Big] z(x) \end{aligned} \tag{7.3}$$

and

$$\begin{aligned} \widehat{\mathbf{h}}_{3\varepsilon}(z)(x) &= \mathbf{h}_{1\varepsilon}(z)(x) - \widetilde{\mathbf{h}}_{2\varepsilon}(z)(x) - \widetilde{P}(x,1)\mathbf{E}z(1) \\ &= \mathbf{h}_{3\varepsilon}(z)(x) + \int_x^1 \widetilde{P}_\xi(x,\xi)\boldsymbol{\mathcal{E}}\left(z(\xi) - \int_\xi^1 Q(\xi,\zeta)z(\zeta)d\zeta\right)d\xi \\ &\quad - \widetilde{P}(x,x)\boldsymbol{\mathcal{E}} \int_x^1 Q(x,\xi)z(\xi)d\xi - \int_x^1 \boldsymbol{\mathcal{E}} Q_x(x,\xi)z(\xi)d\xi \\ &\quad + \widetilde{P}(x,x)\boldsymbol{\mathcal{E}} z(x) + \boldsymbol{\mathcal{E}} Q(x,x)z(x). \end{aligned} \tag{7.4}$$

Note that $\mathbf{h}_{3\varepsilon}(z)(x)$ and $\widehat{\mathbf{h}}_{3\varepsilon}(z)(x)$ correspond to the regular part of the source terms appearing in equations (6.1), in the sense that there exists a positive constant $C > 0$ such that

$$\|\mathbf{h}_{3\varepsilon}\|_{\mathcal{L}(L^\infty)} \leqslant C\max(|\varepsilon_1|,|\varepsilon_2|), \quad \|\widehat{\mathbf{h}}_{3\varepsilon}\|_{\mathcal{L}(L^\infty)} \leqslant C\max(|\varepsilon_1|,|\varepsilon_2|), \tag{7.5}$$

and

$$\|\mathbf{h}_{3\varepsilon}\|_{\mathcal{L}(L^2)} \leqslant C\max(|\varepsilon_1|,|\varepsilon_2|), \quad \|\widehat{\mathbf{h}}_{3\varepsilon}\|_{\mathcal{L}(L^2)} \leqslant C\max(|\varepsilon_1|,|\varepsilon_2|). \tag{7.6}$$

The aim of this section is to prove the following theorem.

Theorem 7. Assume that $\rho_2(\widehat{\mathbf{K}}) \geqslant 1$. Then, for any $\varepsilon_0 \in (0, \frac{1}{2}\min(\lambda_1, \lambda_2))$ and any $\boldsymbol{\epsilon} > 0$, there exist non-zero perturbations $(\varepsilon_1, \varepsilon_2)$ with $\max(|\varepsilon_1|, |\varepsilon_2|) \leqslant \varepsilon_0$ for which there are eigenvalues $\mu \in \mathbb{C}$, solutions of the eigenvalue problem (7.2) which satisfy the inequality

$$\Re(\mu) > \gamma^* - \boldsymbol{\epsilon} \quad \text{with} \quad \gamma^* = 2\frac{\lambda_1\lambda_2}{\lambda_1 + \lambda_2}\ln(\rho_2(\widehat{\mathbf{K}})). \tag{7.7}$$

□

The proof of the theorem is organized in four successive steps.

7.1. **Step 1:** In this subsection, we derive the characteristic equation associated to the eigenvalue problem (7.2) which, on the domain $\{\mu \in \mathbb{C} \mid -\kappa < \Re(\mu) < \kappa\}$, has the global form

$$\det\Big(M_0(\mu, \boldsymbol{\mathcal{E}}) + M_I(\mu, \boldsymbol{\mathcal{E}}) + M_1(\mu, \boldsymbol{\mathcal{E}})\Big) = 0 \tag{7.8}$$

provided $(\max(|\varepsilon_1|, |\varepsilon_2|)$ is small enough. The subsection 7.1 will focus primarily on the explicitation of the three matrices M_0, M_I and M_1 as functions of μ and $\boldsymbol{\mathcal{E}}$ given by (7.25), (7.26) and (7.51) respectively.

7.2. **Step 2:** For any $\boldsymbol{\epsilon} > 0$, we build a sequence $\{\boldsymbol{\mathcal{E}}_n : n \geqslant n_0\}$ of perturbations and a sequence $\{\mu_n^0 : n \geqslant n_0\}$ of solutions to the auxiliary characteristic equations

$$\det\Big(M_0(\mu, \boldsymbol{\mathcal{E}}_n)\Big) = 0 \quad \forall n \geqslant n_0 \tag{7.9}$$

which have the following properties:

$$\Re(\mu_n^0) > \gamma^* - \boldsymbol{\epsilon} \ \ \forall n \geqslant n_0, \quad \lim_{n\to+\infty} |\mu_n^0| = +\infty, \quad \lim_{n\to+\infty} \boldsymbol{\mathcal{E}}_n = \mathbf{0}, \tag{7.10}$$

and where $n_0 \in \mathbb{N}_+$ is a positive constant which depends on $\boldsymbol{\epsilon}$.

7.3. **Step 3:** Defining a well chosen rectangular neighbourhood $\Omega_n \subset \mathbb{C}$ of μ_n^0, we show that

$$\lim_{n\to+\infty} \sup_{\mu\in\Omega_n} |M_1(\mu, \boldsymbol{\mathcal{E}}_n)| = 0 \quad \text{and} \quad \lim_{n\to+\infty} \sup_{\mu\in\Omega_n} |M_I(\mu, \boldsymbol{\mathcal{E}}_n)| = 0. \tag{7.11}$$

7.4. **Step 4:** Using the degree theory (see [43, Appendix B]) or, equivalently, the Cauchy's argument principle, we show that the characteristic equation (7.8) has at least one solution located inside the domain Ω_n $\forall n \geqslant n_0$. This allows to complete the proof of the theorem.

7.1 Step 1: Derivation of the characteristic equation

The aim of this subsection is to derive the characteristic equation of the eigenvalue problem (7.2), according to the following proposition.

Proposition 2. For any $\kappa > 0$, there exists $\bar{\varepsilon}_0(\kappa)$ such that, if $\max(|\varepsilon_1|, |\varepsilon_2|) < \bar{\varepsilon}_0(\kappa)$, $\mu \in \mathbb{C}$ with $\Re(\mu) \in (-\kappa, \kappa)$ is a solution to the eigenvalue problem (7.2) if and only if

$$\det\Big(M_0(\mu, \boldsymbol{\mathcal{E}}) + M_I(\mu, \boldsymbol{\mathcal{E}}) + M_1(\mu, \boldsymbol{\mathcal{E}})\Big) = 0 \tag{7.12}$$

where the matrices $M_0(\mu, \boldsymbol{\mathcal{E}})$, $M_I(\mu, \boldsymbol{\mathcal{E}})$ and $M_1(\mu, \boldsymbol{\mathcal{E}})$ are given below by (7.25), (7.26) and (7.51) respectively.

Proof. We define the following diagonal functional matrices:

$$\widehat{\mathbf{E}}(x) := \begin{pmatrix} \exp\left(\frac{-\mu x}{\lambda_1}\right) & 0 \\ 0 & \exp\left(\frac{\mu x}{\lambda_2}\right) \end{pmatrix}, \quad \mathbf{E}(x) := \begin{pmatrix} \exp\left(\frac{-\mu x}{\lambda_1+\varepsilon_1}\right) & 0 \\ 0 & \exp\left(\frac{\mu x}{\lambda_2+\varepsilon_2}\right) \end{pmatrix} \tag{7.13}$$

and

$$\widehat{\mathbf{E}}_0(x) = \Lambda^{-1}\widehat{\mathbf{E}}(x) = \begin{pmatrix} \frac{1}{\lambda_1}\exp\left(\frac{-\mu x}{\lambda_1}\right) & 0 \\ 0 & -\frac{1}{\lambda_2}\exp\left(\frac{\mu x}{\lambda_2}\right) \end{pmatrix}, \tag{7.14}$$

$$\mathbf{E}_0(x) = (\Lambda+\boldsymbol{\mathcal{E}})^{-1}\mathbf{E}(x) = \begin{pmatrix} \frac{1}{\lambda_1+\varepsilon_1}\exp\left(\frac{-\mu x}{\lambda_1+\varepsilon_1}\right) & 0 \\ 0 & -\frac{1}{\lambda_2+\varepsilon_2}\exp\left(\frac{\mu x}{\lambda_2+\varepsilon_2}\right) \end{pmatrix}. \tag{7.15}$$

Then the differential equations (7.2a) and (7.2b) are equivalent to

$$z(x) = \mathbf{E}(x)z(0) + \int_0^x \mathbf{E}_0(x-\xi)\Big(\mathbf{h}_{3\varepsilon}(z)(\xi)\Big)d\xi + \left[\int_0^x \mathbf{E}_0(x-\xi)P(\xi,1)d\xi\right]\boldsymbol{\mathcal{E}}z(1), \tag{7.16a}$$

$$\hat{z}(x) = \widehat{\mathbf{E}}(x)\hat{z}(0) + \int_0^x \widehat{\mathbf{E}}_0(x-\xi)\Big(\widehat{\mathbf{h}}_{3\varepsilon}(z)(\xi)\Big)d\xi + \left[\int_0^x \widehat{\mathbf{E}}_0(x-\xi)\widehat{P}(\xi,1)d\xi\right]\boldsymbol{\mathcal{E}}z(1). \tag{7.16b}$$

We introduce the following linear integral operators $\boldsymbol{\psi}$ and $\widehat{\boldsymbol{\psi}}$ which, for any $z \in L^2((0,1);\mathbb{R}^2)$, are defined by

$$\boldsymbol{\psi}(z)(x) = \int_0^x \mathbf{E}_0(x-\xi)\Big(\mathbf{h}_{3\varepsilon}(z)(\xi)\Big)d\xi, \tag{7.17a}$$

$$\widehat{\boldsymbol{\psi}}(z)(x) = \int_0^x \widehat{\mathbf{E}}_0(x-\xi)\Big(\widehat{\mathbf{h}}_{3\varepsilon}(z)(\xi)\Big)d\xi. \tag{7.17b}$$

We define also the following functional matrices

$$\mathbf{D}(x) = \int_0^x \Big(\mathbf{E}_0(x-\xi)P(\xi,1)\boldsymbol{\mathcal{E}}\Big)d\xi, \tag{7.18a}$$

$$\widehat{\mathbf{D}}(x) = \int_0^x \Big(\widehat{\mathbf{E}}_0(x-\xi)\widehat{P}(\xi,1)\boldsymbol{\mathcal{E}}\Big)d\xi. \tag{7.18b}$$

With these notations, we can now rewrite (7.16) under the compact form:

$$z(x) = \mathbf{E}(x)z(0) + \boldsymbol{\psi}(z)(x) + \mathbf{D}(x)z(1), \tag{7.19a}$$

$$\hat{z}(x) = \widehat{\mathbf{E}}(x)\hat{z}(0) + \widehat{\boldsymbol{\psi}}(z)(x) + \widehat{\mathbf{D}}(x)z(1). \tag{7.19b}$$

<u>Taking account of the boundary conditions</u>

The boundary conditions (7.2c) are rewritten as follows:

$$\begin{pmatrix} \mathbf{K}_{01} & \widehat{\mathbf{K}}_{01} \\ \mathbf{0} & \widehat{\mathbf{K}}_{02} \end{pmatrix}\begin{pmatrix} z(0) \\ \hat{z}(0) \end{pmatrix} + \begin{pmatrix} \mathbf{K}_{11} & \mathbf{0} \\ \mathbf{K}_{12} & \widehat{\mathbf{K}}_{12} \end{pmatrix}\begin{pmatrix} z(1) \\ \hat{z}(1) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}\int_0^1 f(\zeta)\Big(z(\zeta)-\hat{z}(\zeta)\Big)d\zeta = \mathbf{0} \tag{7.20}$$

with

$$\mathbf{K}_{01} = \begin{pmatrix} -1 & k_1 \\ 0 & 0 \end{pmatrix}, \quad \widehat{\mathbf{K}}_{01} = \begin{pmatrix} 0 & -k_c \\ 0 & 0 \end{pmatrix}, \quad \widehat{\mathbf{K}}_{02} = \begin{pmatrix} -1 & k_1 - k_c \\ 0 & 0 \end{pmatrix},$$
$$\mathbf{K}_{11} = \begin{pmatrix} 0 & 0 \\ k_2 & -1 \end{pmatrix}, \quad \mathbf{K}_{12} = \begin{pmatrix} 0 & 0 \\ k_0 & 0 \end{pmatrix}, \quad \widehat{\mathbf{K}}_{12} = \begin{pmatrix} 0 & 0 \\ k_2 - k_0 & -1 \end{pmatrix}. \tag{7.21}$$

From (7.19), we have directly

$$\begin{pmatrix} z(1) \\ \hat{z}(1) \end{pmatrix} = \begin{pmatrix} \mathbf{E}(1) & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{E}}(1) \end{pmatrix} \begin{pmatrix} z(0) \\ \hat{z}(0) \end{pmatrix} + \begin{pmatrix} \boldsymbol{\psi}(z)(1) \\ \widehat{\boldsymbol{\psi}}(z)(1) \end{pmatrix} + \begin{pmatrix} \mathbf{D}(1) \\ \widehat{\mathbf{D}}(1) \end{pmatrix} z(1). \tag{7.22}$$

Moreover, using again (7.19), the integral term of (7.20) is explicited as

$$\int_0^1 f(\zeta)\Big(z(\zeta) - \hat{z}(\zeta)\Big)d\zeta = \int_0^1 f(\zeta)\Big(\mathbf{E}(\zeta) \mid -\hat{\mathbf{E}}(\zeta)\Big)d\zeta \begin{pmatrix} z(0) \\ \hat{z}(0) \end{pmatrix}$$
$$+ \int_0^1 f(\zeta)\Big(\boldsymbol{\psi}(z)(\zeta) - \widehat{\boldsymbol{\psi}}(z)(\zeta)\Big)d\zeta + \left(\int_0^1 f(\zeta)\Big(\mathbf{D}(\zeta) - \widehat{\mathbf{D}}(\zeta)\Big)d\zeta\right) z(1). \tag{7.23}$$

Then, by injecting (7.22) and (7.23) directly into the boundary condition (7.20), we get the simple global form

$$\Big(M_0(\mu, \boldsymbol{\mathcal{E}}) + M_I(\mu, \boldsymbol{\mathcal{E}})\Big) \begin{pmatrix} z(0) \\ \hat{z}(0) \end{pmatrix} + \mathcal{OT}(\mu, \boldsymbol{\mathcal{E}}) = \mathbf{0} \tag{7.24}$$

where the 4×4 matrices M_0 and M_I are

$$M_0(\mu, \boldsymbol{\mathcal{E}}) = \begin{pmatrix} \mathbf{K}_{01} + \mathbf{K}_{11}\mathbf{E}(1) & \widehat{\mathbf{K}}_{01} \\ \mathbf{K}_{12}\mathbf{E}(1) & \widehat{\mathbf{K}}_{02} + \widehat{\mathbf{K}}_{12}\hat{\mathbf{E}}(1) \end{pmatrix} = \begin{pmatrix} -1 & k_1 & 0 & -k_c \\ k_2 e^{-\frac{\mu}{\lambda_1+\varepsilon_1}} & -e^{\frac{\mu}{\lambda_2+\varepsilon_2}} & 0 & 0 \\ 0 & 0 & -1 & k_1 - k_c \\ k_0 e^{-\frac{\mu}{\lambda_1+\varepsilon_1}} & 0 & (k_2 - k_0)e^{-\frac{\mu}{\lambda_1}} & -e^{\frac{\mu}{\lambda_2}} \end{pmatrix} \tag{7.25}$$

and

$$M_I(\mu, \boldsymbol{\mathcal{E}}) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \int_0^1 f(\zeta)\Big(\mathbf{E}(\zeta) \Big| -\hat{\mathbf{E}}(\zeta)\Big)d\zeta \tag{7.26}$$

while $\mathcal{OT}(\mu, \boldsymbol{\mathcal{E}})$ stands for all the "other terms" that are included in (7.24) and will be made explicit later on in Equation (7.47). Remark that we highlight the dependence of the various terms on the perturbation $\boldsymbol{\mathcal{E}}$. The important point here is that when $\boldsymbol{\mathcal{E}} = \mathbf{0}$, Equation (7.24) reduces to

$$\Big(M_0(\mu, \mathbf{0}) + M_I(\mu, \mathbf{0})\Big) \begin{pmatrix} z(0) \\ \hat{z}(0) \end{pmatrix} = 0. \tag{7.27}$$

In other words, $\mathcal{OT}(\mu, \boldsymbol{\mathcal{E}})$ in (7.24) is a disturbance term that cancels out in the absence of perturbation of the characteristic velocities. In order to further analyze the condition (7.24), we need to derive an alternative expression for $z(x)$ given in the next paragraph.

An alternative expression for $z(x)$.

For the norm of the operator $\boldsymbol{\psi}$, from the definition (7.17a), we have the following inequality:

$$\|\boldsymbol{\psi}\|_{\mathcal{L}(L^\infty)} \leqslant \left[\sup_{x\in[0,1]} \int_0^x \left|\mathbf{E}_0(x-\xi)\right| d\xi\right] \|\mathbf{h}_{3\varepsilon}\|_{\mathcal{L}(L^\infty)}. \tag{7.28}$$

Let us now assume that $-\kappa < \Re(\mu) \leqslant \kappa$ for some real $\kappa > 0$. Under this assumption, together with the theorem assumption that $|\varepsilon_i| < \lambda_i/2$ $(i = 1, 2)$, it can be shown that

$$\sup_{x\in[0,1]} \int_0^x \left|\mathbf{E}_0(x-\xi)\right| d\xi \ \leqslant 2\left[\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right] e^{2\kappa/\lambda_2}. \tag{7.29}$$

Moreover, from definition (7.3), the norm of the operator $\mathbf{h}_{3\varepsilon}$ is bounded (see (7.5)). Then, combining (7.5), (7.28) and (7.29), we obtain

$$\|\boldsymbol{\psi}\|_{\mathcal{L}(L^\infty)} \leqslant C \max(|\varepsilon_1|, |\varepsilon_2|) e^{2\kappa/\lambda_2} \tag{7.30}$$

where C is a real positive constant. Similarly, for the norm of the operator $\boldsymbol{\psi}$ in $L^2(0,1)$, we have the inequality:

$$\|\boldsymbol{\psi}\|_{\mathcal{L}(L^2)} \leq \left(\int_0^1 \int_0^x \left|\mathbf{E}_0(x-\xi)\right|^2 d\xi dx\right)^{1/2} \|\mathbf{h}_{3\varepsilon}\|_{\mathcal{L}(L^2)} \leq C \max(|\varepsilon_1|, |\varepsilon_2|) e^{2\kappa/\lambda_2} \tag{7.31}$$

where C is another real positive constant.

Note that, using similar arguments, we can also show the estimates

$$\|\widehat{\boldsymbol{\psi}}\|_{\mathcal{L}(L^\infty)} \ + \ \|\widehat{\boldsymbol{\psi}}\|_{\mathcal{L}(L^2)} \leqslant C \max(|\varepsilon_1|, |\varepsilon_2|) e^{2\kappa/\lambda_2} \tag{7.32}$$

and, from (7.18),

$$\|\mathbf{D}(\cdot)\|_{L^\infty} \ + \ \|\mathbf{D}(\cdot)\|_{L^2} \ + \ \|\widehat{\mathbf{D}}(\cdot)\|_{L^\infty} \ + \ \|\widehat{\mathbf{D}}(\cdot)\|_{L^2} \leqslant C \max(|\varepsilon_1|, |\varepsilon_2|) e^{2\kappa/\lambda_2} \tag{7.33}$$

where C denotes real positive constants.

Let us now rewrite (7.19a) under the form

$$(\mathbf{Id} - \boldsymbol{\psi})(z)(x) = \ \mathbf{E}(x)z(0) + \mathbf{D}(x)z(1) \tag{7.34}$$

where $\mathbf{Id}$ denotes the identity operator. Let us assume that

$$\max(|\varepsilon_1|, |\varepsilon_2|) < (2Ce^{2\kappa/\lambda_2})^{-1} := \bar{\varepsilon}_0(\kappa) \tag{7.35}$$

such that $\|\boldsymbol{\psi}\| < 1/2$ and therefore such that the operator $(\mathbf{Id} - \boldsymbol{\psi})$ is invertible from L^2 into itself or from L^∞ into itself with

$$\|(\mathbf{Id} - \boldsymbol{\psi})^{-1}\| < (1 - \|\boldsymbol{\psi}\|)^{-1} < 2 \tag{7.36}$$

(see e.g. [44, Theorem 8.25]).

Now, since $(\mathbf{Id} - \boldsymbol{\psi})$ is invertible, the solutions of Equation (7.34) have solutions of the general form

$$z(x) = \mathbf{A}(x)z(0) + \mathbf{B}(x)z(1) \tag{7.37}$$

where the 2×2 functional matrices $\mathbf{A}$ and $\mathbf{B}$ have to be determined. By injecting this expression into (7.34), we have

$$z(x) = \psi\Big(\mathbf{A}(\cdot)z(0) + \mathbf{B}(\cdot)z(1)\Big)(x) + \mathbf{E}(x)z(0) + \mathbf{D}(x)z(1). \tag{7.38}$$

We introduce the operator $\Psi : L^2\Big((0,1);\mathbb{R}^{2\times 2}\Big) \to L^2\Big((0,1);\mathbb{R}^{2\times 2}\Big)$ which, for any functional matrix $\mathbf{A} \in L^2\Big((0,1);\mathbb{R}^{2\times 2}\Big)$ and any vector $z \in \mathbb{R}^2$, is defined by

$$\Psi(\mathbf{A})(x)z = \boldsymbol{\psi}\Big(\mathbf{A}(\cdot)z\Big)(x). \tag{7.39}$$

With this definition (7.38) is rewritten as

$$z(x) = \Big(\Psi(\mathbf{A})(x) + \mathbf{E}(x)\Big)z(0) + \Big(\Psi(\mathbf{B})(x) + \mathbf{D}(x)\Big)z(1). \tag{7.40}$$

Then, because the invertibility of $(\mathbf{Id} - \boldsymbol{\psi})$ implies that $(\mathbf{Id} - \Psi)$ is also invertible, by unicity of $\mathbf{A}$ and $\mathbf{B}$, it follows that

$$\mathbf{A}(x) = (\mathbf{Id} - \Psi)^{-1}(\mathbf{E})(x) \quad \text{and} \quad \mathbf{B}(x) = (\mathbf{Id} - \Psi)^{-1}(\mathbf{D})(x). \tag{7.41}$$

Moreover, from (7.33), (7.35) and (7.37), $(I - \mathbf{B}(1))$ is invertible provided $\bar{\varepsilon}_0(\kappa)$ is sufficiently small and we have

$$z(1) = (I - \mathbf{B}(1))^{-1}\mathbf{A}(1)z(0). \tag{7.42}$$

Then by substituting this expression in (7.40) and using (7.41) , we get

$$z(x) = \mathcal{G}(x)z(0) \tag{7.43}$$

where $\mathcal{G}$ is a 2×2 functional matrix defined by

$$\mathcal{G}(1) = (I - \mathbf{B}(1))^{-1}\mathbf{A}(1) \quad \text{and} \quad \mathcal{G}(x) = \mathbf{A}(x) + \mathbf{B}(x)\mathcal{G}(1). \tag{7.44}$$

Consequently, using (7.43) and the definition (7.39) of Ψ, we deduce

$$\boldsymbol{\psi}(z)(x) = \boldsymbol{\psi}(\mathcal{G}(.)z(0))(x) = \Psi(\mathcal{G})(x)z(0). \tag{7.45}$$

Similarly, we can write

$$\widehat{\boldsymbol{\psi}}(z)(x) = \widehat{\boldsymbol{\psi}}(\mathcal{G}(.)z(0))(x) = \widehat{\Psi}(\mathcal{G})(x)z(0) \tag{7.46}$$

with an obvious analog definition of the operator $\widehat{\Psi} : L^2\Big((0,1);\mathbb{R}^2 \times \mathbb{R}^2\Big) \to L^2\Big((0,1);\mathbb{R}^2 \times \mathbb{R}^2\Big)$.

Back to Equation (7.24)

Using (7.22) and (7.23), the third term of (7.24) is

$$\begin{aligned}
\mathcal{OT}(\mu, \boldsymbol{\mathcal{E}}) = &\begin{pmatrix} \mathbf{K}_{01} & \mathbf{0} \\ \mathbf{K}_{12} & \widehat{\mathbf{K}}_{12} \end{pmatrix} \left[\begin{pmatrix} \boldsymbol{\psi}(z)(1) \\ \widehat{\boldsymbol{\psi}}(z)(1) \end{pmatrix} + \begin{pmatrix} \mathbf{D}(1) \\ \widehat{\mathbf{D}}(1) \end{pmatrix} z(1) \right] \\
&+ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \left[\int_0^1 f(\zeta)\Big(\boldsymbol{\psi}(z)(\zeta) - \widehat{\boldsymbol{\psi}}(z)(\zeta)\Big)d\zeta + \left(\int_0^1 f(\zeta)\Big(\mathbf{D}(\zeta) - \widehat{\mathbf{D}}(\zeta)\Big)d\zeta\right) z(1) \right].
\end{aligned} \tag{7.47}$$

Then, using (7.43), (7.45) and (7.46), this quantity $\mathcal{OT}(\mu, \boldsymbol{\mathcal{E}})$ may be written in the following simple form:

$$\mathcal{OT}(\mu, \boldsymbol{\mathcal{E}}) = \Big(M_{11}(\mu, \boldsymbol{\mathcal{E}}) + M_{12}(\mu, \boldsymbol{\mathcal{E}})\Big)z(0) \tag{7.48}$$

where the 4×2 functional matrices M_{11} and M_{12} are

$$M_{11}(\mu, \boldsymbol{\mathcal{E}}) = \begin{pmatrix} \mathbf{K}_{01} & \mathbf{0} \\ \mathbf{K}_{12} & \widehat{\mathbf{K}}_{12} \end{pmatrix} \begin{pmatrix} \mathbf{D}(1) \\ \widehat{\mathbf{D}}(1) \end{pmatrix} \mathcal{G}(1) + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \left[\int_0^1 f(\zeta)\Big(\mathbf{D}(\zeta) - \widehat{\mathbf{D}}(\zeta)\Big)\mathcal{G}(1) d\zeta \right], \tag{7.49a}$$

$$M_{12}(\mu, \boldsymbol{\mathcal{E}}) = \begin{pmatrix} \mathbf{K}_{01} & \mathbf{0} \\ \mathbf{K}_{12} & \widehat{\mathbf{K}}_{12} \end{pmatrix} \begin{pmatrix} \Psi(\mathcal{G})(1) \\ \hat{\Psi}(\mathcal{G})(1) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \left[\int_0^1 f(\zeta)\Big(\Psi(\mathcal{G})(\zeta) - \hat{\Psi}(\mathcal{G})(\zeta)\Big) d\zeta \right]. \tag{7.49b}$$

It follows that (7.24) can now be written in the general form

$$\Big(M_0(\mu, \boldsymbol{\mathcal{E}}) + M_I(\mu, \boldsymbol{\mathcal{E}}) + M_1(\mu, \boldsymbol{\mathcal{E}})\Big) \begin{pmatrix} z(0) \\ \hat{z}(0) \end{pmatrix} = \mathbf{0} \tag{7.50}$$

with the 4×4 matrix $M_1(\mu, \boldsymbol{\mathcal{E}})$ defined as

$$M_1(\mu, \boldsymbol{\mathcal{E}}) = \Big(M_{11}(\mu, \boldsymbol{\mathcal{E}}) + M_{12}(\mu, \boldsymbol{\mathcal{E}}) \,\Big|\, \mathbf{0}\Big). \tag{7.51}$$

Then it follows straightforwardly from (7.50) that there exists a nontrivial solution to the eigenvalue problem (7.2) if and only if (7.12) holds. This concludes the proof of Proposition 2. $\square$

7.2 Step 2: A sequence of solutions to $\det(M_0(\mu, \mathcal{E}) = 0$

From the definition (7.25) of the matrix $M_0(\mu, \boldsymbol{\mathcal{E}})$, it can be checked that

$$\begin{aligned} \det(M_0(\mu, \boldsymbol{\mathcal{E}}))e^{-\mu\left(\frac{1}{\lambda_2} + \frac{1}{\lambda_2+\varepsilon_2}\right)} &= \det\left(\operatorname{diag}\Big(e^{-\frac{\mu}{\lambda_1+\varepsilon_1}}, e^{-\frac{\mu}{\lambda_2+\varepsilon_2}}, e^{-\frac{\mu}{\lambda_1}}, e^{-\frac{\mu}{\lambda_2}}\Big)\widehat{\mathbf{K}} - I\right) \\ &= \Big(k_1k_2e^{-\mu\left(\frac{1}{\lambda_1+\varepsilon_1} + \frac{1}{\lambda_2+\varepsilon_2}\right)} - 1\Big)\Big((k_1 - k_c)(k_2 - k_0)e^{-\mu\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)} - 1\Big) + k_0k_ce^{-\mu\left(\frac{1}{\lambda_1+\varepsilon_1} + \frac{1}{\lambda_2}\right)}. \end{aligned} \tag{7.52}$$

Our objective, in this subsection, is stated in the following proposition.

Proposition 3. For any $\boldsymbol{\epsilon} > 0$, we build a sequence $\{\boldsymbol{\mathcal{E}}_n : n \geqslant n_0\}$ of perturbations and a sequence $\{\mu_n^0 : n \geqslant n_0\}$ of solutions to the auxiliary characteristic equations

$$\det\Big(M_0(\mu, \boldsymbol{\mathcal{E}}_n)\Big) = 0 \quad \forall n \geqslant n_0 \tag{7.53}$$

which have the following properties:

$$\Re(\mu_n^0) > 2\frac{\lambda_1\lambda_2}{\lambda_1 + \lambda_2} \ln(\rho_2(\widehat{\mathbf{K}})) - \boldsymbol{\epsilon} \;\; \forall n \geqslant n_0, \quad \lim_{n\to+\infty} |\mu_n^0| = +\infty, \quad \lim_{n\to+\infty} \boldsymbol{\mathcal{E}}_n = \mathbf{0}, \tag{7.54}$$

and where $n_0 \in \mathbb{N}_+$ is a positive constant which depends on $\boldsymbol{\epsilon}$.

Proof. We introduce the following functions α_i (i=1,2,3,4):

$$\alpha_1(\mu,\varepsilon_1) = \exp\left[-\mu\Big(\frac{1}{2}\Big(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\Big)+\Big(\frac{1}{\lambda_1+\varepsilon_1}-\frac{1}{\lambda_1}\Big)\Big)\right], \tag{7.55a}$$

$$\alpha_2(\mu,\varepsilon_2) = \exp\left[-\mu\Big(\frac{1}{2}\Big(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\Big)+\Big(\frac{1}{\lambda_2+\varepsilon_2}-\frac{1}{\lambda_2}\Big)\Big)\right], \tag{7.55b}$$

$$\alpha_3(\mu) = \exp\left[-\mu\Big(\frac{1}{2}\Big(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\Big)\Big)+id\right], \tag{7.55c}$$

$$\alpha_4(\mu) = \exp\left[-\mu\Big(\frac{1}{2}\Big(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\Big)\Big)-id\right], \tag{7.55d}$$

where $i=\sqrt{-1}$ and $d\in\mathbb{R}$ is to be determined. With these functions, we observe that

$$\alpha_1(\mu,\varepsilon_1)\alpha_2(\mu,\varepsilon_2) = e^{-\mu\left(\frac{1}{\lambda_1+\varepsilon_1}+\frac{1}{\lambda_2+\varepsilon_2}\right)}, \tag{7.56a}$$

$$\alpha_3(\mu)\alpha_4(\mu) = e^{-\mu\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)}, \tag{7.56b}$$

$$\alpha_1(\mu,\varepsilon_1)\alpha_4(\mu) = e^{-\mu\left(\frac{1}{\lambda_1+\varepsilon_1}+\frac{1}{\lambda_2}\right)} \tag{7.56c}$$

which are exactly the exponential coefficients that appear in equation (7.52). It can therefore be shown that

$$\det\Big(\mathcal{A}(\mu,\varepsilon_1,\varepsilon_2)\widehat{\mathbf{K}}-I\Big) = \det\Big(\mathrm{diag}\Big(e^{-\frac{\mu}{\lambda_1+\varepsilon_1}},e^{-\frac{\mu}{\lambda_2+\varepsilon_2}},e^{-\frac{\mu}{\lambda_1}},e^{-\frac{\mu}{\lambda_2}}\Big)\widehat{\mathbf{K}}-I\Big). \tag{7.57}$$

with the matrix notation

$$\mathcal{A}(\mu,\varepsilon_1,\varepsilon_2) = \mathrm{diag}\Big(\alpha_1(\mu,\varepsilon_1),\alpha_2(\mu,\varepsilon_2),\alpha_3(\mu),\alpha_4(\mu)\Big). \tag{7.58}$$

We will now build the sequences $\{\mu_n^0 : n\geqslant n_0\}$ and $\{\boldsymbol{\mathcal{E}}_n : n\geqslant n_0\}$ announced in the statement of Proposition 3 by first imposing the following relationships:

$$\exp\left(-i\Im(\mu_n^0)\left[\frac{1}{2}\Big(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\Big)+\Big(\frac{1}{\lambda_1+\varepsilon_{1,n}}-\frac{1}{\lambda_1}\Big)\right]\right) = e^{-i\theta_1}, \tag{7.59a}$$

$$\exp\left(-i\Im(\mu_n^0)\left[\frac{1}{2}\Big(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\Big)+\Big(\frac{1}{\lambda_2+\varepsilon_{2,n}}-\frac{1}{\lambda_2}\Big)\right]\right) = e^{-i\theta_2}, \tag{7.59b}$$

$$\exp\left(-i\Im(\mu_n^0)\Big(\frac{1}{2}\Big(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\Big)\Big)+id\right) = e^{-i\theta_3}, \tag{7.59c}$$

$$\exp\left(-i\Im(\mu_n^0)\Big(\frac{1}{2}\Big(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\Big)\Big)-id\right) = e^{-i\theta_4} \tag{7.59d}$$

where (see (B.38) in Appendix B)

$$\theta_1 = -\frac{1}{\sigma_2}\frac{\pi}{2},\ \theta_2 = \frac{\sigma_2}{\sigma_1}\frac{\pi}{2},\ \theta_3 = \sigma_3\frac{\pi}{2},\ \theta_4 = -\frac{\pi}{2} \tag{7.60}$$

with

$$\sigma_1 = \text{sign}(k_1 k_2), \quad \sigma_2 = \text{sign}(k_0 k_c), \quad \sigma_3 = \text{sign}\Big((k_1 - k_c)(k_2 - k_0)\Big). \tag{7.61}$$

The system (7.59) is equivalent to the existence of $l_1,\ l_2, l_3, l_4 \in \mathbb{Z}$ such that

$$\frac{\Im(\mu_n^0)}{2}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) - \Im(\mu_n^0)\hat{\varepsilon}_{1,n} = \theta_1 + 2l_1\pi \tag{7.62a}$$

$$\frac{\Im(\mu_n^0)}{2}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) - \Im(\mu_n^0)\hat{\varepsilon}_{2,n} = \theta_2 + 2l_2\pi \tag{7.62b}$$

$$\frac{\Im(\mu_n^0)}{2}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) - d = \theta_3 + 2l_3\pi, \tag{7.62c}$$

$$\frac{\Im(\mu_n^0)}{2}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) + d = \theta_4 + 2l_4\pi, \tag{7.62d}$$

where

$$\hat{\varepsilon}_{i,n} = \frac{\varepsilon_{i,n}}{\lambda_i(\lambda_i + \varepsilon_{i,n})}, \quad i = 1, 2. \tag{7.63}$$

We want to construct a family of l_i and $\hat{\varepsilon}_{i,n}$ that is solution to (7.62). For such a solution we deduce from (7.62c) and (7.62d) that

$$\frac{\Im(\mu_n^0)}{2}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) = \theta_3 + 2l_3\pi + d = \theta_4 + 2l_4\pi - d. \tag{7.64}$$

In order to satisfy this relationship, the parameters l_3, l_4 and d are selected such that

$$l_3 = l_4 \quad \text{and} \quad \theta_3 + d = \theta_4 - d. \tag{7.65}$$

In the case where $\sigma_3 = 1$ (see (7.60)), we select $d = -\theta_3 = \theta_4 = -\pi/2$. In the case where $\sigma_3 = -1$, we know from (7.60) that $\theta_3 = \theta_4 = -\pi/2$ and we select $d = 0$ to satisfy (7.65). Moreover, in both cases we select

$$l_3 = l_4 = n \tag{7.66}$$

where $n \in \mathbb{N}$ ($n \geqslant n_0$) is the index of the sequence $\{\mu_n^0 : n \geqslant n_0\}$. With these choices, we have from (7.62c) and (7.62d) that

$$\Im(\mu_n^0) = 4n\pi\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)^{-1} \quad \text{if} \ \ \sigma_3 = 1, \tag{7.67}$$

and

$$\Im(\mu_n^0) = (4n+1)\pi\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)^{-1} \quad \text{if} \ \ \sigma_3 = -1. \tag{7.68}$$

We observe that in both cases

$$\lim_{n \to +\infty} |\Im(\mu_n^0)| = +\infty. \tag{7.69}$$

Using (7.67), we have from (7.62a) that

$$\hat{\varepsilon}_{1,n} = \frac{2(n - l_1)\pi - \theta_1}{4n\pi}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) \quad \text{if} \ \ \sigma_3 = 1, \tag{7.70}$$

and

$$\hat{\varepsilon}_{1,n} = \frac{2(n - l_1)\pi - \theta_1 + \pi/2}{(4n+1)\pi}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) \quad \text{if} \ \ \sigma_3 = -1. \tag{7.71}$$

We select $l_1 = n-1$ so that

$$\hat{\varepsilon}_{1,n} = \frac{2\pi - \theta_1}{4n\pi}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) \quad \text{if } \ \sigma_3 = 1, \tag{7.72}$$

and

$$\hat{\varepsilon}_{1,n} = \frac{5\pi/2 - \theta_1}{(4n+1)\pi}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) \quad \text{if } \ \sigma_3 = -1. \tag{7.73}$$

Similarly, by selecting $l_2 = n-1$, we have from (7.62b) that

$$\hat{\varepsilon}_{2,n} = \frac{2\pi - \theta_2}{4n\pi}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) \quad \text{if } \ \sigma_3 = 1, \tag{7.74}$$

and

$$\hat{\varepsilon}_{2,n} = \frac{5\pi/2 - \theta_2}{(4n+1)\pi}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) \quad \text{if } \ \sigma_3 = -1. \tag{7.75}$$

Considering the different values of the parameters θ_i (see (7.60)), we obtain the following explicit expressions:

$$\text{for } \ \theta_i = +\frac{\pi}{2} \ \ (i=1,2) \ \text{ and } \ \theta_3 = +\frac{\pi}{2}: \quad \hat{\varepsilon}_{i,n} = \frac{3}{8n}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right), \tag{7.76a}$$

$$\text{for } \ \theta_i = -\frac{\pi}{2} \ \ (i=1,2) \ \text{ and } \ \theta_3 = +\frac{\pi}{2}: \quad \hat{\varepsilon}_{i,n} = \frac{5}{8n}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right), \tag{7.76b}$$

$$\text{for } \ \theta_i = +\frac{\pi}{2} \ \ (i=1,2) \ \text{ and } \ \theta_3 = -\frac{\pi}{2}: \quad \hat{\varepsilon}_{i,n} = \frac{2}{4n+1}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right), \tag{7.76c}$$

$$\text{for } \ \theta_i = -\frac{\pi}{2} \ \ (i=1,2) \ \text{ and } \ \theta_3 = -\frac{\pi}{2}: \quad \hat{\varepsilon}_{i,n} = \frac{3}{4n+1}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right). \tag{7.76d}$$

By inverting the function (7.63), $\varepsilon_{i,n}$ is computed from $\hat{\varepsilon}_{i,n}$ as:

$$\varepsilon_{i,n} = \frac{\lambda_i^2 \hat{\varepsilon}_{i,n}}{1 - \lambda_i \hat{\varepsilon}_{i,n}} \qquad i = 1,2. \tag{7.77}$$

We assume that the constant n_0 is selected sufficiently large to ensure that

$$\hat{\varepsilon}_{i,n} < \lambda_i^{-1} \quad \text{and} \quad \varepsilon_{i,n} < \varepsilon_0 \quad \forall n \geqslant n_0, \quad i = 1,2. \tag{7.78}$$

In this case, we observe that both $\{\hat{\varepsilon}_{i,n} : n \geqslant n_0\}$ and $\{\varepsilon_{i,n} : n \geqslant n_0\}$ are strictly decreasing sequences such that

$$\lim_{n\to+\infty} \hat{\varepsilon}_{i,n} = 0 \quad \text{and} \quad \lim_{n\to+\infty} \varepsilon_{i,n} = 0, \quad i = 1,2. \tag{7.79}$$

We introduce the matrix notation

$$e^{i\Theta} = \operatorname{diag}\Big(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4}\Big). \tag{7.80}$$

With this notation, using (7.55), (7.58), (7.59) and (7.63), we can write

$$\mathcal{A}(\mu_n^0, \varepsilon_{1,n}, \varepsilon_{2,n}) = \exp\left(\frac{\Re(\mu_n^0)}{2}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)\right) \operatorname{diag}\Big(e^{\Re(\mu_n^0)\hat{\varepsilon}_{1,n}}, e^{\Re(\mu_n^0)\hat{\varepsilon}_{2,n}}, 1, 1\Big) e^{i\Theta}. \tag{7.81}$$

We define the matrix

$$\mathcal{M} = \Big(\mathcal{A}(\mu_n^0, \varepsilon_{1,n}, \varepsilon_{2,n}) - \frac{e^{i\Theta}}{\rho_2(\widehat{\mathbf{K}})}\Big)\widehat{\mathbf{K}} \tag{7.82}$$

which, from (7.81), can be considered as a function of $\Re(\mu_n^0)$ and $1/n$ that can be written as follows:

$$\mathcal{M}(\Re(\mu_n^0), 1/n) = \frac{e^{i\Theta}}{\rho_2(\widehat{\mathbf{K}})}\left[\frac{\rho_2(\widehat{\mathbf{K}})}{\exp\left(\frac{\Re(\mu_n^0)}{2}\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)\right)}\mathrm{diag}\Big(e^{\Re(\mu_n^0)\hat{\varepsilon}_{1,n}}, e^{\Re(\mu_n^0)\hat{\varepsilon}_{2,n}}, 1, 1\Big) - I\right]\widehat{\mathbf{K}}. \tag{7.83}$$

From the definition of γ^* given by (7.7), we observe that

$$\mathcal{M}(\gamma^*, 0) = 0. \tag{7.84}$$

We define the function

$$f_{\mathcal{M}} : (x, y) \mapsto \det\left(\frac{e^{i\Theta}}{\rho_2(\widehat{\mathbf{K}})}\widehat{\mathbf{K}} - I + \mathcal{M}(x, y)\right). \tag{7.85}$$

where the function $\mathcal{M}(x, y)$ is simply defined from (7.83) by replacing $\Re(\mu_n^0)$ with x in (7.83) and $(1/n)$ with y in (7.83) and in the expressions (7.76) of $\hat{\varepsilon}_{i,n}$.

Now, because it is assumed here that $\rho_2(\widehat{\mathbf{K}}) > 1$, from the computation of $\rho_2(\widehat{\mathbf{K}})$ in Appendix B, and especially from Equations (B.36) and (B.38), we know that

$$\det\Big(e^{i\Theta}D^*\widehat{\mathbf{K}}(D^*)^{-1} - \rho_2(\widehat{\mathbf{K}})I\Big) = 0 \tag{7.86}$$

where the matrix D^* is defined in (B.13). Since D^* is positive diagonal, (7.86) is equivalent to

$$\det\left(\frac{e^{i\Theta}}{\rho_2(\widehat{\mathbf{K}})}\widehat{\mathbf{K}} - I\right) = 0. \tag{7.87}$$

Therefore, using (7.84), we have

$$f_{\mathcal{M}}(\gamma^*, 0) = 0. \tag{7.88}$$

First let us consider the special case $k_0 = k_c = 0$. In this case, the expression of $\widehat{\mathbf{K}}$ simplifies such that from (7.52)

$$\det(M_0(\mu, \boldsymbol{\mathcal{E}}))e^{-\mu\left(\frac{1}{\lambda_2}+\frac{1}{\lambda_2+\varepsilon_2}\right)} = \Big(k_1k_2e^{-\mu\left(\frac{1}{\lambda_1+\varepsilon_1}+\frac{1}{\lambda_2+\varepsilon_2}\right)} - 1\Big)\Big((k_1k_2e^{-\mu\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)} - 1\Big) \tag{7.89}$$

and it is clear that there is a sequence of solution $\mu_n^0 = \left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)^{-1}(\ln(|k_1k_2|) + i(2n+m)\pi)$, independent of $\boldsymbol{\mathcal{E}}$, where $m = 0$ or π depending on the sign of k_1k_2. Since, in this case, $|k_1k_2| = \rho_2(\widehat{\mathbf{K}})$, Proposition 3 holds with any sequence $\{\boldsymbol{\mathcal{E}}_n \;:\; n \geq n_0\}$ converging to 0.

Let us now assume that $k_0 \neq 0$ or $k_c \neq 0$. Using the implicit function theorem, we are going to prove that there exist neighborhoods $[0, \eta^*)$ and $\Gamma_\delta := (\gamma^* - \delta, \gamma^* + \delta)$, and a map $\gamma_{\mathcal{M}} \in C^1([0, \eta^*), \Gamma_\delta)$ such that $\gamma_{\mathcal{M}}(0) = \gamma^*$ and $f_{\mathcal{M}}(\gamma_{\mathcal{M}}(\eta), \eta) = 0$ for all $\eta \in [0, \eta^*)$.

It is actually sufficient to show the following Lemma.

Lemma 4. $\partial_x f_{\mathcal{M}}(\gamma^*, 0) \neq 0$.

Proof. From the classical formula of the differential of the determinant, and setting

$$\mathbf{G} = e^{i\Theta}\widehat{\mathbf{K}}/\rho_2(\widehat{\mathbf{K}}), \tag{7.90}$$

we have (with "com" being the abbreviation of "comatrix")

$$\partial_x f_{\mathcal{M}}(\gamma^*, 0) = \mathrm{Tr}(\mathrm{com}(\mathbf{G} - I)^{\mathsf{T}}\partial_x\mathcal{M}(\gamma^*, 0)) = -\frac{1}{2}\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)\mathrm{Tr}(\mathrm{com}(\mathbf{G} - I)^{\mathsf{T}}\mathbf{G}). \tag{7.91}$$

Since

$$\begin{aligned}\operatorname{Tr}(\operatorname{com}(\mathbf{G}-I)^{\mathsf{T}}\mathbf{G}) &= \operatorname{Tr}(\operatorname{com}(\mathbf{G}-I)^{\mathsf{T}}(\mathbf{G}-I)) + \operatorname{Tr}(\operatorname{com}(\mathbf{G}-I)^{\mathsf{T}}) \\ &= \operatorname{Tr}((\mathbf{G}-I)\operatorname{com}(\mathbf{G}-I)^{\mathsf{T}}) + \operatorname{Tr}(\operatorname{com}(\mathbf{G}-I)^{\mathsf{T}}) \\ &= 4\det(\mathbf{G}-I) + \operatorname{Tr}(\operatorname{com}(\mathbf{G}-I)^{\mathsf{T}}),\end{aligned} \tag{7.92}$$

and since $\det(\mathbf{G}-I)=0$, we have

$$\partial_x f_{\mathcal{M}}(\gamma^*,0) = -\frac{1}{2}\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)\operatorname{Tr}(\operatorname{com}(\mathbf{G}-I)^{\mathsf{T}}). \tag{7.93}$$

We can now check that the matrix $(\mathbf{G}-I)$ is of rank 3. Indeed

a) if $k_c \neq 0$ the following 3×3 submatrix of $(\mathbf{G}-I)$ has rank 3:

$$\begin{pmatrix} k_1 e^{i\theta_1}\rho_2^{-1}(\widehat{\mathbf{K}}) & 0 & -k_c e^{i\theta_1}\rho_2^{-1}(\widehat{\mathbf{K}}) \\ -1 & 0 & 0 \\ 0 & -1 & (k_1-k_c)e^{i\theta_3}\rho_2^{-1}(\widehat{\mathbf{K}}) \end{pmatrix}; \tag{7.94}$$

b) if $k_0 \neq 0$ the following 3×3 submatrix of $(\mathbf{G}-I)$ has rank 3:

$$\begin{pmatrix} k_2 e^{i\theta_2}\rho_2^{-1}(\widehat{\mathbf{K}}) & -1 & 0 \\ 0 & 0 & -1 \\ k_0 e^{i\theta_4}\rho_2^{-1}(\widehat{\mathbf{K}}) & 0 & (k_2-k_0)e^{i\theta_4}\rho_2^{-1}(\widehat{\mathbf{K}}) \end{pmatrix}. \tag{7.95}$$

With these two possibilities, we have addressed all possible cases since we assumed that $k_0 \neq 0$ or $k_c \neq 0$.

Thus $\operatorname{rank}(\mathbf{G}-I)=3$ in which case, because

$$(\mathbf{G}-I)\operatorname{com}(\mathbf{G}-I)^{\mathsf{T}} = \det(\mathbf{G}-I)I = 0, \tag{7.96}$$

it follows that each column of $\operatorname{com}(\mathbf{G}-I)^{\mathsf{T}}$ belongs to $\operatorname{Ker}(\mathbf{G}-I)$ which has dimension 1 so that $\operatorname{rank}(\operatorname{com}(\mathbf{G}-I)^{\mathsf{T}})$ is at most equal to 1. And, since $\operatorname{rank}(\mathbf{G}-I)=3$, there is at least one principal minor that is non-zero and the coefficients of $\operatorname{com}(\mathbf{G}-I)^{\mathsf{T}}$ are the determinant of the principal minors of $(\mathbf{G}-I)$ by definition, then $\operatorname{rank}(\operatorname{com}(\mathbf{G}-I)^{\mathsf{T}})=1$. As a consequence the matrix $\operatorname{com}(\mathbf{G}-I)^{\mathsf{T}}$ has only one non-zero real eigenvalue and from (7.93) $\partial_x f_{\mathcal{M}}(\gamma^*,0)\neq 0$. □

Using this lemma, from the implicit function theorem, we conclude that there exists $\eta^*>0$, $\delta>0$ and a map $\gamma_{\mathcal{M}} \in C^1([0,\eta^*),\Gamma_\delta)$ such that $\gamma_{\mathcal{M}}(0)=\gamma^*$ and for any $\eta\in[0,\eta^*)$

$$f_{\mathcal{M}}(\gamma_{\mathcal{M}}(\eta),\eta)=0. \tag{7.97}$$

Let us define

$$\Re(\mu_n^0) = \gamma_{\mathcal{M}}(1/n) \quad \text{and} \quad \boldsymbol{\mathcal{E}}_n = \begin{pmatrix} \varepsilon_{1,n} & 0 \\ 0 & -\varepsilon_{2,n}\end{pmatrix}. \tag{7.98}$$

Together with (7.67), this fully determines the sequences $\{\mu_n^0 : n \geqslant n_0\}$ and $\{\boldsymbol{\mathcal{E}}_n : n\geqslant n_0\}$ provided $n_0 > (\eta^*)^{-1}$ is taken sufficiently large to ensure that

$$\Re(\mu_n^0) > \gamma^* - \boldsymbol{\epsilon} \ \ \forall n \geqslant n_0, \quad \lim_{n\to+\infty}|\mu_n^0| = +\infty, \quad \lim_{n\to+\infty}\boldsymbol{\mathcal{E}}_n = \mathbf{0}. \tag{7.99}$$

From (7.97), using the definition (7.82) of the matrix $\mathcal{M}$ and the definition (7.85) of the function $f_{\mathcal{M}}$, we deduce that

$$f_{\mathcal{M}}(\Re(\mu_n^0), 1/n) = \det\Big(\mathcal{A}(\mu_n^0, \varepsilon_{1,n}, \varepsilon_{2,n})\widehat{\mathbf{K}} - I\Big) = 0 \quad \forall n \geqslant n_0. \tag{7.100}$$

This, together with (7.52) and (7.57), directly implies that

$$\det\Big(M_0(\mu_n^0, \boldsymbol{\mathcal{E}}_n)\Big) = 0 \quad \forall n \geqslant n_0. \tag{7.101}$$

This completes the proof of Proposition 3. □

7.3 Step 3: Estimates of $M_1(\mu,\mathcal{E}_n)$ and $M_I(\mu,\mathcal{E}_n)$

Note that it suffices to prove Theorem 7 for $\boldsymbol{\epsilon} > 0$ small enough since if the theorem holds for $\boldsymbol{\epsilon} > 0$ it holds for every $\boldsymbol{\epsilon}' \geqslant \boldsymbol{\epsilon}$. In this section 7.3 and in the next section 7.4 we select an arbitrary $\kappa > \gamma^*$ and we fix $\boldsymbol{\epsilon} \in (0, \kappa)$ that we will choose to be sufficiently small (see (7.138) below). Note that some constants which appear in these two sections, as, for example, n_0 and $\boldsymbol{\delta} > 0$ may depend on $\boldsymbol{\epsilon}$ which is considered as given.

Let us consider the sequences $\{\mu_n^0 : n \geqslant n_0\}$ and $\{\boldsymbol{\mathcal{E}}_n : n \geqslant n_0\}$ which have been constructed in the previous subsection. For $\kappa > \gamma^*$, $c \in (0, \pi)$ and $\boldsymbol{\epsilon} \in (0, \kappa)$, we define the following rectangular domain in the complex plane:

$$\Omega_n = \Big\{\mu \in \mathbb{C} : \gamma^* - \boldsymbol{\epsilon} \leqslant \Re(\mu) \leqslant \kappa \;\text{ and }\; \Im(\mu_n^0) - m_1 \leqslant \Im(\mu) \leqslant \Im(\mu_n^0) + m_1\Big\} \tag{7.102}$$

where

$$m_1 = (2\pi + c)\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)^{-1}, \tag{7.103}$$

and n_0 is assumed to be sufficiently large to ensure that $\mu_n^0 \in \Omega_n$, $\forall n \geqslant n_0$.

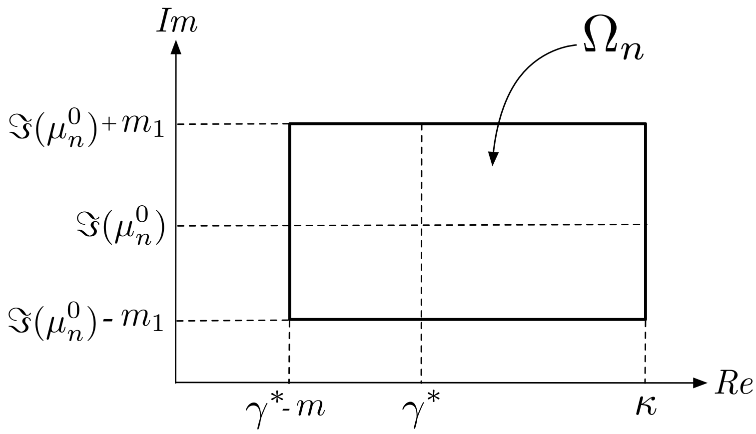


Figure 2 – Ω_n domain.

Under this assumption, we shall now prove the following two propositions.

Proposition 4.

$$\lim_{n\to+\infty} \sup_{\mu\in\Omega_n} |M_1(\mu, \boldsymbol{\mathcal{E}}_n)| = 0. \tag{7.104}$$

Proposition 5.

$$\lim_{n\to+\infty}\sup_{\mu\in\Omega_n}|M_I(\mu,\boldsymbol{\mathcal{E}}_n)|=0. \tag{7.105}$$

Proof of Proposition 4. From the definition of the functional matrix $\mathcal{G}(x)$ in (7.44), with the functional matrices $\mathbf{A}$ and $\mathbf{B}$ given in (7.41), we have

$$\mathcal{G}(1)=\Big(\mathbf{Id}-(\mathbf{Id}-\Psi)^{-1}(\mathbf{D})(1)\Big)^{-1}\Big((\mathbf{Id}-\Psi)^{-1}(\mathbf{E})(1)\Big) \tag{7.106}$$

and

$$\mathcal{G}(x)=(\mathbf{Id}-\Psi)^{-1}(\mathbf{E})(x)+(\mathbf{Id}-\Psi)^{-1}(\mathbf{D})(x)\mathcal{G}(1). \tag{7.107}$$

Using (7.33) and (7.36), we have the estimate

$$|(\mathbf{Id}-\Psi)^{-1}(\mathbf{D})(1)|\leqslant C\max(|\varepsilon_{1,0}|,|\varepsilon_{2,0}|)e^{2\kappa/\lambda_2}. \tag{7.108}$$

We assume that n_0 is taken sufficiently large to ensure that

$$\max(|\varepsilon_{1,n}|,|\varepsilon_{2,n}|)<\Big(2Ce^{2\kappa/\lambda_2}\Big)^{-1}\quad \forall n\geqslant n_0, \tag{7.109}$$

and therefore

$$|(\mathbf{Id}-\Psi)^{-1}(\mathbf{D})(1)|<\frac{1}{2}, \tag{7.110}$$

which implies that

$$\Big|\mathbf{Id}-(\mathbf{Id}-\Psi)^{-1}(\mathbf{D})(1)\Big|^{-1}<2. \tag{7.111}$$

Then, using again (7.36), from (7.106) and (7.111), we can deduce the following estimate:

$$|\mathcal{G}(1)|\leqslant Ce^{2\kappa/\lambda_2}. \tag{7.112}$$

Moreover, for the functional matrix $\mathcal{G}(x)$ in (7.107), using (7.33), (7.36) and (7.112), we have for all $n\geqslant n_0$:

$$\|\mathcal{G}(\cdot)\|_{L^2}\leqslant\|(\mathbf{Id}-\Psi)^{-1}\|_{\mathcal{L}(L^2)}\Big[\|\mathbf{E}(\cdot)\|_{L^2}+\|\mathbf{D}(\cdot)\|_{L^2}|\mathcal{G}(1)|\Big]\leqslant C\max(|\varepsilon_{1,n}|,|\varepsilon_{2,n}|)e^{4\kappa/\lambda_2}. \tag{7.113}$$

Using these estimates together with (7.30), (7.32) and (7.33), it follows from the definitions of M_{11} and M_{12} given in (7.49) that we finally have the following estimates for all $\mu\in\Omega_n$ and all $n\geqslant n_0$:

$$|M_{11}(\mu,\boldsymbol{\mathcal{E}}_n)|\leqslant C\max(|\varepsilon_{1,n}|,|\varepsilon_{2,n}|)e^{4\kappa/\lambda_2} \tag{7.114}$$

and

$$|M_{12}(\mu,\boldsymbol{\mathcal{E}}_n)|\leqslant C\Big(\max(|\varepsilon_{1,n}|,|\varepsilon_{2,n}|)e^{4\kappa/\lambda_2}+(\max(|\varepsilon_{1,n}|,|\varepsilon_{2,n}|)^2e^{6\kappa/\lambda_2}\Big).$$

Then since

$$|M_1(\mu,\boldsymbol{\mathcal{E}}_n)|\leqslant|M_{11}(\mu,\boldsymbol{\mathcal{E}}_n)|+|M_{12}(\mu,\boldsymbol{\mathcal{E}}_n)|, \tag{7.115}$$

Proposition 4 follows readily because $\max(|\varepsilon_{1,n}|,|\varepsilon_{2,n}|)\to 0$ when $n\to+\infty$ (see (7.99)). □

Proof of Proposition 5. The matrix M_I defined in (7.26) is written in the following form

$$M_I(\mu,\boldsymbol{\mathcal{E}})=-\begin{pmatrix} I_1(\mu,\boldsymbol{\mathcal{E}}) & I_2(\mu,\boldsymbol{\mathcal{E}}) & I_3(\mu) & I_4(\mu)\\ 0&0&0&0\\ I_1(\mu,\boldsymbol{\mathcal{E}}) & I_2(\mu,\boldsymbol{\mathcal{E}}) & I_3(\mu) & I_4(\mu)\\ 0&0&0&0\end{pmatrix} \tag{7.116}$$

with

$$
\begin{aligned}
I_1(\mu, \boldsymbol{\mathcal{E}}) &= \int_0^1 f_1(\zeta) e^{-\frac{\mu}{\lambda_1+\varepsilon_1}\zeta} d\zeta, \\
I_2(\mu, \boldsymbol{\mathcal{E}}) &= \int_0^1 f_2(\zeta) e^{\frac{\mu}{\lambda_2+\varepsilon_2}\zeta} d\zeta, \\
I_3(\mu) &= \int_0^1 f_1(\zeta) e^{-\frac{\mu}{\lambda_1}\zeta} d\zeta, \\
I_4(\mu) &= \int_0^1 f_2(\zeta) e^{\frac{\mu}{\lambda_2}\zeta} d\zeta.
\end{aligned}
\tag{7.117}
$$

Let $\mu \in \Omega_n$. For any $n \geqslant n_0$ we have

$$
I_2(\mu, \boldsymbol{\mathcal{E}}_n) = \int_0^1 f_2(\zeta) e^{\frac{\mu}{\lambda_2+\varepsilon_{2,}}\zeta} d\zeta. \tag{7.118}
$$

Since $f_2 \in H^1((0,1);\mathbb{R})$, we can write

$$
\begin{aligned}
|I_2(\mu, \boldsymbol{\mathcal{E}}_n)| &\leq \left| \int_0^1 f_2'(\zeta) \frac{(\lambda_2 + \varepsilon_{2,n})}{\mu} e^{\frac{\mu\zeta}{\lambda_2+\varepsilon_{2,n}}} d\zeta \right| + \left| f_2(0) \frac{(\lambda_2 + \varepsilon_{2,n})}{\mu} \right| + \left| f_2(1) \frac{(\lambda_2 + \varepsilon_{2,n})}{\mu} \right| e^{\frac{\kappa}{\lambda_2+\varepsilon_{2,n}}} \\
&\leq \|f_2\|_{H^1} \frac{(\lambda_2 + \varepsilon_{2,n})}{|\mu|} \left[\left(\int_0^1 e^{\frac{2\Re(\mu)\zeta}{\lambda_2+\varepsilon_{2,n}}} d\zeta \right)^{1/2} + (1 + e^{\frac{\kappa}{\lambda_2+\varepsilon_{2,n}}}) \right] \\
&\leq \frac{9}{2} \|f_2\|_{H^1} \frac{\lambda_2}{\Im(\mu)} e^{2\kappa/\lambda_2},
\end{aligned}
\tag{7.119}
$$

where the first inequality uses integration by parts, the second inequality uses Schwarz inequality and the third inequality uses $|\varepsilon_{2,n}| \leqslant \lambda_2/2$. On the other hand, combining the definition (7.67) of $\Im(\mu_n^0)$ and the definition (7.103) of m_1, since $n_0 \geqslant 1$, we have

$$
m_1 < \frac{2\pi + c}{3\pi} \Im(\mu_n^0) \quad \forall n \geqslant n_0. \tag{7.120}
$$

Then, using this in (7.119), we deduce that

$$
|I_2(\mu, \boldsymbol{\mathcal{E}}_n)| \leq \|f_2\|_{H^1} \frac{12\pi\lambda_2}{(\pi - c)\Im(\mu_n^0)} e^{3\kappa/2\lambda_2}. \tag{7.121}
$$

The same can be done for I_1, I_3 and I_4 and, as a consequence,

$$
|M_I(\mu, \boldsymbol{\mathcal{E}}_n)| \leq \frac{C}{\Im(\mu_n^0)} e^{3\kappa/2\lambda_2} \tag{7.122}
$$

where C is a positive constant which depends only on the system parameters. Hence, because $\Im(\mu_n^0) \to +\infty$ when $n \to +\infty$ (see (7.99)), Proposition 5 follows readily. □

7.4 Step 4: Conclusion based on degree theory

Let us now define the matrix

$$
M(\mu, \boldsymbol{\mathcal{E}}) = M_0(\mu, \boldsymbol{\mathcal{E}}) + M_I(\mu, \boldsymbol{\mathcal{E}}) + M_1(\mu, \boldsymbol{\mathcal{E}}). \tag{7.123}
$$

In order to complete the proof of Theorem 7, we shall now show that for $\boldsymbol{\epsilon}$ sufficently small there exist n_0 such that the characteristic equation $\det(M(\mu, \boldsymbol{\mathcal{E}}_n)) = 0$ has at least one solution located inside the domain Ω_n $\forall n \geqslant n_0$. To establish this result, we will use the degree theory (see [43, Appendix B]) or, equivalently, the Cauchy's argument principle.

We start the analysis with the following preliminary lemma.

Lemma 5. There exist n_0 and $\boldsymbol{\delta}$ (independent of n) such that

$$|\det(M_0(\mu, \boldsymbol{\mathcal{E}}_n))| \geqslant \boldsymbol{\delta} > 0 \quad \forall \mu \in \partial\Omega_n \quad \text{and} \quad \forall n \geqslant n_0. \tag{7.124}$$

Proof. From (7.52) and (7.57), we know that

$$\det(M_0(\mu, \boldsymbol{\mathcal{E}}_n)) = e^{\mu\left(\frac{1}{\lambda_2} + \frac{1}{\lambda_2 + \varepsilon_{2,n}}\right)} \det\left(\mathcal{A}(\mu, \varepsilon_{1,n}, \varepsilon_{2,n})\widehat{\mathbf{K}} - I\right) \tag{7.125}$$

with $\mathcal{A}(\mu, \varepsilon_1, \varepsilon_2)$ defined by (7.58).

We introduce the following functions

$$\sigma(\mu) = \frac{1}{2}\Big(\gamma^* - \Re(\mu)\Big)\left[\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right], \tag{7.126}$$

$$\bar{\phi}(\mu, n) = \Big(\Im(\mu) - \Im(\mu_n^0)\Big), \tag{7.127}$$

$$\phi(\mu, n) = -\frac{1}{2}\bar{\phi}(\mu, n)\left[\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right], \tag{7.128}$$

which, using (7.102) and (7.103), imply that

$$\mu \in \Omega_n \quad \iff \quad \begin{cases} -(\pi + c/2) \leqslant \phi(\mu, n) \leqslant \pi + c/2, \\ \frac{1}{2}(\gamma^* - \kappa)\left[\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right] \leqslant \sigma(\mu) \leqslant \frac{1}{2}\boldsymbol{\epsilon}\left[\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right]. \end{cases} \tag{7.129}$$

With these notations, we have

$$\mathcal{A}(\mu, \varepsilon_{1,n}, \varepsilon_{2,n}) = e^{\sigma(\mu) + i\phi(\mu,n)}\mathcal{B}(\mu, \varepsilon_{1,n}, \varepsilon_{2,n})\frac{e^{i\Theta}}{\rho_2(\widehat{\mathbf{K}})} \tag{7.130}$$

where

$$\mathcal{B}(\mu, \varepsilon_{1,n}, \varepsilon_{2,n}) = \operatorname{diag}\Big(e^{-(\Re(\mu) + i\bar{\phi}(\mu,n))\hat{\varepsilon}_{1,n}}, e^{-(\Re(\mu) + i\bar{\phi}(\mu,n))\hat{\varepsilon}_{2,n}}, 1, 1\Big). \tag{7.131}$$

Then, denoting by $b_j(\mu, \varepsilon_{1,n}, \varepsilon_{2,n})$ ($j \in \{1, 2, 3, 4\}$) the eigenvalues of the matrix

$$\mathcal{B}(\mu, \varepsilon_{1,n}, \varepsilon_{2,n})\frac{e^{i\Theta}}{\rho_2(\widehat{\mathbf{K}})}\widehat{\mathbf{K}}, \tag{7.132}$$

we have

$$\det\left(\mathcal{A}(\mu, \varepsilon_{1,n}, \varepsilon_{2,n})\widehat{\mathbf{K}} - I\right) = \prod_{j=1}^{4} \underbrace{\left(e^{\sigma(\mu) + i\phi(\mu,n)}\; b_j(\mu, \varepsilon_{1,n}, \varepsilon_{2,n}) - 1\right)}_{\mathbf{b}_j(\mu, \varepsilon_{1,n}, \varepsilon_{2,n})}. \tag{7.133}$$

From this expression, we see that the lemma will be proved if we can show that there exist n_0 and β_j (independent of n) such that

$$|\mathbf{b}_j(\mu, \varepsilon_{1,n}, \varepsilon_{2,n})| \geqslant \beta_j > 0 \quad j \in \{1, 2, 3, 4\} \quad \forall \mu \in \partial\Omega_n \quad \text{and} \quad \forall n \geqslant n_0. \tag{7.134}$$

From (7.131) and (7.132), we observe that $\lim_{n\to\infty} \mathcal{B}(\mu, \varepsilon_{1,n}, \varepsilon_{2,n}) = I$ and that $b_j^0 = b_j(\mu, 0, 0)$ ($j \in \{1, 2, 3, 4\}$) are the eigenvalues of the matrix $e^{i\Theta}\widehat{\mathbf{K}}/\rho_2(\widehat{\mathbf{K}})$. We can deduce that, for any $\eta_0 > 0$, n_0 can be taken sufficiently large to ensure that $\forall n \geqslant n_0$ and $\forall \mu \in \Omega_n$:

$$\begin{gathered} |b_j^0| - \eta_0 \leqslant |b_j(\mu, \varepsilon_{1,n}, \varepsilon_{2,n})| \leqslant |b_j^0| + \eta_0, \\ |\arg\Big(b_j(\mu, \varepsilon_{1,n}, \varepsilon_{2,n})\Big)| \leqslant |\arg(b_j^0)| + \arctan\Big(\eta_0/|b_j^0|\Big). \end{gathered} \tag{7.135}$$

with the convention that $\arctan(\eta_0/|b_j^0|) = \pi/2$ if $b_j^0 = 0$. We also know from Equation (7.87) that the matrix $e^{i\Theta}\widehat{\mathbf{K}}/\rho_2(\widehat{\mathbf{K}})$ has an eigenvalue (denoted b_1^0) which is equal to 1 or, equivalently, that $\rho_2(\widehat{\mathbf{K}})$ is an eigenvalue of $e^{i\Theta}\widehat{\mathbf{K}}$. Moreover, from Theorem 3.12 in [41], we know that $\rho_2(\widehat{\mathbf{K}})$ is the spectral radius of the matrix $e^{i\Theta}\widehat{\mathbf{K}}$. It follows that

$$b_1^0 = 1 \quad \text{and} \quad |b_j^0| \leqslant 1 \quad \forall j \in \{2, 3, 4\}. \tag{7.136}$$

We denote by $\mathcal{S}_0$ and $\mathcal{S}_1$ the sets defined by

$$j \in \mathcal{S}_1 \Leftrightarrow |b_j^0| = 1 \quad \text{and} \quad \mathcal{S}_0 = \{1, 2, 3, 4\} \setminus \mathcal{S}_1. \tag{7.137}$$

We will now consider separately the case where $j \in \mathcal{S}_0$ and the case where $j \in \mathcal{S}_1$.

Let us thus assume that $j \in \mathcal{S}_0$, then $|b_j^0| < 1$. We impose that $\boldsymbol{\epsilon} \in (0, \kappa)$ be small enough so that

$$e^{\frac{1}{2}\boldsymbol{\epsilon}\left[\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right]}|b_j^0| < 1 \tag{7.138}$$

Then, from (7.129) and (7.135), $\eta_0 > 0$ can be selected sufficiently small to ensure, provided n_0 is taken sufficiently large, that

$$\begin{aligned} |e^{\sigma(\mu)+i\phi(\mu,n)}\, b_j(\mu, \varepsilon_{1,n}, \varepsilon_{2,n})| &= e^{\sigma(\mu)}\, |b_j(\mu, \varepsilon_{1,n}, \varepsilon_{2,n})| \\ &\leqslant e^{\frac{1}{2}\boldsymbol{\epsilon}\left[\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right]}(|b_j^0| + \eta_0) < 1, \end{aligned} \tag{7.139}$$

$$\forall j \in \mathcal{S}_0, \quad \forall n \geqslant n_0 \quad \text{and} \quad \forall \mu \in \partial\Omega_n.$$

Defining

$$\beta_{0j} = 1 - e^{\frac{1}{2}\boldsymbol{\epsilon}\left[\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right]}(|b_j^0| + \eta_0) > 0 \tag{7.140}$$

it follows directly that

$$|\mathbf{b}_j(\mu, \varepsilon_{1,n}, \varepsilon_{2,n})| = \left|e^{\sigma(\mu)+i\phi(\mu,n)}\, b_j(\mu, \varepsilon_{1,n}, \varepsilon_{2,n}) - 1\right| \geqslant \beta_{0j} > 0 \tag{7.141}$$

$$\forall j \in \mathcal{S}_0, \quad \forall n \geqslant n_0 \quad \text{and} \quad \forall \mu \in \Omega_n.$$

Let us now assume that $j \in \mathcal{S}_1$ with $|b_j^0| = 1$. In that case, we will evaluate $|\mathbf{b}_j(\mu, \varepsilon_{1,n}, \varepsilon_{2,n})|$ as μ traverses the contour $\partial\Omega_n$ of the rectangular domain Ω_n. We have the following three cases.

- First case: along the right boundary of Ω_n

$$\Re(\mu) = \kappa > \gamma^*, \quad \Im(\mu) \in [\Im(\mu_n^0) - m_1, \Im(\mu_n^0) + m_1]. \tag{7.142}$$

  In that case, from (7.126) and (7.135), η_0 can be selected sufficiently small to ensure, provided n_0 is taken sufficiently large, that

$$\begin{aligned} |e^{\sigma(\mu)+i\phi(\mu,n)}\, b_j(\mu, \varepsilon_{1,n}, \varepsilon_{2,n})| &= e^{\sigma(\mu)}\, |b_j(\mu, \varepsilon_{1,n}, \varepsilon_{2,n})| \\ &\leqslant e^{\frac{1}{2}(\gamma^*-\kappa)\left[\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right]}(1 + \eta_0) < 1 \end{aligned} \tag{7.143}$$

  for all $n \geqslant n_0$ and for all μ on the right boundary (7.142).

  Defining

$$\beta_{1j} = 1 - e^{\frac{1}{2}(\gamma^*-\kappa)\left[\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right]}(1 + \eta_0) > 0 \tag{7.144}$$

  it follows directly that

$$|\mathbf{b}_j(\mu, \varepsilon_{1,n}, \varepsilon_{2,n})| = \left|e^{\sigma(\mu)+i\phi(\mu,n)}\, b_j(\mu, \varepsilon_{1,n}, \varepsilon_{2,n}) - 1\right| \geqslant \beta_{1j} > 0 \tag{7.145}$$

  for all $j \in \mathcal{S}_1$, for all $n \geqslant n_0$ and for all μ on the right boundary (7.142).

- Second case: along the left boundary of Ω_n

$$\Re(\mu) = \gamma^* - \boldsymbol{\epsilon}, \quad \Im(\mu) \in [\Im(\mu_n^0) - m_1, \Im(\mu_n^0) + m_1]. \tag{7.146}$$

In that case, η_0 can be selected sufficiently small to ensure, provided n_0 is taken sufficiently large, that

$$\begin{aligned} |e^{\sigma(\mu)+i\phi(\mu,n)}\ b_j(\mu, \varepsilon_{1,n}, \varepsilon_{2,n})| &= e^{\sigma(\mu)}\ |b_j(\mu, \varepsilon_{1,n}, \varepsilon_{2,n})| \\ &\geqslant e^{\frac{1}{2}\boldsymbol{\epsilon}\left[\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right]}(1-\eta_0) > 1 \end{aligned} \tag{7.147}$$

for all $n \geqslant n_0$ and for all μ on the left boundary (7.146).

Defining

$$\beta_{2j} = e^{\frac{1}{2}\boldsymbol{\epsilon}\left[\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right]}(1-\eta_0) - 1 > 0 \tag{7.148}$$

it follows directly that

$$|\mathbf{b}_j(\mu, \varepsilon_{1,n}, \varepsilon_{2,n})| = \left|e^{\sigma(\mu)+i\phi(\mu,n)}\ b_j(\mu, \varepsilon_{1,n}, \varepsilon_{2,n}) - 1\right| \geqslant \beta_{2j} > 0 \tag{7.149}$$

for all $j \in \mathcal{S}_1$, for all $n \geqslant n_0$ and for all μ on the left boundary (7.146).

- Third case: along the upper and lower boundaries of Ω_n, where

$$\Re(\mu) \in [\gamma^* - \boldsymbol{\epsilon}, \kappa], \quad \Im(\mu) = \Im(\mu_n^0) \pm m_1, \tag{7.150}$$

for which, from the definition of m_1 given in (7.103) and using (7.128),

$$\phi(\mu, n) = \pm(\pi + \frac{c}{2}), \tag{7.151}$$

where $(\pi + c/2)$ corresponds to the lower boundary and $-(\pi + c/2)$ to the upper boundary. We introduce the notations

$$\phi_j(\mu, n) = \arg\Big(b_j(\mu, \varepsilon_{1,n}, \varepsilon_{2,n})\Big), \quad \phi_j^0 = \arg(b_j^0) \quad \text{and} \quad \Delta\phi_j(\mu, n) = \phi_j(\mu, n) - \phi_j^0. \tag{7.152}$$

where $\arg : \mathbb{C} \to (-\pi, \pi]$ and therefore $\phi_j^0 \in (-\pi, \pi]$. With these notations, using (7.151), we have

$$\phi(\mu, n) + \phi_j(\mu, n) = \phi_j^0 \pm (\pi + \frac{c}{2}) + \Delta\phi_j(\mu, n). \tag{7.153}$$

Then, using (7.129) and (7.135), η_0 can be selected sufficiently small to ensure, provided n_0 is taken sufficiently large, that

$$\begin{aligned} |\mathbf{b}_j(\mu, \varepsilon_{1,n}, \varepsilon_{2,n})| &= |e^{\sigma(\mu)+i\phi(\mu,n)}\ b_j(\mu, \varepsilon_{1,n}, \varepsilon_{2,n}) - 1| \\ &= \left|e^{\sigma(\mu)}|b_j(\mu, \varepsilon_{1,n}, \varepsilon_{2,n})|e^{i(\phi(\mu,n)+\phi_j(\mu,n))} - 1\right| \\ &\geqslant\ e^{\sigma(\mu)}|b_j(\mu, \varepsilon_{1,n}, \varepsilon_{2,n})|\ \left|\sin\Big(\phi(\mu, n) + \phi_j(\mu, n)\Big)\right| \\ &\geqslant e^{\frac{1}{2}(\gamma^*-\kappa)\left[\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right]}(1-\eta_0)\left|\sin\Big(\phi_j^0 \pm (\pi + \frac{c}{2}) + \Delta\phi_j(\mu, n)\Big)\right|. \end{aligned} \tag{7.154}$$

for all $j \in \mathcal{S}_1$, for all $n \geqslant n_0$ and for all μ on the boundary (7.150).

We will now select c such that, in addition to $c \in (0, \pi)$, we have, for any $j \in \mathcal{S}_1$:

$$\begin{cases} \frac{\pi}{32} \leq \phi_j^0 + (\pi + \frac{c}{2}) (\text{mod } 2\pi) \leq \pi - \frac{\pi}{32} \\ \text{or} \\ \pi + \frac{\pi}{32} \leq \phi_j^0 + (\pi + \frac{c}{2}) (\text{mod } 2\pi) \leq 2\pi - \frac{\pi}{32}. \end{cases} \tag{7.155}$$

and

$$\begin{cases} \frac{\pi}{32} \leq \phi_j^0 - (\pi + \frac{c}{2}) (\text{mod } 2\pi) \leq \pi - \frac{\pi}{32} \\ \text{or} \\ \pi + \frac{\pi}{32} \leq \phi_j^0 - (\pi + \frac{c}{2}) (\text{mod } 2\pi) \leq 2\pi - \frac{\pi}{32}. \end{cases} \tag{7.156}$$

We first note (see (7.136)) that $\phi_1^0 = 0$ and therefore, for any $c/2 \in (\pi/32, \pi/2)$ and for $j = 1$, the third line of (7.155) and the first line of (7.156) hold, which implies that (7.155) and (7.156) hold. For each of the three other ϕ_j^0 ($j \in \{2, 3, 4\}$), in order to satisfy both (7.155) and (7.156), there are two arcs of length $\pi/16$ which must be avoided (modulo 2π) for the selection of $c/2$. Hence, for the three ϕ_j^0 together, the total length of the six forbidden arcs is at most equal to $6\pi/16$. Therefore, since $\pi/2 - \pi/32 > 6\pi/16$, it is clear that the set of forbidden angles does not fully cover the arc $(\pi/32, \pi/2)$ and, consequently, that there is always room for the selection of $c/2$ into $(\pi/32, \pi/2)$ such that (7.155) and (7.156) hold.

Using (7.135), (7.155), and (7.156), it follows that η_0 can be selected sufficiently small to ensure, provided n_0 is taken sufficiently large, that

$$\left| \sin\left(\phi_j^0 \pm (\pi + \frac{c}{2}) + \Delta\phi_j(\mu, n) \right) \right| \geqslant \frac{1}{2} \sin\left(\frac{\pi}{32} \right). \tag{7.157}$$

Then from (7.154), we deduce that

$$|\mathbf{b}_j(\mu, \varepsilon_{1,n}, \varepsilon_{2,n})| \geqslant \frac{1}{2} e^{\frac{1}{2}(\gamma^* - \kappa)\left[\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right]} (1 - \eta_0) \sin\left(\frac{\pi}{32} \right) =: \beta_{3j} > 0 \tag{7.158}$$

for all $j \in \mathcal{S}_1$, for all $n \geqslant n_0$ and for all μ on the boundary (7.150)

We can now observe that, based on their definitions, the four inequalities (7.139), (7.143), (7.147), (7.158) can be simultaneously verified with a single (sufficiently small) value of η_0 and a single (sufficiently large) value of n_0.

This implies readily that we can define β_j ($j \in \{1, 2, 3, 4\}$) independent of n such that

$$|\mathbf{b}_j(\mu, \varepsilon_{1,n}, \varepsilon_{2,n})| \geqslant \beta_j > 0, \quad j \in \{1, 2, 3, 4\}, \quad \forall \mu \in \partial\Omega_n \text{ and } \forall n \geqslant n_0, \tag{7.159}$$

with

$$\beta_j = \beta_{0j} \text{ if } j \in \mathcal{S}_0 \quad \text{and} \quad \beta_j = \min(\beta_{1j}, \beta_{2j}, \beta_{3j}) \text{ if } j \in \mathcal{S}_1. \tag{7.160}$$

Then, from (7.133), it follows that

$$\left| \det\left(\mathcal{A}(\mu, \varepsilon_{1,n}, \varepsilon_{2,n}) \widehat{\mathbf{K}} - I \right) \right| \geqslant \beta := \prod_{j=1}^{4} \beta_j > 0 \quad \forall \mu \in \partial\Omega_n \quad \text{and} \quad \forall n \geqslant n_0. \tag{7.161}$$

Moreover, from (7.125), we have

$$|\det(M_0(\mu, \boldsymbol{\mathcal{E}}_n))| = \left| e^{\mu\left(\frac{1}{\lambda_2} + \frac{1}{\lambda_2 + \varepsilon_{2,n}} \right)} \right| \left| \det\left(\mathcal{A}(\mu, \varepsilon_{1,n}, \varepsilon_{2,n}) \widehat{\mathbf{K}} - I \right) \right|. \tag{7.162}$$

Because $\varepsilon_{2,n} < \lambda_2/2$ and $\Re(\mu) \geqslant \gamma^* - \boldsymbol{\epsilon}$ (with $\gamma^* \geqslant 0$) by assumption, it follows that

$$|\det(M_0(\mu, \boldsymbol{\mathcal{E}}_n))| \geqslant e^{-3\boldsymbol{\epsilon}/\lambda_2} \beta =: \boldsymbol{\delta} > 0 \quad \forall \mu \in \partial\Omega_n \quad \text{and} \quad \forall n \geqslant n_0. \tag{7.163}$$

This completes the proof of Lemma 5. ☐

Therefore, since the function $\det(M_0(\mu, \boldsymbol{\mathcal{E}}_n))$ is holomorphic over Ω_n and does not vanish on the boundary $\partial\Omega_n$, since it has at least one zero (μ_n^0) inside the domain Ω_n, this implies, according to the degree theory, that the image of the function (in the complex plane) encircles the origin at least once (at a distance, from Lemma 5, at least equal to $\boldsymbol{\delta}$) when μ traverses the contour $\partial\Omega_n$.

Let us now come back to the characteristic equation

$$\det(M(\mu, \boldsymbol{\mathcal{E}}_n)) = 0. \tag{7.164}$$

Since the determinant is a polynomial function, and using Propositions 4 and 5, we can write

$$\det(M(\mu, \boldsymbol{\mathcal{E}}_n)) = \det(M_0(\mu, \boldsymbol{\mathcal{E}}_n)) + r_n^0(\mu) \tag{7.165}$$

where $r_n^0(\mu)$ satisfies

$$\lim_{n \to +\infty} \sup_{\mu \in \Omega_n} |r_n^0(\mu)| = 0. \tag{7.166}$$

We consider the function

$$(n, \mu, t) \mapsto \det(M_0(\mu, \boldsymbol{\mathcal{E}}_n)) + t\, r_n^0(\mu) \tag{7.167}$$

which is continuous on the domain $\{n \geqslant n_0, \mu \in \partial\Omega_n, t \in [0,1]\}$. Now, for this continuous function, it follows from (7.166) that n_0 can be selected sufficiently large to ensure that

$$|\det(M_0(\mu, \boldsymbol{\mathcal{E}}_n)) + t\, r_n^0(\mu)| \geqslant \boldsymbol{\delta}/2 \quad \forall n \geqslant n_0,\ \forall \mu \in \partial\Omega_n,\ \forall t \in [0,1]. \tag{7.168}$$

Moreover, using definition (7.167), we deduce that the two functions $\det(M_0(\mu, \boldsymbol{\mathcal{E}}_n))$ and $\det(M(\mu, \boldsymbol{\mathcal{E}}_n))$ are homotopic, or, otherwise stated, that $t \mapsto \det(M_0(\mu, \boldsymbol{\mathcal{E}}_n)) + t\, r_n^0(\mu)$ is an homotopic function that continuously deforms $\det(M_0(\mu, \boldsymbol{\mathcal{E}}_n))$ (for $t = 0$) into $\det(M(\mu, \boldsymbol{\mathcal{E}}_n))$ (for $t = 1$). Then, from the homotopy invariance of the degree (see [43, Proposition B.8]), it follows that, for each $n \geqslant n_0$, the function $\det(M(\mu, \boldsymbol{\mathcal{E}}_n))$ has the same degree as $\det(M_0(\mu, \boldsymbol{\mathcal{E}}_n))$ and therefore has at least one zero (denoted μ^n) inside Ω_n.

Thus, based on our demonstration above, it is now clear that for any $\varepsilon_0 \in (0, \frac{1}{2}\min(\lambda_1, \lambda_2))$ and for any $\boldsymbol{\epsilon} > 0$, taking n_0 sufficiently large, there exists an infinite sequence of non-zero perturbations $(\varepsilon_{1,n}, \varepsilon_{2,n})$ with

$$\max(|\varepsilon_{1,n}|, |\varepsilon_{2,n}|) \leqslant \varepsilon_0 \quad \forall n \geqslant n_0 \tag{7.169}$$

for which there are eigenvalues $\mu_n \in \mathbb{C}$, solutions to the eigenvalue problem (7.2), such that

$$\mu_n \in \Omega_n \quad \text{with} \quad \Re(\mu^n) > 2\frac{\lambda_1\lambda_2}{\lambda_1 + \lambda_2}\ln(\rho_2(\widehat{\mathbf{K}})) - \boldsymbol{\epsilon} \quad \forall n \geqslant n_0. \tag{7.170}$$

This completes the proof of Theorem 7. □

8 Concluding remarks

In this article, we analysed the robustness of observer-controllers that stabilize hyperbolic systems represented by (1.1) under uncertainties in characteristic velocities λ_i. These uncertainties are relevant not only from a mathematical standpoint but also from a physical one. In fact, many linear hyperbolic systems arise as linearizations of physical models that are inherently quasilinear. However, it is well known that the stability of a linearized system does not necessarily imply the stability of the corresponding nonlinear system, even locally. This limitation stems precisely from the lack of robustness of the linearized dynamics with respect to uncertainties in the characteristic velocities.

We focused on the single-input-single-output setting, considering separately the cases of co-located and anti-located sensing and actuation. In both situations, a stabilizing observer-controller was designed using the backstepping technique, whose principle was recalled in Section 2.

For the case of co-located sensing and actuation, we recalled in Section 3 that output feedback stabilization by an observer-controller is always robust to small perturbations of the characteristic velocities. However, it should be noted that this does not imply robustness against all types of infinitesimal static or dynamic disturbances. For instance, it is well known that robustness against parasitic delays in the loop holds only if $\mathcal{D}_b = |k_1 k_2| < 1$ (see e.g. Auriol et al. [29, Section III]).

Our main contribution concerns the case of anti-located sensing and actuation for which the observer-controller design was presented in Section 4. In this setting, we have proved in Sections 5, 6 and 7 that robustness against small perturbations in characteristic velocities is guaranteed if and only if the following fairly technical condition is verified:

$$\rho_2(\widehat{\mathbf{K}}) < 1. \tag{8.1}$$

From a practical point of view, a more relevant question is whether the control tuning parameters k_c and k_0 can be chosen to satisfy this condition. As explained in Remark 1, this is possible if and only if

$$\mathcal{D}_b = |k_1 k_2| < 1 \tag{8.2}$$

which is thus the necessary and sufficient condition to allow the existence of tuning parameters k_c and k_0 that guarantee robustness against disturbances in characteristic velocities in the anti-located configuration.

Consequently, when the open-loop boundary damping factor satisfies $\mathcal{D}_b = |k_1 k_2| \geqslant 1$, output-feedback stabilization, while theoretically feasible, may fail to be robust in the anti-located case. We believe, however, that this conclusion should be interpreted with some caution. Indeed, as argued in our recent works [45]–[46], unconditional robustness is very likely to persist in the presence of small, unmodeled diffusion effects in the system that are neglected in the purely hyperbolic model (1.1). Establishing a rigorous and comprehensive proof of this conjecture remains a challenging and largely open problem in the field.

A Appendix: Kernel equations

A.1 For the full state feedback

A) The matrix

$$P(x,\xi) = \begin{pmatrix} P_{(1)}(x,\xi) \\ P_{(2)}(x,\xi) \end{pmatrix} = \begin{pmatrix} p_{11}(x,\xi) & p_{12}(x,\xi) \\ p_{21}(x,\xi) & p_{22}(x,\xi) \end{pmatrix} \tag{A.1}$$

is the solution of the matrix PDE

$$P_\xi(x,\xi)\mathbf{\Lambda} + \mathbf{\Lambda} P_x(x,\xi) - P(x,\xi)\mathbf{C} = 0, \tag{A.2}$$

with the boundary condition

$$p_{12}(x,x) = \frac{c_1}{\lambda_1 + \lambda_2}, \qquad p_{21}(x,x) = \frac{-c_2}{\lambda_1 + \lambda_2}, \tag{A.3a}$$

$$\lambda_1 p_{11}(x,1) - k_2 \lambda_2 p_{12}(x,1) = 0, \qquad \lambda_1 p_{21}(x,1) - k_2 \lambda_2 p_{22}(x,1) = 0. \tag{A.3b}$$

B) The matrix

$$Q(x,\xi) = \begin{pmatrix} Q_{(1)}(x,\xi) \\ Q_{(2)}(x,\xi) \end{pmatrix} = \begin{pmatrix} q_{11}(x,\xi) & q_{12}(x,\xi) \\ q_{21}(x,\xi) & q_{22}(x,\xi) \end{pmatrix} \tag{A.4}$$

is the solution of the matrix PDE

$$Q_\xi(x,\xi)\mathbf{\Lambda} + \mathbf{\Lambda} Q_x(x,\xi) + \mathbf{C}Q(x,\xi) = 0, \tag{A.5}$$

with the boundary condition

$$q_{12}(x,x) = \frac{-c_1}{\lambda_1+\lambda_2}, \quad q_{21}(x,x) = \frac{c_2}{\lambda_1+\lambda_2}, \tag{A.6a}$$

$$\lambda_1 q_{11}(x,1) - k_2\lambda_2 q_{12}(x,1) = 0, \quad \lambda_1 q_{21}(x,1) - k_2\lambda_2 q_{22}(x,1) = 0. \tag{A.6b}$$

A.2 For the observer with co-located input/output

A) The matrix

$$\widetilde{P}(x,\xi) = \begin{pmatrix} \widetilde{P}_{(1)}(x,\xi) \\ \widetilde{P}_{(2)}(x,\xi) \end{pmatrix} = \begin{pmatrix} \tilde{p}_{11}(x,\xi) & \tilde{p}_{12}(x,\xi) \\ \tilde{p}_{21}(x,\xi) & \tilde{p}_{22}(x,\xi) \end{pmatrix} \tag{A.7}$$

is the solution of the matrix PDE

$$\widetilde{P}_\xi(x,\xi)\mathbf{\Lambda} + \mathbf{\Lambda}\widetilde{P}_x(x,\xi) - \widetilde{P}(x,\xi)\mathbf{C} = 0 \tag{A.8}$$

with the boundary conditions

$$\tilde{p}_{12}(x,x) = \frac{-c_1}{\lambda_1+\lambda_2}, \quad \tilde{p}_{21}(x,x) = \frac{c_2}{\lambda_1+\lambda_2}, \tag{A.9a}$$

$$(k_1-k_0)\tilde{p}_{11}(x,0)\lambda_1 - \tilde{p}_{12}(x,0)\lambda_2 = 0, \quad (k_1-k_0)\tilde{p}_{21}(x,0)\lambda_1 - \tilde{p}_{22}(x,0)\lambda_2 = 0. \tag{A.9b}$$

B) The matrix

$$\widetilde{Q}(x,\xi) = \begin{pmatrix} \widetilde{Q}_{(1)}(x,\xi) \\ \widetilde{Q}_{(2)}(x,\xi) \end{pmatrix} = \begin{pmatrix} \tilde{q}_{11}(x,\xi) & \tilde{q}_{12}(x,\xi) \\ \tilde{q}_{21}(x,\xi) & \tilde{q}_{22}(x,\xi) \end{pmatrix} \tag{A.10}$$

is the solution of the matrix PDE

$$\widetilde{Q}_\xi(x,\xi)\mathbf{\Lambda} + \mathbf{\Lambda}\widetilde{Q}_x(x,\xi) + \mathbf{C}\widetilde{Q}(x,\xi) = 0 \tag{A.11}$$

with the boundary conditions

$$\tilde{q}_{12}(x,x) = \frac{c_1}{\lambda_1+\lambda_2}, \quad \tilde{q}_{21}(x,x) = \frac{-c_2}{\lambda_1+\lambda_2}, \tag{A.12a}$$

$$k_2\tilde{q}_{21}(0,\xi) - \tilde{q}_{11}(0,\xi) = 0, \quad k_2\tilde{q}_{22}(0,\xi) - \tilde{q}_{12}(0,\xi) = 0. \tag{A.12b}$$

C) Moreover, the internal output injection gains $\upsilon_1(x)$ and $\upsilon_2(x)$ are

$$\upsilon_1(x) = \lambda_2\tilde{q}_{12}(x,0) - \lambda_1\tilde{q}_{11}(x,0)(k_1-k_0), \tag{A.13a}$$

$$\upsilon_2(x) = \lambda_2\tilde{q}_{22}(x,0) - \lambda_1\tilde{q}_{21}(x,0)(k_1-k_0). \tag{A.13b}$$

A.3 For the observer with an anti-located input/ output

A) The matrix

$$\widetilde{P}(x,\xi) = \begin{pmatrix} \widetilde{P}_{(1)}(x,\xi) \\ \widetilde{P}_{(2)}(x,\xi) \end{pmatrix} = \begin{pmatrix} \tilde{p}_{11}(x,\xi) & \tilde{p}_{12}(x,\xi) \\ \tilde{p}_{21}(x,\xi) & \tilde{p}_{22}(x,\xi) \end{pmatrix} \tag{A.14}$$

is the solution of the matrix PDE

$$\widetilde{P}_{\xi}(x,\xi)\boldsymbol{\Lambda} + \boldsymbol{\Lambda}\widetilde{P}_x(x,\xi) - \widetilde{P}(x,\xi)\mathbf{C} = 0 \tag{A.15}$$

with the boundary conditions

$$\tilde{p}_{12}(x,x) = \frac{c_1}{\lambda_1+\lambda_2}, \quad \tilde{p}_{21}(x,x) = \frac{-c_2}{\lambda_1+\lambda_2}, \tag{A.16a}$$

$$k_1\tilde{p}_{21}(0,\xi) - \tilde{p}_{11}(0,\xi) = 0, \quad k_1\tilde{p}_{22}(0,\xi) - \tilde{p}_{12}(0,\xi) = 0. \tag{A.16b}$$

B) The matrix

$$\widetilde{Q}(x,\xi) = \begin{pmatrix} \widetilde{Q}_{(1)}(x,\xi) \\ \widetilde{Q}_{(2)}(x,\xi) \end{pmatrix} = \begin{pmatrix} \tilde{q}_{11}(x,\xi) & \tilde{q}_{12}(x,\xi) \\ \tilde{q}_{21}(x,\xi) & \tilde{q}_{22}(x,\xi) \end{pmatrix} \tag{A.17}$$

is the solution of the matrix PDE

$$\widetilde{Q}_{\xi}(x,\xi)\boldsymbol{\Lambda} + \boldsymbol{\Lambda}\widetilde{Q}_x(x,\xi) + \mathbf{C}\widetilde{Q}(x,\xi) = 0 \tag{A.18}$$

with the boundary conditions

$$\tilde{q}_{12}(x,x) = \frac{c_1}{\lambda_1+\lambda_2}, \quad \tilde{q}_{21}(x,x) = \frac{c_2}{\lambda_1+\lambda_2}, \tag{A.19a}$$

$$k_1\tilde{q}_{21}(0,\xi) - \tilde{q}_{11}(0,\xi) = 0, \quad k_1\tilde{q}_{22}(0,\xi) - \tilde{q}_{12}(0,\xi) = 0. \tag{A.19b}$$

C) Moreover, the internal output injection gains $\upsilon_1(x)$ and $\upsilon_2(x)$ are

$$\upsilon_1(x) = \lambda_1\tilde{q}_{11}(x,1) - \lambda_2\tilde{q}_{12}(x,1)(k_2-k_0), \tag{A.20a}$$

$$\upsilon_2(x) = \lambda_1\tilde{q}_{21}(x,1) - \lambda_2\tilde{q}_{22}(x,1)(k_2-k_0). \tag{A.20b}$$

B Appendix: Proof of Proposition 1

Consider the matrix

$$\widehat{\mathbf{K}} = \begin{pmatrix} 0 & k_1 & 0 & -k_c \\ k_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_1-k_c \\ k_0 & 0 & k_2-k_0 & 0 \end{pmatrix}. \tag{B.1}$$

The goal is to compute

$$\rho_2(\widehat{\mathbf{K}}) = \inf_D \|D\widehat{\mathbf{K}}D^{-1}\|_2 \tag{B.2}$$

where

$$D = \mathrm{diag}\{d_1, d_2, d_3, d_4\} \tag{B.3}$$

is a positive diagonal matrix.

We introduce the notations

$$\delta_1 = \frac{d_1}{d_2}, \delta_2 = \frac{d_1}{d_4}, \delta_3 = \frac{d_3}{d_4}, \tag{B.4}$$

and we compute the matrices $\mathbf{M} = D\widehat{\mathbf{K}}D^{-1}$ and $\mathbf{M}^\mathsf{T}\mathbf{M}$:

$$\mathbf{M} = D\widehat{\mathbf{K}}D^{-1} = \begin{pmatrix} 0 & k_1\delta_1 & 0 & -k_c\delta_2 \\ k_2\delta_1^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & (k_1 - k_c)\delta_3 \\ k_0\delta_2^{-1} & 0 & (k_2 - k_0)\delta_3^{-1} & 0 \end{pmatrix} \tag{B.5}$$

$$\mathbf{M}^\mathsf{T}\mathbf{M} = \begin{pmatrix} k_2^2\delta_1^{-2} + k_0^2\delta_2^{-2} & 0 & k_0(k_2 - k_0)\delta_2^{-1}\delta_3^{-1} & 0 \\ 0 & k_1^2\delta_1^2 & 0 & -k_1k_c\delta_1\delta_2 \\ k_0(k_2 - k_0)\delta_2^{-1}\delta_3^{-1} & 0 & (k_2 - k_0)^2\delta_3^{-2} & 0 \\ 0 & -k_1k_c\delta_1\delta_2 & 0 & k_c^2\delta_2^2 + (k_1 - k_c)^2\delta_3^2 \end{pmatrix}. \tag{B.6}$$

Defining

$$\delta := (\delta_1, \delta_2, \delta_3), \tag{B.7}$$

it can be verified that the eigenvalues of the matrix $\mathbf{M}^\mathsf{T}\mathbf{M}$ are:

$$\mu_1(\delta) = \frac{1}{2}\left[\frac{k_2^2}{\delta_1^2} + \frac{k_0^2}{\delta_2^2} + \frac{(k_2 - k_0)^2}{\delta_3^2} + \sqrt{\left(\frac{k_2^2}{\delta_1^2} + \frac{k_0^2}{\delta_2^2} + \frac{(k_2 - k_0)^2}{\delta_3^2}\right)^2 - 4\frac{k_2^2}{\delta_1^2}\frac{(k_2 - k_0)^2}{\delta_3^2}}\right], \tag{B.8a}$$

$$\mu_2(\delta) = \frac{1}{2}\left[\frac{k_2^2}{\delta_1^2} + \frac{k_0^2}{\delta_2^2} + \frac{(k_2 - k_0)^2}{\delta_3^2} - \sqrt{\left(\frac{k_2^2}{\delta_1^2} + \frac{k_0^2}{\delta_2^2} + \frac{(k_2 - k_0)^2}{\delta_3^2}\right)^2 - 4\frac{k_2^2}{\delta_1^2}\frac{(k_2 - k_0)^2}{\delta_3^2}}\right], \tag{B.8b}$$

$$\mu_3(\delta) = \frac{1}{2}\left[k_1^2\delta_1^2 + k_c^2\delta_2^2 + (k_1 - k_c)^2\delta_3^2 + \sqrt{\left(k_1^2\delta_1^2 + k_c^2\delta_2^2 + (k_1 - k_c)^2\delta_3^2\right)^2 - 4k_1^2(k_1 - k_c)^2\delta_1^2\delta_3^2}\right], \tag{B.8c}$$

$$\mu_4(\delta) = \frac{1}{2}\left[k_1^2\delta_1^2 + k_c^2\delta_2^2 + (k_1 - k_c)^2\delta_3^2 - \sqrt{\left(k_1^2\delta_1^2 + k_c^2\delta_2^2 + (k_1 - k_c)^2\delta_3^2\right)^2 - 4k_1^2(k_1 - k_c)^2\delta_1^2\delta_3^2}\right]. \tag{B.8d}$$

Then, by definition, we have

$$\rho_2(\widehat{\mathbf{K}}) = \inf_D \|D\widehat{\mathbf{K}}D^{-1}\|_2 = \inf_\delta \sqrt{\max\left\{\mu_1(\delta), \mu_3(\delta)\right\}}. \tag{B.9}$$

We first assume that

$$k_1 \neq 0, \quad k_c \neq 0, \quad k_c \neq k_1, \quad k_0 \neq 0, \quad k_0 \neq k_2. \tag{B.10}$$

We shall now prove that the infimum with respect to δ in (B.9) is reached for a value δ^* such that $\mu_1(\delta^*) = \mu_3(\delta^*)$ which is given by

$$\delta_1^* = \sqrt{\left|\frac{k_2}{k_1}\right|}, \quad \delta_2^* = \sqrt{\left|\frac{k_0}{k_c}\right|}, \quad \delta_3^* = \sqrt{\left|\frac{k_2 - k_0}{k_1 - k_c}\right|}. \tag{B.11}$$

It can indeed be verified that

$$\sqrt{\mu_1(\delta^*)} = \sqrt{\mu_3(\delta^*)} =$$

$$\sqrt{\begin{aligned}&\tfrac{1}{2}\Big[|k_1k_2| + |k_0k_c| + |(k_2-k_0)(k_1-k_c)|\Big] \\ &\quad + \tfrac{1}{2}\sqrt{\Big(|k_1k_2| + |k_0k_c| + |(k_2-k_0)(k_1-k_c)|\Big)^2 - 4|k_1k_2||(k_2-k_0)(k_1-k_c)|}\end{aligned}} \tag{B.12}$$

which is the required form for $\rho_2(\widehat{\mathbf{K}})$.

Let us select a diagonal matrix

$$D^* = \mathrm{diag}(d_1^*, d_2^*, d_3^*, d_4^*) \tag{B.13}$$

such that

$$\frac{d_1^*}{d_2^*} = \delta_1^* = \sqrt{\left|\frac{k_2}{k_1}\right|}, \quad \frac{d_1^*}{d_4^*} = \delta_2^* = \sqrt{\left|\frac{k_0}{k_c}\right|}, \quad \frac{d_3^*}{d_4^*} = \delta_3^* = \sqrt{\left|\frac{k_2-k_0}{k_1-k_c}\right|}. \tag{B.14}$$

We compute the matrix

$$\mathbf{M}^* = D^*\widehat{\mathbf{K}}(D^*)^{-1} = \begin{pmatrix} 0 & k_1\sqrt{\left|\frac{k_2}{k_1}\right|} & 0 & -k_c\sqrt{\left|\frac{k_0}{k_c}\right|} \\ k_2\sqrt{\left|\frac{k_1}{k_2}\right|} & 0 & 0 & 0 \\ 0 & 0 & 0 & (k_1-k_c)\sqrt{\left|\frac{k_2-k_0}{k_1-k_c}\right|} \\ k_0\sqrt{\left|\frac{k_c}{k_0}\right|} & 0 & (k_2-k_0)\sqrt{\left|\frac{k_1-k_c}{k_2-k_0}\right|} & 0 \end{pmatrix}. \tag{B.15}$$

We introduce the notations

$$\omega_1 = k_1\sqrt{\left|\frac{k_2}{k_1}\right|}, \quad \omega_2 = k_0\sqrt{\left|\frac{k_c}{k_0}\right|}, \quad \omega_3 = (k_1-k_c)\sqrt{\left|\frac{k_2-k_0}{k_1-k_c}\right|}, \tag{B.16}$$

$$\sigma_1 = \mathrm{sign}(k_1k_2), \quad \sigma_2 = \mathrm{sign}(k_0k_c), \quad \sigma_3 = \mathrm{sign}\Big((k_1-k_c)(k_2-k_0)\Big). \tag{B.17}$$

Using these notations, the matrix $\mathbf{M}^*$ is rewritten

$$\mathbf{M}^* = \begin{pmatrix} 0 & \omega_1 & 0 & -\sigma_2\omega_2 \\ \sigma_1\omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_3 \\ \omega_2 & 0 & \sigma_3\omega_3 & 0 \end{pmatrix} \tag{B.18}$$

and the matrix $\mathbf{Q}^* = \mathbf{M}^{*\mathsf{T}}\mathbf{M}^*$ is

$$\mathbf{Q}^* = \mathbf{M}^{*\mathsf{T}}\mathbf{M}^* = \begin{pmatrix} \omega_1^2+\omega_2^2 & 0 & \sigma_3\omega_2\omega_3 & 0 \\ 0 & \omega_1^2 & 0 & -\sigma_2\omega_1\omega_2 \\ \sigma_3\omega_2\omega_3 & 0 & \omega_3^2 & 0 \\ 0 & -\sigma_2\omega_1\omega_2 & 0 & \omega_2^2+\omega_3^2 \end{pmatrix}. \tag{B.19}$$

We observe that this matrix $\mathbf{Q}^*$ is equivalent by permutation to the following block-diagonal matrix:

$$\begin{pmatrix} \omega_1^2+\omega_2^2 & \sigma_3\omega_2\omega_3 & 0 & 0 \\ \sigma_3\omega_2\omega_3 & \omega_3^2 & 0 & 0 \\ 0 & 0 & \omega_1^2 & -\sigma_2\omega_1\omega_2 \\ 0 & 0 & -\sigma_2\omega_1\omega_2 & \omega_2^2+\omega_3^2 \end{pmatrix}. \tag{B.20}$$

It can be verified that the two blocks have the same characteristic polynomial:

$$\begin{vmatrix} \mu-(\omega_1^2+\omega_2^2) & -\sigma_3\omega_2\omega_3 \\ -\sigma_3\omega_2\omega_3 & \mu-\omega_3^2 \end{vmatrix} = \begin{vmatrix} \mu-\omega_1^2 & \sigma_2\omega_1\omega_2 \\ \sigma_2\omega_1\omega_2 & \mu-(\omega_2^2+\omega_3^2) \end{vmatrix} = (\mu-\omega_1^2)(\mu-\omega_3^2)-\mu\omega_2^2 \tag{B.21}$$

and therefore the same eigenvalues. It follows that the matrix $\mathbf{Q}^*$ has two eigenvalues (each with multiplicity 2)

$$\mu_1^* = \frac{(\omega_1^2+\omega_2^2+\omega_3^2)+\sqrt{(\omega_1^2+\omega_2^2+\omega_3^2)^2-4\omega_1^2\omega_3^2}}{2} \tag{B.22}$$

and

$$\mu_2^* = \frac{(\omega_1^2+\omega_2^2+\omega_3^2)-\sqrt{(\omega_1^2+\omega_2^2+\omega_3^2)^2-4\omega_1^2\omega_3^2}}{2} \tag{B.23}$$

which obviously satisfy

$$\mu_1^* = \mu_1(\delta^*) = \mu_3(\delta^*) \quad \text{and} \quad \mu_2^* = \mu_2(\delta^*) = \mu_4(\delta^*). \tag{B.24}$$

The eigenvectors corresponding to μ_1^* are

$$\boldsymbol{v_1} = \begin{pmatrix} 0 \\ -\sigma_2\omega_1\omega_2 \\ 0 \\ \mu_1^*-\omega_1^2 \end{pmatrix}, \quad \boldsymbol{v_2} = \begin{pmatrix} \mu_1^*-\omega_3^2 \\ 0 \\ \sigma_3\omega_2\omega_3 \\ 0 \end{pmatrix}. \tag{B.25}$$

We compute $\mathbf{M}^*\boldsymbol{v_1}$ and $\mathbf{M}^*\boldsymbol{v_2}$:

$$\mathbf{M}^*\boldsymbol{v_1} = \begin{pmatrix} -\sigma_2\omega_2\mu_1^* \\ 0 \\ \omega_3(\mu_1^*-\omega_1^2) \\ 0 \end{pmatrix}, \quad \mathbf{M}^*\boldsymbol{v_2} = \begin{pmatrix} 0 \\ \sigma_1\omega_1(\mu_1^*-\omega_3^2) \\ 0 \\ \omega_2\mu_1^* \end{pmatrix}. \tag{B.26}$$

We define the vector Υ:

$$\Upsilon = \boldsymbol{v_1} + \vartheta\, i\, \boldsymbol{v_2} \tag{B.27}$$

where ϑ is a constant to be determined and $i=\sqrt{-1}$. We have

$$\Upsilon = \boldsymbol{v_1} + \vartheta\, i\, \boldsymbol{v_2} = \begin{pmatrix} \vartheta\, i(\mu_1^*-\omega_3^2) \\ -\sigma_2\omega_1\omega_2 \\ \vartheta\, i\, \sigma_3\omega_2\omega_3 \\ \mu_1^*-\omega_1^2 \end{pmatrix} \quad \text{and} \quad \mathbf{M}^*\Upsilon = \begin{pmatrix} -\sigma_2\mu_1^*\omega_2 \\ \vartheta\, i\, \sigma_1\omega_1(\mu_1^*-\omega_3^2) \\ \omega_3(\mu_1^*-\omega_1^2) \\ \vartheta\, i\mu_1^*\omega_2 \end{pmatrix}. \tag{B.28}$$

We define the matrix E:

$$E = \mathrm{diag}\{e_1,e_2,e_3,e_4\} \quad \text{with} \quad e_\ell \in \{1,-1,i,-i\}, \quad \ell=1,2,3,4. \tag{B.29}$$

We compute

$$\mathbf{M}^*\Upsilon - \sqrt{\mu_1^*}\,E\Upsilon = \begin{pmatrix} -\sigma_2\mu_1^*\omega_2 - \sqrt{\mu_1^*}e_1\vartheta\, i(\mu_1^* - \omega_3^2) \\ \vartheta\, i\,\sigma_1\omega_1(\mu_1^* - \omega_3^2) + \sqrt{\mu_1^*}e_2\sigma_2\omega_1\omega_2 \\ \omega_3(\mu_1^* - \omega_1^2) - \sqrt{\mu_1^*}e_3\vartheta\, i\,\sigma_3\omega_2\omega_3 \\ \vartheta\, i\mu_1^*\omega_2 - \sqrt{\mu_1^*}e_4(\mu_1^* - \omega_1^2) \end{pmatrix}. \tag{B.30}$$

We shall show that the parameters e_ℓ and ϑ can be selected such that $\mathbf{M}^*\Upsilon - \sqrt{\mu_1^*}\,E\Upsilon = 0$. First the parameters e_ℓ are selected as follows:

$$e_1 = \sigma_2\, i, \quad e_2 = -\frac{\sigma_1}{\sigma_2}\, i, \quad e_3 = -\frac{i}{\sigma_3}, \quad e_4 = i. \tag{B.31}$$

With these values of e_ℓ, we have

$$\mathbf{M}^*\Upsilon - \sqrt{\mu_1^*}\,E\Upsilon = \begin{pmatrix} -\sigma_2\sqrt{\mu_1^*}\Big[\sqrt{\mu_1^*}\omega_2 - \vartheta(\mu_1^* - \omega_3^2)\Big] \\ \sigma_1\, i\,\omega_1\Big[\vartheta(\mu_1^* - \omega_3^2) - \sqrt{\mu_1^*}\omega_2\Big] \\ \omega_3\Big[(\mu_1^* - \omega_1^2) - \vartheta\sqrt{\mu_1^*}\omega_2\Big] \\ i\sqrt{\mu_1^*}\Big[\vartheta\sqrt{\mu_1^*}\omega_2 - (\mu_1^* - \omega_1^2)\Big] \end{pmatrix}. \tag{B.32}$$

The parameter ϑ is then selected as follows:

$$\vartheta = \frac{\omega_2\sqrt{\mu_1^*}}{\mu_1^* - \omega_3^2} = \frac{\mu_1^* - \omega_1^2}{\omega_2\sqrt{\mu_1^*}}. \tag{B.33}$$

Remark that the second equality directly follows from the characteristic equation (B.21). Then, with this value of ϑ, it is readily verified that all components of the vector $M^*\Upsilon - \sqrt{\mu_1^*}E\Upsilon = 0$.

Let us now recall the definitions of the maps $\bar{\rho}$ and ρ_2:

$$\bar{\rho}(\widehat{\mathbf{K}}) := \max\Big\{\rho\Big(\mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4})\widehat{\mathbf{K}}\Big)\,;\theta_i \in \mathbb{R}\Big\}, \quad \rho_2(\widehat{\mathbf{K}}) := \inf_D \|D\widehat{\mathbf{K}}D^{-1}\|_2 \tag{B.34}$$

where $\rho(\cdot)$ denotes the spectral radius and D belongs to the set of positive real diagonal matrices.

From these definitions and Theorem 3.11 in [41], we know that

$$\bar{\rho}(\widehat{\mathbf{K}}) \leqslant \rho_2(\widehat{\mathbf{K}}) \leqslant \sqrt{\mu_1^*}. \tag{B.35}$$

Moreover, we have

$$\mathbf{M}^*\Upsilon - \sqrt{\mu_1^*}\,E\Upsilon = 0 \quad \Longrightarrow \quad E^{-1}\mathbf{M}^*\Upsilon - \sqrt{\mu_1^*}\,\Upsilon = 0 \tag{B.36}$$

which implies that

$$\sqrt{\mu_1^*} \leqslant \rho(E^{-1}\mathbf{M}^*) \leqslant \bar{\rho}(\mathbf{M}^*) \leqslant \rho_2(\mathbf{M}^*) \tag{B.37}$$

because

$$E^{-1} = \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4}) \quad \text{with} \quad \theta_1 = -\frac{1}{\sigma_2}\frac{\pi}{2},\ \theta_2 = \frac{\sigma_2}{\sigma_1}\frac{\pi}{2},\ \theta_3 = \sigma_3\frac{\pi}{2},\ \theta_4 = -\frac{\pi}{2}. \tag{B.38}$$

We define the diagonal matrix Δ such that $D = \Delta D^*$. Then, we have

$$\rho_2(\widehat{\mathbf{K}}) = \inf_D \|D\widehat{\mathbf{K}}D^{-1}\|_2 = \inf_\Delta \|\Delta D^*\widehat{\mathbf{K}}(D^*\Delta)^{-1}\|_2 = \inf_\Delta \|\Delta\mathbf{M}^*\Delta^{-1}\|_2 = \rho_2(\mathbf{M}^*). \tag{B.39}$$

Finally, combining inequalities (B.35) and (B.37), this implies that we have

$$\rho_2(\widehat{\mathbf{K}}) \leqslant \sqrt{\mu_1^*} \leqslant \rho_2(\mathbf{M}^*) = \rho_2(\widehat{\mathbf{K}}) \tag{B.40}$$

and, therefore,

$$\rho_2(\widehat{\mathbf{K}}) = \sqrt{\mu_1^*}. \tag{B.41}$$

Using (B.12) and (B.24), this completes the proof of Proposition 1 under assumption (B.10). The general case follows from the continuity of $\rho_2(\widehat{\mathbf{K}})$ with respect to $\widehat{\mathbf{K}}$, a continuity which is proved in [47, Proposition A.2], and the density in $\mathbb{R}^4$ of the $(k_0, k_1, k_2, k_c)^\intercal$ satisfying (B.10). □

Remark 5. The use of $\boldsymbol{v_1}$, $\boldsymbol{v_2}$ and Υ to prove Proposition 1 is inspired from [47, Appendix B]. Let us take the opportunity of this remark to mention two typos in [47, Appendix B]: in this article, $E_{ij} := Y_{ij}^2 - X_{ij}^2$ in (B.10) has to be replaced by $E_{ij} := Y_{ij}^2 - \rho_1(K)^2 X_{ij}^2$ and $A_j := \sqrt{t_j} X_j$ in (B.12) has to be replaced by $A_j := \sqrt{t_j}\rho_1(K) X_j$.

C Appendix: Well-posedness of the system Σ_{cl}

The well-posedness follows from the application of Lumer-Phillips theorem. We define the operator $\mathcal{A}_{cl} : \mathrm{D}(\mathcal{A}_{cl}) \subset L^2((0,1);\mathbb{R}^4) \to L^2((0,1);\mathbb{R}^4)$ by

$$\begin{gathered} \mathcal{A}_{cl} = -\begin{pmatrix} \Lambda + \mathcal{E} & 0 \\ 0 & \Lambda \end{pmatrix} \partial_x + \begin{pmatrix} \mathbf{h}_{1\varepsilon} & 0 \\ \mathbf{h}_{1\varepsilon} - \widetilde{\mathbf{h}}_{2\varepsilon} & 0 \end{pmatrix}, \\ \mathrm{D}(\mathcal{A}_{cl}) = \left\{ (z, \tilde{z}) \in H^1((0,1);\mathbb{R}^4) \mid \text{(6.1c) holds} \right\}. \end{gathered} \tag{C.1}$$

With this definition, the system Σ_{cl} is simply written as the evolution system

$$\partial_t (z, \hat{z})^T = \mathcal{A}_{cl} (z, \hat{z})^T. \tag{C.2}$$

To apply Lumer-Phillips theorem and show that this system is well-posed in the space $L^2((0,1);\mathbb{R}^4)$ it is sufficient to show that $\mathcal{A}_{cl}$ is quasi-dissipative and that there exists $\lambda_0 \in \mathbb{R}$ such that $(\mathcal{A}_{cl} - \lambda_0 \mathbf{I_d})$ is surjective (from $\mathrm{D}(\mathcal{A}_{cl})$ to $L^2((0,1);\mathbb{R}^4)$). The quasi-dissipativity follows directly from the existence of the Lyapunov function given in (6.55) (note that this Lyapunov function is equivalent to the square of the L^2 norm of $(z, \hat{z})$), and only surjectivity remains to be shown for some $\lambda_0 \in \mathbb{R}$. Let $\bar{f} \in L^2((0,1);\mathbb{R}^4)$, we want to show that there exists $\bar{g} \in \mathrm{D}(\mathcal{A}_{cl})$ such that

$$(\mathcal{A}_{cl} - \lambda_0 \mathbf{I_d})\bar{g} = \bar{f}. \tag{C.3}$$

In fact, such a $\bar{g}$ can be found with $\lambda_0 = 0$ using the computations of the Steps 1-3 in Section 6. Indeed, define

$$\bar{g} = \mathcal{T}\bar{f} = (\mathbf{I_d} - \mathcal{A}_0)^{-1}\left[\mathcal{B}_0(\bar{f}) + \mathcal{C}\right], \tag{C.4}$$

where

$$\mathcal{C} = \mathcal{D}_{\boldsymbol{\varepsilon}}^{-1}\mathcal{J}(\mathbf{I_d} - \mathcal{A}_0)^{-1}\mathcal{B}_0(\bar{f}). \tag{C.5}$$

Then, from (6.5) and (6.6), we have

$$\mathcal{A}_{cl}\bar{g} = \bar{f}. \tag{C.6}$$

It remains to show that $\bar{g} \in \mathrm{D}(\mathcal{A}_{cl})$. This is directly a consequence of the computation of Step 1 and 3 in Section 6. Indeed, the fact that $g \in H^1$ follows from the fact that $\mathcal{B}_0(L^2((0,1);\mathbb{R}^4)) \subset H^1((0,1);\mathbb{R}^4)$ and (6.11). With (C.4), the boundary conditions (6.1c) for g hold if and only if (see (6.17) and (6.19) together with (6.15) and (6.26))

$$\mathcal{D}_{\boldsymbol{\varepsilon}}\mathcal{C} = \mathcal{J}(\mathbf{I_d} - \mathcal{A}_0)^{-1}\mathcal{B}_0(\bar{f}) \tag{C.7}$$

which is exactly given by (C.5). The well-posedness in $H^1((0,1);\mathbb{R}^4)$ can be shown in the same way.

D Appendix: Lyapunov function independent of the perturbations

.
We define

$$\mathcal{T}^0(z,\hat{z}) = \begin{pmatrix} \mathcal{T}^0_{(1)}(z,\hat{z}) \\ \mathcal{T}^0_{(2)}(z,\hat{z}) \end{pmatrix} = \mathcal{T}\Big|_{(\varepsilon=0)}(z,\hat{z}) \tag{D.1}$$

where, in this Appendix, the notation ε is a shorthand for $(\varepsilon_1, \varepsilon_2)$.

Proposition 6. The function

$$\mathcal{V}(z,\hat{z}) = \mathbf{W}_1(\mathcal{T}^0_{(1)}(z,\hat{z})) + M\mathbf{W}_2(\mathcal{T}^0_{(1)}(z,\hat{z}) - \mathcal{T}^0_{(2)}(z,\hat{z})) + \mathbf{W}_3(z) + \mathbf{W}_4(\hat{z}), \tag{D.2}$$

is a Lyapunov function (for the L^2 norm) of the system (6.1).

Proof. We intend to proceed as in the proof of Theorem 5. Let $(z,\hat{z}) \in C^0([0,+\infty); L^2)$ be a solution of the system (6.1). We begin by studying the dynamics of $\mathcal{T}^0(z,\hat{z})$ where, by definition,

$$\mathcal{T}^0 = (\mathbf{I_d} + \mathcal{D}_0^{-1}\mathcal{J})\mathcal{B}_0\Big|_{(\varepsilon=0)}. \tag{D.3}$$

We will first prove the following lemma.

Lemma 6. For any $|\varepsilon| \in (0,\varepsilon_0)$ and for any solution $(z,\hat{z})$ of (6.1), there exists $\varepsilon_0 > 0$ such that

$$\begin{aligned} \partial_t(\mathcal{T}^0(z(t,\cdot),\hat{z}(t,\cdot))) + \begin{pmatrix} \mathbf{\Lambda} & 0 \\ 0 & \mathbf{\Lambda} \end{pmatrix} \partial_x(\mathcal{T}^0(z(t,\cdot),\hat{z}(t,\cdot))) \\ = O_{L^2}(\varepsilon; (z,\hat{z})) + O(\varepsilon; (z(t,0), z(t,1))), \end{aligned} \tag{D.4}$$

where $O_{L^2}(\varepsilon;(z,\hat{z}))$ and $O(\varepsilon;(z(t,0),z(t,1)))$ refer to terms that are bounded respectively in L^2 and L^∞ norms and tend to 0 when $\varepsilon \to 0$, that is there exists $C(\varepsilon) > 0$ such that for any $y, \hat{y} \in L^2$

$$\|O_{L^2}(\varepsilon;(z,\hat{z}))\|_{L^2} \le C(\varepsilon)\|y,\hat{y}\|_{L^2}, \quad \sup_{x\in[0,1]} |O(\varepsilon;(z(t,0),z(t,1)))| \le C(\varepsilon)|z(t,1)|, \tag{D.5}$$

and $\lim_{\varepsilon\to 0} C(\varepsilon) = 0$. Moreover, $\mathcal{T}^0(z(t,\cdot),\hat{z}(t,\cdot))$ satisfies the boundary conditions (6.1c).

Proof. One has

$$\partial_t(\mathcal{T}^0(z(t,\cdot),\hat{z}(t,\cdot))) = (Id + D_0^{-1}\mathcal{J})\mathcal{B}_0|_{(\varepsilon=0)}(\partial_t(z,\hat{z}))(t,\cdot). \tag{D.6}$$

Using the definition (6.9) of $\mathcal{B}_0$ we have

$$\overline{\mathcal{B}}_0 := \mathcal{B}_0|_{(\varepsilon=0)}(\partial_t(z,\hat{z}))(t,x) = \begin{pmatrix} \frac{(\lambda_1+\varepsilon_1)}{\lambda_1}\int_0^x \partial_s z_1(t,s)ds - \lambda_1^{-1}\int_0^x h_{11\varepsilon}(z(t,\cdot)(s)ds \\ -\frac{(\lambda_2+\varepsilon_2)}{\lambda_2}\int_x^1 \partial_s z_2(t,s)ds - \lambda_2^{-1}\int_x^1 h_{12\varepsilon}(z(t,\cdot)(s)ds \\ \int_0^x \partial_s \hat{z}_1(t,s)ds - \lambda_1^{-1}\int_0^x (h_{11\varepsilon}(z(t,\cdot)(s) - \tilde{h}_{21\varepsilon}(z(t,\cdot)(s))ds \\ -\int_x^1 \partial_s \hat{z}_2(t,s)ds - \lambda_2^{-1}\int_x^1 (h_{12\varepsilon}(z(t,\cdot)(s) - \tilde{h}_{22\varepsilon}(z(t,\cdot)(s))ds \end{pmatrix} \tag{D.7}$$

As a consequence, from the definition (6.26) of $\mathcal{J}$, we have

$$\begin{aligned}\mathcal{J}\overline{\mathcal{B}}_0 = \widehat{\mathbf{K}} &\begin{pmatrix} z_1(t,1) - z_1(t,0) - \lambda_1^{-1}\int_0^1 h_{11\varepsilon}(z(t,\cdot))(s)ds \\ z_2(t,0) - z_2(t,1) - \lambda_2^{-1}\int_0^1 h_{12\varepsilon}(z(t,\cdot))(s)ds \\ \hat{z}_1(t,1) - \hat{z}_1(t,0) - \lambda_1^{-1}\int_0^1 (h_{11\varepsilon}(z(t,\cdot)(s) - \tilde{h}_{21\varepsilon}(z(t,\cdot)(s))ds \\ \hat{z}_2(t,0) - \hat{z}_2(t,1) - \lambda_2^{-1}\int_0^1 (h_{12\varepsilon}(z(t,\cdot)(s) - \tilde{h}_{22\varepsilon}(z(t,\cdot)(s))ds \end{pmatrix} \\ &+ \widehat{\mathbf{K}} \begin{pmatrix} \frac{\varepsilon_1}{\lambda_1}(z_1(t,1) - z_1(t,0)) \\ \frac{\varepsilon_2}{\lambda_2}(z_2(t,0) - z_2(t,1)) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \int_0^1 f(\zeta) \begin{pmatrix} \overline{\mathcal{B}}_{01} - \overline{\mathcal{B}}_{03} \\ \overline{\mathcal{B}}_{02} - \overline{\mathcal{B}}_{04} \end{pmatrix} d\zeta, \end{aligned} \tag{D.8}$$

where $\overline{\mathcal{B}}_{0j}$ ($j \in \{1,2,3,4\}$) denote the components of $\overline{\mathcal{B}}_0$. Therefore

$$\begin{aligned}\mathcal{J}\overline{\mathcal{B}}_0 = \ \widehat{\mathbf{K}} &\begin{pmatrix} z_1(t,1) - z_1(t,0) \\ z_2(t,0) - z_2(t,1) \\ \hat{z}_1(t,1) - \hat{z}_1(t,0) \\ \hat{z}_2(t,0) - \hat{z}_2(t,1) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \int_0^1 f(\zeta)\mathcal{F} \begin{pmatrix} z_1(t,x) - z_1(t,0) \\ z_2(t,x) - z_2(t,1) \\ \hat{z}_1(t,x) - \hat{z}_1(t,0) \\ \hat{z}_2(t,x) - \hat{z}_2(t,1) \end{pmatrix} d\zeta \\ &+ O_{L^2}(\varepsilon; (z,\hat{z})) + O(\varepsilon; (z(t,0), z(t,1))) \end{aligned} \tag{D.9}$$

where $\mathcal{F}$ is defined in (6.18). Using the boundary conditions (6.1c) we get

$$\begin{aligned}\mathcal{J}\overline{\mathcal{B}}_0 = (\mathbf{I_d} - \widehat{\mathbf{K}}) &\begin{pmatrix} z_1(t,0) \\ z_2(t,1) \\ \hat{z}_1(t,0) \\ \hat{z}_2(t,1) \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \int_0^1 f(\zeta)\mathcal{F} \begin{pmatrix} z_1(t,0) \\ z_2(t,1) \\ \hat{z}_1(t,0) \\ \hat{z}_2(t,1) \end{pmatrix} d\zeta \\ &+ O_{L^2}(\varepsilon; (z,\hat{z})) + O(\varepsilon; (z(t,0), z(t,1))), \end{aligned} \tag{D.10}$$

From the definition (6.22) of $\mathcal{D}_0$ we have for any $(\varphi, \hat{\varphi})^{\mathsf{T}} \in \mathbb{R}^4$

$$\mathcal{D}_0(\varphi,\hat{\varphi})^{\mathsf{T}} = (\mathbf{I_d} - \widehat{\mathbf{K}}) \begin{pmatrix} \varphi \\ \hat{\varphi} \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \int_0^1 f(\zeta)\mathcal{F} \begin{pmatrix} \varphi \\ \hat{\varphi} \end{pmatrix} d\zeta. \tag{D.11}$$

Using this together with (D.9) we have

$$\mathcal{J}\overline{\mathcal{B}}_0 = \mathcal{D}_0(z_1(t,0), z_2(t,1), \hat{z}_1(t,0), \hat{z}_2(1))^{\mathsf{T}} + O_{L^2}(\varepsilon; (z,\hat{z})) + O(\varepsilon; (z(t,0), z(t,1))), \tag{D.12}$$

Therefore, using the definition of $\overline{\mathcal{B}}_0$ given by (D.7),

$$\begin{aligned}(\mathbf{I_d} + \mathcal{D}_0^{-1}\mathcal{J})\overline{\mathcal{B}}_0 &= \overline{\mathcal{B}}_0 + (z_1(t,0), z_2(t,1), \hat{z}_1(t,0), \hat{z}_2(t,1))^{\mathsf{T}} + O_{L^2}(\varepsilon; (z,\hat{z})) + O(\varepsilon; (z(t,0), z(t,1))) \\ &= (z(t,x), \hat{z}(t,x))^{\mathsf{T}} + O_{L^2}(\varepsilon; (z,\hat{z})) + O(\varepsilon; (z(t,0), z(t,1))). \end{aligned} \tag{D.13}$$

Using (D.6), (D.7) we have

$$\partial_t(\mathcal{T}^0(z(t,\cdot), \hat{z}(t,\cdot))) = (z(t,x), \hat{z}(t,x))^{\mathsf{T}} + O_{L^2}(\varepsilon; (z,\hat{z})) + O(\varepsilon; (z(t,0), z(t,1))). \tag{D.14}$$

Let us now look at $\partial_x(\mathcal{T}^0(z(t,\cdot),\hat{z}(t,\cdot)))$:

$$\partial_x(\mathcal{T}^0(z(t,\cdot),\hat{z}(t,\cdot))) = \partial_x(\mathcal{B}_0|_{\varepsilon=0}(z(t,\cdot),\hat{z}(t,\cdot))) = -\begin{pmatrix}\mathbf{\Lambda}^{-1} & 0\\ 0 & \mathbf{\Lambda}^{-1}\end{pmatrix}(z(t,\cdot),\hat{z}(t,\cdot))^{\mathsf{T}} \tag{D.15}$$

where we use in the first equality that $\mathcal{D}_0^{-1}\mathcal{J}\in\mathcal{L}(H^1;\mathbb{R}^4)$. Overall, combining with (D.14), we obtain

$$\partial_t(\mathcal{T}^0(z(t,\cdot),\hat{z}(t,\cdot))) + \begin{pmatrix}\mathbf{\Lambda} & 0\\ 0 & \mathbf{\Lambda}\end{pmatrix}\partial_x(\mathcal{T}^0(z(t,\cdot),\hat{z}(t,\cdot))) = O_{L^2}(\varepsilon;(z,\hat{z})) + O(\varepsilon;(z(t,0),z(t,1))). \tag{D.16}$$

As expected, the dynamics satisfied by $\mathcal{T}^0(z(t,\cdot),\hat{z}(t,\cdot))$ are close to the dynamics of the system up to terms that are sufficiently regular and small.

Let us now look at the boundary conditions satisfied by $\mathcal{T}^0(z(t,\cdot),\hat{z}(t,\cdot))$. We are going to show that $\mathcal{T}^0(z(t,\cdot),\hat{z}(t,\cdot))$ satisfies the boundary conditions (6.1c), or equivalently that

$$\mathcal{F}_1\mathcal{T}^0(z(t,\cdot),\hat{z}(t,\cdot)) - \mathcal{J}\mathcal{T}^0(z(t,\cdot),\hat{z}(t,\cdot)) = 0, \tag{D.17}$$

where $\mathcal{F}_1$ is defined in (6.18). First note that, by definition (6.9) of $\mathcal{B}_0$, we have

$$\mathcal{F}_1(\mathcal{B}_0|_{\varepsilon=0}(z,\hat{z})) = (0,0,0,0)^T. \tag{D.18}$$

Therefore

$$\mathcal{F}_1((\mathbf{I_d}+\mathcal{D}_0^{-1}\mathcal{J})\mathcal{B}_0|_{\varepsilon=0}(z,\hat{z})) = \mathcal{F}_1(\mathcal{D}_0^{-1}J\mathcal{B}_0|_{\varepsilon=0}(z,\hat{z})) = \mathcal{D}_0^{-1}J\mathcal{B}_0|_{\varepsilon=0}(z,\hat{z}), \tag{D.19}$$

where we used that $\mathcal{D}_0^{-1}$ is valued in $\mathbb{R}^4$ (which can be seen as constant functions) and that for constant functions $(\varphi,\hat{\varphi})$, $\mathcal{F}_1(\varphi,\hat{\varphi}) = (\varphi,\hat{\varphi})$. Then,

$$\mathcal{J}(\mathbf{I_d}+\mathcal{D}_0^{-1}\mathcal{J})\mathcal{B}_0|_{\varepsilon=0}(z,\hat{z}) = \mathcal{J}\mathcal{B}_0|_{\varepsilon=0}(z,\hat{z}) + \mathcal{J}\mathcal{D}_0^{-1}\mathcal{J}\mathcal{B}_0|_{\varepsilon=0}(z,\hat{z}). \tag{D.20}$$

The key is to observe, again, that $\mathcal{D}_0^{-1}$ is valued in $\mathbb{R}^4$ and that for a constant fonction y

$$\mathcal{J}y = y - \mathcal{D}_0 y. \tag{D.21}$$

As a consequence, from (D.19)–(D.21), we get

$$\begin{aligned}&\mathcal{F}_1\mathcal{T}^0(z,\hat{z}) - \mathcal{J}\mathcal{T}^0(z,\hat{z})\\ &\quad= \mathcal{D}_0^{-1}\mathcal{J}\mathcal{B}_0|_{\varepsilon=0}(z,\hat{z}) - \mathcal{J}\mathcal{B}_0|_{\varepsilon=0}(z,\hat{z}) - \mathcal{D}_0^{-1}\mathcal{J}\mathcal{B}_0|_{\varepsilon=0}(z,\hat{z}) + \mathcal{J}\mathcal{B}_0|_{\varepsilon=0}(z,\hat{z}) = 0.\end{aligned} \tag{D.22}$$

Hence $\mathcal{T}^0(z(t,\cdot),\hat{z}(t,\cdot))$ satisfies the boundary conditions (6.1c) and this completes the proof of Lemma 6. □

The proof of Proposition 6 follows from Lemma 6. Let us set $\mathcal{V}_0 := \mathbf{W}_1(Z(t,\cdot)) + M\mathbf{W}_2(\hat{Z}(t,\cdot) - Z(t,\cdot))$, $(Z,\hat{Z}) := \mathcal{T}^0(z(t,\cdot),\hat{z}(t,\cdot))$. By performing exactly the same calculation as in Section 5 for $\mathbf{V}_0$ with $(Z,\hat{Z}-Z)$ instead of $(z,\tilde{z})$, we obtain overall

$$\begin{aligned}\frac{d\mathcal{V}_0}{dt} \le -\mathcal{X}_0^T(t)F\mathcal{X}_0(t) + \lambda_1\bar{a}_1\Big[\boldsymbol{\theta}^2(\hat{Z}-Z) + (2(k_1-k_c)z_2(t,0) + 2k_c\tilde{z}_2(t,0))\boldsymbol{\theta}(\tilde{z}(t,\cdot))\Big]&\\ +\tilde{\mathcal{I}}_1 + M\tilde{\mathcal{I}}_2 + C(\varepsilon)\|Z,\hat{Z}\|_{L^2}(\|z\|_{L^2} + |z(t,0),z(t,1)|),&\end{aligned} \tag{D.23}$$

where

- $C(\varepsilon)$ does not depend on z and $C(\varepsilon)\to 0$ when $\varepsilon\to 0$;

- $\tilde{\mathcal{I}}_1$ and $\tilde{\mathcal{I}}_2$ are the same as $\mathcal{I}_1$ and $\mathcal{I}_2$ but with $(Z, \hat{Z}-Z)$ instead of $(z, \tilde{z})$ and $(z, \hat{z})$ instead of $(s, \hat{s})$ and $\varepsilon = 0$;
- $\mathcal{X}_0$ is the same as $\mathbf{X}_0$ but with $(Z, \hat{Z}-Z)$ instead of $(z, \tilde{z})$.

We can now perform all the other computations of Section 5 identically with $(Z, \hat{Z}-Z)$ instead of $(z, \tilde{z})$ and $(z, \hat{z})$ instead of $(s, \hat{s})$ (note that $(z, \hat{z})$ satisfy the same equations as $(s, \hat{s})$). We finally obtain that there exists $\varepsilon^* > 0$ and $\nu > 0$ such that for any $\varepsilon = (\varepsilon_1, \varepsilon_2)$ with $|\varepsilon| < \varepsilon^*$

$$\frac{d\mathcal{V}(z, \hat{z})}{dt} \leq -\nu \mathcal{V}(z, \hat{z}). \tag{D.24}$$

Note that $\mathcal{V}$ is equivalent to the square of the L^2 norm. Indeed, $\mathbf{W}_1$ and $\mathbf{W}_2$ are positive, hence there exists $c > 0$ such that for any $(z, \hat{z}) \in L^2$,

$$\mathcal{V}(z, \hat{z}) \geq c\|z, \hat{z}\|_{L^2}^2, \tag{D.25}$$

and looking at the definition of $\mathcal{T}^0$ and $\mathcal{B}_0$ and using the equivalence of $\mathbf{W}_1 + M\mathbf{W}_2$ with the square of the L^2 norm, there exists C such that for any $(z, \hat{z}) \in L^2$,

$$\mathcal{V}(z, \hat{z}) \leq C\|z, \hat{z}\|_{L^2}^2. \tag{D.26}$$

This concludes the proof of Proposition 6. □

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