Global boundary stabilization of 1d systems of scalar conservation laws

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Abstract

We study a system of several one-dimensional scalar conservation laws coupled through boundary feedback conditions that combine physical boundary constraints with static feedback control laws. Our first contribution establishes the well-posedness of the system in the space of L^∞ entropy solutions. Our second contribution provides a set of sufficient dissipative conditions on the boundary coupling that ensure global exponential stability in L^1 and L^∞ norms.
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Contents

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1 Introduction

Boundary feedback stabilization of hyperbolic balance laws is a classical theme in control theory, motivated by applications where actuation and sensing are naturally available at the boundaries: open-channel flow, gas pipelines, traffic flow, and more generally transport phenomena on networks. In the smooth (classical) regime, a large body of work relates exponential stability to *dissipative* boundary conditions, typically expressed through a contraction property of the boundary map in suitable norms. Reference can be made to the pioneering work of Greenberg–Li [19], Qin [29], Zhao [32], and Li [25] as well as to the modern synthesis and extensions in [11, 4, 20]. In parallel, an approach through time-delay systems allows for sharp dissipativity conditions in $W^{2,p}$-norm [15].

In these works the solution of the hyperbolic balance laws is required to be classical and the stability is usually studied in either C^1 or H^2 norm. In many applications, however, discontinuities may occur in finite time, and the relevant notion of solution is that of *entropy solutions*. For scalar conservation laws, entropy well-posedness in $\mathbb{R}$ goes back to Kružkov [23], and the initial–boundary value problem requires a careful entropy formulation of boundary conditions. The Bardos–Le Roux–Nédélec (BLN) theory [2] provides the canonical admissibility condition at the boundary; alternative and complementary viewpoints include the viscosity/Riemann-problem approach of Dubois–LeFloch [17] and subsequent developments in the general boundary-value framework [1]. A crucial technical ingredient to study boundary feedback stabilization is the existence of (strong) boundary traces for bounded entropy solutions: this is by now well established in broad generality, notably through the works of Kwon–Vasseur and Panov [24, 27] and, for bounded domains with Dirichlet-type data, through the analysis of Coclite–Karlsen–Kwon [9]. We also refer to standard monographs for background on hyperbolic conservation laws, BV estimates, and front-tracking/semigroup methods; see e.g. [8, 22].

On the control side, stability and boundary stabilization results for non-classical solutions are comparatively more recent. For scalar laws, we can mention the asymptotic stabilization of entropy solutions (in the case of a convex flux) in [28] and [7], and the stabilization of a shock steady-state for the Burgers equation in [6]. For systems of scalar conservation laws coupled by boundary interconnections, the semi-global exponential stability in BV norm using saturated controls is obtained in [18]. For 2×2 systems of conservation laws, the local stabilization in BV norms is addressed in [13] while the local stabilization of H^2-solutions with a single shock is addressed in [5].

In the present paper, we consider the following *system of one-dimensional scalar conservation laws*

$$u_t + (f(u))_x = 0, \qquad t \in (0, +\infty),\ x \in (0, 1), \tag{1.1}$$

$$u(t, 0) = G(u(t, 1)), \tag{1.2}$$

where $u = (u_1, \dots, u_n)^\top$, the flux f belongs to $C^2(\mathbb{R}^n; \mathbb{R}^n)$, and the map $G : \mathbb{R}^n \to \mathbb{R}^n$ is globally Lipschitz with $G(0) = 0$.

We assume that the flux is diagonal i.e. $f(u) = (f_1(u_1), \dots, f_n(u_n))^\top$, which implies that (1.1) represents n scalar conservation laws

$$\partial_t u_i + \partial_x f_i(u_i) = 0, \quad i \in \{1, \dots, n\} \tag{1.3}$$

which are decoupled on the interior spatial domain $x \in (0,1)$ but coupled through the boundary conditions (1.2). We also assume that the characteristic velocities are strictly positive:

$$\exists\, a > 0 \quad \text{such that} \quad f_i'(s) \geq a, \quad \forall s \in \mathbb{R},\ \forall i \in \{1, ..., n\}. \tag{1.4}$$

This assumption is adopted throughout our article and, in particular, in each of the theorems even though it is not explicitly mentioned.

The coupling of the n scalar conservation laws is induced by the boundary condition (1.2) with the function G representing a combination of physical boundary constraints and potential static feedback control laws. Such couplings arise naturally in networked transport models as shown for instance in [3] and [4, Section 4.2] with an example of ramp-metering control in road traffic networks. It should also be noted that the positivity condition (1.4) on the direction of the characteristics allows to consider that the system (1.1)–(1.2) is a closed-loop interconnection of two causal input-output systems as represented in Figure 1.

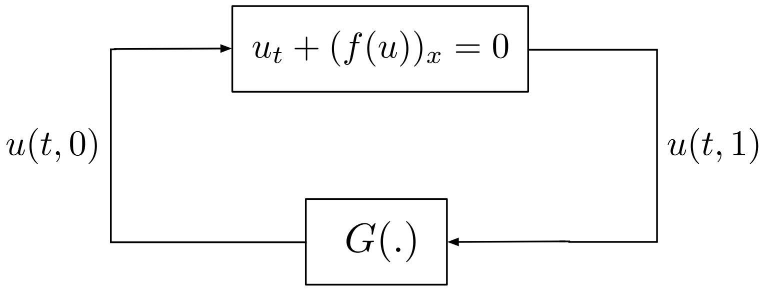

Figure 1: A closed-loop interconnection of two causal input-output systems.

Well-posedness with a nonlocal feedback. Our first contribution (Theorem 2.1) establishes global well-posedness of the system (1.1)–(1.2), under a global Lipschitz assumption on f: it is shown that the system has a unique entropy solution with $u(t,0)$ and $u(t,1)$ denoting, respectively, the incoming and outgoing strong boundary traces of the solution that are shown to exist. The main difficulty to prove the theorem is that the boundary condition (1.2) is non-local in space. However, since scalar conservation laws are delay-type systems, the outgoing trace $u(t,1)$ is independent, over sufficiently small time intervals, of the most recent values of the incoming signal $u(t,0)$. This allows to construct an iterative approach for proving the theorem where, for each successive time slab, the outgoing signal $u(t,1)$ is first computed as the output of an open-loop system which is known to be well-posed (in the BLN framework [2, 9, 1] together with strong traces [24, 27, 9]), and then injected in the feedback loop through the function G to initiate the next step. In particular, with this approach, the non local boundary condition (1.2) can be enforced without any smallness or contraction argument in the entropy setting.

Global exponential stability in L^1 and L^∞. Our second contribution is a set of sufficient conditions on G guaranteeing global exponential stability in the L^1 and L^∞ norms respectively. Note that global exponential stability is usually impossible when using norms which are too regular (such as C^1 or H^2 norms). For this reason, almost all existing results deal with *local* exponential stability [11, 4, 20, 29, 15]. However, note that global exponential stability has been established for some particular semilinear systems in [21].

For the L^1-norm, we introduce a weighted entropy/Lyapunov functional inspired by entropy-based network analyses [3] and we derive a dissipation inequality in which the boundary term naturally involves f_i (Theorem 2.4). This yields a flux-dependent stability criterion, which is qualitatively different from the classical smooth theory where the Jacobian $G'(0)$ and matrix norms dominate the condition [4, 15]. We then identify an important subclass (notably concave fluxes) for which the criterion reduces to a flux-independent weighted contraction property of G for the $|\cdot|_1$-norm on $\mathbb{R}^n$ (Theorem 2.6).

For the L^∞-norm, we prove global exponential stability under a weighted ℓ^∞ contraction of G (Theorem 2.7), in the spirit of dissipative boundary conditions for hyperbolic systems [11, 4]. A notable feature is that, unlike the global well-posedness, the L^∞ stability does *not* require global Lipschitzness of f: the boundary contraction prevents amplitude growth along successive traversals of the domain and allows global-in-time control of the solution through a truncation argument.

Local exponential stability in L^∞. The global exponential stability result requires a somewhat strong assumption on the f_i in that $f_i'(s) \geq a > 0$. Nevertheless, a consequence of our result is that one can remove this constraint if we only look at the local exponential stability (see Corollary 2.10).

Organization of the paper. Section 3 recalls the entropy formulation and strong trace properties. Section 4 is devoted to proving the well-posedness of the closed-loop system. The argument relies on the well-posedness and stability estimates for the open-loop system (Theorem 4.1 and Proposition 4.2 below), which are based on [9], as well as on the finite propagation speed of information (see Lemma 4.3 below). This property allows us to treat the closed-loop system as a sequence of open-loop systems, enabling the iterative application of the open-loop results. Finally, Sections 5 and 6 develop Lyapunov arguments yielding global exponential stability in L^1 and L^∞ under the proposed dissipativity conditions on G. Theorems 2.4 and 2.6 are proved in Section 5, and Theorem 2.7 is proved in Section 6.

2 Main results

Let $T > 0$, our functional framework is

$$u \in C([0,T]; L^1(0,1)) \cap L^\infty((0,T) \times (0,1)). \tag{2.1}$$

In (2.1) and in what follows, we use the abbreviations $L^1(0,1) = L^1((0,1); \mathbb{R}^n)$ and $L^\infty((0,1) \times (0,T)) = L^\infty((0,1) \times (0,T); \mathbb{R}^n)$. Similarly, $L^p(0,1) = L^p((0,1); \mathbb{R}^n)$, $L^p(0,T) = L^p((0,T); \mathbb{R}^n)$, etc.

Our first result is:

Theorem 2.1 (Well-posedness of the closed-loop system)**.** *Let $T > 0$ and assume $u_0 \in L^\infty(0,1)$. If f is globally Lipschitz, then there exists a unique entropy solution u of (1.1)–(1.2) on $(0,T) \times (0,1)$ (in the sense of Definition 3.1) with initial condition*

$$u(0,x) = u_0(x) \qquad \textit{for a.e. } x \in (0,1), \tag{2.2}$$

such that:

- *$u \in C([0,T];L^1(0,1)) \cap L^\infty((0,T)\times(0,1))$;*
- *u admits strong traces $u(\cdot,0), u(\cdot,1) \in L^\infty(0,T)$ which satisfy the boundary condition (1.2) a.e. on $(0,T)$.*

Moreover, $u_0 \mapsto u$ is continuous from $L^1(0,1) \cap L^\infty(0,1)$ endowed with the L^1 norm to $C^0([0,T];L^1(0,1))$.

Remark 2.2. Note that the global Lipschitz assumption on f is necessary in the absence of additional information on G. Otherwise, even in the scalar case $n = 1$, blow-up may occur. For instance, consider $G(U) = 2U$ and $f(U) = U^2/2$ for $U \geq 1$. Let $T > 0$. Subject to the initial data

$$u_0(x) = \frac{2-x}{T}, \tag{2.3}$$

the corresponding solution to (1.1), (1.2), and (2.2) is given by

$$u(t,x) = \frac{2-x}{T-t} \geq 1, \quad t \in (0,T), \quad x \in [0,1] \tag{2.4}$$

and it blows up in time T.

The proof of Theorem 2.1 (Section 4) will use two ingredients:

- An analysis of the (open-loop) system for the initial boundary value problem (1.1) with boundary conditions of the form

$$u(t,0) = w(t), \tag{2.5}$$

  where $w \in L^\infty(0,T)$ is given (Theorem 4.1 and Proposition 4.2 below).
- A finite speed of propagation/delay lemma (Lemma 4.3) showing that, because the outgoing trace $u(t,1)$ at time t is not influenced by the most recent values of the incoming signal $u(t,0)$, the system (1.1) of scalar conservation laws can be considered as an open-loop system over a sufficiently short time interval.

Our next contribution is to give conditions on the map G which guarantee the global exponential stability of systems of scalar conservation laws of the form (1.1)–(1.2) in L^1 and L^∞ norms, according to the following definition.

Definition 2.3. The system (1.1)–(1.2) is said globally exponentially stable in the L^1-norm (resp. in the L^∞-norm), if there exist $C > 0$ and $\gamma > 0$ such that for any $u_0 \in L^\infty(0,1)$ and for any $T > 0$, there exists a unique entropy solution u of (1.1)–(1.2) with initial condition u_0 in the sense of Definition 3.1 with

- $u \in C([0,T];L^1(0,1)) \cap L^\infty((0,T)\times(0,1))$,
- u admits strong traces $u(\cdot,0), u(\cdot,1) \in L^\infty(0,T)$ which satisfy the boundary condition (1.2) a.e. on $(0,T)$,

and

$$\|u(t,\cdot)\|_{L^1(0,1)} \leq Ce^{-\gamma t}\|u_0\|_{L^1(0,1)}, \quad \forall t \in [0,+\infty), \tag{2.6}$$

$$\text{(resp. } \|u(t,\cdot)\|_{L^\infty(0,1)} \leq Ce^{-\gamma t}\|u_0\|_{L^\infty(0,1)}, \quad \forall t \in [0,+\infty)). \tag{2.7}$$

Our main results are the following.

Theorem 2.4 (Global exponential stability in the L^1-norm)**.** *Assume that f is globally Lipschitz and that there exist $\mu > 0$ and $p_i > 0$ such that*

$$\sum_{i=1}^{n} p_i \left(|f_i(y_i) - f_i(0)|e^{-\mu} - |f_i(G_i(y)) - f_i(0)| \right) \geq 0, \quad \forall y = (y_1, ..., y_n)^\top \in \mathbb{R}^n. \tag{2.8}$$

Then the system (1.1)–(1.2) *is globally exponentially stable for the L^1 norm (in the sense of Definition 2.3)*

Remark 2.5. The condition (2.8) for *global stability* depends explicitly on the functions f_i. It is therefore instructive to compare it with the *local stability* conditions previously established in the literature. For instance in [15], the system (1.1)–(1.2) is shown to be *locally* exponentially stable (for the $W^{2,p}$-norm) if there exists $\Delta \in \mathcal{D}^n$ such that

$$|\Delta G'(0)\Delta^{-1}|_p < 1 \tag{2.9}$$

where $\mathcal{D}^n$ denotes the set of diagonal $n \times n$ matrices with strictly positive diagonal entries and $|\cdot|_p$ ($p \in [1, +\infty]$) denotes the classical matrix norm[1]. The same condition is found also in [13, Theorem 1.2] and [18, Theorem 3.4] to obtain the stability in the BV-norm. An important point to note is that the condition (2.9) is equivalent to the existence of $\mu > 0$ and $\Delta \in \mathcal{D}^n$ such that

$$|\Delta G'(0)\Delta^{-1}|_p \leq e^{-\mu}, \tag{2.10}$$

which is equivalent to

$$|\Delta G'(0)z|_p \leq e^{-\mu}|\Delta z|_p, \quad \forall z \in \mathbb{R}^n \setminus \{0\}. \tag{2.11}$$

which, in turn, can be recovered from (2.8) locally around $y = 0$ for $p = 1$. Indeed, a first-order expansion of (2.8) around $y = 0$ corresponds to

$$\sum_{i=1}^{n} p_i |f_i'(0)G_i'(0)y| \leq \sum_{i=1}^{n} p_i |f_i'(0)y_i| e^{-\mu}, \quad \forall y = (y_1, \ldots, y_n)^\top \in \mathbb{R}^n. \tag{2.12}$$

For each $i \in \{1, \ldots, n\}$, we know that $p_i f_i'(0)$ is strictly positive by (1.4). Defining the diagonal matrix $\Delta := \mathrm{diag}(p_1 f_1'(0), \ldots, p_n f_n'(0)) \in \mathcal{D}^n$, we can rewrite (2.12) as

$$|\Delta G'(0)y|_1 \leq e^{-\mu}|\Delta y|_1, \quad \forall y \in \mathbb{R}^n, \tag{2.13}$$

which directly implies (2.11) for $p = 1$ with Δ depending on the $f_i'(0)$. Hence the condition (2.8) for global stability in L^1 corresponds around $y = 0$ to the condition (2.9) for the local stability in $W^{2,1}$.

Let us now come back to the presentation of our main results. First, under a convexity/concavity assumption on the f_i, a direct global extension of (2.13) gives a sufficient condition for the global exponential stability in the L^1-norm which does not depend on the functions f_i. This is stated in the following theorem.

[1] $|M|_p := \max\{|Mz|_p;\ z \in \mathbb{R}^n \text{ such that } |z|_p \leq 1\}$, $\forall\, p \in [1, +\infty]$, with, for $z = (z_1, \ldots, z_n)^\top \in \mathbb{R}^n$, $|z|_p := \left(\sum_{i=1}^{n} |z_i|^p\right)^{1/p}$ if $p \in [1, +\infty)$ and $|z|_\infty := \max\{|z_i|;\ i \in \{1, \ldots, n\}\}$.

Theorem 2.6. *Assume that f is globally Lipschitz, that the f_i are convex on $(-\infty, 0]$ and concave on $[0, +\infty)$, and that there exist $\mu > 0$ and $\Delta \in \mathcal{D}^n$ such that*

$$|\Delta G(z)|_1 \leq e^{-\mu}|\Delta z|_1, \quad \forall z \in \mathbb{R}^n. \tag{2.14}$$

Then the system (1.1)–(1.2) is globally exponentially stable for the L^1 norm (in the sense of Definition 2.3).

Then, interestingly, this result also holds in the L^∞ norm without any convexity/concavity assumption on the f_i.

Theorem 2.7 (Global exponential stability in the L^∞-norm)**.** *Assume that there exist $\mu > 0$ and $\Delta \in \mathcal{D}^n$ such that*

$$|\Delta G(z)|_\infty \leq e^{-\mu}|\Delta z|_\infty, \quad \forall z \in \mathbb{R}^n. \tag{2.15}$$

Then the system (1.1)–(1.2) is globally exponentially stable for the L^∞ norm (in the sense of Definition 2.3).

Remark 2.8 (Linear feedback)*.* If the feedback law is linear, i.e. if $G(z) = Kz$ for some matrix $K \in \mathbb{R}^{n\times n}$, then (2.15) is equivalent to

$$\rho_\infty(K) < 1, \tag{2.16}$$

where

$$\rho_\infty(K) := \inf\{|\Delta K \Delta^{-1}|_\infty;\ \Delta \in \mathcal{D}^n\}. \tag{2.17}$$

Note that it is proved in [19], [29], [32], [25, Theorem 1.3], [11], and [4, Section 4.1] that $\rho_\infty(G'(0)) < 1$ is a sufficient condition for the local exponential stability of the system (1.1)–(1.2) in the C^k-norm with $k \geq 1$. It is also proved in [18] that the condition (2.16) ensures a semi-global exponential stability in the BV norm.

Remark 2.9. Note that, in contrast with the well-posedness result of Theorem 2.1, the global Lipschitzness of f is not needed for the exponential stability results of Theorem 2.7. This is allowed thanks to the assumption (2.15) on G, which implies that the system (1.1)–(1.2) is well-posed for all time $T > 0$.

In many physical cases, the positivity condition (1.4) is too strong (one can think of Burgers, LWR or Buckley-Leverett equations, for instance). Let us however remark that this assumption can be removed if we look at the local exponential stability. Indeed a direct consequence of Theorem 2.7 is the following Corollary.

Corollary 2.10 (Local stability in L^∞)**.** *Assume that f satisfies*

$$f_i'(0) > 0,\ \forall i \in \{1, ..., n\}, \tag{2.18}$$

instead of (1.4) and that

$$\rho_\infty(G'(0)) < 1, \tag{2.19}$$

then the system (1.1)–(1.2) is (locally) exponentially stable for the L^∞ norm. Moreover, the exponential estimate holds for any decay rate μ satisfying

$$\mu < \min_{i\in\{1,..,n\}} f_i'(0)\left[-\ln(\rho_\infty(G'(0)))\right]. \tag{2.20}$$

3 Entropy solutions and boundary traces

We recall the definition of an entropy solution.

Definition 3.1 (Kružkov entropy inequalities in the interior [23])**.** A function $u \in L^\infty((0,T)\times(0,1))$ is an *entropy solution of* (1.1) *in the interior* if for every $k \in \mathbb{R}$ and every nonnegative $\varphi \in C_c^1([0,T)\times(0,1))$,

$$\int_0^T \int_0^1 \Big(|u_i - k|\varphi_t + \operatorname{sgn}(u_i - k)\big(f_i(u_i) - f_i(k)\big)\varphi_x \Big)\, dx\, dt \\ + \int_0^1 |u_{0,i}(x) - k|\varphi(0,x)\, dx \geq 0, \quad \forall\, i \in \{1, ..., n\}. \tag{3.1}$$

Remark 3.2. If $u \in L^\infty((0,T)\times(0,1))$ is an entropy solution of (1.1) in the interior, then one has for any convex η (see [23]) of class C^1,

$$\int_0^T \int_0^1 \Big(\eta(u_i)\varphi_t + q_i(u_i)\varphi_x \Big)\, dx\, dt + \int_0^1 \eta(u_{0,i})\varphi(0,x) dx \geq 0, \quad \forall\, i \in \{1, ..., n\}, \tag{3.2}$$

for every function $\varphi \in C_c^1([0,T)\times(0,1);[0,+\infty))$ provided that

$$q_i'(z) = \eta'(z) f_i'(z), \quad \forall z \in \mathbb{R}. \tag{3.3}$$

The function q_i is called the entropy flux associated to the entropy η.

The boundary traces of an entropy solution are defined in the following proposition.

Proposition 3.3 (Strong traces)**.** *Let $u \in L^\infty((0,T)\times(0,1))$ be an entropy solution in the sense of Definition 3.1. Then there exist (strong) boundary traces $u(\cdot,0), u(\cdot,1) \in L^\infty(0,T)$ such that (up to redefining u on a null set),*

$$\lim_{\varepsilon\to 0,\, \varepsilon>0} \int_0^T |u(t,\varepsilon) - u(t,0)|\, dt = 0, \qquad \lim_{\varepsilon\to 0,\, \varepsilon>0} \int_0^T |u(t,1-\varepsilon) - u(t,1)|\, dt = 0. \tag{3.4}$$

By virtue of (1.4), Proposition 3.3 follows from [24, Theorem 1] (where $\eta(u) := u$ and $H(u) := f_i(u)$) or [27, Theorem 1.4] (where k is chosen large enough so that $\psi_k(u) = f_i(k) - f_i(u)$). For both approaches, the natural open set is $\Omega := (0,+\infty)\times(0,1)$. However, Ω does not satisfy the boundary regularity conditions required by these theorems, namely C^2 for [24] and C^1 for [27]. To resolve this, it suffices to smooth the corners to obtain a sequence of regularized domains Ω_l that appropriately converge to Ω. Applying the theorems to Ω_l yields Proposition 3.3. We omit the explicit details of this approximation. Finally, because the problem is spatially one-dimensional, an alternative approach is to exchange the variables t and x, allowing for the application of a trace theorem at $t = 0$, such as [26, Theorem 1.2].

4 Well-posedness via a step-by-step method

This section is devoted to the proof of Theorem 2.1. This proof relies on the following:

1. The analysis of the open-loop system (Theorem 4.1 and Proposition 4.2 below).

2. Assumption (1.4), which guarantees that information propagates to the right at a finite speed (Lemma 4.3 below). This allows us to treat the *closed-loop* system as a sequence of *open-loop* systems, to which the above analysis can be applied iteratively.

Let us start with the analysis of the *open-loop* system with a prescribed boundary datum w:

$$\begin{cases} u_t + (f(u))_x = 0 & \text{in } (0,T)\times(0,1), \\ u(0,x) = u_0(x) & \text{for a.e. } x \in (0,1), \\ u(t,0) = w(t) & \text{for a.e. } t \in (0,T). \end{cases} \tag{4.1}$$

Note that imposing the boundary condition $u(t,0) = w(t)$ a priori makes sense under the positivity condition (1.4) on the direction of the characteristics. See Bardos–Le Roux–Nédélec (BLN) [2] and modern treatments such as [9, 1] in more general cases.

Theorem 4.1 (Open-loop well-posedness). *Assume $u_0 \in L^\infty(0,1)$, $w \in L^\infty(0,T)$ and $f \in C^1(\mathbb{R}^n)$. Then there exists a unique entropy solution u to (4.1) such that $u \in C([0,T]; L^1(0,1)) \cap L^\infty((0,T)\times(0,1))$ and u admits strong boundary traces $u(t,0) = w(t)$ and $u(t,1)$. Moreover, for two data (u_0, w) and (v_0, z) with corresponding solutions u, v one has the L^1 stability estimate*

$$\|u_i(t,\cdot) - v_i(t,\cdot)\|_{L^1(0,1)} \le \|u_{0,i} - v_{0,i}\|_{L^1(0,1)} + \int_0^t |f(w_i(s)) - f(z_i(s))|\, ds, \\ \forall t \in [0,T],\ \forall\, i \in \{1, ..., n\}. \tag{4.2}$$

This theorem is essentially a consequence of [9, Theorem 1.1–1.2] and its proof is given in Appendix B.

Proposition 4.2 (Maximum principle). *Under the assumptions of Theorem 4.1,*

$$\|u\|_{L^\infty((0,T)\times(0,1))} \le \max\Big\{ \|u_0\|_{L^\infty(0,1)}, \|w\|_{L^\infty(0,T)} \Big\}. \tag{4.3}$$

This proposition is shown in Appendix B.

Our next lemma, proved in Appendix C, addresses the finite rightward propagation of information. For the statement of the lemma, we introduce the following notations:

$$\bar{a}(w) = \max\{ f_i'(v) \mid i \in \{1, ..., n\},\ |v| \le \|u_0\|_{L^\infty(0,1)} + \|w\|_{L^\infty(0,1)} \}, \tag{4.4}$$

$$\delta(w) := \frac{1}{\bar{a}(w)}. \tag{4.5}$$

Lemma 4.3 (Output independent of input for short times). *Let u^w, u^z be the open-loop entropy solutions of (4.1) with the* same *initial datum $u_0 \in L^\infty(0,1)$ and two (possibly different) input boundary data $w, z \in L^\infty(0,T)$. Then their outgoing traces coincide on $(0, \bar{\delta})$ with $\bar{\delta} = \min(\delta(w), \delta(z))$:*

$$u^w(t,1) = u^z(t,1) \qquad \text{for a.e. } t \in (0, \bar{\delta}). \tag{4.6}$$

More generally, if $w = z$ a.e. on $(0, \tilde{t})$ for some $\tilde{t} \in (0,T)$, then $u^w(t,1) = u^z(t,1)$ for a.e. t in $(0, \tilde{t} + \bar{\delta}) \cap (0,T)$.

Let us now explain how this lemma can be used to reduce the closed-loop system as a sequence of open-loop systems. First note that when f is globally Lipschitz, then, for any $w \in L^\infty$

$$\bar{a}(w) \leq C_f, \tag{4.7}$$

where C_f is the Lipschitz constant of f. As a consequence, denoting $\tau = C_f^{-1}$,

$$0 < \tau \leq \delta(w), \ \forall w \in L^\infty. \tag{4.8}$$

Let $T > 0$ and $N \in \mathbb{N}$ such that $N\tau > T$ and denote

$$t_n = \min(n\tau, T). \tag{4.9}$$

We inductively construct an input function $w \in L^\infty(0,T)$ and the corresponding open-loop solution u of (4.1) such that $w(t) = G(u(t,1))$ a.e. in $(0,T)$.

Step $n = 1$. Choose an arbitrary "dummy" input $\widetilde{w}^{(1)} \in L^\infty(0,T)$ and let $\widetilde{u}^{(1)}$ be the open-loop solution of (4.1) with input $\widetilde{w}^{(1)}$. Define

$$w(t) \coloneqq G(\widetilde{u}^{(1)}(t,1)) \qquad \text{for a.e. } t \in (0,t_1). \tag{4.10}$$

By Proposition 3.3 and the Lipschitz continuity of G, we have $w \in L^\infty(0,t_1)$.

Now let $u^{(1)}$ be the open-loop solution on $(0,t_1)$ with input w (which is unique from Theorem 4.1). By Lemma 4.3 (with $t_0 = 0$), the corresponding outgoing traces $\widetilde{u}^{(1)}(t,1)$ and $u^{(1)}(t,1)$ coincide a.e. on $(0,t_1)$, hence

$$w(t) = G(u^{(1)}(t,1)) \qquad \text{for a.e. } t \in (0,t_1), \tag{4.11}$$

so the boundary feedback condition (1.2) is satisfied on $(0,t_1)$.

Induction step. Assume that we have defined w on $(0,t_n)$ and constructed $u^{(n)}$, the open-loop solution on $(0,t_n)$ with input w, such that $w(t) = G(u^{(n)}(t,1))$ for a.e. $t \in (0,t_n)$. Choose any extension $\widetilde{w}^{(n+1)} \in L^\infty(0,T)$ such that $\widetilde{w}^{(n+1)} = w$ a.e. on $(0,t_n)$.

Let $\widetilde{u}^{(n+1)}$ solve (4.1) (on $(0,T)$) with input $\widetilde{w}^{(n+1)}$. Define for a.e. $t \in (t_n,t_{n+1})$:

$$w(t) \coloneqq G(\widetilde{u}^{(n+1)}(t,1)). \tag{4.12}$$

Let $\widetilde{u}^{(n+1)}$ be the open-loop solution on $(0,t_{n+1})$ with this input w. By Lemma 4.3 (applied with $\tau = t_n$), $\widetilde{u}^{(n+1)}(t,1)$ and $u^{(n+1)}(t,1)$ coincide on (t_n,t_{n+1}) since w and $\widetilde{w}^{(n+1)}$ coincide on $(0,t_n)$ by definition. Hence,

$$w(t) = G(u^{(n+1)}(t,1)) \tag{4.13}$$

holds a.e. on (t_n,t_{n+1}).

After at most N steps we obtain w on $(0,T)$ and an entropy solution u on $(0,T) \times (0,1)$ of the system (4.1) which satisfies the boundary feedback relation (1.2). It is therefore an entropy solution of the system (1.1)–(1.2) since, from Theorem 4.1, $u \in C([0,T];L^1(0,1)) \cap L^\infty((0,T) \times (0,1))$.

Let us now prove the uniqueness of the solution to the closed-loop system. Let u, v be two entropy solutions of the closed-loop problem (1.1)–(1.2) with initial condition

$u_0 \in L^\infty(0,1)$. Set $w_u(t) := G(u(t,1))$ and $w_v(t) := G(v(t,1))$. On $(0,t_1)$, Lemma 4.3 (with $\tilde{t}=0$) implies that $u(\cdot,1)=v(\cdot,1)$ a.e. hence $w_u = w_v$ a.e. on $(0,t_1)$. By open-loop uniqueness (Theorem 4.1), we infer $u=v$ on $(0,t_1)\times(0,1)$.

Assume inductively that $u=v$ on $(0,t_n)\times(0,1)$. Then $w_u=w_v$ a.e. on $(0,t_n)$, and Lemma 4.3 (with $\tilde{t}=t_n$) yields $u(\cdot,1)=v(\cdot,1)$ a.e. on (t_n,t_{n+1}). Thus $w_u=w_v$ a.e. on (t_n,t_{n+1}), and open-loop uniqueness gives $u=v$ on $(0,t_{n+1})\times(0,1)$. This ends the proof of the induction and, after finitely many steps, $u\equiv v$ on $(0,T)\times(0,1)$.

This completes the proof of Theorem 2.1.

5 Exponential stability in the L^1-norm

We now prove Theorem 2.4. Let $T>0$ and u be an entropy solution of (1.1)–(1.2). We define

$$V(t)=\int_0^1\sum_{i=1}^n p_i e^{-\mu x}|u_i(t,x)|dx, \tag{5.1}$$

which from Theorem 2.1 belongs to $C([0,T];\mathbb{R}_+)$. Let $\phi\in C_c^1((0,T);\mathbb{R}_+)$,

$$\int_0^T V(t)\phi_t dt=\sum_{i=1}^n\int_0^T\int_0^1|u_i(t,x)|p_ie^{-\mu x}\phi_t\,dx\,dt. \tag{5.2}$$

Let us denote

$$W_i=\int_0^T\int_0^1|u_i(t,x)|p_ie^{-\mu x}\phi_t\,dx\,dt, \tag{5.3}$$

and define

$$W_{i,\varepsilon}=\int_0^T\int_0^1|u_i(t,x)|\chi_\varepsilon(x)p_ie^{-\mu x}\phi_t\,dx\,dt, \tag{5.4}$$

where χ_ε is defined by

$$\chi_\varepsilon(x)=\begin{cases} x/\varepsilon \quad \text{for } x\in(0,\varepsilon] \\ 1 \quad \text{for } x\in(\varepsilon,1-\varepsilon] \\ (1-x)/\varepsilon \quad \text{for } x\in(1-\varepsilon,1]. \end{cases} \tag{5.5}$$

Using Definition 3.1 with $k=0$, this gives[2]

$$\begin{aligned} W_{i,\varepsilon}\geq &-\int_0^T\int_0^1 \operatorname{sgn}(u_i)\Big(f_i(u_i)-f_i(0)\Big)\left(\chi_\varepsilon'(x)-\mu\chi_\varepsilon(x)\right)p_ie^{-\mu x}\phi(t)\,dx\,dt \\ =&-\int_0^T\left(\frac{1}{\varepsilon}\int_0^\varepsilon \operatorname{sgn}(u_i)\Big(f_i(u_i)-f_i(0)\Big)p_ie^{-\mu x}\,dx\right)\phi(t)dt \\ &+\int_0^T\left(\frac{1}{\varepsilon}\int_{1-\varepsilon}^1 \operatorname{sgn}(u_i)\Big(f_i(u_i)-f_i(0)\Big)p_ie^{-\mu x}\,dx\right)\phi(t)dt \\ &+\int_0^T\int_0^1 \operatorname{sgn}(u_i)\Big(f_i(u_i)-f_i(0)\Big)\mu\chi_\varepsilon(x)p_ie^{-\mu x}\phi(t)\,dx\,dt. \end{aligned} \tag{5.6}$$

[2] Strictly speaking the definition of entropy solutions requires $\varphi : (t,x)\mapsto\chi_\varepsilon(x)p_ie^{-\mu x}\phi(t)$ to belong to $C_c^1((0,T)\times(0,1);\mathbb{R}_+)$ and therefore $\chi_\varepsilon\in C_c^1((0,1);\mathbb{R}_+)$. However the function χ_ε can be approximated by functions $(\chi_{\varepsilon,k})_{k\in\mathbb{N}}$ in $C_c^1((0,1);\mathbb{R}_+)$ such that $|\chi_{\varepsilon,k}'|_{L^\infty(0,1)}\leq 2/\varepsilon$ and $\lim_{k\to+\infty}\chi_{\varepsilon,k}'(x)=\chi_\varepsilon'(x)$, for every $x\in(0,1)\setminus\{\varepsilon,1-\varepsilon\}$. Then (5.6) holds with χ_ε replaced by $\chi_{\varepsilon,k}$ and letting $k\to+\infty$ in these (5.6) with $\chi_{\varepsilon,k}$, one gets the desired (5.6).

Letting $\varepsilon \to 0$ in the above inequality and in (5.4), together with (3.4) we obtain

$$\begin{aligned} W_i \geq \mu \int_0^T \int_0^1 \operatorname{sgn}(u_i)\big(f_i(u_i) - f_i(0)\big)p_i e^{-\mu x}\phi(t)\,dx\,dt \\ + \int_0^T \Big[\operatorname{sgn}(u_i(t,1))\big(f_i(u_i(t,1)) - f_i(0)\big)e^{-\mu} \\ - \operatorname{sgn}(u_i(t,0))\big(f_i(u_i(t,0)) - f_i(0)\big)\Big]p_i\phi(t)dt. \end{aligned} \tag{5.7}$$

Using the fact that u satisfies (1.2) and (1.4)

$$\begin{aligned} W_i \geq a\mu \int_0^T \int_0^1 |u_i| p_i e^{-\mu x}\phi(t)\,dx\,dt \\ + \int_0^T \Big(|f_i(u_i(t,1)) - f_i(0)|e^{-\mu} - |f_i(G_i(u(t,1))) - f_i(0)|\Big)p_i\phi(t)dt, \end{aligned} \tag{5.8}$$

where $G_i(u)$ is the i-th component of $G(u)$ and where we used the fact that, from (1.4), f_i is strictly increasing and therefore $\operatorname{sgn}(f_i(u) - f_i(0)) = \operatorname{sgn}(u)$. Finally, summing and using (5.1) we obtain

$$\begin{aligned} \int_0^T V(t)\phi_t(t)dt \geq a\mu \int_0^T V(t)\phi(t)dt \\ + \int_0^T \sum_{i=1}^n \Big(|f_i(u_i(t,1)) - f_i(0)|e^{-\mu} - |f_i(G_i(u(t,1))) - f_i(0)|\Big)p_i\phi(t)dt. \end{aligned} \tag{5.9}$$

Using (2.8) we obtain

$$\int_0^T V(t)\phi_t(t)dt \geq a\mu \int_0^T V(t)\phi(t)dt, \tag{5.10}$$

which holds for any $\phi \in C_c^1((0,T);\mathbb{R}_+)$ and implies that

$$V(t) \leq V(0)e^{-a\mu t}, \forall t \in [0,T]. \tag{5.11}$$

From (5.1) we obtain directly that there exists C depending only on p_i and μ such that

$$\|u(t,\cdot)\|_{L^1(0,1)} \leq Ce^{-a\mu t}\|u_0\|_{L^1(0,1)}, \forall t \in [0,T]. \tag{5.12}$$

This completes the proof of Theorem 2.4.

We now prove Theorem 2.6. We observe that, instead of considering the Lyapunov function candidate (5.1), we could also consider the more general function

$$V(t) = \int_0^1 \sum_{i=1}^n p_i e^{-\mu x}\eta_i(u_i(t,x))dx, \tag{5.13}$$

where η_i is a convex function which satisfies

$$\alpha|U| \leq \eta_i(U) \leq \beta|U|, \quad \forall U \in \mathbb{R}, \tag{5.14}$$

where α and β are two positive constants. We choose

$$\eta_i(U) = \left|\int_0^U \frac{1}{f_i'(s)}ds\right|. \tag{5.15}$$

From (1.4) and the convexity/concavity assumption on the f_i, it follows that the η_i are convex. Moreover, by (1.4) and since f is globally Lipschitz, the η_i satisfy also (5.14). Proceeding as previously, applying (3.2) with entropy η_i and associated entropy flux $q_i(U) = |U|$, for any $\phi \in C_c^1([0,T);\mathbb{R}_+)$ we have

$$\int_0^T V(t)\phi_t(t)dt \geq a\mu \int_0^T V(t)\phi(t)dt + \int_0^T \sum_{i=1}^n \Big(|u_i(t,1)|e^{-\mu} - |G_i(u(t,1))|\Big)\, p_i \phi(t)dt. \tag{5.16}$$

Selecting now $p_i = \Delta_i$, under the assumption (2.14), we have

$$\sum_{i=1}^n \Big(|u_i(t,1)|e^{-\mu} - |G_i(u(t,1))|\Big)\, p_i \geq 0, \tag{5.17}$$

hence

$$\int_0^T V(t)\phi_t(t)dt \geq a\mu \int_0^T V(t)\phi(t)dt, \tag{5.18}$$

and similarly as previously the exponential stability in the L^1 norm holds. This completes the proof of Theorem 2.6.

Remark 5.1. The weight $e^{-\mu x}$ appearing in (5.1) (see also (6.2) below) is now a standard choice for equations where the transport component is significant. For instance, we refer to the definition of the Lyapunov function V given in [10] for the Euler equations of incompressible fluids. In the context of smooth solutions to 1d hyperbolic systems (or non-smooth solutions to linear hyperbolic systems), one may consult [31, 12, 4]. Furthermore, for entropy solutions of 1d hyperbolic systems or conservation laws, we refer the reader to [14, 16].

6 Exponential stability in the L^∞-norm

We now prove Theorem 2.7. Our proof is partly inspired by [11] (see also [4, Section 4.1]). Let $T > 0$ and let

$$\nu \in (0, \mu). \tag{6.1}$$

Let us first assume that f is globally Lipschitz. Then, according to Theorem 2.1, (1.1)–(1.2) has a unique entropy solution u. Given $\Delta_i > 0$ for $i \in \{1, \dots, n\}$, and letting $\Delta = \operatorname{diag}(\Delta_1, \dots, \Delta_n)$, we define $V : [0,T] \to [0, +\infty)$ by (compare with (5.1))

$$V(t) = \| \Big(\Delta_1 e^{-\nu x}|u_1(t,x)|, \dots, \Delta_n e^{-\nu x}|u_n(t,x)|\Big)^\top \|_{L^\infty(0,1)}, \tag{6.2}$$

which from Theorem 2.1 belongs to $L^\infty((0,T);\mathbb{R}_+)$. In contrast with the V of section 5, this new V may be discontinuous. However, since $u \in C([0,T]; L^1(0,1))$, $V(t)$ is well defined for every $t \in [0,T]$ and not only for almost every $t \in [0,T]$. In (6.2), $\mathbb{R}^n$ is equipped with the $|\cdot|_\infty$-norm. Hence

$$V(t) = \max\{\Delta_i \|e^{-\nu x} u_i(t,x)\|_{L^\infty(0,1)};\ i \in \{1, \dots, n\}\}. \tag{6.3}$$

For a (strictly) positive integer m, let us define $V_m : [0,T] \to [0, +\infty)$ by

$$V_m(t) := \left(\int_0^1 \sum_{i=1}^n \Delta_i^{2m} e^{-2m\nu x} u_i^{2m}(t,x)\, dx\right)^{\frac{1}{2m}}. \tag{6.4}$$

From Theorem 2.1 we know that $u \in C([0,T];L^1(0,1)) \cap L^\infty((0,T)\times(0,1))$ from which we gain directly that $u \in C([0,T];L^{2m}(0,1))$ and therefore V_m now belongs to $C([0,T];\mathbb{R}_+)$.

Let us prove that

$$\lim_{m\to+\infty} V_m(t) = V(t), \quad \forall t \in (0,T). \tag{6.5}$$

For a fixed $t \in [0,T]$, let $M : [0,1] \to [0,+\infty)$ be defined by

$$M(x) := \max\{\Delta_i e^{-\nu x}|u_i(t,x)|;\ 1 \le i \le n\}. \tag{6.6}$$

One has

$$M^{2m}(x) = \max\{\Delta_i^{2m} e^{-2m\nu x} u_i^{2m}(t,x);\ 1 \le i \le n\} \le \sum_{i=1}^{n} \Delta_i^{2m} e^{-2m\nu x} u_i^{2m}(t,x). \tag{6.7}$$

Integrating (6.7) on $[0,1]$, one gets

$$\|M\|_{L^{2m}(0,1)} := \left(\int_0^1 M^{2m}(x)\,dx\right)^{\frac{1}{2m}} \le V_m(t). \tag{6.8}$$

For the upper-bound of V_m, one has

$$\sum_{i=1}^{n} \Delta_i^{2m} e^{-2m\nu x} u_i^{2m}(t,x) \le n \max\{\Delta_i^{2m} e^{-2m\nu x} u_i^{2m}(t,x);\ 1 \le i \le n\} = nM^{2m}(x). \tag{6.9}$$

Integrating this inequality on $[0,1]$ and taking the $(2m)$-th root, we get

$$V_m \le n^{\frac{1}{2m}} |M|_{L^{2m}(0,1)}. \tag{6.10}$$

Finally (6.5) follows from (6.8), (6.10) and the following classical result

$$\lim_{m\to+\infty} \|M\|_{L^{2m}(0,1)} = \|M\|_{L^\infty(0,1)}. \tag{6.11}$$

Let $\phi \in C_c^1((0,T);\mathbb{R}_+)$. One has

$$\int_0^T V_m^{2m}(t)\phi_t dt = \sum_{i=1}^{n} W_{m,i} \tag{6.12}$$

with

$$W_{m,i} := \int_0^T \int_0^1 \Delta_i^{2m} e^{-2m\nu x} u_i^{2m}(t,x)\phi_t\,dx\,dt. \tag{6.13}$$

Let us define, for $\varepsilon \in (0,1/2]$,

$$W_{m,i,\varepsilon} := \int_0^T \int_0^1 u_i^{2m}(t,x)\chi_\varepsilon(x)\Delta_i^{2m} e^{-2m\nu x}\phi_t\,dx\,dt, \tag{6.14}$$

where χ_ε is again defined by (5.5). Let $q_{m,i} : \mathbb{R} \to \mathbb{R}$ be such that

$$\frac{dq_{m,i}}{dz}(z) = 2mz^{2m-1} f_i'(z), \quad \forall z \in \mathbb{R}, \text{ and } q_{m,i}(0) = 0. \tag{6.15}$$

Recalling that u is an entropy solution and applying (3.2) with $\eta(z) = z^{2m}$ and $\varphi(t,x) = \phi(t)\chi_\varepsilon(x)e^{-2m\nu x}$, one has

$$\begin{aligned} W_{m,i,\varepsilon} \geq & - \int_0^T \int_0^1 q_{m,i}(u_i(t,x)) \left(\chi_\varepsilon'(x) - 2m\nu\chi_\varepsilon(x)\right) \Delta_i^{2m} e^{-2m\nu x}\phi(t)\, dx\, dt \\ = & - \int_0^T \left(\frac{1}{\varepsilon}\int_0^\varepsilon q_{m,i}(u_i(t,x))\Delta_i^{2m}e^{-2m\nu x}\, dx\right)\phi(t)dt \\ & + \int_0^T \left(\frac{1}{\varepsilon}\int_{1-\varepsilon}^1 q_{m,i}(u_i(t,x))\Delta_i^{2m}e^{-2m\nu x}\, dx\right)\phi(t)dt \\ & + 2m\nu \int_0^T \int_0^1 q_{m,i}(u_i(t,x))\Delta_i^{2m}e^{-2m\nu x}\chi_\varepsilon(x)\phi(t)\, dx\, dt. \end{aligned} \tag{6.16}$$

Letting $\varepsilon \to 0$ in (6.16) and in (6.14) and using (3.4), we obtain,

$$W_{m,i} \geq 2m\nu \int_0^T \int_0^1 q_{m,i}(u_i(t,x))\Delta_i^{2m}e^{-2m\nu x}\phi(t)\, dx\, dt + \int_0^T B_{m,i}(t)\phi(t)\, dt \tag{6.17}$$

with

$$B_{m,i}(t) := \Delta_i^{2m}\left(q_{m,i}(u_i(t,1))e^{-2m\nu} - q_{m,i}(u_i(t,0))\right). \tag{6.18}$$

Using the fact that u satisfies (1.2), we have

$$B_{m,i}(t) := \Delta_i^{2m}\left(q_{m,i}(u_i(t,1))e^{-2m\nu} - q_{m,i}(G_i(u(t,1)))\right) \tag{6.19}$$

where, again, $G_i(u)$ is the i-th component of $G(u)$. Let $\bar{\mathbf{f}}' > 0$ be such that

$$f_i'(z) \leq \bar{\mathbf{f}}', \quad \forall z \in [-\|u_i\|_{L^\infty((0,T)\times(0,1))}, \|u_i\|_{L^\infty((0,T)\times(0,1))}],\ \forall i \in \{1,\dots,n\}. \tag{6.20}$$

From (1.4), (6.15), and (6.20), one has

$$a|\xi|^{2m} \leq q_{m,i}(\xi) \leq \bar{\mathbf{f}}'|\xi|^{2m}, \quad \forall \xi \in \mathbb{R},\ \forall m \in \mathbb{N}\setminus\{0\},\ \forall i \in \{1,\dots,n\}. \tag{6.21}$$

From (6.19) and (6.21), one has

$$B_m(t) \geq \left(a\left(\sum_{i=1}^n \Delta_i^{2m}u_i^{2m}(t,1)e^{-2m\nu}\right) - \bar{\mathbf{f}}'\left(\sum_{i=1}^n \Delta_i^{2m}G_i(u(t,1))^{2m}\right)\right), \tag{6.22}$$

with

$$B_m(t) := \sum_{i=1}^n B_{m,i}(t). \tag{6.23}$$

Let us prove the existence of $m_0 \in \mathbb{N}\setminus\{0\}$ such that

$$B_m(t) \geq 0, \quad \forall t \in [0,T],\ \forall m \geq m_0. \tag{6.24}$$

Let

$$\alpha_m(t) := \left(a\left(\sum_{i=1}^n \Delta_i^{2m}u_i^{2m}(t,1)e^{-2m\nu}\right)\right)^{\frac{1}{2m}}, \tag{6.25}$$

$$\beta_m(t) := \left(\bar{\mathbf{f}}'\left(\sum_{i=1}^n \Delta_i^{2m}G_i(u(t,1))^{2m}\right)\right)^{\frac{1}{2m}}. \tag{6.26}$$

Note that

$$0 \leq \left(\sum_{i=1}^{n} z_i^{2m}\right)^{\frac{1}{2m}} - |z|_\infty \leq (n^{\frac{1}{2m}} - 1)|z|_\infty, \quad \forall z = (z_1, \dots z_n)^\top \in \mathbb{R}^n. \tag{6.27}$$

From (6.25), (6.26), and (6.27), one has

$$\alpha_m(t) \geq a^{\frac{1}{2m}} e^{-\nu} \max\left\{\Delta_i |u_i(t,1)|;\, i \in \{1, \dots, n\}\right\}, \tag{6.28}$$

$$\beta_m(t) \leq n^{\frac{1}{2m}} \bar{\mathbf{f}}'^{\frac{1}{2m}} \max\left\{\Delta_i |G_i(u(t,1))|;\, i \in \{1, \dots, n\}\right\}. \tag{6.29}$$

From (2.15) and (6.29)

$$\beta_m(t) \leq n^{\frac{1}{2m}} \bar{\mathbf{f}}'^{\frac{1}{2m}} e^{-\mu} \max\left\{\Delta_i |u_i(t,1))|;\, i \in \{1, \dots, n\right\}. \tag{6.30}$$

Note that

$$\lim_{m\to+\infty} a^{\frac{1}{2m}} = 1, \ \lim_{m\to+\infty} \bar{\mathbf{f}}'^{\frac{1}{2m}}(t) = 1, \text{ and } \lim_{m\to+\infty} n^{\frac{1}{2m}} = 1. \tag{6.31}$$

From (6.1), (6.28), (6.30), and (6.31), we get the existence of $m_0 \in \mathbb{N} \setminus \{0\}$ such that

$$\alpha_m(t) \geq \beta_m(t), \quad \forall t \in [0,T],\ \forall m \geq m_0 \tag{6.32}$$

a property which, thanks to (6.22), (6.25) and (6.26), implies (6.24).

From now on we assume that $m \geq m_0$. From (6.12), (6.17), (6.21) , (6.23), and (6.24), we get

$$\int_0^T V_m^{2m}(t)\phi_t dt \geq 2m\nu a \sum_{i=1}^{n} \int_0^T \int_0^1 \Delta_i^{2m} e^{-2m\nu x} u_i^{2m}(t,x)\phi(t)\, dx\, dt. \tag{6.33}$$

Now from (6.33), for every $\phi \in C_c^1((0,T);\mathbb{R}_+)$ and for every $0 \leq t_1 \leq t_2 \leq T$, it can be checked that

$$V_m^{2m}(t_2) - V_m^{2m}(t_1) \leq -2m\nu a \int_{t_1}^{t_2} V_m^{2m}(t)\, dt. \tag{6.34}$$

Indeed, one can proceed as for the proof of (5.7) (see in particular the footnote of page 11): with a standard approximation procedure, one gets that, for every $\varepsilon \in (0, (t_2 - t_1)/2]$, (6.33) holds for

$$\phi(t) = \phi_\varepsilon(t) = \begin{cases} 0 \ \text{ for } t \in (0, t_1], \\ (t - t_1)/\varepsilon \ \text{ for } t \in (t_1, t_1 + \varepsilon], \\ 1 \ \text{ for } t \in (t_1 + \varepsilon, t_2 - \varepsilon), \\ (t_2 - t)/\varepsilon \ \text{ for } t \in [t_2 - \varepsilon, t_2], \\ 0 \ \text{ for } t \in (t_2, T], \end{cases} \tag{6.35}$$

and, letting $\varepsilon \to 0^+$ in (6.33) with this ϕ, one gets (6.34) if $t_2 > t_1$, while (6.34) is trivial if $t_1 = t_2$. Let us point out that (6.34) implies that

$$V_m \text{ is a non-increasing function on } [0,T]. \tag{6.36}$$

Note that, by the convexity of $\xi \in \mathbb{R} \to \xi^{2m}$,

$$V_m^{2m}(t_2) - V_m^{2m}(t_1) \geq 2m V_m^{2m-1}(t_1)\left(V_m(t_2) - V_m(t_1)\right). \tag{6.37}$$

From (6.34), (6.37), and (6.36), one gets

$$V_m^{2m-1}(t_1)\left(V_m(t_2)-V_m(t_1)\right) \leq -\nu a \int_{t_1}^{t_2} V_m^{2m}(t)\, dt \leq -\nu a\,(t_2-t_1)\, V_m^{2m}(t_2). \tag{6.38}$$

One has the following lemma, whose proof is given in Appendix A.

Lemma 6.1. *Let $\theta : [0,T] \to [0,+\infty)$ be a continuous non-increasing function such that, for some positive integer $m \geq 1$ and for some constant $c > 0$,*

$$\theta^{2m-1}(t_1)\left(\theta(t_2)-\theta(t_1)\right) \leq -c(t_2-t_1)\theta^{2m}(t_2), \quad \forall\, 0 \leq t_1 \leq t_2 \leq T. \tag{6.39}$$

Then,

$$\theta(t) \leq e^{-ct}\theta(0), \quad \forall t \in [0,T]. \tag{6.40}$$

By (6.36) and (6.38), the assumptions of this lemma are satisfied for $\theta := V_m$ and $c := \nu a$. Hence, by this lemma,

$$V_m(t) \leq e^{-\nu a t} V_m(0), \quad \forall t \in [0,T], \quad \forall m \geq m_0. \tag{6.41}$$

Letting $m \to +\infty$ in (6.41) and using (6.5), we get

$$V(t) \leq e^{-\nu a t} V(0), \quad \forall t \in [0,T], \tag{6.42}$$

which concludes the proof of Theorem 2.7 if f is globally Lipschitz since, by (6.3), there exists $C \geq 1$ independent of u, t, and T such that

$$\frac{1}{C}\|u(t,\cdot)\|_{L^\infty(0,1)} \leq V(t) \leq C\|u(t,\cdot)\|_{L^\infty(0,1)}. \tag{6.43}$$

If f is not globally Lipschitz, one can no longer get the existence of the solution u of (1.1)–(1.2) as a direct application of Theorem 2.1. To show, without this assumption of global Lipschitzness, the existence and uniqueness of the solution u (and that it satisfies the exponential stability estimate) on $[0,+\infty)$, one can proceed as follows. For u_0 given and $T > 0$ arbitrary, let $F > 0$ to be chosen and define $\tilde{f}_i$ that coincides with f_i on $\{y \in \mathbb{R}^n \mid |y| < F\}$ and is extended on $\mathbb{R}^n$ in a globally Lipschitz way such that its Lipschitz constant is

$$C_{\tilde{f}_i} \leq 2 \sup_{|y|\leq F}\{|\partial_u f_i(y)|\}. \tag{6.44}$$

The system (1.1), (1.2) with $\tilde{f}_i$ instead of f_i satisfies the assumption of Theorem 2.1 and there exists a unique entropy solution $u \in L^\infty((0,T)\times(0,1))$ which satisfies, from (2.7),

$$\|u\|_{L^\infty((0,T)\times(0,1))} \leq C\|u_0\|_{L^\infty}, \tag{6.45}$$

where C only depends on the Δ_i. Thus if we choose $F > C\|u_0\|_{L^\infty}$ (note that u might depend on F but u_0 does not), then u is also a solution of (1.1), (1.2) with f_i (indeed, $f_i(u(t,x)) = \tilde{f}_i(u(t,x))$ for a.e. $(t,x) \in (0,T)\times(0,1)$). This shows that (1.1), (1.2) admits an entropy solution in $L^\infty((0,T)\times(0,1))$ with strong boundary traces and satisfies (2.7) on $[0,T)$ instead of $[0,+\infty)$ where C does not depend on T. Extending the solution on $[0,+\infty)$ can be done classically as in [11, 20].

Remark 6.2. Since (6.42) holds for every $\nu \in (0,\mu)$, letting $\nu \to \mu^-$ leads to

$$\widetilde{V}(t) \leq e^{-\mu a t}\widetilde{V}(0), \quad \forall t \in [0,T], \tag{6.46}$$

for $\widetilde{V}$ defined by

$$\widetilde{V}(t) := \left\| \left(\Delta_1 e^{-\mu x}|u_1(t,x)|, \ldots, \Delta_n e^{-\mu x}|u_n(t,x)|\right)^\top \right\|_{L^\infty(0,1)}. \tag{6.47}$$

7 Conclusion

In this paper, we have established the well-posedness and the global exponential stability of 1D systems of scalar conservation laws closed by a nonlocal boundary feedback. While the existing literature on the boundary stabilization of hyperbolic systems mostly focuses on local stability for classical solutions in regular functional spaces (such as C^1 or H^2), our work addresses the natural framework of weak entropy solutions, where shocks can form in finite time. A key novelty of our approach are the resulting explicit dissipativity conditions on the feedback map to guarantee global exponential stability in both L^1 and L^∞ norms. Notably, we do not require the solutions to have bounded variation (BV functions) and our L^∞ stability framework succeeds in guaranteeing exponential decay without requiring the flux to be globally Lipschitz. We also provide a local exponential stability result in L^∞ norm under weaker assumptions. This work paves the way for several subsequent open problems. Among others, future research directions include: Investigating the existence and characterization of forward invariant sets for the closed-loop system under boundary feedback; Removing the diagonal assumption on the fluxes to address general systems of conservation laws that are fully coupled in the interior domain; Relaxing the strict wave speed assumption (1.4) in the global exponential stability to encompass systems where characteristic velocities may vanish or change sign; Extending these well-posedness and boundary stabilization strategies to scalar conservation laws in multi-dimensional spatial domains.

A Proof of Lemma 6.1

Since θ is non-negative and non-increasing, if there exists some $t_0 \in [0,T]$ such that $\theta(t_0) = 0$, then $\theta(t) = 0$ for all $t \in [t_0, T]$. In this case, the inequality $\theta(t) \le \theta(0)e^{-ct}$ holds trivially for every $t \in [t_0, T]$. Thus, we may assume $\theta(t) > 0$ for every $t \in [0, T)$.

Fix $t \in [0,T)$ and let $h > 0$ such that $t + h \le T$. Setting $t_1 = t$ and $t_2 = t + h$ in (6.39), we have:

$$\theta^{2m-1}(t)(\theta(t+h) - \theta(t)) \le -ch\theta^{2m}(t+h). \tag{A.1}$$

Dividing both sides by $h\theta^{2m-1}(t)$ (which is positive), we obtain:

$$\frac{\theta(t+h) - \theta(t)}{h} \le -c\frac{\theta^{2m}(t+h)}{\theta^{2m-1}(t)}. \tag{A.2}$$

We consider the upper right Dini derivative, defined as

$$D^+\theta(t) = \limsup_{h\to 0^+} \frac{\theta(t+h) - \theta(t)}{h}. \tag{A.3}$$

Taking the limit as $h \to 0^+$ on the right hand side and using the continuity of θ as well as (A.2), we get

$$D^+\theta(t) \le \lim_{h\to 0^+} \left(-c\frac{\theta^{2m}(t+h)}{\theta^{2m-1}(t)}\right) = -c\frac{\theta^{2m}(t)}{\theta^{2m-1}(t)} = -c\theta(t). \tag{A.4}$$

Let us now define the auxiliary function $g(t) = \theta(t)e^{ct}$. We compute the upper right Dini derivative of $g(t)$:

$$D^+g(t) := \limsup_{h\to 0^+} \frac{\theta(t+h)e^{c(t+h)} - \theta(t)e^{ct}}{h}. \tag{A.5}$$

Using the expansion $e^{ch} = 1 + ch + o(h)$, we have

$$D^+ g(t) = e^{ct} \limsup_{h \to 0^+} \left[\frac{\theta(t+h) - \theta(t)}{h} + c\theta(t+h) \right], \tag{A.6}$$

which, with (A.4), leads to

$$D^+ g(t) \le e^{ct}[-c\theta(t) + c\theta(t)] = 0. \tag{A.7}$$

Since $g(t)$ is continuous and its upper right Dini derivative is non-positive, $g(t)$ is non-increasing on $[0, T]$ (see, for example, [30, Chaper 5, Section 1, Proposition p. 99]). Therefore, we have $g(t) \le g(0)$, which gives

$$\theta(t)e^{ct} \le \theta(0)e^{c(0)} = \theta(0). \tag{A.8}$$

This concludes the proof of Lemma 6.1. □

B Proof of Theorem 4.1 and Proposition 4.2

We start with the proof of Theorem 4.1. Since the conservation laws of (4.1) are decoupled in open-loop, one only needs to establish the result for a single scalar equation. Therefore, in the following, index i is dropped to simplify the notations. For a scalar equation, the existence of a unique entropy solution in the sense of Definition 3.1 which has strong boundary traces is given in [9, Theorem 1.1–1.2]. Deriving the estimate is relatively classical: using a doubling of variable [23, Proof of Theorem 1; (3.2)] and from the definition of entropy solutions one obtains the following, for any $\phi \in C_c^1([0,T);\mathbb{R}_+)$ and $\chi \in \{C^1((0,1);\mathbb{R}_+) \mid \chi(0) = \chi(1) = 0\}$,

$$\int_0^T \int_0^1 \Big(|u(t,x) - v(t,x)|\phi'(t)\chi(x) + \operatorname{sgn}(u(t,x) - v(t,x)) \\ [f(u(t,x)) - f(v(t,x))]\chi'(x)\phi(t) \Big) dxdt + \int_0^1 |u_0 - v_0|\chi(x)\phi(0)dx \ge 0. \tag{B.1}$$

Note that equation (B.1) also holds when χ is replaced by χ_ε, defined in (5.5) (see (5.6)). We then obtain by taking $\varepsilon \to 0$ and using the existence of strong traces of the solution together with (4.1) (see also (5.6)–(5.7))

$$\begin{aligned} &\int_0^T \int_0^1 |u(t,x) - v(t,x)|\phi'(t)\, dx\, dt + \int_0^1 |u_0 - v_0|\phi(0)dx \\ &- \int_0^T \operatorname{sgn}(u(t,1) - v(t,1))(f(u(t,1)) - f(v(t,1)))\phi(t)dt \\ &+ \int_0^T \operatorname{sgn}(w(t) - z(t))(f(w(t)) - f(z(t)))\phi(t)dt \ge 0. \end{aligned} \tag{B.2}$$

Since f is non-decreasing, we get

$$\int_0^T \int_0^1 |u(t,x) - v(t,x)|\phi'(t)\, dx\, dt + \int_0^1 |u_0 - v_0|\phi(0)dx + \int_0^T |f(w(t)) - f(z(t))|\phi(t)dt \ge 0. \tag{B.3}$$

For t given, choosing ϕ as an approximation of $\mathbf{1}_{[0,t]}$, the indicator function of $[0,t]$ and letting it converge to $\mathbf{1}_{[0,t]}$, we have exactly (4.2). This ends the proof of Theorem 4.1.

For Proposition 4.2, one only needs to note that, thanks to Remark 3.2, one can choose the entropy $\eta : z \to (z-k)_+ := \max(z-k,0)$ for any $k \in \mathbb{R}$. The associated entropy flux is $q : z \to (f(z)-f(k))_+$ because f is increasing. Then, proceeding as above, we obtain, for any $k \in \mathbb{R}$ and for any $\phi \in C_c^1([0,T);\mathbb{R}_+)$,

$$\int_0^T \int_0^1 (u(t,x)-k)_+\phi'(t)\,dx\,dt + \int_0^1 (u_0(x)-k)_+\phi(0)dx \\ + \int_0^T (f(w(t))-f(k))_+\phi(t)dt \geq 0. \quad \text{(B.4)}$$

Thus, for any $t \in (0,T)$, choosing ϕ as an approximation of the characteristic function of $[0,t)$, we obtain

$$\int_0^1 (u(t,x)-k)_+dx \leq \int_0^1 (u_0(x)-k)_+dx + \int_0^t (f(w(s))-f(k))_+dt. \quad \text{(B.5)}$$

Now, assume that (4.3) does not hold, then one can choose[3]

$$k \in \Big(\max\Big\{\|u_0\|_{L^\infty(0,1)}, \|w\|_{L^\infty(0,T)}\Big\}, \|u\|_{L^\infty((0,T)\times(0,1))}\Big)$$

and get that

$$\int_0^1 (u(t,x)-k)_+dx \leq 0, \quad \text{(B.6)}$$

which implies that $u(t,x) \leq k$ a.e. on $(0,T)\times(0,1)$. The same can be done with $(-k-u(t,x))_+$ to deduce that $-u(t,x) \leq k$ and this contradicts the fact that $k < \|u\|_{L^\infty((0,T)\times(0,1))}$.

C Proof of Lemma 4.3

Proof. Lemma 4.3 is a finite speed of propagation statement: data at the input boundary $x=0$ cannot influence a neighborhood of $x=1$ before the travel time $\bar{\delta}$. Let us deal with the scalar case, and since (4.1) is diagonal the non-scalar case follows directly. Let $I(t) := (\bar{\delta}^{-1}(t-\tilde{t}),1)$ defined for $t \in [0,\bar{\delta})$, we want to study

$$E(t) := \int_{I(t)\cap(0,1)} |u^w(t,x)-u^z(t,x)|dx. \quad \text{(C.1)}$$

Note that E is continuous since u^w and u^z belong to $C([0,T];L^1(\mathbb{R}))$. Using the same doubling of variables as in Appendix B (see also [23, Theorem 1, (3.2)]) but with $\chi_\varepsilon(t,x)$ defined by

$$\chi_\varepsilon(t,x) := \begin{cases} 0 \ \text{ for } x \in [0,\bar{\delta}^{-1}(t-\tilde{t})_+], \\ (x-\bar{\delta}^{-1}(t-\tilde{t})_+)/\varepsilon \ \text{ for } x \in [\bar{\delta}^{-1}(t-\tilde{t})_+, \bar{\delta}^{-1}(t-\tilde{t})_+ + \varepsilon], \\ 1 \ \text{ for } x \in [\bar{\delta}^{-1}(t-\tilde{t})_+ + \varepsilon, 1-\varepsilon), \\ (1-x)/\varepsilon \text{ otherwise,} \end{cases} \quad \text{(C.2)}$$

[3]Again, this entropy is not C^1, however (3.2) still hold for continuous, convex and piecewise C^1 functions by approaching them weakly by convex C^1 functions.

one has

$$\int_0^T \int_{\bar\delta^{-1}(t-\tilde t)_+}^1 |u^w(t,x)-u^z(t,x)|\phi'(t)\,dx\,dt - \bar\delta^{-1}\int_0^T |u^w(t,\bar\delta^{-1}(t-\tilde t)_+)-u^z(t,\bar\delta^{-1}(t-\tilde t)_+)|\phi(t)dt$$
$$-\int_0^T \operatorname{sgn}(u^w(t,1)-v^z(t,1))(f(u^w(t,1))-f(u^z(t,1)))\phi(t)dt$$
$$+\int_0^T \operatorname{sgn}(u^w(t,\bar\delta^{-1}(t-\tilde t)_+)-u^z(t,\bar\delta^{-1}(t-\tilde t)_+)$$
$$(f(u^w(t,\bar\delta^{-1}(t-\tilde t)_+))-f(u^z(t,\bar\delta^{-1}(t-\tilde t)_+)))\phi(t)dt \geq 0. \quad \text{(C.3)}$$

where we used that $\partial_t\chi_\varepsilon(t,x) = -(\bar\delta\varepsilon)^{-1}\mathbf{1}_{\left(\bar\delta^{-1}(t-\tilde t)_+,\bar\delta^{-1}(t-\tilde t)_++\varepsilon\right)}(x)\mathbf{1}_{(\tilde t,T)}(t)$ (here $\mathbf{1}_{[a,b]}$ refers to the indicator function of $[a,b]$) and

$$\lim_{\varepsilon\to 0} -\frac{\bar\delta^{-1}}{\varepsilon}\mathbf{1}_{t>\tilde t}\int_0^T\int_{\bar\delta^{-1}(t-\tilde t)_+}^{\bar\delta^{-1}(t-\tilde t)_++\varepsilon} |u^w(t,x)-u^z(t,x)|\phi(t)dxdt =$$
$$-\bar\delta^{-1}\int_0^T |u^w(t,\bar\delta^{-1}(t-\tilde t)_+)-u^z(t,\bar\delta^{-1}(t-\tilde t)_+)|\phi(t)dt. \quad \text{(C.4)}$$

Using now that $\bar\delta^{-1} > |f'(s)|$ for any $s\in[-M,M]$ with

$$M := \max\{\|u^w\|_{L^\infty((0,T)\times(0,1))}, \|u^z\|_{L^\infty((0,T)\times(0,1))}\}$$

(see Proposition 4.2 and the definition of $\bar\delta$ given by (4.5)), we have

$$\int_{\tilde t}^T \Big[\operatorname{sgn}(u^w(t,\bar\delta^{-1}(t-\tilde t)_+)-u^z(t,\bar\delta^{-1}(t-\tilde t)_+)$$
$$(f(u^w(t,\bar\delta^{-1}(t-\tilde t)_+))-f(u^z(t,\bar\delta^{-1}(t-\tilde t)_+)))$$
$$-\bar\delta^{-1}|u^w(t,\bar\delta^{-1}(t-\tilde t)_+)-u^z(t,\bar\delta^{-1}(t-\tilde t)_+)|\Big]\phi(t)dt \leq 0, \quad \text{(C.5)}$$

and, since $w=z$ a.e. on $(0,\tilde t)$ by assumption, and using (4.1)

$$\int_0^{\tilde t} \operatorname{sgn}(u^w(t,0)-u^z(t,0))(f(u^w(t,0))-f(u^z(t,0)))\phi(t)dt = 0. \quad \text{(C.6)}$$

As a consequence from (C.5), (C.6), choosing ϕ as an approximation of $\mathbf{1}_{[0,t)}$ in (C.3), we have

$$\int_{\bar\delta^{-1}(t-\tilde t)}^1 |u^w(t,x)-u^z(t,x)|dx$$
$$\leq -\int_0^t \operatorname{sgn}(u^w(t,1)-v^z(t,1))(f(u^w(t,1))-f(u^z(t,1)))dt \leq 0. \quad \text{(C.7)}$$

Hence, $u^w = u^z$ a.e. on $(\bar\delta^{-1}(t-\tilde t)_+, 1)$ as long as $t<\tilde t+\bar\delta$ and since both solutions admit strong traces at $x=1$ we deduce that $u^w(t,1)=u^z(t,1)$ and this holds for a.e. $t\in(0,\tilde t+\bar\delta)$. □

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