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Improving the performance of low-gain designs for bounded control of linear systems ☆

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Abstract

Several solutions of the problem of stabilizing linear systems with bounded control rely on a one-parameter family of low-gain linear control laws $u = K(\varepsilon)x$. This paper presents an online scheduling of the parameter ensuring, in addition to closed-loop stability, the fastest possible transient between two extreme values of ε, chosen for stability and performance, respectively.

Keywords: Bounded control; Saturation; Low-gain design; Gain scheduling; Lyapunov design

1. Introduction

The design of feedback control laws for linear systems subject to magnitude constraint on the control variable,

$$\dot{x} = Ax + bu, \quad u \in \mathbb{R}, \ |u| \leqslant 1$$

is recognized as a significant nonlinear control problem, both theoretically and practically. Recently, special emphasis has been put on the design of stabilizing control laws that guarantee "large" regions of attraction (Sontag & Sussmann, 1990; Teel, 1992; Lin & Saberi, 1993; Megretski, 1996). Several of the proposed solutions rely on a one-parameter family of linear control laws $u = K(\varepsilon)x$. As the parameter $\varepsilon \to 0$, the norm $\|K(\varepsilon)\|$ decreases (hence the name "low-gain" designs), so that the magnitude constraint $|u| \leqslant 1$ is satisfied in a large domain. At the same time, the guaranteed region of attraction of $x = 0$ increases and may tend to the entire state space if A has no eigenvalue with strictly positive real part.

In such designs, the tuning of the low-gain parameter ε involves two conflicting objectives: on the one hand, large regions of attraction require small values of $\varepsilon > 0$ so that the input bound is never attained along the solutions. Obviously, this leads to cautious designs, resulting in slow convergence. In contrast, local performance dictates a larger value of ε, resulting in a reduced guaranteed region of attraction.

Based on the rationale that a small value ε_0 is needed far from the origin and that a larger value ε_f is needed close to the origin, the present paper proposes an online adaptation of $\varepsilon(t)$ aimed at the fastest possible evolution from ε_0 to ε_f while guaranteeing closed-loop stability for the a priori selected set of initial conditions. Without loss of generality, ε_f can be normalized to one ($\varepsilon_f = 1$).

Our design combines two ideas that have been recently discussed in the same context: the high–low gain philosophy of Saberi, Lin and Teel (1996), which explicitly uses the infinite gain margin of Riccati based linear control laws $u = K(\varepsilon)x$, and the gain scheduling proposed in Megretski (1996), where $\varepsilon(x)$ is chosen at every point so that x lies on the boundary of the guaranteed region of attraction.

Our algorithm enforces invariance of the manifold $K(\varepsilon)x = 0$ while enforcing the fastest possible increase of ε. An analytical example and simulations suggest that these heuristics lead to accelerated convergence of the closed-loop solutions.

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This paper is organized as follows. In Section 2, we expose methods to generate low-gain control laws. The general algorithm is presented in Section 3, while the design of the scheduling controller is detailed in Section 4. The algorithm is then illustrated on the double integrator in Section 5, with a comparison with the earlier scheme proposed in Megretski (1996) and the time-optimal solution. Finally, we give some conclusions.

2. Design and tuning of low-gain control laws

In this paper, we restrict our attention to saturated linear low-gain control laws in the form

$$u = -\mathrm{sat}(b^{\mathrm{T}}P(\varepsilon)x) \tag{1}$$

with $\mathrm{sat}(y) = \mathrm{sign}(y)\min(|y|, 1)$, based on the quadratic Control Lyapunov Functions $V(x, \varepsilon) = x^{\mathrm{T}}P(\varepsilon)x$.

The ε-family of control laws (1) can be generated in various ways; one method uses the Riccati equation

$$P(\varepsilon)A + A^{\mathrm{T}}P(\varepsilon) - P(\varepsilon)bb^{\mathrm{T}}P(\varepsilon) = -Q(\varepsilon), \quad \varepsilon \in (0, \infty) \tag{2}$$

with $Q(\varepsilon) > 0$ (positive definite), continuous, such that $\lim_{\varepsilon\to 0} Q(\varepsilon) = 0$ and $\mathrm{d}Q(\varepsilon)/\mathrm{d}\varepsilon > 0$ (Lin, Stoorvogel, & Saberi, 1996; Teel, 1995). Usually $Q(\varepsilon) = \varepsilon I$.

If (A, b) is asymptotically null controllable, that is (A, b) is stabilizable and all the eigenvalues of A are in the closed left half-plane, then $\lim_{\varepsilon\to 0} P(\varepsilon) = 0$, $A - bb^{\mathrm{T}}P(\varepsilon)$ is Hurwitz for all $\varepsilon > 0$, and $\mathrm{d}P(\varepsilon)/\mathrm{d}\varepsilon > 0$ (Lin, 1998). In the remainder of the paper, null controllability of the considered linear system will always be assumed. Efficient constructions of $P(\varepsilon) > 0$, that do not require the online solution of (2), are discussed in Megretski (1996).

For a fixed $\varepsilon > 0$, the maximal level set of $V(x, \varepsilon)$ inside which $|b^{\mathrm{T}}P(\varepsilon)x| \leqslant 1$ provides a guaranteed region of attraction for $x = 0$. This level is the solution of the optimization problem

$$\bar{V}(\varepsilon) = \min_{x\in\mathbb{R}^n} x^{\mathrm{T}}P(\varepsilon)x \quad \text{such that } b^{\mathrm{T}}P(\varepsilon)x = 1, \tag{3}$$

which yields $\bar{V}(\varepsilon) = 1/b^{\mathrm{T}}P(\varepsilon)b$ and $|b^{\mathrm{T}}P(\varepsilon)x| \leqslant 1$ within the set

$$\mathscr{T}(\varepsilon) = \left\{ x \in \mathbb{R}^n | x^{\mathrm{T}}P(\varepsilon)x \leqslant \frac{1}{b^{\mathrm{T}}P(\varepsilon)b} \right\}.$$

Finding $\varepsilon(x_0)$ then amounts to look for the largest ε such that x_0 lies within the set $\mathscr{T}(\varepsilon)$, that is

$$\varepsilon(x_0) = \max\{\eta \in (0, 1] : (x_0^{\mathrm{T}}P(\eta)x_0)(b^{\mathrm{T}}P(\eta)b) \leqslant 1\}. \tag{4}$$

The family of *low-gain* control laws (1) achieves semiglobal stabilization because $\mathscr{T}(\varepsilon) \to \mathbb{R}^n$ as $\varepsilon \to 0$. Because large regions of attraction require small values of the parameter, the resulting design is "cautious" and slow: it uses little actuation near the origin.

A first improvement of the above low-gain design is the "high–low" gain modification based on the observation that the control laws (1) have infinite gain margin. As a consequence, the region of attraction achieved with the control law $u = -\mathrm{sat}(kb^{\mathrm{T}}P(\varepsilon)x)$ still includes the set $\mathscr{T}(\varepsilon(x_0))$ with any gain $k \geqslant 1$. The limiting case for $k \to \infty$ results in the sliding mode control

$$u = -\mathrm{sign}(b^{\mathrm{T}}P(\varepsilon)x). \tag{5}$$

In this situation, full actuation is used throughout. However, the motion along the sliding surface $b^{\mathrm{T}}P(\varepsilon)x = 0$ induces chattering and can be very slow when ε is small.

On the other hand, the idea that ε should be small far from the origin (for stability) and larger near the origin (for performance) suggests that the performance of low-gain designs will improve with an on-line adaptation of ε. Megretski (1996) proposes the online adaptation of rule (4). It is shown that $V(x, \varepsilon(x))$ then decreases along the solutions in the whole state space. If the initial condition is far from the origin, the parameter $\varepsilon(x)$ will be initially small. However, it will increase as the solution approaches the origin. This gain scheduling can be stopped once ε has reached a value judged acceptable for local performance ($\varepsilon = 1$ in this paper).

The design flexibility of multiplying the control law by any gain $k \geqslant 1$ can be combined with online adaptation of ε, k proportional to $1/\varepsilon$ being suggested in Lin (1998).

3. Algorithm

The online adaptation of ε presented in this paper is based on the following observation: multiplying the low-gain control law by a large gain enforces the near-invariance of $\ker b^{\mathrm{T}}P(\varepsilon)$, at least near the origin. In this region, rendering this subspace truly invariant will be less conservative and more relevant than ensuring $\dot{V} < 0$ in the entire set $\mathscr{T}(\varepsilon(x_0))$. As a consequence, ε will be allowed to increase faster along the solutions.

A key point in this procedure is to characterize a reasonable set of initial conditions that can be steered to the origin under the two constraints $b^{\mathrm{T}}P(\varepsilon)x = 0$ and $|u(t)| \leqslant 1$. For a fixed $\varepsilon > 0$, invariance of $b^{\mathrm{T}}P(\varepsilon)x = 0$ implies convergence to the origin because

$$\dot{V}_{|\ker b^{\mathrm{T}}P(\varepsilon)} = x^{\mathrm{T}}PAx + x^{\mathrm{T}}A^{\mathrm{T}}Px = -x^{\mathrm{T}}Q(\varepsilon)x < 0.$$

For $\varepsilon > 0$ fixed, the following proposition identifies a subset of the subspace $\ker b^{\mathrm{T}}P(\varepsilon)$ that can be made invariant with $|u|$, based on a Lyapunov estimate:

Proposition 1. *Let* $\varepsilon > 0$, *the set*

$$\begin{aligned}\Omega_\varepsilon &= \ker b^{\mathrm{T}}P(\varepsilon) \cap \{x \mid q(x, \varepsilon) \leqslant g(\varepsilon)\} \\ &= \ker b^{\mathrm{T}}P(\varepsilon) \cap \{x \mid V(x, \varepsilon) \leqslant \bar{V}(\varepsilon)\}\end{aligned} \tag{6}$$

with

$$q(x,\varepsilon) = ((b^{\mathrm{T}}PAP^{-1}A^{\mathrm{T}}Pb)(b^{\mathrm{T}}Pb) - (b^{\mathrm{T}}PAb)^2)(x^{\mathrm{T}}Px),$$
$$g(\varepsilon) = (b^{\mathrm{T}}Pb)^3, \tag{7}$$
$$\bar{V}(\varepsilon) = \frac{(b^{\mathrm{T}}Pb)^3}{(b^{\mathrm{T}}PAP^{-1}A^{\mathrm{T}}Pb)(b^{\mathrm{T}}Pb) - (b^{\mathrm{T}}PAb)^2}$$

can be made controlled invariant with $|u| \leqslant 1$.

Proof. Invariance of the subspace $\ker b^{\mathrm{T}}P(\varepsilon)$ means

$$\frac{\mathrm{d}}{\mathrm{d}t}(b^{\mathrm{T}}P(\varepsilon)x) = b^{\mathrm{T}}P(\varepsilon)Ax + b^{\mathrm{T}}P(\varepsilon)bu = 0,$$

which imposes the control $u = -b^{\mathrm{T}}P(\varepsilon)Ax/b^{\mathrm{T}}P(\varepsilon)b$. When $\ker b^{\mathrm{T}}P(\varepsilon)$ is A-invariant, the whole subspace $\ker b^{\mathrm{T}}P(\varepsilon)$ can be made controlled invariant with $u = 0$ ($\leqslant 1$). Also, A-invariance implies $b^{\mathrm{T}}P(\varepsilon) = \alpha b^{\mathrm{T}}P(\varepsilon)A$ for some $\alpha \neq 0$. Then $q(x,\varepsilon) = 0$ and $\Omega_\varepsilon = \ker b^{\mathrm{T}}P(\varepsilon)$.

Otherwise, let $\bar{V}(\varepsilon)$ be the maximum level set of $V(x,\varepsilon)$ such that, when $V(x,\varepsilon) \leqslant \bar{V}(\varepsilon)$ and $b^{\mathrm{T}}P(\varepsilon)x = 0$, we have $|-b^{\mathrm{T}}P(\varepsilon)Ax/b^{\mathrm{T}}P(\varepsilon)b| \leqslant 1$. This amounts to find the minimal level set where the bound $|-b^{\mathrm{T}}P(\varepsilon)Ax/b^{\mathrm{T}}P(\varepsilon)b| = 1$ is attained when $b^{\mathrm{T}}P(\varepsilon)x = 0$, that is

$$\bar{V}(\varepsilon) = \min_{x\in\mathbb{R}^n} x^{\mathrm{T}}P(\varepsilon)x \tag{8}$$

s.t.

$$b^{\mathrm{T}}P(\varepsilon)Ax = b^{\mathrm{T}}P(\varepsilon)B \quad \text{and} \quad b^{\mathrm{T}}P(\varepsilon)x = 0$$

The unique solution of (8) is given by (7). □

Presumably, the additional constraint of (8) compared to (3) ensures a larger value of the level $\bar{V}(\varepsilon)$, so that similar performance can be obtained inside a larger set.

In the following, we denote $\Omega = \bigcup_{\varepsilon\in(0,1]} \Omega_\varepsilon = \{x \in \mathbb{R}^n | \exists\varepsilon \in (0,1] : b^{\mathrm{T}}P(\varepsilon)x = 0 \text{ and } V(x,\varepsilon) \leqslant \bar{V}(\varepsilon)\}$.

The proposed algorithm consists of three phases:

- Controller 1 (stabilizing controller): Steer x_0 to the interior of Ω with the static state feedback:

$$u = -\mathrm{sat}(k_0 b^{\mathrm{T}}P(\varepsilon_0)x) \quad k_0 \geqslant 1. \tag{9}$$

  This controller has a guaranteed basin of attraction $\mathscr{T}(\varepsilon_0)$ that can be made arbitrarily large by selecting ε_0 small enough.
- Controller 2 (scheduling controller): for $x \in \Omega$ and $\varepsilon \in (0,1]$, design a Lipschitz continuous dynamic feedback control law:

$$u = \alpha(x,\varepsilon),$$
$$\dot{\varepsilon} = \gamma(x,\varepsilon), \tag{10}$$

  which maximizes $\dot{\varepsilon}$ while making Ω invariant, under the constraints

$$b^{\mathrm{T}}P(\varepsilon)x = 0, \quad |u| \leqslant 1, \quad \varepsilon \leqslant 1.$$

  The construction of controller (10) is detailed in Section 4. Upon initialization with x in the interior of Ω and ε such that $b^{\mathrm{T}}P(\varepsilon)x = 0$, we prove convergence of $x(t)$ to the origin.
- Controller 3 (local controller): For $x \in \mathscr{T}(1) = \{x | x^{\mathrm{T}}P(1)x \leqslant 1/b^{\mathrm{T}}P(1)b\}$, apply the stabilizing feedback $u = -\mathrm{sat}(b^{\mathrm{T}}P(1)x)$. This controller has $\mathscr{T}(1)$ as guaranteed basin of attraction.

Along a closed-loop solution, the control law undergoes at most two discontinuities, determined by the switching times between Controllers 1 and 2, then between Controllers 2 and 3. Controllers 1 and 3 are initialized in $\mathscr{T}(\varepsilon_0)$ and $\mathscr{T}(1)$, respectively.

Initialization of Controller 2: Online checking that $x \in \Omega$ is not an obvious task because it requires to solve the nonlinear equation $b^{\mathrm{T}}P(\eta)x = 0$. One way to initialize Controller 2 properly is to fix some $\tilde{\varepsilon} > 0$ a priori and to wait until $x(T) \in \Omega_{\tilde{\varepsilon}}$ to initialize Controller 2 with $\varepsilon(T) = \tilde{\varepsilon}$. In fact, T is guaranteed to be finite if $\tilde{\varepsilon} = \varepsilon_0$ and $u = -\mathrm{sign}(b^{\mathrm{T}}P(\varepsilon_0)x)$, which is the limit of Controller 1 for $k_0 \to \infty$.

This strategy is feasible, but does not take advantage of the fact that $x(t)$ will usually enter the set Ω before it reaches $\Omega_{\tilde{\varepsilon}}$. Several options are possible to reduce this conservatism; a practical recommendation is to check the condition $x(t) \in \Omega_{\varepsilon_i}$ for a few ε_i in the interval $(0,1]$.

Performance evaluation: The proposed algorithm is designed to have the fastest possible scheduling from the "stabilizing controller" to the "local controller". The heuristics behind this performance criterion are that controllers $u = -\mathrm{sat}(kb^{\mathrm{T}}P(\varepsilon)x)$ are less cautious with large values of ε and perform "better" in that sense. Our algorithm is designed to speed up the transfer of ε from ε_0 to 1 (inside the set Ω). As it will be illustrated in Section 5, the set Ω extends well beyond $\mathscr{T}(1)$ and is not confined to a local neighborhood of the origin, which suggests an improvement of the behavior in a large domain of the state space.

4. The scheduling controller

The scheduling controller (10) is determined as the solution of

$$\max \dot{\varepsilon} \tag{11}$$

$$\text{s.t.} \quad \frac{\mathrm{d}}{\mathrm{d}t}(b^{\mathrm{T}}P(\varepsilon)x) = 0, \tag{12}$$

$$\dot{\varepsilon} \leqslant \mathrm{sat}_{[0,\dot{\varepsilon}_{\max}]}\left(\frac{\dot{\varepsilon}_{\max}}{\delta}(1-\varepsilon)\right)\mathrm{sat}_{[0,1]}\left(\frac{\bar{V}-V}{\delta\bar{V}}\right), \tag{13}$$

$$|u| \leqslant 1. \tag{14}$$

The solution of the problem (11)–(14), under the constraint that V stays smaller or equal to $\bar{V}$, achieves the fastest transfer of ε from ε_0 to 1. This might require $\dot{\varepsilon} = +\infty$ or a discontinuous control function. Constraint (13) and the parameters $\dot{\varepsilon}_{\max} \geqslant 1$ and $0 < \delta \ll 1$ then ensure the Lipschitz continuity of the feedback controller inside Ω. Constraint (13) is imposed so that ε and $V(x,\varepsilon)$ do not reach their upper bound 1 and $\bar{V}(\varepsilon)$ in finite time.

Theorem 2. *The solution of the optimization problem* (11)–(14) *is*

$$\dot{\varepsilon} = \min\left(\begin{array}{c} \mathrm{sat}_{[0,\dot{\varepsilon}_{\max}]}\left(\dfrac{\dot{\varepsilon}_{\max}}{\delta}(1-\varepsilon)\right)\mathrm{sat}_{[0,1]}\left(\dfrac{\bar{V}-V}{\delta\bar{V}}\right), \\ -\dfrac{b^{\mathrm{T}}PAx - b^{\mathrm{T}}Pb\,\mathrm{sign}(b^{\mathrm{T}}(\mathrm{d}P/\mathrm{d}\varepsilon)x)}{b^{\mathrm{T}}(\mathrm{d}P/\mathrm{d}\varepsilon)x}\end{array}\right), \quad (15)$$

$$u = -\frac{b^{\mathrm{T}}P(\varepsilon)Ax + b^{\mathrm{T}}(\mathrm{d}P(\varepsilon)/\mathrm{d}\varepsilon)x\dot{\varepsilon}}{b^{\mathrm{T}}P(\varepsilon)b} \quad (16)$$

When $\varepsilon(0)$ is such that $b^{\mathrm{T}}P(\varepsilon(0))x(0)=0$, the Controller (15)–(16) *renders the origin $x=0$ attractive with the interior of Ω as basin of attraction.*

Proof. The control law (16) is determined by (12). In (15), $\dot{\varepsilon}$ is selected as the minimum value that renders one of Constraints (13)–(14) active.

Invariance of Ω is direct from (12) and (13). Indeed, those ensure that $b^{\mathrm{T}}P(\varepsilon)x=0$, $V(x,\varepsilon)\leqslant\bar{V}(\varepsilon)$, and $\varepsilon\leqslant 1$ stay satisfied.

A problem of definition of solutions could arise on the boundary of Ω because the vector field is not defined outside Ω. Because this vector field is continuous inside Ω, its definition can formally be extended to the whole state space through Tietze's theorem, so that solutions are well defined in the classical sense on the boundary of Ω (invariance of Ω ensures that the exact knowledge of this extension is not necessary because $x(t)$ never quits Ω).

Next, we prove that $\varepsilon(t)$ is not decreasing along the closed-loop solutions. Indeed, $\dot{\varepsilon}=0$ is always admissible. The control law ensuring (12) is then $u=-b^{\mathrm{T}}P(\varepsilon)Ax/b^{\mathrm{T}}P(\varepsilon)b$, which satisfies (14) because $V(x,\varepsilon)\leqslant\bar{V}(\varepsilon)$. This guarantees that $\dot{\varepsilon}$, solution of (11)–(14), is non-negative.

Because $\varepsilon(t)$ is an increasing function with an upper bound, there exists $\bar{\varepsilon}\leqslant 1$ such that $\varepsilon(t)$ converges to $\bar{\varepsilon}\leqslant 1$ as $t\to\infty$.

Next, we prove that a solution $x(t)$ is bounded. The derivative of the Lyapunov function is

$$\dot{V}(x,\varepsilon) = -x^{\mathrm{T}}Q(\varepsilon)x + x^{\mathrm{T}}\frac{\mathrm{d}P(\varepsilon)}{\mathrm{d}\varepsilon}x\dot{\varepsilon}.$$

Continuity of $\mathrm{d}P(\varepsilon)/d\varepsilon>0$ on $[\varepsilon_0,\bar{\varepsilon}]$ implies that one can find $\alpha>0$ such that $x^{\mathrm{T}}(\mathrm{d}P(\varepsilon)/\mathrm{d}\varepsilon)x\leqslant\alpha V(x,\varepsilon)$, which implies

$$\dot{V}(x,\varepsilon)\leqslant\alpha\dot{\varepsilon}V(x,\varepsilon).$$

and

$$V(x(t),\varepsilon(t))\leqslant V(x_0,\varepsilon_0)e^{\int_0^t\alpha\dot{\varepsilon}(\tau)\,\mathrm{d}\tau}.$$

The ensuing upperbound $V(x(t),\varepsilon(t))\leqslant V(x_0,\varepsilon_0)\mathrm{e}^{\alpha(\bar{\varepsilon}-\varepsilon_0)}$ for all $t\geqslant 0$ guarantees boundedness of $x(t)$ along the solutions.

Invariance of $\Omega_{\varepsilon(x(t))}$ implies that $x(t)$ converges to the set $\Omega_{\bar{\varepsilon}}$. Boundedness of $x(t)$ implies that the limit set of $x(t)$ is compact and invariant, and that it is contained in the largest invariant set of $\Omega_{\bar{\varepsilon}}$, which is the origin. This completes the proof. □

Remark 3. The proposed Controller 2 steers every initial condition to the origin. This steering property is not robust to errors on $\varepsilon(0)$, since (15) and (16) enforces $b^{\mathrm{T}}P(\varepsilon(t))x(t)=b^{\mathrm{T}}P(\varepsilon(0))x(0)$ along the closed-loop solution. A remedy to this lack of robustness is to replace constraint (12) by

$$\frac{\mathrm{d}}{\mathrm{d}t}(b^{\mathrm{T}}P(\varepsilon)x) = -\gamma(b^{\mathrm{T}}P(\varepsilon)x),$$

where $\gamma>0$ is a damping parameter. It must also be noted that, in the proposed algorithm, the controller is only required to steer initial conditions in Ω to the set $\mathcal{T}(1)$. This property is robust to small errors on $\varepsilon(0)$, even with the original constraint (12).

We summarize the convergence properties of the proposed algorithm as follows:

- With any of the Controllers 1, 2, and 3, the equilibrium $x=0$ is an attractor of the closed-loop system with basin of attraction $\mathcal{T}(\varepsilon_0)$, Ω, and $\mathcal{T}(1)$, respectively.
- $x=0$ is locally exponentially stable and, for any $x_0\in\mathbb{R}^n$, there exists $\varepsilon_0\in(0,1]$ such that x_0 belongs to the basin of attraction when the following switching algorithm is used: apply Controller 1 until $x\in\Omega$; switch to Controller 2 with $\varepsilon(0)$ such that $b^{\mathrm{T}}P(\varepsilon(0))x=0$ and $V(x,\varepsilon(0))\leqslant\bar{V}(\varepsilon(0))$ and apply Controller 2 until $x\in\mathcal{T}(1)$; then switch to Controller 3 which guarantees local exponential stability.

5. The double integrator

The algorithm is now illustrated on the double integrator

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= u, \quad |u|\leqslant 1.\end{aligned} \quad (17)$$

The simplicity of this second-order system allows for analytical calculations: if we consider that the behavior obtained with the pole placement in $(-\sqrt{3}\pm\mathrm{i})/2$ (which is achieved with $Q=I$) is satisfying, we can solve the Riccati equation (2) with

$$Q(\varepsilon) = \begin{pmatrix}\varepsilon^4 & 0\\ 0 & \varepsilon^2\end{pmatrix} \quad (18)$$

so that the family of low-gain controls is then

$$u = -b^{\mathrm{T}}P(\varepsilon)x = -\varepsilon^2x_1 - \sqrt{3}\varepsilon x_2, \quad \varepsilon>0, \quad (19)$$

which is a typical low-gain control for second-order systems in Brunovski form. We see that the set Ω_ε is characterized by

$$x_2 + \frac{\varepsilon x_1}{\sqrt{3}} = 0 \quad\text{and}\quad x^{\mathrm{T}}P(\varepsilon)x\leqslant\frac{6\sqrt{3}}{\varepsilon} \quad (20)$$

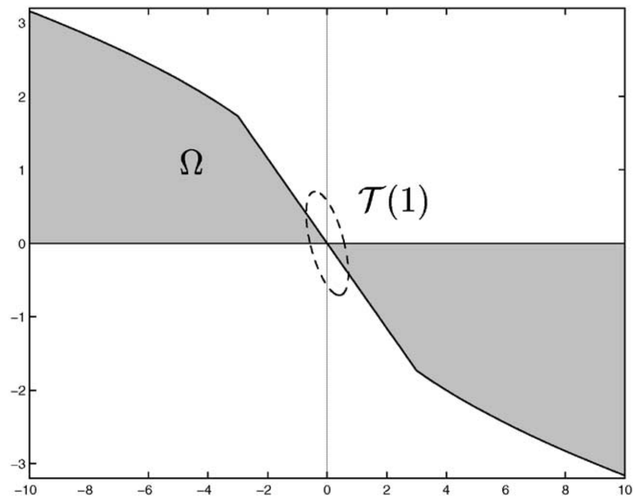

Fig. 1. The regions Ω (shaded) and $\mathcal{T}(1)$.

or, equivalently

$$x_2 + \frac{\varepsilon x_1}{\sqrt{3}} = 0 \quad \text{and} \quad \frac{\varepsilon|x_2|}{\sqrt{3}} \leqslant 1. \tag{21}$$

The boundary of the region Ω is, therefore, characterized by

$$V(x,\varepsilon) = \bar{V}(\varepsilon) : x_1 + x_2|x_2| = 0 \quad \text{and} \quad |x_2| \geqslant \sqrt{3},$$

$$\varepsilon = 1 : x_2 + \frac{x_1}{\sqrt{3}} = 0 \quad \text{and} \quad |x_2| \leqslant \sqrt{3},$$

$$\varepsilon = 0 : x_2 = 0.$$

This results in the region Ω, which is shaded in Fig. 1.

For the first controller, implemented only for initial condition outside Ω, we follow (9) and use

$$u = -\mathrm{sat}\left(k_0\left(x_2 + \frac{\varepsilon(x_0)x_1}{\sqrt{3}}\right)\right)$$

with

$$\varepsilon(x_0) = \max\left\{\begin{array}{l}\eta \in (0,1] : \\ \sqrt{3}\eta^3 x_{01}^2 + 2\eta^2 x_{01}x_{02} + \sqrt{3}\eta x_{02}^2 \leqslant \frac{1}{\sqrt{3}}\varepsilon\end{array}\right\}.$$

For Controller 2, in order to expose the geometry of the solution, we consider the limit case $\dot{\varepsilon}_{\max} = +\infty$ and $\delta = 0$. We use the maximum $\dot{\varepsilon}$, while verifying

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x_2 + \frac{\varepsilon x_1}{\sqrt{3}}\right) = 0 \equiv \frac{x_1\dot{\varepsilon}}{\sqrt{3}} + u + \frac{\varepsilon x_2}{\sqrt{3}}$$

and $V(x,\varepsilon) \leqslant \bar{V}(\varepsilon)$ $(\varepsilon|x_2|/\sqrt{3} \leqslant 1)$.

The control algorithm (15)–(16) is then

- When $V(x,\varepsilon) < \bar{V}(\varepsilon)$ and $\varepsilon < 1$:

$$\dot{\varepsilon} = \min\left\{+\infty, -\frac{-\sqrt{3}\,\mathrm{sign}(x_1) + \varepsilon x_2}{x_1}\right\}, \qquad u = -\frac{\varepsilon x_2 + x_1\dot{\varepsilon}}{\sqrt{3}} = -\mathrm{sign}(x_1). \tag{22}$$

- When $V(x,\varepsilon) = \bar{V}(\varepsilon)$ and $\varepsilon < 1$ (that is $x_1 + x_2|x_2| = 0$):

$$\dot{\varepsilon} = \min\left\{0, -\frac{-\sqrt{3}\,\mathrm{sign}(x_1) + \varepsilon x_2}{x_1}\right\}, \qquad u = -\frac{\varepsilon x_2 + x_1\dot{\varepsilon}}{\sqrt{3}} = -\mathrm{sign}(x_2). \tag{23}$$

- When $V(x,\varepsilon) \leqslant \bar{V}(\varepsilon)$ and $\varepsilon = 1$ (that is $x_2 + (x_1/\sqrt{3}) = 0$):

$$\dot{\varepsilon} = \min\left\{0, -\frac{-\sqrt{3}\,\mathrm{sign}(x_1) + x_2}{x_1}\right\}, \qquad u = -\frac{x_2 + x_1\dot{\varepsilon}}{\sqrt{3}} = -\frac{x_2}{\sqrt{3}}. \tag{24}$$

When $\delta = 0$, the closed-loop vector field becomes discontinuous on $\partial\Omega$ and solutions chatter between (22) and (23) on $\partial\Omega$ (as long as $\varepsilon(t) < 1$). This forces the solution to slide along the manifold $x_1 + x_2|x_2| = 0$ and results in the equivalent control (see Utkin, 1992)

$$\dot{\varepsilon} = \frac{\varepsilon^2}{2\sqrt{3}},$$

$$u = -\tfrac{1}{2}\mathrm{sign}(x_2).$$

The equivalent control is thus the control $u = \pm\frac{1}{2}$ that keeps the manifold $x_1 + x_2|x_2| = 0$ invariant (this equivalent control is what is used to draw Fig. 2). The chattering phenomenon disappears when $\delta > 0$.

If no upper bound is chosen for ε, such that it can diverge towards $+\infty$, a finite escape time is observed for ε, which corresponds to finite time convergence of the solutions to the origin. The resulting control law is reminiscent of the time-optimal solution. The control law is bang–bang and the switching surface is the one of the time-optimal control for a constraint $|u| \leqslant \frac{1}{2}$.

If the adaptation of $\varepsilon(t)$ is stopped once ε reaches 1, controller (24) is used, which forces invariance of $x_2 + (x_1/\sqrt{3}) = 0$, and $x(t)$ converges to $\mathcal{T}(1)$.

Our control algorithm and the control algorithm proposed in Megretski (1996) can be compared based on the time taken by both schemes to reach the set $\mathcal{T}(1)$ (the ellipses on Figs. 1 and 2). Faster convergence of the proposed algorithm results from a faster adaptation of ε (see Fig. 2): for a solution to reach $\mathcal{T}(1)$ from the initial condition $(-10, 0)$, the stars indicate that it takes 6.79 for our control algorithm versus 10.25 for the algorithm in Megretski (1996), to be compared with 5.64 for the time-optimal solution.

6. Conclusion

The control algorithm presented in this paper aims at improving the control performance of linear systems with bounded input. Starting from low-gain designs $u = -b^{\mathrm{T}}P(\varepsilon)x$ with infinite gain margin, we take advantage of the fact that, with high gain, the condition $b^P(\varepsilon)x \approx 0$ holds after a finite time. We explicitly use this observation in the adaptation rule $\dot{\varepsilon}$, which leads to less conservative designs. Our

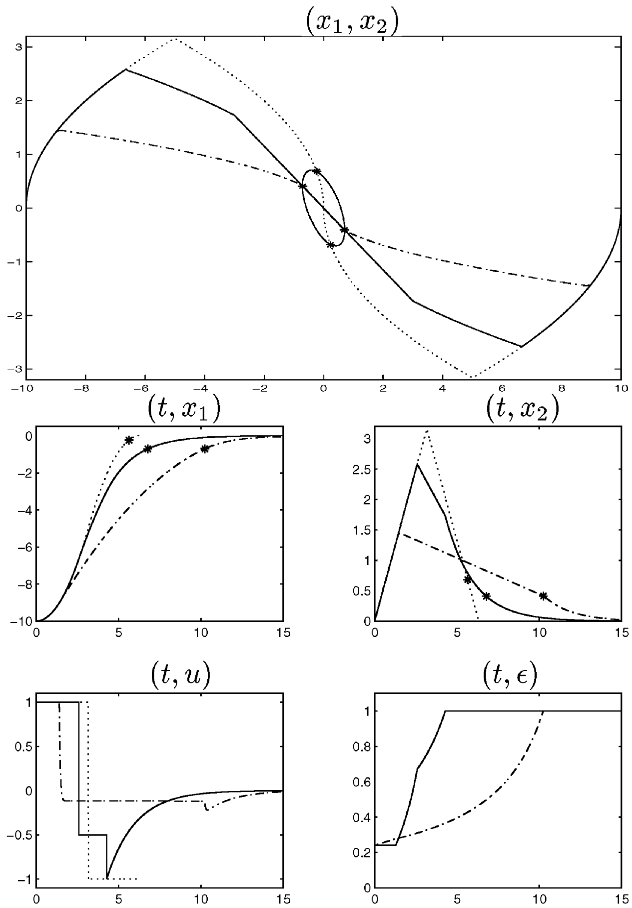

Fig. 2. Phase plane and time evolutions with $(x_{10}, x_{20}) = (-10, 0)$: time-optimal method (dotted), controller of Megretski (1996) with $k = 1/\varepsilon$ (dash–dotted) and proposed controller. Stars indicate the time instants or the points in state space at which the solutions reach $\mathscr{T}(1)$.

design can be seen as a sliding mode design for which the sliding surface is calculated online. Though this paper only deals with single input systems, the design can be extended to multiple inputs systems. This algorithm can also be extended to linear systems subject to other kinds of constraints: input rate and magnitude constraints and affine constraints (see Grognard, Sepulchre, & Bastin, 2000).

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