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Boundary feedback control in networks of open channels☆

J. de Halleux[a,*], C. Prieur[b], J.-M. Coron[c], B. d'Andréa-Novel[d], G. Bastin[a,d]

[a]Centre for Systems Engineering and Applied Mechanics (CESAME), Université Catholique de Louvain, Bâtiment Euler, 4–6, Avenue Georges Lemaitre, 1348 Louvain-la-Neuve, Belgium
[b]CNRS, Laboratoire SATIE, ENS de Cachan, 61, Avenue du President Wilson, 94235 Cachan, France
[c]Département de Mathématiques, Analyse numérique et EDP, Université Paris-Sud, Bâtiment 425 91405 Orsay, France
[d]Centre de Robotique, École des mines de Paris, 60, Bvd Saint Michel 75272 Paris, Cedex 06, France

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Abstract

This article deals with the regulation of water flow in open-channels modelled by Saint-Venant equations. By means of a Riemann invariants approach, we deduce stabilizing control laws for a single horizontal reach without friction. The stability condition is extended to a general class of hyperbolic systems which can describe canal networks with more general topologies. A control law design based on this condition is illustrated with a simple case study: two reaches in cascade. The proof of the main stability theorem is based on a previous result from Li Ta-tsien concerning the existence and decay of classical solutions of hyperbolic systems.

Keywords: Partial differential equations; Hyperbolic systems; Stabilization; Water management

1. Introduction

The so-called *Saint-Venant equations* are the partial differential equations (PDE) that are commonly used in hydraulics to describe the flow of water in open channels (see e.g. the textbooks in Chow, 1954 or Graf, 1998). These equations are a standard tool for solving engineering problems regarding the dynamics of canals and rivers. In this paper we will focus our attention on canals made up of a cascade of reaches delimited by underflow gates. Such systems typically occur in canalized water-ways and irrigation networks. But we shall see that the results of the paper are directly applicable to more complicated networks of canals and other kinds of control gates.

We address the problem of regulating the water level and the water velocity in a channel by using the gate openings as control actions.

This problem has been considered for a long time in the literature as reported in the survey paper Malaterre, Rogers and Schuurmans (1998) which involves a comprehensive bibliography. Starting from rudimentary and heuristic feedback control approaches, various advanced control methods where progressively investigated. Among other relevant references, we may mention for instance:

- LQ control methods which have been especially developed and studied in Balogun, Hubbard, and De Vries (1988), Garcia, Hubbard, and De Vries (1992) and Malaterre (1998). On the basis of finite-dimensional discrete linear approximations of the Saint-Venant equations.
- Robust H_∞ control design techniques which are developed in Litrico and Georges (2001) and Litrico (2001) on the basis of a model approximation by a simple linear diffusive wave equation.
- Boundary PI regulation which is analyzed in Xu and Sallet (1999) on the basis of a linear PDE model around a steady state.

In this paper, we go a step further since the control design is derived and analyzed directly from the nonlinear

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* Corresponding author. Fax: +32-104-723-80.
E-mail addresses: dehalleux@auto.ucl.ac.be (J. de Halleux), christophe.prieur@satie.ens-cachan.fr (C. Prieur), jean-michel.Coron@math.u-psud.fr(J.-M. Coron), andrea@caor.ensmp.fr (B. d'Andréa-Novel), bastin@auto.ucl.ac.be (G. Bastin).

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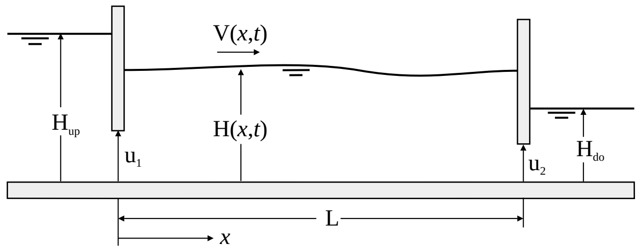

Fig. 1. An horizontal reach delimited by underflow gates.

Saint-Venant partial differential equations without any model approximation, linearization or discretisation. This approach has also been used previously in Coron, d'Andréa-Novel, and Bastin (1999) and Leugering and Schmidt (2002).

In Section 2, we start our analysis with the flow modelling in the special case of a single reach. Two different forms of the model, respectively, in terms of flow velocity and Riemann invariants are successively established. Sufficient conditions for the system stability are then stated in Theorem 1. This theorem is due to Greenberg and Li (1984).

Section 3 deals with the boundary control design in a single horizontal reach without friction. A control law is proposed on the basis of the Riemann invariants whose stabilizability is analyzed as an application of Theorem 1. Some illustrative simulation experiments of the control law are given in Section 4.

The main result of the paper is presented in Section 5. The aim is to generalize the previous result to open channels made up of several interconnected reaches in cascade. For the sake of clarity, we treat the special case of two reaches in cascade. Our stability result is given in Theorem 4 which, as we will see in Appendix, is a consequence of a theorem due to Li (1994).

The theorem provides a sufficient stability condition which can be applied to the stability analysis of canal networks having more general topologies (like for instance the star configurations considered in Leugering & Schmidt, 2002).

Some conclusions are given in Section 6.

2. Modelling in open channels

2.1. Saint-Venant equations

Let us consider a one-dimensional portion of a canal delimited by two underflow gates as depicted in Fig. 1 under the following modelling assumptions:

- the canal is horizontal,
- the canal is prismatic with a constant rectangular cross section and a unit width,
- the friction effects due to walls are neglected.

The dynamics of the system are then described by the Saint-Venant equations Saint-Venant (1871) (also called shallow water equations):

$$\frac{\partial}{\partial t}\begin{pmatrix} H \\ V \end{pmatrix} + A(H,V)\frac{\partial}{\partial x}\begin{pmatrix} H \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad (1)$$

where $A(H,V)$ is the characteristic matrix

$$A(H,V) = \begin{pmatrix} V & H \\ g & V \end{pmatrix},$$

$x \in [0,L]$ is the space coordinate, L is the length of the reach, t is time, $V(x,t)$ and $H(x,t)$ are the water velocity and depth (at point x and time t) and g is the gravity constant.

The control actions are provided by two underflow gates located at the left end ($x=0$) and the right end ($x=L$) of the reach (see Fig. 1). The gate openings are denoted by u_1 and u_2. A standard discharge relationship for under flow gates is as follows:

$$V(0,t)\,|\,V(0,t)\,|\,H^2(0,t) = u_1(H_{\mathrm{up}} - H(0,t)), \qquad (2)$$

$$V(L,t)\,|\,V(L,t)\,|\,H^2(L,t) = u_2(H(L,t) - H_{\mathrm{do}}). \qquad (3)$$

The left and right water levels outside the reach, denoted H_{up} and H_{do}, are supposed to be constant and satisfy the inequality $H_{\mathrm{up}} > H_{\mathrm{do}}$. Eqs. (2) and (3) constitute the boundary conditions at $x=0$ and L, associated with the PDEs (1).

2.2. Steady states

For given constant openings $\bar{u}_1 > 0$ and $\bar{u}_2 > 0$ there exists a constant steady-state solution $(\bar{H},\bar{V})$ of Eq. (1) which satisfies, from (2) and (3), the following relations:

$$\bar{H} = \frac{\bar{u}_1 H_{\mathrm{up}} + \bar{u}_2 H_{\mathrm{do}}}{\bar{u}_1 + \bar{u}_2}, \qquad (4)$$

$$\bar{V} = \frac{1}{\bar{H}}\sqrt{\frac{\bar{u}_1\bar{u}_2}{\bar{u}_1 + \bar{u}_2}(H_{\mathrm{up}} - H_{\mathrm{do}})}. \qquad (5)$$

By inverting these relations, it is interesting to note that any arbitrary steady state $(\bar{H},\bar{V})$ satisfying the conditions

$$H_{\mathrm{do}} < \bar{H} < H_{\mathrm{up}} \quad \text{and} \quad 0 < \bar{V},$$

can be assigned by an appropriate choice of $\bar{u}_1$ and $\bar{u}_2$.

2.3. Characteristic velocities

The eigenvalues c_α and c_β of the characteristic matrix $A(H,V)$,

$$c_\alpha(H,V) = V + \sqrt{gH} \quad \text{and} \quad c_\beta(H,V) = V - \sqrt{gH}, \qquad (6)$$

are called the characteristic velocities of the fluid in the canal. The flow is said to be *subcritical* or *fluvial* if the characteristic velocities are of opposite sign:

$$c_\beta(H,V) < 0 < c_\alpha(H,V),$$

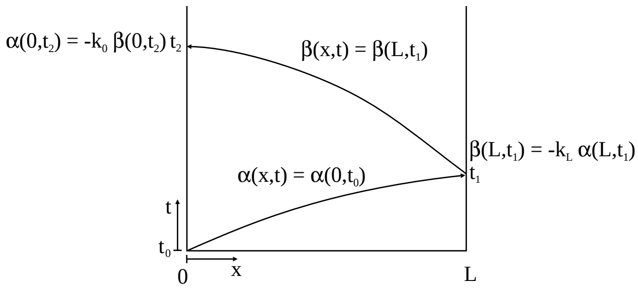

Fig. 2. Following the invariants along the characteristic curves.

which is equivalent to

$|V| < \sqrt{gH}$.

2.4. Model in terms of Riemann invariants

Let us now consider the following change of coordinates:

$$\alpha = V - \bar{V} + 2(\sqrt{gH} - \sqrt{g\bar{H}}),$$
$$\beta = V - \bar{V} - 2(\sqrt{gH} - \sqrt{g\bar{H}}). \tag{7}$$

With these new coordinates (α, β) system (1) is rewritten in the following diagonal form:

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} + \begin{pmatrix}c_\alpha(\alpha,\beta) & 0\\ 0 & c_\beta(\alpha,\beta)\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}, \tag{8}$$

where c_α and c_β are the characteristic velocities now expressed in terms of α, β:

$$c_\alpha(\alpha,\beta) = \tfrac{3}{4}\,\alpha + \tfrac{1}{4}\,\beta + \bar{V} + \sqrt{g\bar{H}},$$

$$c_\beta(\alpha,\beta) = \tfrac{1}{4}\,\alpha + \tfrac{3}{4}\beta + \bar{V} - \sqrt{g\bar{H}}.$$

The solutions $\alpha(x,t)$ and $\beta(x,t)$ of (8) are classically called *Riemann invariants* (see e.g. Serre, 1996, Vol. I, p. 96). The reason is that, for any given smooth solution $H(x,t)$, $V(x,t)$ of (1), the corresponding solutions $\alpha(x,t)$ and $\beta(x,t)$ of (8) are easily shown to be constant along the characteristic curves (see Fig. 2 in Section 3.2) $(x(t),t)$ which are defined as

for α, $\quad \dfrac{dx}{dt} = c_\alpha(\alpha(x,t),\beta(x,t)) > 0,$

for β, $\quad \dfrac{dx}{dt} = c_\beta(\alpha(x,t),\beta(x,t)) < 0.$

Note that the change of coordinates (7) is a bijection.

2.5. Stability analysis

It is obvious that the equilibrium $\bar{H}$, $\bar{V}$ expressed in the α, β coordinates is

$\bar{\alpha} = 0 \quad$ and $\quad \bar{\beta} = 0.$

The stability of the flow in a neighborhood of this steady state in a single reach can be analyzed with the following theorem. The theorem is stated here in a rather general form because it will be used also later on for the control stability analysis.

We consider the Saint-Venant equations (8), expressed in (α,β) coordinates, defined on the domain $(x,t) \in [0,L] \times [0,\infty)$. The boundary conditions are supposed to be given in the following general form:

$f_0(\alpha_0,\beta_0) = 0 \quad$ and $\quad f_L(\alpha_L,\beta_L) = 0,$

where $(\alpha_0,\beta_0) := (\alpha(0,t),\beta(0,t))$ and $(\alpha_L,\beta_L) := (\alpha(L,t),\beta(L,t))$, and with the functions f_0 and f_L being of class C^1. By differentiating these boundary conditions with respect to t and using Eq. (8), we have the following so-called *boundary compatibility conditions* on the initial state $(\alpha(x,0),\beta(x,0)) = (\alpha^\#(x),\beta^\#(x))$:

$$f_0(\alpha^\#(0),\beta^\#(0)) = 0, \qquad f_L(\alpha^\#(L),\beta^\#(L)) = 0,$$

$$\partial_\alpha f_0(\alpha^\#(0),\beta^\#(0))c_\alpha(\alpha^\#(0),\beta^\#(0))\partial_x\alpha^\#(0) + \partial_\beta f_0(\alpha^\#(0),\beta^\#(0))c_\beta(\alpha^\#(0),\beta^\#(0))\partial_x\beta^\#(0) = 0,$$

$$\partial_\alpha f_L(\alpha^\#(L),\beta^\#(L))c_\alpha(\alpha^\#(L),\beta^\#(L))\partial_x\alpha^\#(L) + \partial_\beta f_L(\alpha^\#(L),\beta^\#(L))c_\beta(\alpha^\#(L),\beta^\#(L))\partial_x\beta^\#(L) = 0. \tag{9}$$

To state the following result, we need to define the classical norms on $C^0([0,L])$ and $C^1([0,L])$. Given Φ continuous on $[0,L]$ and Ψ differentiable continuous on $[0,L]$, we denote

$$|\Phi|_{C^0([0,L])} = \max_{x\in[0,L]}|\Phi(x)|,$$

$$|\Psi|_{C^1([0,L])} = |\Psi|_{C^0([0,L])} + |\Psi'|_{C^0([0,L])}.$$

Theorem 1. *Assume that the initial conditions* $(\alpha^\#(x),\beta^\#(x)) \in C^1([0,L])^2$ *satisfy the boundary compatibility conditions* (9) *and that the following inequality holds*:

$A_0A_L < 1$

with

$$A_0 = \left|\frac{(\partial f_0/\partial\beta_0)(0,0)}{(\partial f_0/\partial\alpha_0)(0,0)}\right| \quad \text{and}$$
$$A_L = \left|\frac{(\partial f_L/\partial\alpha_L)(0,0)}{(\partial f_L/\partial\beta_L)(0,0)}\right|. \tag{10}$$

Then, for all $\mu > 0$ *such that*

$$\mu < \frac{(\bar{V}^2 - g\bar{H})\ln(A_0A_L)}{2L\sqrt{g\bar{H}}}, \tag{11}$$

there exist positive constants ε, M, *such that, if the initial condition is small enough*:

$$|\alpha^\#(\cdot)|_{C^1([0,L])} + |\beta^\#(\cdot)|_{C^1([0,L])} \leqslant \varepsilon,$$

there is a unique solution $\alpha(x,t)$, $\beta(x,t)$ *of class* C^1 *on* $[0,L]\times[0,\infty)$ *which decays to zero with an exponential rate*: $\forall t\in[0,+\infty[$,

$$|\alpha(\cdot,t)|_{C^1([0,L])}+|\beta(\cdot,t)|_{C^1([0,L])}\leqslant M\mathrm{e}^{-\mu t}.$$

If the product $A_0A_L=0$, *the right-hand side of* (11) *is equal to* $+\infty$.

This theorem, which is a direct application of Theorem 2 in Greenberg and Li (1984), can be used for instance for the stability analysis of the steady state in a single reach with *constant* gate openings $\bar{u}_1$ and $\bar{u}_2$. In that case, indeed, the functions f_0 and f_L are as follows:

$$f_0(\alpha_0,\beta_0)=\left(\bar{V}+\frac{\alpha_0+\beta_0}{2}\right)^2\left(\frac{(\alpha_0-\beta_0+4\sqrt{g\bar{H}})^2}{16g}\right)^2 -\bar{u}_1\left(H_{\text{up}}-\frac{(\alpha_0-\beta_0+4\sqrt{g\bar{H}})^2}{16g}\right),$$

$$f_L(\alpha_L,\beta_L)=\left(\bar{V}+\frac{\alpha_L+\beta_L}{2}\right)^2\left(\frac{(\alpha_L-\beta_L+4\sqrt{g\bar{H}})^2}{16g}\right)^2 -\bar{u}_2\left(\frac{(\alpha_L-\beta_L+4\sqrt{g\bar{H}})^2}{16g}-H_{\text{do}}\right).$$

It follows that A_0, A_L are given by

$$A_0=\left|\frac{2\bar{V}\bar{H}(\bar{H}\sqrt{g}-\bar{V}\sqrt{\bar{H}})-\bar{u}_1\sqrt{\bar{H}}}{2\bar{V}\bar{H}(\bar{H}\sqrt{g}+\bar{V}\sqrt{\bar{H}})+\bar{u}_1\sqrt{\bar{H}}}\right|,$$

$$A_L=\left|\frac{2\bar{V}\bar{H}(\bar{H}\sqrt{g}+\bar{V}\sqrt{\bar{H}})-\bar{u}_2\sqrt{\bar{H}}}{2\bar{V}\bar{H}(\bar{H}\sqrt{g}-\bar{V}\sqrt{\bar{H}})+\bar{u}_2\sqrt{\bar{H}}}\right|.$$

If the gate openings $\bar{u}_1$ and $\bar{u}_2$ are chosen such that $A_0A_L>1$, the steady state is unstable and the trajectories diverge as it will be illustrated in Section 4.2. In such a case, the stabilization of the steady state may be achieved with the feedback control techniques that are presented hereafter.

3. Feedback control design for a single reach

3.1. Statement of the control problem

The control objective is to regulate system (1) at the set point $(\bar{H},\bar{V})$. The control actions are the two gate openings u_1 and u_2. The water levels $H(0,t)$ and $H(L,t)$ are supposed to be measured online at each time instant t. The external constant water levels H_{up} and H_{do} are known.

3.2. Control design based on Riemann invariants

From (7), it is obvious that the set point $(\bar{H},\bar{V})$ expressed in the (α,β) coordinates is

$$\bar{\alpha}=0 \quad\text{and}\quad \bar{\beta}=0.$$

The control objective can thus be reformulated as the problem of finding boundary controls able to regulate $\alpha(x,t)$ and $\beta(x,t)$ at zero.

Consider a solution $\alpha(x,t)$ along its characteristic curve, starting from $(0,t_0)$. By the invariance property and for $|\alpha(\cdot,0)|_{C^0([0,L])}+|\beta(\cdot,0)|_{C^0([0,L])}$ sufficiently small, there exist obviously a time instant $t_1>t_0$ such that $\alpha(L,t_1)=\alpha(0,t_0)$. Suppose now that we are able to apply a boundary control at the right gate $(x=L)$ such that $\beta(L,t_1)=-k_L\alpha(L,t_1)$ with $0<k_L$. Obviously, there exists a time instant $t_2>t_1$ such that $\beta(0,t_2)=\beta(L,t_1)$. We now apply a boundary control at the left gate such that $\alpha(0,t_2)=-k_0\beta(0,t_2)$ with $0<k_0$ and so on. This implies clearly that, for any arbitrary t_0 there is a monotonically increasing sequence of time instants t_i, $i=0,1,2,\dots$ such that

$$\alpha(0,t_{2j})=(k_0k_L)^j\alpha(0,t_0),$$

$$\beta(L,t_{2j+1})=(k_0k_L)^j\beta(L,t_1),\qquad j=1,2,\dots\,.$$

We choose k_0 and k_L such that $0\leqslant k_0k_L<1$, this allows to understand why this boundary control will guarantee the convergence of $\alpha(x,t)$ and $\beta(x,t)$ to zero. The required boundary control is thus implicitly defined as

$$\alpha(0,t)=-k_0\beta(0,t)\quad\text{and}\quad \beta(L,t)=-k_L\alpha(L,t). \tag{12}$$

By using the change of coordinates (7), we get the following explicit expressions in the (H,V) coordinates:

$$V_0=\bar{V}-\lambda_0(\sqrt{gH_0}-\sqrt{g\bar{H}}),$$

$$V_L=\bar{V}+\lambda_L(\sqrt{gH_L}-\sqrt{g\bar{H}})$$

with

$$\lambda_0=2\,\frac{1-k_0}{1+k_0}\quad\text{and}\quad \lambda_L=2\,\frac{1-k_L}{1+k_L}.$$

The feedback control actions (i.e. the gate openings) are then deduced from (2) and (3):

$$u_1=\frac{(\bar{V}-\lambda_0(\sqrt{gH_0}-\sqrt{g\bar{H}}))^2H_0^2}{H_{\text{up}}-H_0},$$

$$u_2=\frac{(\bar{V}-\lambda_L(\sqrt{gH_L}-\sqrt{g\bar{H}}))^2H_L^2}{H_L-H_{\text{do}}}. \tag{13}$$

Obviously, these control laws are positive and well defined only if $H_{\text{up}}>H_0$ and $H_L>H_{\text{do}}$. The stability of the closed-loop system in a neighborhood of the set point is trivially analyzed with Theorem 1. Indeed, the functions f_0 and f_L representing the boundary conditions are immediately given by (12)

$$f_0(\alpha_0,\beta_0)=\alpha_0+k_0\beta_L,$$

$$f_L(\alpha_L,\beta_L)=\beta_L+k_L\alpha_L.$$

It follows readily that $A_0=k_0$, $A_L=k_L$ and the stability condition $A_0A_L<1$ is satisfied for any positive constants k_0,k_L such that $0\leqslant k_0k_L<1$.

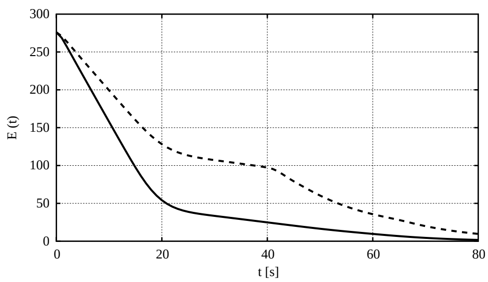

Fig. 3. Evolution of the entropy. Legend: solid line for Riemann control, dashed for open-loop control.

It must be emphasized that the practical implementation of the feedback control law (13) is very simple since it involves on-line measurements of the water level $H_0(t)$ and $H_L(t)$ at the gates only. Thus, neither water level measurements inside the reach nor any water velocity measurements (which are much more difficult to carry on in practice) are needed. Furthermore, the control laws are totally *decentralized*: the control u_1 at the left gate depends only on the local measurement H_0 at the same gate but not on the measurement H_L at the other gate, and vice versa.

4. Simulation experiments

4.1. Comparison with a unit-step open-loop control law

The control design method is illustrated with some realistic simulation experiments. In this section, we consider a small channel which is typical in local irrigation networks. Simulations for larger waterways will be given in the next section. The simulation parameters are $L = 100$ m, width $\ell = 1$ m, $H_{\text{up}} = 2$ m, $H_{\text{do}} = 0.5$ m, $H(x, 0) = 1.4$ m, $Q(x, 0) = 3\ \text{m}^3\ \text{s}^{-1}$, $\bar{H} = 0.7$ m, $\bar{Q} = 1\ \text{m}^3\ \text{s}^{-1}$.

The Saint-Venant equations are integrated numerically using a standard Preissman scheme (see e.g. Graf, 1998, Chapter 5) with a spatial step size $\Delta x = 1$ m, a time step $\Delta t = 1$ s and a weighting coefficient $\theta = 0.57$. Along the solutions, we will keep a subcritical flow with a Froude number around 0.35. The Riemann control is implemented with gain values

$$k_0 = 0.1 \quad \text{and} \quad k_L = 0.45.$$

These gain values have been roughly optimized in order to get reasonably smooth control actions.

The deviation of the water state with respect to the equilibrium is measured by the entropy of the fluid (see Coron et al., 1999):

$$E = \int_0^L H \frac{(V - \bar{V})^2}{2} + g \frac{(H - \bar{H})^2}{2} \, \mathrm{d}x. \tag{14}$$

In Fig. 3, the evolution of the entropy E is represented for the closed-loop system and compared with the open-loop system where unit-step gate openings are applied at the initial time instant (i.e. without any on-line feedback). In this figure, we can first observe that the Riemann control law effectively stabilizes the system at the desired set point. More important, we can appreciate the acceleration of the convergence compared with the open-loop behavior.

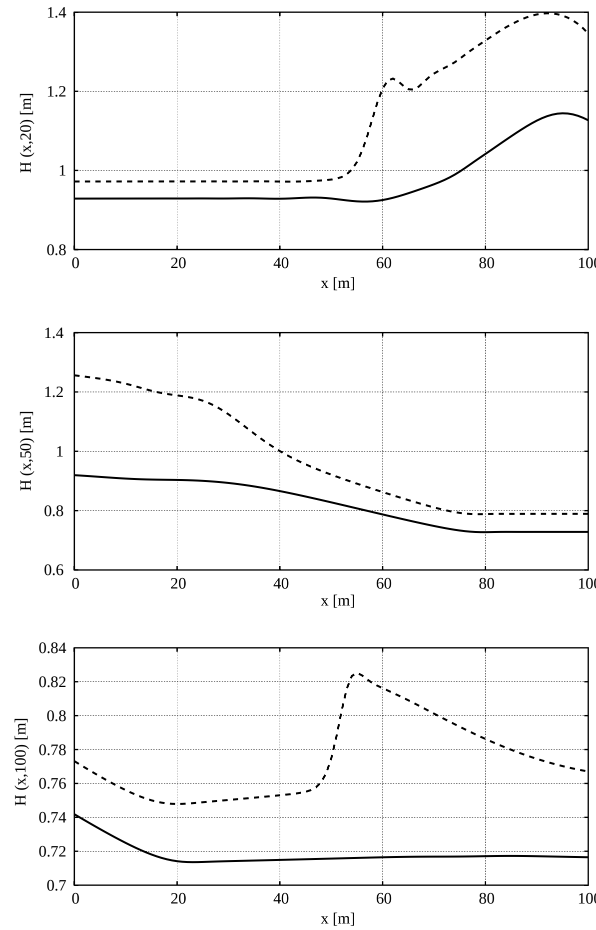

Fig. 4. Water depth curves $H(x, t)$ for $t = 20, 50, 100$ s. Legend: solid line for Riemann control, dashed for open-loop control.

In Fig. 4, the profiles of the water levels in the channel are displayed at time instants $t = 20$, 50 and $t = 100$ s. It can be observed that the closed-loop control strategy provides a fast convergence compared to the open-loop control with waves of relatively small amplitudes.

In Fig. 5, the evolutions of the control actions $u_1(t)$ and $u_2(t)$ are displayed.

4.2. Simulation with an unstable open-loop

We now consider simulation conditions such that $A_0 A_L > 1$ for the open-loop control: $H_{\text{up}} = 2$ m, $H_{\text{do}} = 0.1$ m, $L = 12$ m, $\bar{H} = 1.93$ m, $\bar{V} = 1.76\ \text{ms}^{-1}$. For these settings, the product $A_0 A_L = 1.05$. The open-loop is compared with a closed-loop control with tuning parameters $k_0 = 0.2$ and $k_L = 0.8$.

The evolution of the entropy is depicted in Fig. 6. One can see that the feedback control loop effectively stabilizes the canal while the open loop diverges.

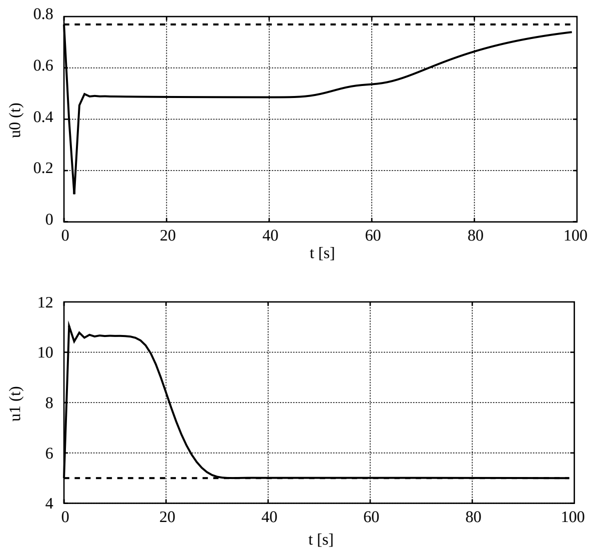

Fig. 5. Gate openings. Up: u_1, Down: u_2. Legend: solid line for Riemann control, dashed for open-loop control.

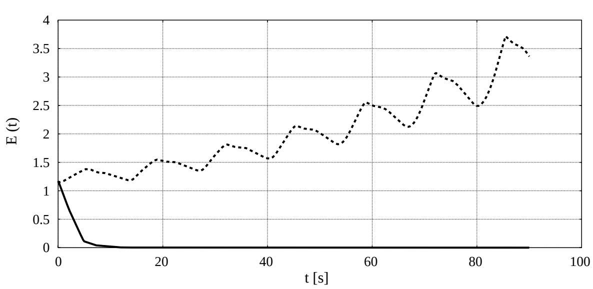

Fig. 6. Evolution of entropy. Legend: solid line for Riemann control, dashed for open-loop control.

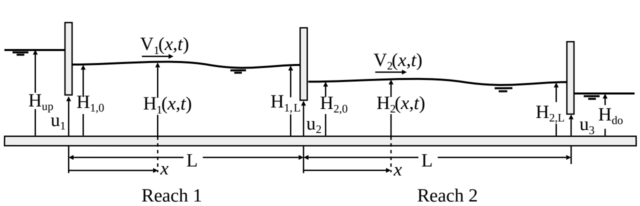

Fig. 7. A canal with two reaches and three underflow gates.

5. Control of multireach canals

The aim of this section is to generalize the previous sufficient stability condition to open channels made up of several interconnected reaches and to show how this condition can be used for control law design.

However, for the sake of clarity, we shall treat explicitly the special case of two reaches in cascade separated by three underflow gates as depicted in Fig. 7.

5.1. Modelling

The following notations are introduced (see Fig. 7 and Section 2.1 above). For the sake of simplicity, we assume without loss of generality that the two reaches have the same length L.

$H_i(x,t)$ is the water level in the ith reach ($i = 1,2$):

$$H_{i,0} = H_i(0,t) \quad \text{and} \quad H_{i,L} = H_i(L,t).$$

$V_i(x,t)$ is the water velocity in the ith reach:

$$V_{i,0} = V_i(0,t) \quad \text{and} \quad V_{i,L} = V_i(L,t).$$

H_{up} and $H_{\text{do}}(< H_{\text{up}})$ are the left and right water levels outside the canal. The Saint-Venant equations (1) are written for each reach $i = 1,2$:

$$\frac{\partial}{\partial t}\begin{pmatrix} H_i \\ V_i \end{pmatrix} + A(H_i,V_i)\frac{\partial}{\partial x}\begin{pmatrix} H_i \\ V_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{15}$$

The flows are sub-critical in each reach:

$$|V_i| < \sqrt{gH_i}. \tag{16}$$

The control actions are provided by the three gate openings denoted u_1 for the left gate, u_2 for the intermediate gate and u_3 for the right gate. The discharge relationships for the three gates are similar to 2, 3. The discharge relationships for the three gates are written as

$$\begin{aligned} &V_{1,0}\,|\,V_{1,0}\,|\,H_{1,0}^2 = u_1(H_{\text{up}} - H_{1,0}), \\ &V_{1,L}\,|\,V_{1,L}\,|\,H_{1,L}^2 = u_2(H_{1,L} - H_{2,0}), \\ &V_{2,L}\,|\,V_{2,L}\,|\,H_{2,L}^2 = u_3(H_{2,L} - H_{\text{do}}). \end{aligned} \tag{17}$$

Note that an equation of flow conservation is involved at the intermediate gate:

$$H_{1,L}V_{1,L} = H_{2,0}V_{2,0}. \tag{18}$$

Eqs. (17) and (18) are the four boundary conditions associated with PDEs (15).

5.2. Steady states

As for the single reach case, for given constant gate openings $\bar{u}_1$, $\bar{u}_2$, $\bar{u}_3$ there exists a steady-state solution $(\bar{H}_1,\bar{H}_2,\bar{V}_1)$ of the Saint-Venant equations. Any arbitrary steady state $(\bar{H}_1,\bar{H}_2,\bar{V}_1)$ satisfying the conditions

$$H_{\text{do}} < \bar{H}_2 < \bar{H}_1 < H_{\text{up}} \quad \text{and} \quad 0 < \bar{H}_1\bar{V}_1$$

can be assigned by an appropriate choice of $\bar{u}_1$, $\bar{u}_2$ and $\bar{u}_3$.

5.3. Statement of the control problem

The control objective is to stabilize the water levels H_1 and H_2 and the velocity V_1 and V_2 at given set points $(\bar{H}_1,\bar{H}_2,\bar{V}_1)$. The control actions are the three gate openings u_1, u_2 and u_3. The levels $H_{1,0}(t)$, $H_{1,L}(t)$, $H_{2,0}(t)$ and $H_{2,L}(t)$ are supposed to be measured on-line at each time instant t. The external water levels H_{up} and H_{do} are known.

5.4. Control analysis from Riemann invariants

The Riemann invariants for the ith reach ($i = 1, 2$) are denoted $\alpha_i(x_i, t)$ and $\beta_i(x_i, t)$ with

$$\alpha_{i,0} = \alpha_i(0, t) \quad \text{and} \quad \beta_{i,L} = \beta_i(L, t).$$

In terms of Riemann invariants, model (15) is written

$$\begin{aligned} &\alpha_{i,0} = \alpha_i(0, t), \alpha_i L = \alpha_i(L, t), \\ &\beta_{i,0} = \beta_i(0, t), \beta_i, L = \beta_i(L, t), \end{aligned} \tag{19}$$

with $c_{i,\beta}(\alpha_i, \beta_i) < 0 < c_{i,\alpha}(\alpha_i, \beta_i)$. As we have seen above, the control objective is to stabilize the system at the origin. We are looking for *decentralized* boundary control laws where the control action at a given gate is a feedback function of the state variables at the same gate only. Let us assume that such decentralized feedback control laws are selected for the three gate openings:

$$u_1(\alpha_{1,0}, \beta_{1,0}), \quad u_2(\alpha_{1,L}, \beta_{1,L}, \alpha_{2,0}, \beta_{2,0}), \quad u_3(\alpha_{2,L}, \beta_{2,L}).$$

Introducing these expressions in (13) and using the change of coordinates $(\alpha_i, \beta_i) \leftrightarrow (H_i, Q_i)$, the four boundary conditions (17) may be formally written as

$$\begin{aligned} &f_1(\alpha_{1,0}, \beta_{1,0}) = 0, \\ &f_2(\alpha_{1,L}, \beta_{1,L}, \alpha_{2,0}, \beta_{2,0}) = 0, \\ &f_3(\alpha_{1,L}, \beta_{1,L}, \alpha_{2,0}, \beta_{2,0}) = 0, \\ &f_4(\alpha_{2,L}, \beta_{2,L}) = 0. \end{aligned} \tag{20}$$

We define the following vectors $\boldsymbol{\xi}_-(x, t)$, $\boldsymbol{\xi}_+(x, t)$, $\boldsymbol{\xi}(x, t)$ by

$$\boldsymbol{\xi}_-(x, t) = \begin{pmatrix} \beta_1(x, t) \\ \beta_2(x, t) \end{pmatrix}, \quad \boldsymbol{\xi}_+(x, t) = \begin{pmatrix} \alpha_1(x, t) \\ \alpha_2(x, t) \end{pmatrix}$$

and

$$\boldsymbol{\xi}(x, t) = \begin{pmatrix} \boldsymbol{\xi}_-(x, t) \\ \boldsymbol{\xi}_+(x, t) \end{pmatrix}.$$

Model (19) is equivalent to

$$\frac{\partial \boldsymbol{\xi}}{\partial t} + \Lambda(\boldsymbol{\xi}) \frac{\partial \boldsymbol{\xi}}{\partial x} = 0, \tag{21}$$

where $\boldsymbol{\xi} \mapsto \Lambda(\boldsymbol{\xi})$ is a diagonal matrix in $\mathbb{R}^{4\times 4}$ defined by

$$\Lambda(\boldsymbol{\xi}) = \operatorname{diag}(c_{1,\beta}(\boldsymbol{\xi}), c_{2,\beta}(\boldsymbol{\xi}), c_{1,\alpha}(\boldsymbol{\xi}), c_{2,\alpha}(\boldsymbol{\xi})).$$

We define the vector of boundary functions $\mathbf{f}$

$$\begin{aligned} &\mathbf{f}(\boldsymbol{\xi}_-(0, t), \boldsymbol{\xi}_-(L, t), \boldsymbol{\xi}_+(0, t), \boldsymbol{\xi}_+(L, t)) \\ &\quad = (f_1,\ f_2,\ f_3,\ f_4)^{\mathrm{T}}. \end{aligned} \tag{22}$$

Note that due to (17), we have

$$\mathbf{f}(0) = 0. \tag{23}$$

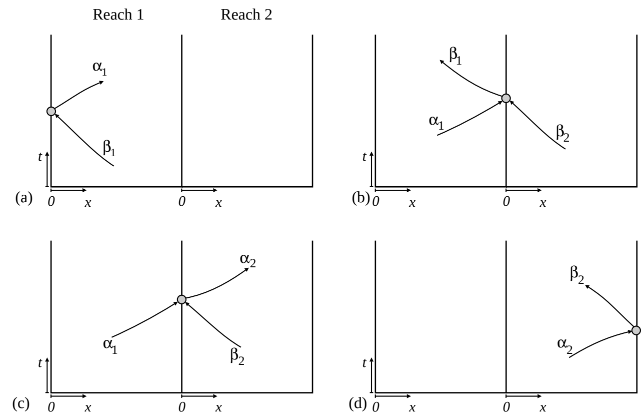

Fig. 8. Illustration of (25).

According to the Inverse Function Theorem (see e.g. Cartan, 1967, Theorem 4.7.1), these four equations are solvable with respect to $(\boldsymbol{\xi}_-(L, t), \boldsymbol{\xi}_+(0, t))$ in a neighborhood of the origin if the functions $\mathbf{f}$ are continuously differentiable and satisfy the following conditions:

$$\det \nabla_{[\boldsymbol{\xi}_-(L,t), \boldsymbol{\xi}_+(0,t)]} \mathbf{f}(0) \neq 0, \tag{24}$$

where $\nabla_{[\boldsymbol{\xi}_-(L,t), \boldsymbol{\xi}_+(0,t)]} \mathbf{f}$ denotes the Jacobian of $\mathbf{f}$ with respect to the vector $[\boldsymbol{\xi}_-(L, t)^{\mathrm{T}}, \boldsymbol{\xi}_+(0, t)^{\mathrm{T}}]^{\mathrm{T}}$.

Then the boundary conditions (20) may be rewritten as follows in a neighborhood of the origin:

$$\begin{pmatrix} \boldsymbol{\xi}_-(L, t) \\ \boldsymbol{\xi}_+(0, t) \end{pmatrix} = \mathbf{g} \begin{pmatrix} \boldsymbol{\xi}_-(0, t) \\ \boldsymbol{\xi}_+(L, t) \end{pmatrix}, \tag{25}$$

where $\mathbf{g} : \mathbb{R}^4 \to \mathbb{R}^4$ is a suitable function. The Jacobian of $\mathbf{g}$ is defined at the equilibrium by

$$\nabla \mathbf{g}(0) = -(\nabla_{[\boldsymbol{\xi}_-(L,t), \boldsymbol{\xi}_+(0,t)]} \mathbf{f}(0))^{-1} \nabla_{[\boldsymbol{\xi}_-(0,t), \boldsymbol{\xi}_+(L,t)]} \mathbf{f}(0),$$

where $\nabla \mathbf{g}$ denotes the Jacobian of $\mathbf{g}$ with respect to the vector $[\boldsymbol{\xi}_-(0, t)^{\mathrm{T}}, \boldsymbol{\xi}_+(L, t)^{\mathrm{T}}]^{\mathrm{T}}$.

Remark 2. A natural justification of the form of Eq. (25) is that, at each boundary, the outgoing invariants $\boldsymbol{\xi}_-(L, t) = (\beta_i(L, t))$ and $\boldsymbol{\xi}_+(0, t) = (\alpha_i(0, t))$ are expressed in terms of the incoming invariants $\boldsymbol{\xi}_-(0, t) = (\beta_i(0, t))$ and $\boldsymbol{\xi}_+(L, t) = (\alpha_i(L, t))$. This is illustrated in Fig. 8 where the arrows denote the characteristic curves and the direction of the invariants.

Differentiating (25) with respect to t and using (21), it can be shown that

$$\begin{aligned} &\begin{pmatrix} \Lambda_-(\boldsymbol{\xi}(L, t)) \partial_x \boldsymbol{\xi}_-(L, t) \\ \Lambda_+(\boldsymbol{\xi}(0, t)) \partial_x \boldsymbol{\xi}_+(0, t) \end{pmatrix} \\ &\quad = \nabla \mathbf{g} \begin{pmatrix} \boldsymbol{\xi}_-(0, t) \\ \boldsymbol{\xi}_+(L, t) \end{pmatrix} \begin{pmatrix} \Lambda_-(\boldsymbol{\xi}(0, t)) \partial_x \boldsymbol{\xi}_-(0, t) \\ \Lambda_+(\boldsymbol{\xi}(L, t)) \partial_x \boldsymbol{\xi}_+(L, t) \end{pmatrix}, \end{aligned} \tag{26}$$

where $\Lambda_-(\boldsymbol{\xi}) = \operatorname{diag}(c_{1,\beta}(\boldsymbol{\xi}), c_{2,\beta}(\boldsymbol{\xi}))$ and $\Lambda_+(\boldsymbol{\xi}) = \operatorname{diag}(c_{1,\alpha}(\boldsymbol{\xi}), c_{2,\alpha}(\boldsymbol{\xi}))$.

For a matrix $A \in \mathbb{R}^{m\times p}$, $A=(a_{ij})$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant p$, we define the following norms:

$$|A| := \max \left\{ \sum_{j=1}^{p} |a_{ij}| \colon i \in \{1,\dots,m\} \right\},$$

$$\mathrm{abs}(A) \in \mathbb{R}^{m\times p} \quad \text{such that } (\mathrm{abs}(A))_{ij} = |a_{ij}|,$$

$$\rho(A) := \lim |A^l|^{1/l} = \max\{|z| \colon z \in \mathbb{C} \quad \text{and} \quad \det(A - zI) = 0\}.$$

Eqs. (25) and (26) lead to the definition of the compatibility condition (C):

Definition 3. The function $\xi^{\#} \in C^1([0,L];\mathbb{R}^4)$ satisfies the compatibility condition (C) if

$$\begin{pmatrix} \xi_-^{\#}(L) \\ \xi_+^{\#}(0) \end{pmatrix} = \mathbf{g} \begin{pmatrix} \xi_-^{\#}(0) \\ \xi_+^{\#}(L) \end{pmatrix},$$

$$\begin{pmatrix} \Lambda_-(\xi^{\#}(L))\partial_x \xi_-^{\#}(L) \\ \Lambda_+(\xi^{\#}(0))\partial_x \xi_+^{\#}(0) \end{pmatrix} = \nabla \mathbf{g} \begin{pmatrix} \xi_-^{\#}(0) \\ \xi_+^{\#}(L) \end{pmatrix} \times \begin{pmatrix} \Lambda_-(\xi^{\#}(0))\partial_x \xi_-^{\#}(0) \\ \Lambda_+(\xi^{\#}(L))\partial_x \xi_+^{\#}(L) \end{pmatrix}.$$

The main result of this section is the following.

Theorem 4. *If*

$$\rho(\mathrm{abs}(\nabla \mathbf{g}(0)) < 1,$$

then there exists $\varepsilon > 0$, $\mu > 0$ *and* $C > 0$ *such that, for all* $\xi^{\#} \in C^1([0,L];\mathbb{R}^4)$ *satisfying condition* (C) *and such that*

$$|\xi^{\#}|_{C^1([0,L])} \leqslant \varepsilon,$$

there exists one and only one function $\xi \in C^1([0,L] \times [0,+\infty[;\mathbb{R}^4)$ *satisfying* (21)–(25) *and*

$$\xi(x,0) = \xi^{\#}(x) \quad \forall x \in [0,L]. \tag{27}$$

Moreover, this function decays to zero with an exponential rate

$$|\xi(.,t)|_{C^1([0,L])} \leqslant C e^{-\mu t} |\xi^{\#}|_{C^1([0,L])} \quad \forall t \geqslant 0.$$

Proof. This theorem is a special case of the more general Theorem 6 which is proved in Appendix A. As explained in the appendix, this theorem is a consequence of a previous result of Li (1994) on the stability of hyperbolic systems. □

5.5. *Example of application*

Our purpose in this section is to illustrate how the above theorem can be used to analyze the stability of a particular control structure for a channel with two reaches in cascade.

We intend to apply the Riemann invariant approach of Section 3.2. We choose to select the gate opening u_2 to stabilize the water velocity of the upstream reach:

$$V_{1,L} = \bar{V}_1 + 2\frac{1-k_2}{1+k_2}(\sqrt{gH_{1,L}} - \sqrt{g\bar{H}_1}),$$

or, expressed in Riemann invariants,

$$f_3(\beta_{1,L},\alpha_{1,L}) = \beta_{1,L} + k_2\alpha_{1,L}$$

for the positive gain k_2 in $[0,1[$ that can be used to tune the control sensitivity.

The other control laws are identical to the single reach case

$$f_1(\beta_{1,0},\alpha_{1,0}) = \alpha_{1,0} + k_1\beta_{1,0},$$

$$f_4(\beta_{2,L},\alpha_{2,L}) = \beta_{2,L} + k_3\alpha_{2,L}.$$

The boundary condition f_2 is given by the flow conservation condition (18):

$$\begin{aligned} f_2(\beta_{2,0},\beta_{1,L},\alpha_{2,0},\alpha_{1,L}) &= (\alpha_{1,L} + \beta_{1,L} + 2\bar{V}_1)(\alpha_{1,L} - \beta_{1,L} + 4\sqrt{g\bar{H}_1})^2 \\ &\quad -(\alpha_{2,0} + \beta_{2,0} + 2\bar{V}_2)(\alpha_{2,0} - \beta_{2,0} + 4\sqrt{g\bar{H}_2})^2. \end{aligned}$$

For these boundary conditions, we can compute $\nabla \mathbf{g}$:

$$\nabla_{[\xi_-(L,t),\xi_+(0,t)]}\mathbf{f}(0) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 16\sqrt{g\bar{H}_1}(\sqrt{g\bar{H}_1} - \bar{V}_1) & 0 & 0 & -16\sqrt{g\bar{H}_2}(\sqrt{g\bar{H}_2} + \bar{V}_2) \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\nabla_{[\xi_-(0,t),\xi_+(L,t)]}\mathbf{f}(0) = \begin{pmatrix} k_1 & 0 & 0 & 0 \\ 0 & -16\sqrt{g\bar{H}_2}(\sqrt{g\bar{H}_2} - \bar{V}_2) & 16\sqrt{g\bar{H}_1}(\sqrt{g\bar{H}_1} + \bar{V}_1) & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & 0 & 0 & k_3 \end{pmatrix},$$

$$\nabla \mathbf{g}(0) = \begin{pmatrix} 0 & 0 & -k_2 & 0 \\ 0 & 0 & 0 & -k_3 \\ -k_1 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{g\bar{H}_2}-\bar{V}_2}{\sqrt{g\bar{H}_2}+\bar{V}_2} & \frac{\sqrt{g\bar{H}_1}(\sqrt{g\bar{H}_1}(1-k_2)+\bar{V}_1(1+k_2))}{\sqrt{g\bar{H}_2}(\sqrt{g\bar{H}_2}+\bar{V}_2)} & 0 \end{pmatrix},$$

and, finally,

$$\rho(\mathrm{abs}(\nabla \mathbf{g}(0))) = \max \left\{ \begin{array}{c} \sqrt{|k_1\, k_2|} \\ \sqrt{|k_3\, \frac{\sqrt{g\bar{H}_2}-\bar{V}_2}{\sqrt{g\bar{H}_2}+\bar{V}_2}|} \end{array} \right\}.$$

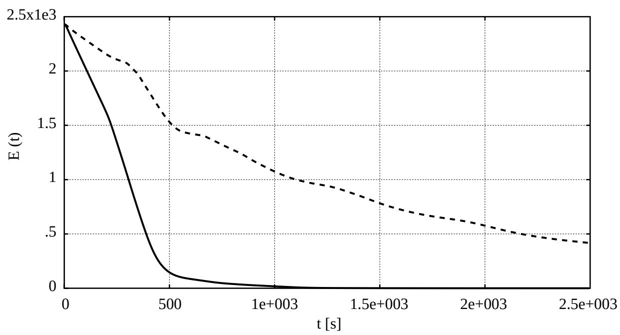

Fig. 9. Evolution of the entropy. Legend: solid line for Riemann control, dashed for open-loop control.

This control design is illustrated with a realistic simulation experiment for a channel with a constant rectangular cross section (unit width) and two reaches of length $L=1000$ m. The two external water levels are $H_{\rm up}=2$ m and $H_{\rm do}=0.2$ m. The set points are selected as

$$(\bar{H}_1,\bar{H}_2)=(1,0.8)\ \text{m} \quad \bar{Q}=0.5\ \text{m}^3\ \text{s}^{-1}.$$

The initial conditions are, for $x\in[0,L]$

$$(H_1(x,0),H_2(x,0))=(1.5,1.2)\ \text{m}, \quad Q(x,0)=1.5\ \text{m}^3\ \text{s}^{-1}.$$

The control gains k_1, k_2 and k_3 are set to 0.2. With these numerical values,

$$\rho(\text{abs}(\nabla\mathbf{g}(\mathbf{0})))=0.89 \quad \text{for the open loop,}$$

$$\rho(\text{abs}(\nabla\mathbf{g}(\mathbf{0})))=0.35 \quad \text{for the closed loop,} \tag{28}$$

which satisfies the required inequality of Theorem 4.

The Saint-Venant equations are integrated numerically with a spatial step $\Delta x=10$ m, a time step $\Delta t=5$ s and a weighting coefficient $\theta=0.6$.

In Fig. 9 the evolution of the entropy function

$$E=\int_0^{L_1}\left[H_1\frac{(V_1-\bar{V}_1)^2}{2}+g\frac{(H_1-\bar{H}_1)^2}{2}\right]\mathrm{d}x + \int_0^{L_2}\left[H_2\frac{(V_2-\bar{V}_2)^2}{2}+g\frac{(H_2-\bar{H}_2)^2}{2}\right]\mathrm{d}x.$$

is represented for the closed-loop system and compared with the open-loop system where unit-step gate openings are applied at the initial time instant. In this figure, we can observe the acceleration of the convergence with respect to the open-loop behavior since the response time in closed-loop is much smaller than in open-loop: around 500 s (8 min 20 s) for the closed-loop simulations and more than 2500 s (41 min 40 s) for the open-loop.

In Fig. 10, the deviations of the water levels in the channel are displayed at time instants $t=500, 1000$ and 1500 s.

6. Conclusions

In this paper, a general sufficient stability condition for water velocities and water levels in open channels has been described and analyzed. A control law design based on this stability condition has been proposed and applied to reaches in cascade.

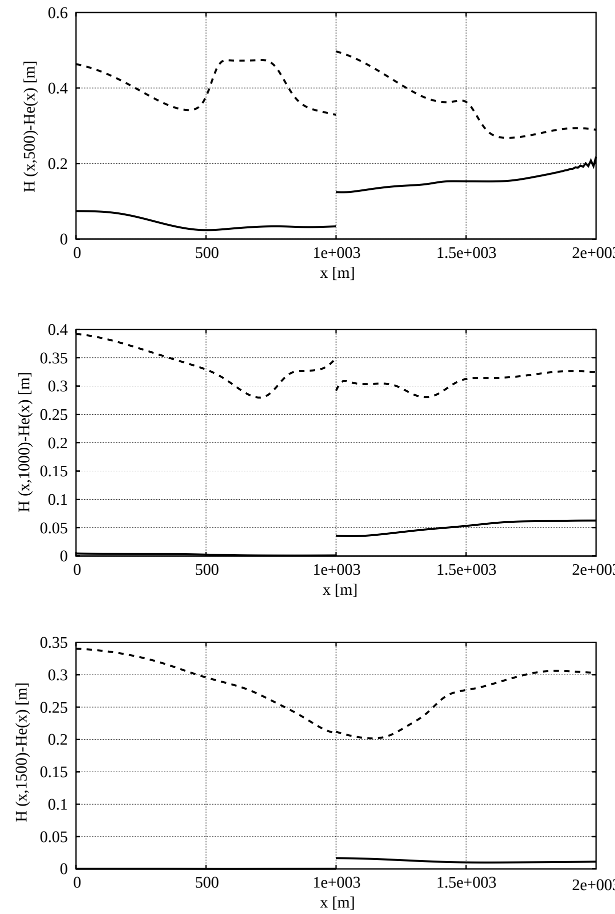

Fig. 10. Deviation of the water depth, $H(x,t)-\bar{H}(x)$, at time instants 500, 1000 and 1500 s. Legend: solid line for Riemann control, dashed for open-loop control.

The main theoretical result of the paper is an application of a previous result of Li Ta-tsien given in Theorem 6. For the sake of simplicity, the theorem has been applied to a prototype canal made up of two horizontal reaches in cascade with a rectangular cross section and under flow gates.

Theorem 6 as such does not give a fully systematic method for the design of stabilizing boundary control laws for general canal networks. It is, however, worth to emphasize that Theorem 6 is valid for any hyperbolic PDE system (A.1) with boundary conditions of the form (A.2) as long as the damping condition (A.3) is satisfied. Hence, as it is clearly illustrated in this paper, Theorem 6 provides a very efficient tool for verifying the stabilizability properties of boundary control laws for canal networks with more general topologies, with reaches having non-rectangular cross sections and other kinds of hydraulic gates like mobile spillways (see e.g. de Halleux & Bastin, 2002), provided they can be cast in the form (A.1) with boundary conditions of the form (A.2).

The control laws analyzed in this paper are based on Riemann invariants. An alternative Lyapunov approach is studied in Coron et al. (1999), de Halleux, d'Andréa-Novel, Coron, and Bastin (2001) with the entropy function introduced in Section 4 of this paper as Lyapunov function.

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Appendix A. Proof of Theorem 4

As mentioned after the statement of Theorem 4, we give in this section a more general result than Theorem 4 and explain how this general result is a direct application of Theorem 1.3 in Li (1994, Chap. 5). Let $0 \leqslant m \leqslant n$ be two integers. Let $\Lambda : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be a continuously differentiable function in a neighborhood of $0 \in \mathbb{R}^n$ such that

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$$

with $\lambda_i(0) < 0,\ \forall i \in \{1, \ldots, m\}$ and $\lambda_i(0) > 0,\ \forall i \in \{m+1, \ldots, n\}$.

Let $\Lambda_- : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ and $\Lambda_+ : \mathbb{R}^n \to \mathbb{R}^{(n-m)\times(n-m)}$ be the two functions defined by

$$\Lambda_- = \mathrm{diag}(\lambda_1, \ldots, \lambda_m), \qquad \Lambda_- = \mathrm{diag}(\lambda_{m+1}, \ldots, \lambda_n).$$

For all $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)^{\mathrm{T}} \in \mathbb{R}^n$, let $\boldsymbol{\xi}_- = (\xi_1, \ldots, \xi_m)^{\mathrm{T}} \in \mathbb{R}^m$ and $\boldsymbol{\xi}_+ = (\xi_{m+1}, \ldots, \xi_n)^{\mathrm{T}} \in \mathbb{R}^{n-m}$. We consider, for $x \in [0, L]$ and $t \in \mathbb{R}$, the hyperbolic system

$$\frac{\partial \xi}{\partial t} + \Lambda(\xi)\frac{\partial \xi}{\partial x} = 0, \tag{A.1}$$

together with the boundary condition

$$\begin{pmatrix} \xi_-(L,t) \\ \xi_+(0,t) \end{pmatrix} = \mathbf{g}\begin{pmatrix} \xi_-(0,t) \\ \xi_+(L,t) \end{pmatrix}, \tag{A.2}$$

where $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function in a neighborhood of $0 \in \mathbb{R}^n$ satisfying $\mathbf{g}(0) = 0$.

Similarly as in Definition 3 for (12) and (25), we define the compatibility condition for (A.1) and (A.2):

Definition 5. The function $\xi^\# \in C^1([0, L]; \mathbb{R}^n)$ satisfies the compatibility condition (C) if

$$\begin{pmatrix} \xi_-^\#(L) \\ \xi_+^\#(0) \end{pmatrix} = \mathbf{g}\begin{pmatrix} \xi_-^\#(0) \\ \xi_+^\#(L) \end{pmatrix},$$

$$\begin{pmatrix} \Lambda_-(\xi^\#(L))\dfrac{\partial \xi_-^\#}{\partial x}(L) \\ \Lambda_+(\xi^\#(0))\dfrac{\partial \xi_+^\#}{\partial x}(0) \end{pmatrix} = \nabla\mathbf{g}\begin{pmatrix} \xi_-^\#(0) \\ \xi_+^\#(L) \end{pmatrix} \begin{pmatrix} \Lambda_-(\xi^\#(0))\dfrac{\partial \xi_-^\#}{\partial x}(0) \\ \Lambda_+(\xi^\#(L))\dfrac{\partial \xi_+^\#}{\partial x}(L) \end{pmatrix}.$$

Theorem 4 is a consequence of the following result by choosing $n = 4$ and $m = 2$.

Theorem 6. *If*

$$\rho(\mathrm{abs}(\nabla\mathbf{g}(0))) < 1, \tag{A.3}$$

then there exists $\varepsilon > 0$, $\mu > 0$ *and* $C > 0$ *such that, for every* $\xi^\# \in C^1([0, L]; \mathbb{R}^n)$ *satisfying condition* (C) *and such that*

$$|\xi^\#|_{C^1([0,L])} \leqslant \varepsilon,$$

there exists one and only one function $\xi \in C^1([0, L] \times [0, +\infty); \mathbb{R}^n)$ *satisfying* (A.1), (A.2) *and*

$$\xi(x, 0) = \xi^\#(x) \quad \forall x \in [0, L]. \tag{A.4}$$

Moreover, this function ξ *satisfies*

$$|\xi(.,t)|_{C^1([0,L])} \leqslant C\mathrm{e}^{-\mu t}|\xi^\#|_{C^1([0,L])} \quad \forall t \geqslant 0. \tag{A.5}$$

This theorem is a special case of (the proof of) Theorem 1.3 in Li (1994, Chapter 5) if the boundary condition (A.2) has the following particular form:

$$\begin{pmatrix} \xi_-(L,t) \\ \xi_+(0,t) \end{pmatrix} = \begin{pmatrix} \mathbf{g}_1(\xi_+(L,t)) \\ \mathbf{g}_2(\xi_-(0,t)) \end{pmatrix}, \tag{A.6}$$

where $\mathbf{g}_1 : \mathbb{R}^{n-m} \to \mathbb{R}^n$ and $\mathbf{g}_2 : \mathbb{R}^m \to \mathbb{R}^n$ are of class C^1 on neighborhoods of $0 \in \mathbb{R}^{n-m}$ and of $0 \in \mathbb{R}^m$ respectively.[1] But one can use Li (1994) to prove Theorem 6 even if (A.6) does not hold by doubling the size of the state as follows. Consider the hyperbolic system

$$\frac{\partial \tilde{\xi}}{\partial t} + \tilde{\Lambda}(\tilde{\xi})\frac{\partial \tilde{\xi}}{\partial x} = 0, \tag{A.7}$$

with

$$\tilde{\xi} = (\xi_{1-}^{\mathrm{T}},\ \xi_{2-}^{\mathrm{T}},\ \xi_{1+}^{\mathrm{T}},\ \xi_{2+}^{\mathrm{T}})^{\mathrm{T}},$$

where $\xi_{1-} \in \mathbb{R}^m$, $\xi_{2-} \in \mathbb{R}^{n-m}$, $\xi_{1+} \in \mathbb{R}^{n-m}$, $\xi_{2+} \in \mathbb{R}^m$ and $\tilde{\Lambda} : \mathbb{R}^{2n} \to \mathbb{R}^{2n\times 2n}$ is defined by

$$\tilde{\Lambda}(\tilde{\xi}) = \mathrm{diag}\begin{pmatrix} \Lambda_-((\xi_{1-}^{\mathrm{T}}, \xi_{1+}^{\mathrm{T}})^{\mathrm{T}}) \\ -\Lambda_+((\xi_{2+}^{\mathrm{T}}, \xi_{2-}^{\mathrm{T}})^{\mathrm{T}}) \\ \Lambda_+((\xi_{1-}^{\mathrm{T}}, \xi_{1+}^{\mathrm{T}})^{\mathrm{T}}) \\ -\Lambda_-((\xi_{2+}^{\mathrm{T}}, \xi_{2-}^{\mathrm{T}})^{\mathrm{T}}) \end{pmatrix}.$$

[1] Let us point out that with the definition of $\|A\|_{\min}$ given in Li (1994, p. 170) one has $\|A\|_{\min} = \|\mathrm{abs}(A)\|_{\min} = \rho(\mathrm{abs}(A))$. See Lemma 2.4 in Li (1994, p. 146).

The boundary condition for (A.7) is defined by

$$\begin{pmatrix} \xi_{1-}(L,t) \\ \xi_{2-}(L,t) \end{pmatrix} = \mathbf{g}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \xi_{1+}(L,t) \\ \xi_{2+}(L,t) \end{pmatrix}\right),$$

$$\begin{pmatrix} \xi_{1+}(0,t) \\ \xi_{2+}(0,t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{g}\begin{pmatrix} \xi_{1-}(0,t) \\ \xi_{2-}(0,t) \end{pmatrix}.$$

This boundary condition can be written in the following form:

$$\begin{pmatrix} \tilde{\xi}_{-}(L,t) \\ \tilde{\xi}_{+}(0,t) \end{pmatrix} = \tilde{\mathbf{g}}\begin{pmatrix} \tilde{\xi}_{+}(L,t) \\ \tilde{\xi}_{-}(0,t) \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{g}}_1(\tilde{\xi}_{+}(L,t)) \\ \tilde{\mathbf{g}}_2(\tilde{\xi}_{-}(0,t)) \end{pmatrix} \tag{A.8}$$

with

$$\tilde{\xi}_{-} = (\xi_{1-}^{\mathrm{T}}, \xi_{2-}^{\mathrm{T}})^{\mathrm{T}}, \quad \tilde{\xi}_{+} = (\xi_{1+}^{\mathrm{T}}, \xi_{2+}^{\mathrm{T}})^{\mathrm{T}}, \tag{A.9}$$

$$\tilde{\mathbf{g}}_1 = \mathbf{g} \circ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{\mathbf{g}}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ \mathbf{g}. \tag{A.10}$$

In particular, the boundary condition for $\tilde{\xi}$ has the special form required by Theorem 1.3 in Li (1994, Chap. 5) and

$$\rho(\mathrm{abs}(\nabla\tilde{\mathbf{g}}(0))) = \rho(\mathrm{abs}(\nabla\mathbf{g}(0)))^2. \tag{A.11}$$

Let $\xi^{\#} \in C^1([0,L];\mathbb{R}^n)$ satisfying condition (C) and such that $|\xi^{\#}|_{C^1([0,L])}$ is small enough. We choose as initial condition for $\tilde{\xi}$ at $t=0$,

$$\xi_{1-}^{\#}(x) = \xi_{-}^{\#}(x), \quad \xi_{2-}^{\#}(x) = \xi_{+}^{\#}(L-x),$$
$$\xi_{1+}^{\#}(x) = \xi_{+}^{\#}(x), \quad \xi_{2+}^{\#}(x) = \xi_{-}^{\#}(L-x). \tag{A.12}$$

One easily sees that $\tilde{\xi}^{\#} := (\xi_{1-}^{\#\mathrm{T}}, \xi_{2-}^{\#\mathrm{T}}, \xi_{1+}^{\#\mathrm{T}}, \xi_{2+}^{\#\mathrm{T}})^{\mathrm{T}}$ satisfies the compatibility condition associated to (A.7) and (A.8). Hence there exists a unique C^1-solution $\tilde{\xi}$ of (A.7) and (A.8) such that

$$\tilde{\xi}(x,0) = \tilde{\xi}^{\#}(x). \tag{A.13}$$

Let

$$\tilde{\xi}^{*}(x,t) = \begin{pmatrix} \xi_{2+}(L-x,t)^{\mathrm{T}} \\ \xi_{1+}(L-x,t)^{\mathrm{T}} \\ \xi_{2-}(L-x,t)^{\mathrm{T}} \\ \xi_{1-}(L-x,t)^{\mathrm{T}} \end{pmatrix}.$$

Then, as one easily checks, $\tilde{\xi}^{*}$ satisfies as $\tilde{\xi}$ hyperbolic system (A.7), boundary condition (A.8) and initial condition (A.13). Hence by the uniqueness of the C^1-solution of the Cauchy problem associated to (A.7) and (A.8), one has

$$\tilde{\xi}^{*} = \tilde{\xi}.$$

In particular,

$$\xi_{1-}(x,t) = \xi_{2+}(L-x,t),$$
$$\xi_{1+}(x,t) = \xi_{2-}(L-x,t). \tag{A.14}$$

Hence, if

$$\xi_{-}(x,t) := \xi_{1-}(x,t), \quad \xi_{+}(x,t) := \xi_{1+}(x,t),$$

then $\xi = (\xi_{-}^{\mathrm{T}}, \xi_{+}^{\mathrm{T}})^{\mathrm{T}}$ satisfies (A.1), (A.2) and (A.4).

Conversely, if $\xi = (\xi_{-}^{\mathrm{T}}, \xi_{+}^{\mathrm{T}})^{\mathrm{T}}$ satisfies (A.1), (A.2) and (A.4), then $\tilde{\xi}$ defined by

$$\xi_{1-}(x,t) := \xi_{-}(x,t), \quad \xi_{1+}(x,t) := \xi_{+}(x,t),$$

$$\xi_{2+}(x,t) := \xi_{-}(L-x,t), \quad \xi_{2-}(x,t) := \xi_{+}(L-x,t),$$

satisfies hyperbolic system (A.7), boundary condition (A.8) and initial condition (A.13). Hence, see also (A.11), Theorem 6 for the hyperbolic system (A.1) and the boundary condition (A.2) is a consequence of (the proof of) Theorem 1.3 in Li (1994, Chap. 5) (see also Qin, 1985; Zhao, 1986) for the hyperbolic system (A.7) and boundary condition (A.8).

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Jonathan de Halleux received the Applied Mathematics engineering degree in 2000 from Université Catholique de Louvain, Louvain-la-Neuve, Belgium. He is presently doing a Ph.D. in the same institution on the stabilization of nonlinear partial differential equations.

Christophe Prieur was born in Essey-les-Nancy, France, in 1974. He graduated in Mathematics from the Ecole Normale Supérieure, France in 2000. He received the Ph.D. degree in 2001 in Applied Mathematics from the Universite Paris-Sud, France. Since 2002 he is research associate CNRS at the Laboratoire SATIE, Ecole Normale Supérieure de Cachan, France. His current research interest include nonlinear control theory, robust control and control of nonlinear partial differential equations.

Jean-Michel Coron obtained the diploma of engineer from Ecole polytechnique in 1978 and from the Corps des Mines in 1981. He got his Thèse d'Etat in 1982. He was researcher at Ecole Nationale Supérieure des Mines de Paris, associate professor at Ecole polytechnique. He is currently Professor at Université Paris-Sud. His research interests include nonlinear partial differential equations, calculus of variations and nonlinear control theory.

Brigitte d'Andréa-Novel graduated from Ecole Supérieure d'Informatique Electronique Automatique in 1984. She received the Doctorate degree from Ecole Nationale Supérieure des Mines de Paris in 1987 and her Habilitation degree from Université Paris-Sud in 1995. She is currently Professor of Systems Control Theory and responsible for a research group on "Advanced Controlled Systems" at the Centre de Robotique—Ecole des Mines de Paris. Her current research interests include nonlinear control theory and applications to underactuated mechanical systems, control of wheeled vehicles with application to automated highways and boundary control of flexible mechanical systems coupling ODEs and PDEs.

Georges Bastin received the electrical engineering degree and the Ph.D. degree, both from Université Catholique de Louvain, Louvain-la-Neuve, Belgium. He is presently Professor in the Center for Systems Engineering and Applied Mechanics (CESAME) at the Université Catholique de Louvain and Associate Professor at the Ecole des Mines de Paris. His main research interests are in system identification and nonlinear control theory with applications to mechanical systems and robotics, biological and chemical processes, environmental problems and communication networks.