A Strict Lyapunov Function for Boundary Control of Hyperbolic Systems of Conservation Laws

Jean-Michel Coron, Brigitte d'Andréa-Novel, and Georges Bastin

***Abstract*—We present a strict Lyapunov function for hyperbolic systems of conservation laws that can be diagonalized with Riemann invariants. The time derivative of this Lyapunov function can be made strictly negative definite by an appropriate choice of the boundary conditions. It is shown that the derived boundary control allows to guarantee the local convergence of the state towards a desired set point. Furthermore, the control can be implemented as a feedback of the state only measured at the boundaries. The control design method is illustrated with an hydraulic application, namely the level and flow regulation in an horizontal open channel.**

***Index Terms*—Boundary control, conservation laws, hyperbolic systems, Lyapunov function, partial differential equations.**

I. Introduction

In THIS paper, we are concerned with systems of conservation laws that are described by partial differential equations, with an independent time variable $t \in [0, +\infty)$ and an independent space variable on a finite interval $x \in [0, L]$. For such systems, the boundary control problem that we consider is the problem of designing control actions at the boundaries (i.e., at $x = 0$ and $x = L$) in order to ensure that the smooth solution of the Cauchy problem converges to a desired steady-state.

This problem has been previously considered in the literature. A first important result of asymptotic stability was presented by Greenberg and Li [1] in the case of second-order systems of conservation laws. This result was later deeply generalized to nth-order systems by Li-Tatsien in [2] (see also [3] for controllability results). These results were established by systematically utilizing the explicit evolution of the Riemann invariants along the characteristics. They have been applied for the control of networks of open channels in our previous papers [4]–[6] and in Leugering and Schmidt [7].

In this paper, a different approach that uses Lyapunov techniques is followed. This approach was first introduced in [4], [5] where an entropy of the system was used as a Lyapunov function having a *semi-negative definite* time derivative. However, with this choice of Lyapunov function, we were not able to prove asymptotic stability due to compactness problems. In this paper, in order to overcome this difficulty we exhibit a strict Lyapunov function which is an extension of the entropy and is stated in terms of Riemann invariants but whose time derivative can be made *strictly negative definite* by an appropriate choice of the boundary controls. This function is related to a Lyapunov function used in [8] for the stabilization of the Euler equation of incompressible fluids. It is also similar to the Lyapunov function used in [9] to analyse the stability of a general class of *linear* symmetric hyperbolic systems. Our contribution in this paper is to show how this kind of Lyapunov function can be extended in order to analyse the stability of *nonlinear* hyperbolic systems of conservation laws. For this class of systems, we give a theorem which shows that the boundary control allows to prove the local convergence (in $H^2(0, L)$-norm) of the system trajectories towards a desired set point. Furthermore, the control can be implemented as a feedback of the state only measured at the boundaries.

The considered class of conservative systems has a wide range of potential engineering applications, including for instance electrical transmission lines [10], gas flow pipelines [11], [12], road traffic models [13] or heat exchangers [9]. In this paper, the control design method is illustrated with an hydraulic application: the regulation of the level and the flow in an horizontal reach of an open channel. For the sake of simplicity, our presentation is limited to second order systems (i.e., systems of two scalar conservation laws). However, as we indicate in the conclusions, the method can be easily extended to higher order systems provided they can be diagonalized with Riemann invariants.

II. Boundary Control of Hyperbolic Systems of Conservation Laws : Statement of the Problem

Let Ω be a nonempty connected open set in $\mathbb{R}^2$. We consider a system of two conservation laws of the general form

$$\partial_t Y + \partial_x f(Y) = 0 \tag{1}$$

where

- t and x are the two independent variables: a time variable $t \in [0, +\infty)$ and a space variable $x \in [0, L]$ on a finite interval;
- $Y = (y_1 \quad y_2)^T : [0, +\infty) \times [0, L] \to \Omega$ is the vector of the two dependent variables;
- $f : \Omega \to \mathbb{R}^2$ is the flux density.

We are concerned with the *smooth* solutions of the Cauchy problem for the system (1) over $[0, +\infty) \times [0, L]$ under an initial condition

$$Y(0, x) = Y_0(x) \quad x \in [0, L]$$

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J.-M. Coron is with the Department of Mathematics, Université Paris-Sud, 91405 Orsay, France (e-mail: Jean-Michel.Coron@math.u-psud.fr).

B. d'Andréa-Novel is with the Centre de Robotique, Ecole des Mines de Paris, 75272 Paris Cedex 06, France (e-mail: andrea@caor.ensmp.fr).

G. Bastin is with the Center for Systems Engineering and Applied Mechanics (CESAME), Université Catholique de Louvain, 1348 Louvain-la-Neuve, Belgium (e-mail: bastin@auto.ucl.ac.be).

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and two boundary conditions of the form

$$g_0(Y(t,0), u_0(t)) = 0 \quad t \in [0, +\infty)$$
$$g_L(Y(t,L), u_L(t)) = 0 \quad t \in [0, +\infty)$$

with $g_0, g_L : \Omega \times I\!R \to I\!R$ and where $u_0, u_L : [0, +\infty) \to I\!R$ are the control actions.

Steady-state: For constant control actions $u_0(t) = \bar{u}_0$ and $u_L(t) = \bar{u}_L$, a steady-state solution is a constant solution $Y(t,x) = \bar{Y} \forall t \in [0, +\infty)$, $\forall x \in [0, L]$ which satisfies (1) and the boundary conditions $g_0(\bar{Y}, \bar{u}_0) = 0$ and $g_L(\bar{Y}, \bar{u}_L) = 0$. Depending on the form of these boundary conditions, the steady-state solution may be stable or unstable.

The *boundary control problem* is then the problem of finding control actions $u_0(t)$ and $u_L(t)$ such that, for any smooth enough initial condition $Y_0(x)$, the Cauchy problem has a unique smooth solution converging towards a desired steady-state $\bar{Y}$ (called *set point*).

In this paper, we consider the special case where the following hold.

1) System (1) is strictly hyperbolic, i.e., the Jacobian matrix of the application f,

$$F(Y) = \frac{\partial f}{\partial Y}$$

has two real distinct eigenvalues $\lambda_1(Y)$ and $\lambda_2(Y)$ for all $Y \in \Omega$; it is furthermore assumed that these two eigenvalues of $F(Y)$ have opposite signs: $\lambda_2(Y) < 0 < \lambda_1(Y) \, \forall Y \in \Omega$.

2) Each control action is expressed as a feedback of the system state at the boundaries

$$u_0(t) = \tilde{u}_0(Y(t,0))$$
$$u_L(t) = \tilde{u}_L(Y(t,L)).$$

III. Characteristic Form and Riemann Invariants

Under the previous assumptions, (1) is rewritten as

$$\partial_t Y + F(Y)\partial_x Y = 0. \tag{2}$$

This system can be diagonalised with the *Riemann invariants* (see, for instance, [14, pp. 34–35]). This means that there exists a change of coordinates $\xi(Y) = (a(Y) b(Y))^T$ whose Jacobian matrix is denoted $D(Y)$

$$D(Y) = \frac{\partial \xi}{\partial Y}$$

and diagonalizes $F(Y)$ in Ω

$$D(Y)F(Y) = \Lambda(Y)D(Y), \qquad Y \in \Omega$$

with

$$\Lambda(Y) = \mathrm{diag}(\lambda_1(Y), \lambda_2(Y)).$$

Finding the change of coordinates $\xi(Y)$ requires to find a solution of the first-order partial differential equation $D(Y)F(Y) = \Lambda(Y)D(Y)$. As it is shown in [14, pp. 34–35], this partial differential equation can be reduced to the integration of *ordinary* differential equations. Moreover, in many cases, these ordinary differential equations can be explicitly solved by using separation of variables, homogenity or symetry properties; see, e.g., [15, pp. 146–147, 152] for examples of computations of Riemann invariants.

In the coordinates $\xi = (a\ b)^T$, the system (2) can then be rewritten in the following (diagonal) *characteristic form*:

$$\partial_t \xi + \Lambda(\xi)\partial_x \xi = 0 \tag{3}$$

or

$$\begin{cases} \partial_t a + c(a,b)\partial_x a = 0 \\ \partial_t b - d(a,b)\partial_x b = 0 \end{cases} \tag{4}$$

with $c(a,b) = \lambda_1(\xi) > 0$ and $-d(a,b) = \lambda_2(\xi) < 0$, the eigenvalues of $F(Y)$ expressed in the $\xi = (a\ b)^T$ coordinates.

We observe that the two quantities $\partial_t a + c(a,b)\partial_x a = 0$ and $\partial_t b - d(a,b)\partial_x b = 0$ can then be viewed as the total time derivatives da/dt and db/dt of the functions $a(t,x)$ and $b(t,x)$ at a point (t,x) of the plane, along two curves having slopes

$$\frac{dx}{dt} = c(a,b) \quad \text{and} \quad \frac{dx}{dt} = -d(a,b).$$

These curves are called *characteristic curves* and the solutions $a(t,x), b(t,x)$ are called *characteristic solutions*. Since $da/dt = 0$ and $db/dt = 0$ on the characteristic curves, it follows that $a(t,x)$ and $b(t,x)$ are constant along the characteristic curves. This explains why the characteristic solutions are called *Riemann invariants*.

The change of coordinates $\xi(Y)$ is clearly defined up to a constant. It can, therefore, be selected in such a way that $\xi(\bar{Y}) = 0$ and the control problem can be restated as the problem of determining the control actions in such a way that the characteristic solutions converge towards the origin. Our contribution in this paper is to propose and analyse a control design method based on a strict Lyapunov function that is presented in the next section.

IV. A Strict Lyapunov Function for Boundary Control Design

Let us consider the linear approximation of the characteristic form (4) around the origin

$$\begin{cases} \partial_t a + \bar{c}\partial_x a = 0 \\ \partial_t b - \bar{d}\partial_x b = 0 \end{cases} \tag{5}$$

with $\bar{c} = c(0,0) > 0$ and $\bar{d} = d(0,0) > 0$.

With a view to the boundary control design, the following candidate Lyapunov function is introduced:

$$U(t) = U_1(t) + U_2(t)$$

with

$$U_1(t) = \frac{A}{\bar{c}} \int_0^L a^2(t,x) e^{-(\mu/\bar{c})x}\, dx$$
$$U_2(t) = \frac{B}{\bar{d}} \int_0^L b^2(t,x) e^{+(\mu/\bar{d})x}\, dx$$

and with positive constant coefficients A, B and μ. As we have mentioned above, this function is related to the Lyapunov function used in [8, Def. of V, p. 1886] for the stabilization of the Euler equation of incompressible fluids. It is also similar to the Lyapunov function used in [9] to analyse the stability of linear symmetric hyperbolic systems.

The time derivative of $U(t)$ along the trajectories of the linear approximation (5) is

$$\dot{U}(t) = -\mu U(t) - \left[Aa^2(t,x)e^{-(\mu/\bar{c})x}\right]_0^L + \left[Bb^2(t,x)e^{+(\mu x/\bar{d})x}\right]_0^L$$

which implies

$$\dot{U}(t) = -\mu U(t) - \left[Ae^{-(\mu/\bar{c})L}a^2(t,L) - Aa^2(t,0)\right] - \left[Bb^2(t,0) - Be^{(\mu/\bar{d})L}b^2(t,L)\right].$$

It can be seen that the two last terms depend only on the Riemann invariants at the two boundaries, i.e., at $x = 0$ and at $x = L$. The control laws $u_0(t)$ and $u_L(t)$ can then be defined in order to make these terms negative along the system trajectories.

A simple solution is to select $u_0(t)$ such that

$$a(t,0) = k_0 b(t,0) \tag{6}$$

and $u_L(t)$ such that

$$b(t,L) = k_L a(t,L) \tag{7}$$

with $|k_0 k_L| < 1$. The time derivative of the Lyapunov function is then written

$$\dot{U}(t) = -\mu(U_1(t) + U_2(t)) + \left(Bk_L^2 e^{(\mu/\bar{d})L} - Ae^{-(\mu/\bar{c})L}\right)a^2(t,L) + \left(Ak_0^2 - B\right)b^2(t,0). \tag{8}$$

Since $|k_0 k_L| < 1$, we can select μ such that

$$k_0^2 k_L^2 < |k_0 k_L| < \sigma = e^{-\mu L[(1/\bar{c})+(1/\bar{d})]}.$$

Then, we can select A and B such that

$$\frac{k_L^2}{\sigma} < \frac{A}{B} < \frac{1}{k_0^2}$$

which implies readily that

$$Ak_0^2 - B < 0 \quad \text{and} \quad Bk_L^2 e^{(\mu/\bar{d})L} - Ae^{-(\mu/\bar{c})L} < 0. \tag{9}$$

Then, it can be seen that $\dot{U}(t) \leq -\mu U(t)$ along the trajectories of the linear approximation (5) and that $\dot{U}(t) = 0$ if and only if $a(t,x) = b(t,x) = 0$ (i.e., at the system equilibrium).

In the next section, we will show that such boundary controls for the linearized system (5) can also be applied to the nonlinear system (4) with the guarantee that the trajectories locally converge to the origin.

V. Convergence Analysis

In the previous section, the inequality $\dot{U}(t) \leq -\mu U$ ensures the convergence in $L^2(0,L)$-norm of the solutions of the linear system (5). As we will see hereafter, in order to extend the analysis to the case of the nonlinear system (4), it will be needed to prove a convergence in $H^2(0,L)$-norm (see, for instance, [16, Ch. 16, Sec. 1]).

Thus, we consider the nonlinear system (4)

$$\begin{cases} \partial_t a + c(a,b)\partial_x a = 0 \\ \partial_t b - d(a,b)\partial_x b = 0. \end{cases} \tag{10}$$

By computing the time derivative of $U(t)$ along the solutions of this system, we get

$$\begin{aligned}\dot{U} = &-\left[A\frac{a^2c}{\bar{c}}e^{-(\mu x/\bar{c})}\right]_0^L + \left[B\frac{b^2d}{\bar{d}}e^{+(\mu x/\bar{d})}\right]_0^L \\ &- \int_0^L \left[A\frac{a^2c\mu}{\bar{c}^2}e^{-(\mu x/\bar{c})} + B\frac{b^2d\mu}{\bar{d}^2}e^{+(\mu x/\bar{d})}\right]dx \\ &+ \int_0^L \left[A\frac{a^2\phi}{\bar{c}}e^{-(\mu x/\bar{c})} - B\frac{b^2\psi}{\bar{d}}e^{+(\mu x/\bar{d})}\right]dx\end{aligned}$$

with

$$\phi := \partial_x c = \partial_x a\frac{\partial c}{\partial a} + \partial_x b\frac{\partial c}{\partial b}$$
$$\psi := \partial_x d = \partial_x a\frac{\partial d}{\partial a} + \partial_x b\frac{\partial d}{\partial b}.$$

In the previous section, we have considered *linear* boundary conditions (6), (7). Here we assume more general nonlinear boundary conditions. More precisely, we assume that the boundary control functions $u_0(t)$ and $u_L(t)$ are chosen such that the boundary conditions have the form

$$a(t,0) = \alpha(b(t,0)) \quad \text{and} \quad b(t,L) = \beta(a(t,L)) \tag{11}$$

whith C^1 functions $\alpha, \beta : I\!R \to I\!R$ and we denote

$$k_0 = \alpha'(0) \quad \text{and} \quad k_L = \beta'(0).$$

Moreover, we introduce the following notations:

$$v(t,x) = \partial_x a(t,x) \text{ and } w(t,x) = \partial_x b(t,x).$$

We then have the following lemma.

Lemma 1: If $|k_0 k_L| < 1$, if the positive real constants μ, A, B satisfy inequalities (9), there exist positive real constants K_1, δ_1, λ_1 such that, if $|a(t,x)| + |b(t,x)| < \delta_1 \forall x \in [0,L]$ then

$$\dot{U} \leq -\lambda_1 U + K_1 \int_0^L [a^2(t,x) + b^2(t,x)][|v(t,x)| + |w(t,x)|]dx$$

along the solutions of (10) with the boundary conditions (11).

Proof: See the Appendix. ■

In contrast with the linear analysis of Section IV, it appears readily from Lemma 1 that we cannot just complete the Lyapunov stability analysis with the function U but that we have to examine the dynamics of the variables $v(t,x)$ and $w(t,x)$ and consequently to extend the definition of the Lyapunov function.

By a time differentiation and using the model (10), it is readily shown that $v(t,x)$ and $w(t,x)$ satisfy the following dynamics:

$$\begin{cases} \partial_t v + c(a,b)\partial_x v + v\phi = 0 \\ \partial_t w - d(a,b)\partial_x w - w\psi = 0 \end{cases} \tag{12}$$

and the associated boundary conditions

$$\begin{cases} c(t,0)v(t,0) = -\alpha'(b(t,0))d(t,0)w(t,0) \\ d(t,L)w(t,L) = -\beta'(a(t,L))c(t,L)v(t,L). \end{cases} \tag{13}$$

Obviously, $c(t,0), d(t,0), c(t,L)$ and $d(t,L)$ are compact notations for the values of the functions c and d evaluated at $(a(t,0), b(t,0))$ and $(a(t,L), b(t,L))$ respectively.

In fact, we will need also to examine the dynamics of the spatial second order derivatives of $a(x,t)$ and $b(x,t)$, which are denoted as

$$q(t,x) = \partial_x v(t,x) = \partial_{xx} a(t,x)$$
$$r(t,x) = \partial_x w(t,x) = \partial_{xx} b(t,x)$$

and which satisfy the following dynamics:

$$\begin{cases} \partial_t q + c(a,b)\partial_x q + 2q\phi + v\partial_x\phi = 0 \\ \partial_t r - d(a,b)\partial_x r - 2r\psi - w\partial_x\psi = 0 \end{cases} \tag{14}$$

with the associated boundary conditions shown in (15) at the bottom of the page, where $\phi(t,0), \psi(t,0), \phi(t,L)$ and $\psi(t,L)$ are compact notations for the values of the functions ϕ and ψ evaluated at $(a(t,0), b(t,0), v(t,0), w(t,0))$ and $(a(t,L), b(t,L), v(t,L), w(t,L))$ respectively while the functions $\eta(t)$ and $\chi(t)$ are defined as

$$\eta(t) := \frac{\alpha'(b(t,0))d(t,0)}{c(t,0)}$$
$$\chi(t) := \frac{\beta'(a(t,L))c(t,L)}{d(t,L)}.$$

A key point for the analysis is that the linear approximations (around zero) of systems (12) and (14) have the following form:

$$\begin{cases} \partial_t v + \bar{c}\partial_x v = 0 \\ \partial_t w - \bar{d}\partial_x w = 0 \end{cases}$$
$$\begin{cases} \partial_t q + \bar{c}\partial_x q = 0 \\ \partial_t r - \bar{d}\partial_x r = 0. \end{cases}$$

Both systems have exactly the same form as the linear approximation (5) of the original system (10). Then, in order to prove that the solutions of the global system (10)–(14) converge to zero, it is quite natural to consider an extended Lyapunov function of the form

$$S(t) = U(t) + V(t) + W(t) \tag{16}$$

where $V(t)$ and $W(t)$ have the format of $U(t)$

$$V(t) = V_1(t) + V_2(t)$$
$$V_1(t) = \bar{c}A\int_0^L v^2(t,x)e^{-(\mu/\bar{c})x}dx$$
$$V_2(t) = \bar{d}B\int_0^L w^2(t,x)e^{+(\mu/\bar{d})x}dx$$
$$\text{and}$$
$$W(t) = W_1(t) + W_2(t)$$
$$W_1(t) = \bar{c}^3A\int_0^L q^2(t,x)e^{-(\mu/\bar{c})x}dx$$
$$W_2(t) = \bar{d}^3B\int_0^L r^2(t,x)e^{+(\mu/\bar{d})x}dx.$$

In the following two lemmas, we then examine the time derivatives of the functions $V(t)$ and $W(t)$ along the solutions of the closed-loop system (10)–(15).

Lemma 2: If $|k_0k_L| < 1$, if the positive real constants μ, A, B satisfy inequalities (9), there exist positive real constants K_2, δ_2, λ_2 such that, if $|a(x,t)| + |b(x,t)| < \delta_2 \forall x \in [0,L]$, then

$$\dot{V} \leq -\lambda_2 V + K_2\int_0^L [v^2(t,x) + w^2(t,x)]^{3/2}dx$$

along the solutions of the systems (10)–(12) with the boundary conditions (11)–(13). ■

Lemma 3: If $|k_0k_L| < 1$, if the positive real constants μ, A, B satisfy inequalities (9), there exist positive real constants K_3, δ_3, λ_3 such that, if $|a(x,t)| + |b(x,t)| < \delta_3\ \forall x \in [0,L]$, then

$$\begin{aligned}\dot{W} \leq& -\lambda_3 W \\ &+ K_3\int_0^L [q^2(t,x) + r^2(t,x)] \\ &\times [|v(t,x) + w(t,x)]\,dx \\ &+ K_3\int_0^L [v^2(t,x) + w^2(t,x)] \\ &[|q(t,x)| + |r(t,x)|]\,dx\end{aligned}$$

along the solutions of the systems (10)–(14) with the boundary conditions (11)–(15). ■

$$\begin{cases} c(t,0)q(t,0) + v(t,0)\phi(t,0) = \eta'(t)w(t,0) + \eta(t)[d(t,0)r(t,0) + w(t,0)\psi(t,0)] \\ d(t,L)r(t,L) + w(t,L)\psi(t,L) = -\chi'(t)v(t,L) + \chi(t)[c(t,L)q(t,L) + v(t,L)\phi(t,L)] \end{cases} \tag{15}$$

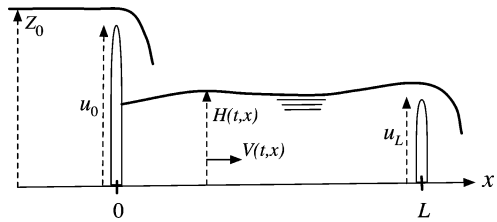

Fig. 1. Reach of an open channel delimited by two adjustable overflow spillways.

The proofs of Lemmas 2 and 3 follow a line that is entirely similar to the proof given in Appendix for Lemma 1. We therefore omit these proofs because we believe that they would be superfluous and would just needlessly lengthen the paper.

We are now in a position to complete our Lyapunov convergence analysis. We start with the analysis of the global Lyapunov function (16): $S = U + V + W$.

Lemma 4: If $|k_0 k_L| < 1$, if the positive real constants μ, A, B satisfy inequalities (9), there exist positive real constants λ_0 and δ_0 such that, if $S(t) < \delta_0$, then

$$\dot{S} \leq -\lambda_0 S$$

along the solutions of the closed-loop system (10)–(15).

Proof: See the Appendix. ■

We then have our main convergence result.

Theorem 1: There exist positive real constants K, δ, λ such that, for any initial conditions $(a_0(x), b_0(x))$ in $H^2(0,L)^2$ satisfying the compatibility conditions

$$a_0(0) = \alpha(b_0(0)) \quad b_0(L) = \beta(a_0(L))$$

$$\begin{cases} c(a_0(0), b_0(0))\partial_x a_0(0) = -\alpha'(b_0(0))d(a_0(0), b_0(0))\partial_x b_0(0) \\ d(a_0(L), b_0(L))\partial_x b_0(L) = -\beta'(a_0(L))c(a_0(L), b_0(L))\partial_x a_0(L) \end{cases}$$

and such that

$$|a_0(x)|_{H^2(0,L)} + |b_0(x)|_{H^2(0,L)} < \delta$$

the closed-loop system (10) with boundary conditions (11) has a unique solution which is continuous from $[0, +\infty)$ into $H^2(0,L)^2$ and satisfies

$$\begin{aligned} &|a(t,x)|_{H^2(0,L)} + |b(t,x)|_{H^2(0,L)} \\ &\qquad < K(|a_0(x)|_{H^2(0,L)} + |b_0(x)|_{H^2(0,L)})e^{-\lambda t}. \end{aligned}$$

Proof: See the Appendix. ■

VI. Application to Level and Flow Control in an Horizontal Reach of an Open Channel

In the field of hydraulics, the flow in open-channels is generally represented by the so-called Saint Venant equations which are a typical example of a system of conservation laws.

We consider the special case of a reach of an open channel delimited by two overflow spillways as represented in Fig. 1.

We assume that

1) the channel is horizontal;
2) the channel is prismatic with a constant rectangular section and a unit width;
3) the friction effects are neglected.

The flow dynamics are described by a system of two laws of conservation (Saint-Venant or shallow water equations), namely a law of mass conservation

$$\partial_t H + \partial_x (HV) = 0$$

and a law of momentum conservation

$$\partial_t V + \partial_x \left(\frac{1}{2} V^2 + gH \right) = 0$$

where $H(t,x)$ represents the water level and $V(t,x)$ the water velocity in the reach while g denotes the gravitation constant. The system is written under the form (2) as follows:

$$\partial_t \begin{pmatrix} H \\ V \end{pmatrix} + A(H,V)\partial_x \begin{pmatrix} H \\ V \end{pmatrix} = 0$$

with the matrix $A(H,V)$ defined as

$$A(H,V) = \begin{pmatrix} V & H \\ g & V \end{pmatrix}.$$

The control actions are the positions u_0 and u_L of the two spillways located at the extremities of the pool and related to the state variables H and V by the following expressions:

$$\begin{aligned} &b_0(H(t,0), V(t,0), u_0(t)) \\ &\quad = H(t,0)V(t,0) - C_0(Z_0 - u_0(t))^{3/2} = 0 \end{aligned} \tag{17}$$

$$\begin{aligned} &b_L(H(t,L), V(t,L), u_L(t)) \\ &\quad = H(t,L)V(t,L) - C_0(H(t,L) - u_L(t))^{3/2} = 0 \end{aligned} \tag{18}$$

where Z_0 denotes the water level above the pool (see Fig. 1) and C_0 is a characteristic constant of the spillways.

For constant spillway positions $\bar{u}_0$ and $\bar{u}_L$, there is a unique steady-state solution which satisfies the following relations:

$$\bar{H} = Z_0 - \bar{u}_0 + \bar{u}_L$$

$$\bar{V} = \frac{C_0(Z_0 - \bar{u}_0)^{3/2}}{Z_0 - \bar{u}_0 + \bar{u}_L}.$$

The control objective is to regulate the level H and the velocity V (or the flow rate $Q = HV$) at the set points $\bar{H}$ and $\bar{V}$ (or $\bar{Q} = \bar{H}\bar{V}$), by acting on the spillway positions u_0 and u_L.

The eigenvalues of the Jacobian matrix $A(H,V)$

$$\lambda_1(H,V) = V + \sqrt{gH}$$

$$\lambda_2(H,V) = V - \sqrt{gH}$$

are generally called *characteristic velocities*. The flow is said to be *fluvial* (or subcritical) when the characteristic velocities have opposite signs

$$\lambda_2(H,V) < 0 < \lambda_1(H,V).$$

The Riemann invariants can be defined as follows:

$$a = V - \bar{V} + 2(\sqrt{gH} - \sqrt{g\bar{H}})$$
$$b = V - \bar{V} - 2(\sqrt{gH} - \sqrt{g\bar{H}}).$$

By using the relations (6) and (7) for the control definition, combined with the spillway characteristics (17) and (18), the following boundary control laws are obtained:

$$u_0 = Z_0 - \sqrt[3]{\left(\frac{H_0^2}{C_0^2}\right)\left(\bar{V} - 2\sqrt{g}\frac{1+k_0}{1-k_0}(\sqrt{H_0} - \sqrt{\bar{H}})\right)^2}$$
$$u_L = H_L - \sqrt[3]{\left(\frac{H_L^2}{C_0^2}\right)\left(\bar{V} + 2\sqrt{g}\frac{1+k_L}{1-k_L}(\sqrt{H_L} - \sqrt{\bar{H}})\right)^2}$$

where $H_0 = H(t,0)$, $H_L = H(t,L)$.

It can be seen that both controls have the form of a state feedback at the two boundaries. In addition, it can be emphasized that the implementation of the controls is particularly simple since only measurements of the levels $H(t,0)$ and $H(t,L)$ at the two spillways are required. This means that the feedback implementation does not require neither level measurements inside the pool nor any velocity or flow rate measurements.

VII. Conclusion

In this paper, we have presented a strict Lyapunov function which can be used for the boundary control design for second order systems of conservation laws and to analyse the convergence of the closed-loop system towards the equilibrium. We have the following additional comments:

1) The Lyapunov function U that we have used in this paper is similar to the Lyapunov function used in [9]. It should, however, be stressed that in [9] this Lyapunov function is used to analyse the stability of a special class of *linear* symmetric hyperbolic systems. Our contribution in this paper has been to show how the Lyapunov function U can be extended to $S = U + V + W$ in order to analyse the stability of *nonlinear* hyperbolic systems of conservation laws by proving a convergence in $H^2(0,L)$ norm.
2) In the special case where $\mu = 0$, the Lyapunov function U is just an entropy function of the system under characteristic form linearised in the space of the Riemann coordinates. In [4] and [5], the interested reader will find an alternative approach of the boundary control design where the entropy is used as such (i.e., without linearisation) as a Lyapunov function inside the space of the system physical coordinates. It must however be emphasized that the entropy is not a strict Lyapunov function because its time derivative is not negative definite but only semi negative definite [as we can see by setting $\mu = 0$ in (8)].
3) Theorem 1 (with the $C^1([0,L])$-norm instead of the $H^2(0,L)$-norm) follows from the previous works [1] and [6]. In the latter reference however the convergence analysis is different: It does not make use of a Lyapunov function but is obtained from a general theorem on the stability of the classical solutions of quasilinear hyperbolic systems. As it is well known, an interest of having an explicit Lyapunov function is that it is a guarantee of control robustness. Indeed, we could extend our analysis to the more general system $\partial_t Y + F(x,Y)\partial_x Y + G(x,Y) = 0$ with $G(x,0) = 0$ and for small enough perturbations $G(x,Y)$ and $\partial F(x,Y)/\partial x$. This kind of generalisation would certainly be more difficult to address with the approach of [1] and [6].
4) In order to solve the control problem, we have selected the particular simple boundary conditions (6) and (7). However, obviously many other forms are admissible provided they make $\dot{U}$ negative. For instance it can be interesting to use controls at a boundary which depend on the state at the other boundary, hence introducing some useful feedforward action in the control (see e.g., [18]).
5) For the sake of simplicity, our presentation was restricted to second order systems of conservation laws. From our analysis, it is however very clear that the approach can be directly extended to any system of conservation laws which can be diagonalised with Riemann invariants. It is in particular the case for networks where the flux on each arc is modelled by a system of two conservation laws (see, e.g., [17]).

Appendix

A. Proof of Lemma 1

Lemma 1: If $|k_0 k_L| < 1$, if the positive real constants μ, A, B satisfy inequalities (9), there exist positive real constants K_1, δ_1, λ_1 such that, if $|a(x,t)| + |b(x,t)| < \delta_1 \forall x \in [0,L]$, then

$$\dot{U} \leq -\lambda_1 U + K_1 \int_0^L [a^2(t,x) + b^2(t,x)] \times [|v(t,x)| + |w(t,x)|]\, dx$$

along the solutions of the system (10) with the boundary conditions (11).

Proof: By computing the time derivative of U along the C^1 solutions of (10), we get

$$\dot{U} = -\left[A\frac{a^2 c}{\bar{c}}e^{-(\mu x/\bar{c})}\right]_0^L + \left[B\frac{b^2 d}{\bar{d}}e^{+(\mu x/\bar{d})x}\right]_0^L - \int_0^L \left[A\frac{a^2 c\mu}{\bar{c}^2}e^{-(\mu x/\bar{c})} + B\frac{b^2 d\mu}{\bar{d}^2}e^{+(\mu x/\bar{d})x}\right]dx + \int_0^L \left[A\frac{a^2\phi}{\bar{c}}e^{-(\mu x/\bar{c})} - B\frac{b^2\psi}{\bar{d}}e^{+(\mu x/\bar{d})}\right]dx.$$

The three terms of this expression are successively considered.

First term:

$$\mathcal{T}_1 = -\left[A\frac{a^2 c}{\bar{c}}e^{-(\mu x/\bar{c})}\right]_0^L + \left[B\frac{b^2 d}{\bar{d}}e^{+(\mu x/\bar{d})x}\right]_0^L = \left[\frac{A}{\bar{c}}\alpha(b(t,0))^2 c(t,0) - \frac{A}{\bar{c}}a^2(t,L)c(t,L)e^{-(\mu/\bar{c})L}\right] + \left[\frac{B}{\bar{d}}(\beta(a(t,L))^2 d(t,L)e^{(\mu/\bar{d})L} - \frac{B}{\bar{d}}b^2(t,0)d(t,0)\right].$$

This expression is written as

$$\mathcal{T}_1 = \mathcal{T}_1^0 + \Delta\mathcal{T}_1$$

where $\mathcal{T}_1^0$ denotes a quadratic approximation of $\mathcal{T}_1$ obtained for small $a(t,L)$ and $b(t,0)$ with

$$c(a,b) \simeq \bar{c} \quad d(a,b) \simeq \bar{d}$$
$$\alpha(b(t,0)) \simeq k_0 b(t,0) \quad \beta(a(t,L)) \simeq k_L a(t,L).$$

We have

$$\mathcal{T}_1^0 = \left(Bk_L^2 e^{(\mu/\bar{d})L} - Ae^{-(\mu/\bar{c})L}\right) a^2(t,L) + \left(Ak_0^2 - B\right) b^2(t,0)$$

which is negative ($\mathcal{T}_1^0 < 0$) as we have already shown in Section III. Moreover, the residual part $\Delta\mathcal{T}_1$ is at least cubic w.r.t $a(t,L)$ and $b(t,0)$ (i.e., $\Delta\mathcal{T}_1 \approx O(a^3(t,L), b^3(t,0))$ for small $|a(t,L)|$ and $|b(t,0)|$).

Second term:

$$\mathcal{T}_2 = -\int_0^L \left[A\frac{a^2 c\mu}{\bar{c}^2} e^{-(\mu x/\bar{c})} + B\frac{b^2 d\mu}{\bar{d}^2} e^{+(\mu x/\bar{d})x}\right] dx.$$

We introduce the notations

$$c(a,b) = \bar{c} + \tilde{c}(a,b) \quad d(a,b) = \bar{d} + \tilde{d}(a,b).$$

Then, we have

$$\mathcal{T}_2 = -\mu U + \Delta\mathcal{T}_2$$

with

$$\Delta\mathcal{T}_2 = -\mu \int_0^L \left[A\frac{a^2\tilde{c}}{\bar{c}^2} e^{-(\mu x/\bar{c})} + B\frac{b^2\tilde{d}}{\bar{d}^2} e^{+(\mu x/\bar{d})x}\right] dx.$$

Clearly, from the definitions of $\tilde{c}$ and $\tilde{d}$, we have that $\Delta\mathcal{T}_2$ is at least cubic w.r.t $a(t,x)$ and $b(t,x)$

$$\Delta\mathcal{T}_2 \approx O\left(\int_0^L |a^3(t,x)|\, dx, \int_0^L |b^3(t,x)|\, dx\right)$$
$$\text{for small } \int_0^L |a^3(t,x)|\, dx \quad \text{and} \quad \int_0^L |b^3(t,x)|\, dx.$$

It follows readily that for a small enough real positive δ_1 there exists a large enough real positive $\lambda_1 > \mu$ such that

$$\mathcal{T}_1 + \mathcal{T}_2 \le -\lambda_1 U \qquad \forall t$$

as long as

$$|a(x,t)| + |b(x,t)| < \delta_1 \quad \forall x \in [0,L].$$

Third term:

$$\mathcal{T}_3 = \int_0^L \left[A\frac{a^2\phi}{\bar{c}} e^{-(\mu x/\bar{c})} - B\frac{b^2\psi}{\bar{d}} e^{+(\mu x/\bar{d})}\right] dx$$

with

$$\phi = v\frac{\partial c}{\partial a} + w\frac{\partial c}{\partial b}$$
$$\psi = v\frac{\partial d}{\partial a} + w\frac{\partial d}{\partial b}.$$

Obviously, under the assumptions of the lemma, there exists a large enough positive real K_1 such that

$$\mathcal{T}_3 \le K_1 \int_0^L [a^2(t,x) + b^2(t,x)][|v(t,x)| + |w(t,x)|]dx.$$

This completes the proof of Lemma 1. ■

B. Proof of Lemma 4

In the proof of Lemma 4, we will use the following inequalities that hold for L^2 functions $f, g : [0,L] \to I\!R$ and for some positive real constant C

$$\int_0^L f^2(x)|g(x)|dx \le |g|_{L^\infty} \int_0^L f^2(x)\, dx \tag{19}$$

$$\int_0^L f^2(x)|g(x)|dx \le |f|_{L^\infty}^2 \int_0^L |g(x)|\, dx \tag{20}$$

$$|f|_{L^\infty} \le C\left[\left(\int_0^L f^2(x)\, dx\right)^{\frac{1}{2}} + \left(\int_0^L (f'(x))^2 dx\right)^{\frac{1}{2}}\right] \tag{21}$$

$$\int_0^L |f(x)|\, dx \le \sqrt{L}\left(\int_0^L f^2(x) dx\right)^{\frac{1}{2}}. \tag{22}$$

Inequalities (19) and (20) are obvious. Inequality (21) is the usual Sobolev inequality (see, for instance, [19, Th. VII, p. 129]) where the constant C depends on L only. Inequality (22) results from a straightforward application of the Cauchy–Schwarz inequality.

Lemma 4: If $|k_0 k_L| < 1$, if the positive real constants μ, A, B satisfy inequalities (9), there exist positive real constants λ_0 and δ_0 such that, if $S(t) < \delta_0$, then

$$\dot{S} \le -\lambda_0 S$$

along the solutions of the closed-loop system (10)–(15).

Proof: From Lemmas 1–3, the time derivative of the global Lyapunov function $S = U + V + W$ satisfies the following inequality:

$$\begin{aligned}\dot{S} \le &-\lambda_4 S \\ &+ K_1 \int_0^L [(a^2 + b^2)(|v| + |w|)]dx \\ &+ K_2 \int_0^L (v^2 + w^2)^{\frac{3}{2}}\, dx \\ &+ K_3 \int_0^L [(q^2 + r^2)(|v| + |w|) \\ &+ (v^2 + w^2)(|q| + |r|)]\, dx\end{aligned}$$

with $\lambda_4 = \min(\lambda_1, \lambda_2, \lambda_3)$.

We shall now compute upper bounds for the integral terms that appear in this expression. There exist real positive constants C_1, C_2 (independent from a, b, v, w, q, r) such that

$$\begin{aligned}
&\int_0^L [(a^2+b^2)(|v|+|w|)]\,dx \\
&\le (|v|_{L^\infty}+|w|_{L^\infty})\int_0^L (a^2+b^2)\,dx \\
&\le C_1\left[\left(\int_0^L v^2\,dx\right)^{\frac{1}{2}}+\left(\int_0^L q^2\,dx\right)^{\frac{1}{2}}+\left(\int_0^L w^2 dx\right)^{\frac{1}{2}}\right. \\
&\qquad\left.+\left(\int_0^L r^2 dx\right)^{\frac{1}{2}}\right]\left[\int_0^L (a^2+b^2)dx\right] \\
&\le C_2\left[V_1^{\frac{1}{2}}+W_1^{\frac{1}{2}}+V_2^{\frac{1}{2}}+W_2^{\frac{1}{2}}\right][U_1+U_2]. \qquad (23)
\end{aligned}$$

The first inequality is an application of (19). The second inequality is an application of (21). The third inequality results from the definitions of the Lyapunov functions U_i, V_i and W_i.

There exist real positive constants C_1' and C_2' (independent from a, b, v, w, q, r) such that

$$\begin{aligned}
&\int_0^L (v^2+w^2)^{\frac{3}{2}}\,dx = \int_0^L [(v^2+w^2)^{\frac{1}{2}}(v^2+w^2)]\,dx \\
&\le \int_0^L [(|v|+|w|)(v^2+w^2)]\,dx \\
&\le (|v|_{L^\infty}+|w|_{L^\infty})\int_0^L (v^2+w^2)\,dx \\
&\le C_1'\left[\left(\int_0^L v^2 dx\right)^{\frac{1}{2}}+\left(\int_0^L q^2 dx\right)^{\frac{1}{2}}+\left(\int_0^L w^2 dx\right)^{\frac{1}{2}}\right. \\
&\qquad\left.+\left(\int_0^L r^2 dx\right)^{\frac{1}{2}}\right]\left[\int_0^L (v^2+w^2)dx\right] \\
&\le C_2'\left[V_1^{\frac{1}{2}}+W_1^{\frac{1}{2}}+V_2^{\frac{1}{2}}+W_2^{\frac{1}{2}}\right][V_1+V_2]. \qquad (24)
\end{aligned}$$

The first inequality is obvious. The second inequality is an application of (19). The third inequality is an application of (21). The fourth inequality results from the definitions of the Lyapunov functions V_i and W_i.

There exist real positive constants C_1'' and C_2'' (independent from a, b, v, w, q, r) such that

$$\begin{aligned}
&\int_0^L [(q^2+r^2)(|v|+|w|)]\,dx \\
&\le (|v|_{L^\infty}+|w|_{L^\infty})\int_0^L (q^2+r^2)\,dx \\
&\le C_1''\left[\left(\int_0^L v^2 dx\right)^{\frac{1}{2}}+\left(\int_0^L q^2 dx\right)^{\frac{1}{2}}\right. \\
&\qquad\left.+\left(\int_0^L w^2 dx\right)^{\frac{1}{2}}+\left(\int_0^L r^2 dx\right)^{\frac{1}{2}}\right] \\
&\qquad\times\left[\int_0^L (q^2+r^2)dx\right] \\
&\le C_2''\left(V_1^{\frac{1}{2}}+W_1^{\frac{1}{2}}+V_2^{\frac{1}{2}}+W_2^{\frac{1}{2}}\right)(W_1+W_2). \qquad (25)
\end{aligned}$$

The first inequality is an application of (19). The second inequality is an application of (21). The third inequality results from the definitions of the Lyapunov functions V_i and W_i.

There exist real positive constants C_1''' and C_2''' (independent from a, b, v, w, q, r) such that

$$\begin{aligned}
&\le \int_0^L [(v^2+w^2)(|q|+|r|)]\,dx \\
&\le \left(|v|_{L^\infty}^2+|w|_{L^\infty}^2\right)\int_0^L (|q|+|r|)\,dx \\
&\le C_1'''\left[\int_0^L (v^2+q^2+w^2+r^2)dx\right]\left[\left(\int_0^L q^2 dx\right)^{\frac{1}{2}}\right. \\
&\qquad\left.+\left(\int_0^L r^2 dx\right)^{\frac{1}{2}}\right] \\
&\le C_2'''(V_1+W_1+V_2+W_2)\left(W_1^{\frac{1}{2}}+W_2^{\frac{1}{2}}\right). \qquad (26)
\end{aligned}$$

The first inequality is an application of (20). The second inequality results from a simultaneous application of both (21) and (22). The third inequality results from the definitions of the Lyapunov functions V_i and W_i.

Now, using inequalities (23)–(26) that we have just established, there exist a positive real constant K_4 such the time derivative of the global Lyapunov function $S = U + V + W$ may be further upperbounded as

$$\dot{S} \le -\lambda_4 S + K_4 S^{\frac{3}{2}}.$$

Then, for any λ_0 such that $0 < \lambda_0 < \lambda_4$, there exists δ_0 such that

$$K_4\left|S^{\frac{3}{2}}\right| < (\lambda_4-\lambda_0)S \qquad \forall S < \delta_0$$

which, in turn, implies that

$$\dot{S} < -\lambda_0 S \quad \forall S < \delta_0. \qquad (27)$$

In addition we may observe that if δ_0 is taken small enough then (27) implies that $|a(t,x)| + |b(x,t)| < \min(\delta_1, \delta_2, \delta_3)\ \forall x \in [0, L]$. This makes the use of Lemmas 1–3 legitimate in the proof of Lemma 4. ■

C. Proof of Theorem 1

Theorem 1: There exist positive real constants K, δ, λ such that, for any initial conditions $(a_0(x), b_0(x))$ in $H^2(0,L)^2$ satisfying the compatibility conditions shown in the equation at the top of the next page, and such that

$$|a_0(x)|_{H^2(0,L)} + |b_0(x)|_{H^2(0,L)} < \delta$$

the closed-loop system (10) with boundary conditions (11) has a unique continuous solution in $H^2(0,L)^2$ for $t \in [0, +\infty)$ which satisfies

$$\begin{aligned}
&|a(t,x)|_{H^2(0,L)} + |b(t,x)|_{H^2(0,L)} \\
&\qquad < K(|a_0(x)|_{H^2(0,L)} + |b_0(x)|_{H^2(0,L)})e^{-\lambda t}.
\end{aligned}$$

$$(CC)\begin{cases} c(a_0(0),b_0(0))\partial_x a_0(0) = \alpha'(b_0(0))d(a_0(0),b_0(0))\partial_x b_0(0) \\ d(a_0(L),b_0(L))\partial_x b_0(L) = \beta'(a_0(L))c(a_0(L),b_0(L))\partial_x a_0(L) \end{cases}$$

Proof: We thus consider the *smooth* solutions of the Cauchy problem (10), (11) with initial condition $(a_0(x), b_0(x))$ in $H^2(0,L)^2$. By a smooth solution we mean a map

$$(a,b) \in C^0(I, H^2(0,L))^2 \cap C^1(I, H^1(0,L))^2 \\ \cap C^2(I, L^2(0,L))^2$$

satisfying the closed-loop system (10) with boundary conditions (11) and initial condition $a(0,x) = a_0(x), b(0,x) = b_0(x)$, with $I \subset \mathbb{R}$ being an interval containing 0.

From [16, Ch. 16, Prop. 1.18, p. 364], we know that two smooth solutions of the Cauchy problem (10), (11) with the same initial condition are equal on the intersection of their interval of definition. (Actually, [16] deals with $\mathbb{R}$ instead of $[0, L]$ but the proof can be easily adapted.)

Furthermore, concerning the existence of smooth solutions to the Cauchy problem, we have the following result from [16, Ch. 16, Prop. 1.5, p. 365]. There exists $\delta_4 > 0$ such that, for every initial condition $(a_0(x), b_0(x)) \in H^2(0,L)^2$ satisfying the above compatibility condition (CC), if every solution (a,b) of the Cauchy problem (10), (11) with the initial condition $(a(0,\cdot), b(0,\cdot)) = (a_0, b_0)$ in $C^0([0,T), H^2(0,L))^2 \cap C^1([0,T), H^1(0,L))^2 \cap C^2([0,T), L^2(0,L))^2$ satisfies $S(t) < \delta_4$ for every $t \in [0,T)$, then this Cauchy problem has a solution defined on $[0,+\infty)$. (Again, the proof in [16] deals with $\mathbb{R}$ instead of $[0,L]$ but it can be easily adapted.)

Let $M > 0$ such that

$$\frac{1}{M} S(t) \leq |a(t,.)|_{H^2(0,L)} + |b(t,.)|_{H^2(0,L)} \leq M S(t).$$

Then it follows from Lemma 4 that Theorem 1 holds with

$$\delta := \frac{1}{M}\min(\delta_0, \delta_4) \quad K := M^2 \quad \lambda := \lambda_0.$$

■

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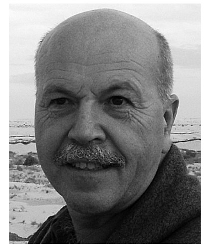
J.-M. Coron was born in Paris, France, in 1956. He received the Diplôme of engineer from Ecole Polytechnique in 1978 and from the Corps des Mines in 1981. He received the Thèse d'état in 1982.

He has been a Researcher at Ecole Nationale Supérieure des Mines de Paris, then Associate Professor at the Ecole Polytechnique. He is currently a Professor at Université Paris-Sud 11 and at the Institut Universitaire de France. His research interests include nonlinear partial differential equations and nonlinear control theory.

B. d'Andréa-Novel graduated from Ecole Supérieure d'Informatique, Electronique, Automatique, in 1984. She received the Ph. D. degree from Ecole Nationale Supérieure des Mines de Paris in 1987 and the Habilitation degree from Université Paris- Sud in 1995.

She is currently a Professor of Systems Control Theory and responsible for the research group in Advanced Control Systems at the Centre de Robotique—Ecole des Mines de Paris. Her current research interests include nonlinear control theory and applications to underactuated mechanical systems, control of wheeled vehicles with applications to automated highways, and boundary control of dynamical systems coupling ODEs and PDEs.



G. Bastin received the electrical engineering and Ph.D. degrees, both from Université Catholique de Louvain, Louvain-la-Neuve, Belgium.

He is presently a Professor in the Department of Mathematical Engineering at Université Catholique de Louvain and an Associate Professor at Ecole des Mines de Paris. His main research interests are in nonlinear control of compartmental systems and boundary control of hyperbolic systems with applications in biology, robotics, communication networks, and environmental systems.