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SHORT COMMUNICATION

Maximal stability region of a perturbed nonnegative matrix

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SUMMARY

For a class of positive matrices $A+K$ with a stable positive nominal part A and a structured positive perturbation part K, we address the problem of finding the largest set of admissible perturbations such that the global matrix remains stable. Theoretical bounds are derived and an algorithm for constructing this set is presented. As an example, this algorithm is applied to the regulation of water flow in open channels. Copyright © 2008 John Wiley & Sons, Ltd.

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1. INTRODUCTION

A linear time-invariant discrete-time system

$$x(k+1) = Ax(k) \tag{1}$$

is known to be stable if and only if $\rho(A) < 1$.

Positive systems are dynamic systems in which the relevant variables assume nonnegative values. These systems are quite common in applications where variables represent positive quantities such as populations, goods, money, time, data packet flows, densities of chemical species, probabilities, etc. The development of theoretical models that take into account this positivity requirement has been a very active field of research for a long time (see, e.g. the proceedings of recent symposia on positive systems [1, 2] and the references therein).

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A dynamical system (1) is called *positive* if any trajectory of the system starting in the nonnegative orthant $\mathbb{R}^n_{0,+}$ remains in $\mathbb{R}^n_{0,+}$. This is the case if and only if the matrix A has only real nonnegative entries. In many cases, it may be useful to consider systems with a known stable 'nominal' part A ($\rho(A)<1$) and an unknown part K, which may represent uncertainty

$$x(k+1)=(A+K)x(k) \tag{2}$$

The robustness of (2) will then depend on the size of the set S such that

$$\rho(A+K)<1 \quad \forall K\in S$$

One particular approach consists of considering structured matrices $K=E_1\Delta E_2^{\mathrm{T}}$, where Δ is a square diagonal matrix involving unknown parameters k_i on the diagonal (see later for a precise definition) and E_1 and E_2 are fixed matrices. The problem is then to find the *stability radius of A with respect to nonnegative perturbations of structure* (E_1, E_2^{T}), which is defined by

$$r_{\mathbb{R}_+}(A;E_1,E_2^{\mathrm{T}}):=\inf\{\|\Delta\|;\Delta\geqslant 0,\rho(A+E_1\Delta E_2^{\mathrm{T}})\geqslant 1\}$$

All perturbations in the following set:

$$S:=\{E_1\Delta E_2^{\mathrm{T}}\,|\,\|\Delta\|<r_{\mathbb{R}_+}(A;E_1,E_2^{\mathrm{T}})\}$$

are then shown to yield a stable system $A+K$. This problem is solved for one parameter in [3] and later for several parameters in [4], where a computable formula is provided.

In this paper we extend these results into a particular direction. We will only consider perturbations matrices Δ in the set $\mathscr{D}$ of nonnegative *diagonal* matrices $\mathscr{D}=\{\mathrm{diag}\{k_1,\ldots,k_m\}|k_i\geqslant 0\}$. The parameters k_i are the so-called free parameters occurring in the matrix K, E_1 and E_2 are two matrices placing the elements in appropriate positions in K. The two matrices E_1 and E_2^{T} have the following properties: there is a nonzero element in row i and column j of E_1 if k_j is present in row i of K and of E_2^{T} if k_i is present in column j of K. We clarify this by an example: if

$$K=\begin{pmatrix}2k_1 & 0 & 0\\ 0 & 0 & k_2\\ k_1 & 0 & 0\end{pmatrix}$$

then

$$\Delta=\begin{pmatrix}k_1 & 0\\ 0 & k_2\end{pmatrix},\quad E_1=\begin{pmatrix}2 & 0\\ 0 & 1\\ 1 & 0\end{pmatrix},\quad E_2^{\mathrm{T}}=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$$

We will restrict ourselves to matrices K for which both E_1 and E_2 are nonnegative as well: $E_1\geqslant 0$, $E_2\geqslant 0$. Notice that if one of the parameters appears in several rows and columns, it will be repeated several times in the diagonal matrix Δ.

The problem is to find the biggest set $S_{\mathscr{D}}\subseteq\{E_1\Delta E_2^{\mathrm{T}}|\Delta\in\mathscr{D}\}$ containing the origin and all the perturbations such that the system remains stable

$$S_{\mathscr{D}}:=\{K=E_1\Delta E_2^{\mathrm{T}}|\Delta\in\mathscr{D},\rho(A+K)<1\}$$

where A, E_1, E_2, Δ are nonnegative matrices. Let us point out that this is in fact a starlike set.

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Theorem 1.1
The set $S_{\mathscr{D}}$ is a starlike set.

Proof
From [5], we know that if $A, B \geqslant 0$ then $\rho(A+B) \geqslant \rho(A)$. It implies that if $K \in S_{\mathscr{D}}$

$$\rho(A+K)<1 \;\Rightarrow\; \rho(A+\alpha K)<1 \quad \forall \alpha \;\; 0\leqslant\alpha\leqslant 1$$

and $\alpha K \in S_{\mathscr{D}}$. If $K \notin S_{\mathscr{D}}$ then

$$\rho(A+K)\geqslant 1 \;\Rightarrow\; \rho(A+\beta K)\geqslant 1 \quad \forall \beta \;\; 1\leqslant\beta$$

and $\beta K \notin S_{\mathscr{D}}$. □

As the spectral radius is a continuous function of the parameter K, the boundary of the set $S_{\mathscr{D}}$ is described by

$$\partial S_{\mathscr{D}} = \{K \,|\, \exists i, k_i = 0 \,\&\, \rho(A+K) \leqslant 1 \text{ or } \rho(A+K)=1 \,\&\, K \geqslant 0\}$$

Examples later show that this set is in general not convex.

On the one side, the problem solved in [4] is more general because it does not assume that the perturbation Δ is diagonal. However, on the other side, when Δ is diagonal, their problem is more restrictive than the one addressed in this paper. All the operator norms induced by an arbitrary monotonic norm on $\mathbb{R}^n$ of a diagonal matrix are equal to the maximum of the elements of the matrix (see [6]). It means that the set S considered in [4] is a box with $k_i \leqslant k_i^m$ and $k_1^m = \cdots = k_n^m$ and hence a convex subset of $S_{\mathscr{D}}$. We will show that there exists a maximum starlike set $S_{\mathscr{D}} = \{E_1 \Delta E_2^{\mathrm{T}} \,|\, \Delta \in \mathscr{D}\}$ for which all matrices K in $S_{\mathscr{D}}$ lead to stable $A+K$ and we will describe the boundary of this set in terms of polynomial equations.

2. MAXIMAL PERTURBATION OF NONNEGATIVE MATRICES

First we develop some new theoretical results and we then present an algorithm for computing $\partial S_{\mathscr{D}}$.

2.1. Theoretical results

In this section we show that the problem may be decoupled in smaller subproblems involving only a subset of the parameters k_i. To each of these subproblems there corresponds a starlike set $S_{\mathscr{D}_i}$ for which we obtain an analytical expression. The set $S_{\mathscr{D}}$ is the intersection of the sets $S_{\mathscr{D}_i}$.

As $K = E_1 \Delta E_2^{\mathrm{T}}$ is nonnegative and as the eigenvalues are continuous functions of the matrix elements, we have that the critical switch between $\rho<1$ and $\rho\geqslant 1$ will occur when

$$\rho(A+E_1\Delta E_2^{\mathrm{T}})=1$$

Working only with positive matrices, we have that the spectral radius is also an eigenvalue and hence the above condition is equivalent to

$$\det(A+E_1\Delta E_2^{\mathrm{T}}-I)=0$$

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and

$$\det(E_1\Delta E_2^{\mathrm{T}}-(I-A))=0$$

As $\det(I-A)\neq 0$ ($\rho(A)<1$) we can multiply the previous equation by $\det(I-A)^{-1}$ to obtain

$$\det((I-A)^{-1}E_1\Delta E_2^{\mathrm{T}}-I)=0$$

Using the well-known property (see [4])

$$\det(MN-I)=0\Leftrightarrow\det(NM-I)=0$$

this is also equivalent to

$$\det(E_2^{\mathrm{T}}(I-A)^{-1}E_1\Delta-I)=0$$

where $M:=E_2^{\mathrm{T}}(I-A)^{-1}E_1$ is nonnegative as $(I-A)^{-1}=\sum_{i=1}^{\infty}A^i$ and E_1, E_2 are nonnegative. We can use Lemma 2.1 (see [7]) to transform M to a normal form $\hat{M}$.

Lemma 2.1

Every nonnegative matrix M has a normal form that can obtained under congruent permutations

$$\hat{M}=PMP^{\mathrm{T}}=\begin{pmatrix}\hat{M}_{11} & & 0\\ & \ddots & \\ * & & \hat{M}_{mm}\end{pmatrix} \tag{3}$$

where each diagonal block $\hat{M}_{ii}$ is square, irreducible or just a 1×1 zero bock.

Applying the same permutation to Δ, we define $\hat{\Delta}=P\Delta P^{\mathrm{T}}$ and have

$$\det(\hat{M}\hat{\Delta}-I)=0$$

This clearly decomposes into a number of decoupled problems

$$\det(\hat{M}_{ii}\hat{\Delta}_{ii}-I)=0$$

This polynomial equation describes a part of $\partial S_{\mathscr{D}_i}$. The intersection of all these sets leads to the admissible set $S_{\mathscr{D}}$. Let us solve the subproblems

- Let $\hat{M}_{ii}=0$ then $\det(\hat{M}_{ii}\hat{\Delta}_{ii}-I)\neq 0$ for all *bounded* $\hat{\Delta}_{ii}$.
- Let $\hat{M}_{ii}\neq 0$ and irreducible. If the size n_i of $\hat{M}_{ii}=[m_{r,c}]_{r,c=1}^{n_i}$ is small enough, the problem can be exactly solved.
  - If $n_i=1$, the solution is trivial, i.e. $\det(m_{11}k_1-1)=0$ for $k_1=m_{11}^{-1}$.
  - If $n_i=2$, we have

$$\det\left(\begin{pmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix}\begin{pmatrix}k_1 & 0\\ 0 & k_2\end{pmatrix}-I\right)=0$$

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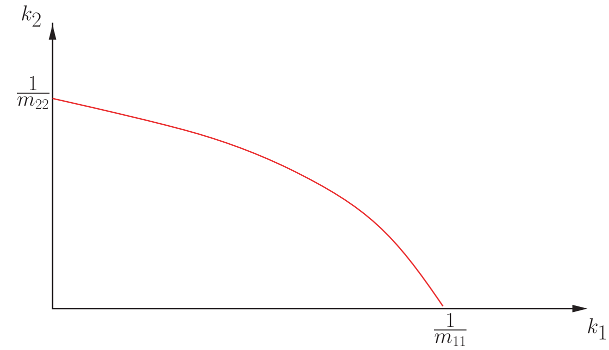

Figure 1. The largest set of the parameters k_1 and k_2 containing the origin such that $A+E_1\Delta E_2^{\mathrm{T}}$ is stable.

or equivalently

$$(m_{11}m_{22}-m_{12}m_{21})k_1k_2-m_{11}k_1-m_{22}k_2+1=0 \tag{4}$$

The stable region for the k_1, k_2 is thus a starlike set with boundary defined by (4) and $k_{1,2}=0$ (see Figure 1).

- If $n_i=3$, we have

$$\det(\hat{M}_{ii}\hat{\Delta}_{ii}-I)=0$$

or equivalently

$$\det(\hat{M}_{ii})k_1k_2k_3+(m_{21}m_{12}-m_{11}m_{22})k_1k_2+(m_{13}m_{31}-m_{11}m_{33})k_1k_3$$
$$+(m_{23}m_{32}-m_{22}m_{33})k_2k_3+m_{11}k_1+m_{22}k_2+m_{33}k_3-1=0 \tag{5}$$

The stable region for the k_i is thus also a starlike set whose boundaries are defined by (5) and $k_{1,2,3}=0$ (see Figure 2).

- It may happen that a coefficient k_i appears in different blocks $\hat{\Delta}_{ii}$. For example, if

$$A=\begin{pmatrix} a_{11} & a_{12} & & \\ a_{21} & a_{22} & & \\ & & a_{33} & a_{34} \\ & & a_{43} & a_{44} \end{pmatrix}, \quad K=\begin{pmatrix} k_1 & & & \\ & k_2 & & \\ & & k_3 & \\ & & & k_1 \end{pmatrix}$$

then

$$\hat{\Delta}_{11}=\begin{pmatrix} k_1 & \\ & k_2 \end{pmatrix}, \quad \hat{\Delta}_{22}=\begin{pmatrix} k_3 & \\ & k_1 \end{pmatrix}$$

In this case, the admissible set for (k_1, k_2, k_3) is simply the intersection of the two sets obtained by analysing the two subproblems. This is illustrated in Figure 3.

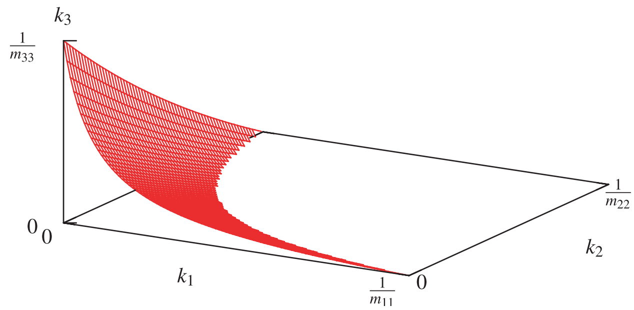

Figure 2. The boundary of the largest set (k_1, k_1, k_3) containing the origin such that $A + E_1 \Delta E_2^{\mathrm{T}}$ is stable.

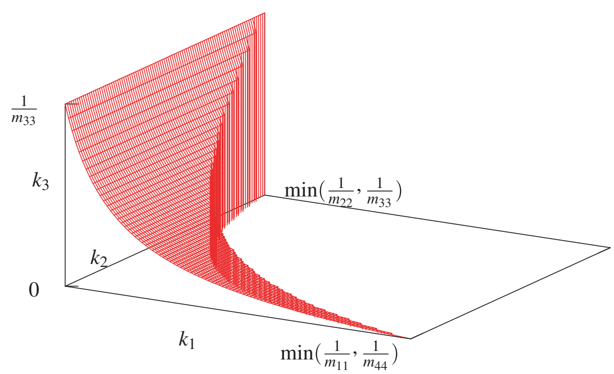

Figure 3. The admissible set is the intersection of the two admissible sets.

2.2. Modified problem

In the above analysis, the sets $S_{\mathscr{D}_i}$ can become quite complex to describe if n_i becomes large, as one has to solve polynomial equation of degree n_i in several variables. It may be easier to consider only necessary conditions that ensure stability (i.e. a subset of $S_{\mathscr{D}}$). Such an approach has been taken in [4, 8, 9] where convex subsets of $S_{\mathscr{D}}$ are described.

We describe here two approximations of the original problem that are easier to compute.

For example, one can freeze one particular k_i and express the conditions on the remaining parameters. This subset will be a slice of $S_{\mathscr{D}_i}$. This subset is still starlike in the leftover parameter.

Another possibility is to express a condition on the maximum of the k_i in the same way as [4]. The following result is a refinement of the global bound obtained in [4].

Lemma 2.2
Let $\hat{M}_{i,i}$ be as defined in Lemma 2.1 and let ρ_i be its spectral radius, then

$$\det(\hat{M}_{ii}\hat{\Delta}_{ii} - I) \neq 0 \quad \text{for } \hat{\Delta}_{ii} < \rho_i^{-1} I$$

$$\det(\hat{M}_{ii}\hat{\Delta}_{ii} - I) = 0 \quad \text{for } \hat{\Delta}_{ii} = \rho_i^{-1} I$$

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Proof

Let x_i be the Perron vector of the irreducible matrix $\hat{M}_{ii}$. It is well known (see [7]) that for an irreducible matrix the so-called Perron vector x_i corresponding to the (positive) Perron root ρ_i is strictly positive. Therefore,

$$\hat{M}_{ii} x_i = \rho_i x_i, \quad x_i > 0$$

then clearly

$$(\hat{M}_{ii} \rho_i^{-1} I - I) x_i = 0, \quad \hat{\Delta}_{ii} = \rho_i^{-1} I$$

Also for $\hat{\Delta}_{ii} < \rho_i^{-1} I$

$$\det(\hat{M}_{ii} \hat{\Delta}_{ii} - I) \neq 0$$

as there exists a scaling

$$\| D^{-1} \hat{M}_{ii} D \|_\infty = \rho_i$$

and clearly

$$\| D^{-1} \hat{M}_{ii} \hat{\Delta}_{ii} D \|_\infty = \| D^{-1} \hat{M}_{ii} D \hat{\Delta}_{ii} \|_\infty < 1$$ □

Therefore, we can claim that all matrices Δ in the following set

$$S = \left\{ \Delta | \begin{pmatrix} \hat{\Delta}_{11} & & \\ & \ddots & \\ & & \hat{\Delta}_{mm} \end{pmatrix} = P \Delta P^{\mathrm{T}}, \quad \hat{\Delta}_{ii} < \begin{cases} \text{any bounded value} & \text{if } \rho_i = 0 \\ \rho_i^{-1} I & \text{if } \rho_i \neq 0 \end{cases} \right\}$$

are such that

$$\rho(A + E_1 \Delta E_2^{\mathrm{T}}) < 1$$

The problem may thus be split into several subproblems. If the subproblems are small enough, we may have some analytical necessary and sufficient conditions. If the subproblems are more complex, to ensure that $\rho(A + E_1 \Delta E_2^{\mathrm{T}}) < 1$, sufficient conditions may be used such as freezing a k_i or imposing that for each $\hat{M}_{ii} \neq 0$, $\hat{\Delta}_{ii} < \rho_i^{-1} I$.

2.3. Algorithm

The results presented in the previous section can be used to construct the set $S_{\mathscr{D}}$, in the following manner:

1. Compute the matrix $M := E_2^{\mathrm{T}} (I - A)^{-1} E_1$ and perform permutations to put it under the normal form (3). This can be done by applying the following algorithm:
   (a) Use Tarjan's algorithm [10] to find the set of strongly connected subgraphs associated with the graph G defined by the adjacency matrix M^{ad} ($M^{\mathrm{ad}}_{i,j} = 1$ if $M_{i,j} \neq 0$, $M^{\mathrm{ad}}_{i,j} = 0$ otherwise).

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(b) Consider a new graph G' whose nodes represent the strongly connected subgraphs: two nodes i and j of G' are connected if there exists one edge between a node of G in the subgraph i and a node of G in the subgraph j. The adjacency matrix of this new graph G' can be computed simply from M^{ad} by first summing up the rows corresponding to the same subgraph and then summing up the columns corresponding to the same subgraph.
(c) Identify a leaf i of the graph G' (which always exists because there is no cycle in G') and permute the columns and the rows of M corresponding to the subgraph i at the beginning of the matrix. Suppress node i from G'. Repeat 1(c) until M is in the canonical form (3).

2. For each of the $\hat{M}_{ii}$ blocks, express the condition $\det(\hat{M}_{ii}\hat{\Delta}_{ii} - I) = 0$ which describes a part of the boundary of $S_{\mathscr{D}}$. If the size of $\hat{M}_{ii}$ is too high, a more restrictive condition such as first freezing a k_i or a condition in terms of $\rho(\hat{M}_{ii})$ can be used.

It can be now claimed that, if

$$(k_1, \ldots, k_n) \in S_{\mathscr{D}}$$

then

$$\rho\left(A + E_1 \begin{pmatrix} k_1 & & \\ & \ddots & \\ & & k_n \end{pmatrix} E_2^{\mathrm{T}}\right) < 1$$

3. APPLICATION TO THE CONTROL OF HYPERBOLIC SYSTEMS OF CONSERVATION LAWS

As a matter of illustration, we present in this section an application to the control design for hyperbolic systems of conservation laws, with a typical example from waterways networks management (see, e.g. [11]).

In the field of hydraulics, the flow in open channels is generally represented by the so-called Saint Venant equations. We consider the special case of open channels delimited by two overflow spillways as represented in Figure 4. We assume that

1. the channels are horizontal;
2. the channels are prismatic with a constant rectangular section and a unit width;
3. the friction effects are neglected.

The flow dynamics in a channel are described by a system of two laws of conservation, namely the law of mass conservation

$$\partial_t H_i + \partial_x Q_i = 0 \tag{6}$$

and the law of momentum conservation

$$\partial_t Q_i + \partial_x \left(\frac{Q_i^2}{H_i} + g \frac{H_i^2}{2} \right) = 0 \tag{7}$$

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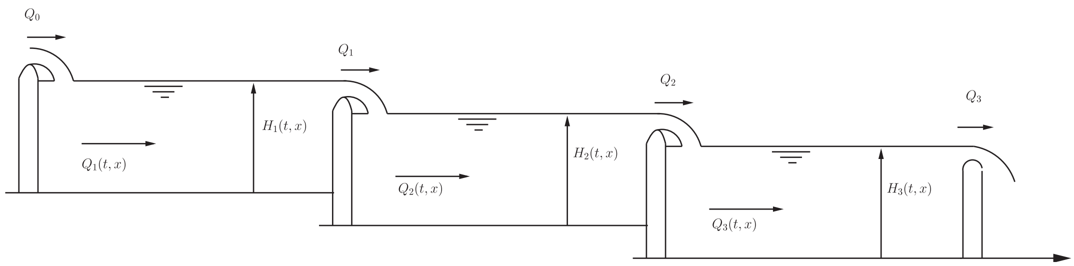

Figure 4. A canal with three reaches and four gates.

where $H_i(t,x)$ represents the water level and $Q_i(t,x)$ the water flow rate in the ist reach while g denotes the gravitation constant.

The control actions are the vertical positions of the spillways located at the extremities of the pools. By manipulating these spillways, the flows between different pools can be controlled. For constant spillway positions and a constant inflow rate Q_0, there is a unique steady-state solution.

Let us consider the deviations of $H_i(t,x)$ and $Q_i(t,x)$ with respect to the steady-state values $\bar{H}_i$ and $\bar{Q}_i$

$$h_i(t,x) = H_i(t,x) - \bar{H}_i$$

$$q_i(t,x) = Q_i(t,x) - \bar{Q}_i$$

By linearizing the model equations (6)–(7) around the steady state $(\bar{H}_i, \bar{Q}_i)$, we get the linear model

$$\partial_t h_i(t,x) + \partial_x q_i(t,x) = 0$$

$$\partial_t q_i(t,x) + \left(g\bar{H}_i - \frac{\bar{Q}_i^2}{\bar{H}_i^2} \right) \partial_x h_i(t,x) + 2\frac{\bar{Q}_i}{\bar{H}_i} \partial_x q_i(t,x) = 0 \quad \text{for } i = 1, \ldots, 3$$

with the six boundary conditions

$$q_1(t,0) = u_0$$

$$q_1(t,L) = u_1$$

$$q_2(t,L) = u_2$$

$$q_3(t,L) = u_3$$

$$q_1(t,L) = q_2(t,0)$$

$$q_2(t,L) = q_3(t,0)$$

The first four conditions are imposed by the controls. The last two conditions express the flow conservation.

As shown, e.g. in [11], it is convenient to work with the *Riemann coordinates* defined by the following change of coordinates:

$$a_i(t,x) = q_i(t,x) + \beta_i h_i(t,x)$$

$$b_i(t,x) = q_i(t,x) - \alpha_i h_i(t,x)$$

where

$$\alpha_i = \sqrt{g\bar{H}_i} + \frac{\bar{Q}_i}{\bar{H}_i}$$

$$\beta_i = \sqrt{g\bar{H}_i} - \frac{\bar{Q}_i}{\bar{H}_i}$$

With these coordinates, the linear system is rewritten under the following diagonal form:

$$\partial_t \begin{pmatrix} a_i(t,x) \\ b_i(t,x) \end{pmatrix} + \begin{pmatrix} +\alpha & 0 \\ 0 & -\beta \end{pmatrix} \partial_x \begin{pmatrix} a_i(t,x) \\ b_i(t,x) \end{pmatrix} = 0 \quad \forall i \in 1, \ldots, 3 \tag{8}$$

and the boundary conditions are expressed as

$$\frac{\alpha a_1(t,0) + \beta b_1(t,0)}{\alpha+\beta} = u_0$$

$$\frac{\alpha a_1(t,L) + \beta b_1(t,L)}{\alpha+\beta} = u_1$$

$$\frac{\alpha a_2(t,L) + \beta b_2(t,L)}{\alpha+\beta} = u_2$$

$$\frac{\alpha a_3(t,L) + \beta b_3(t,L)}{\alpha+\beta} = u_3$$

$$\frac{\alpha a_1(t,L) + \beta b_1(t,L)}{\alpha+\beta} = \frac{\alpha a_2(t,0) + \beta b_2(t,0)}{\alpha+\beta}$$

$$\frac{\alpha a_2(t,L) + \beta b_2(t,L)}{\alpha+\beta} = \frac{\alpha a_3(t,0) + \beta b_3(t,0)}{\alpha+\beta}$$

We consider the situation where each control $u_i(t)$ is a linear function of only one state variable, as follows:

$$u_0 \text{ function of } b_1(t,0), \quad u_0 = k'_0 b_1(t,0)$$

$$u_1 \text{ function of } a_1(t,L), \quad u_1 = k'_1 a_1(t,L)$$

$$u_2 \text{ function of } a_2(t,L), \quad u_2 = k'_2 a_2(t,L)$$

$$u_3 \text{ function of } a_3(t,L), \quad u_3 = k'_3 a_3(t,L)$$

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With the following reparametrization:

$$k_0 = -\frac{\beta}{\alpha} + \frac{(\alpha+\beta)}{\alpha} k_0'$$

$$k_i = -\frac{\alpha}{\beta} + \frac{(\alpha+\beta)}{\beta} k_i', \quad i = 1, \ldots, 3$$

the boundary conditions are expressed as

$$\begin{pmatrix} b_1(t,L) \\ b_2(t,L) \\ b_3(t,L) \\ a_1(t,0) \\ a_2(t,0) \\ a_3(t,0) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 0 & k_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_3 \\ k_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\beta}{\alpha} & 0 & 1+\frac{\beta}{\alpha}(k_1) & 0 & 0 \\ 0 & 0 & -\frac{\beta}{\alpha} & 0 & 1+\frac{\beta}{\alpha}(k_2) & 0 \end{pmatrix}}_{A+K} \cdot \begin{pmatrix} b_1(t,0) \\ b_2(t,0) \\ b_3(t,0) \\ a_1(t,L) \\ a_2(t,L) \\ a_3(t,L) \end{pmatrix} \tag{9}$$

where A is fixed and K has a fixed structure but the values of its nonzero entries are linear combinations of the 'free parameters' k_i. In this case, K is the part of the matrix, which reflects the choice made by the operator for the control parameters and A reflects the conservation of the flow.

The problem studied here is to find the largest range of values for the control parameters k_i such that the system remains stable. Here stability means that, from any smooth enough initial condition, the Cauchy problem for system (8) with boundary conditions (9) has a unique classical solution that exponentially converges to the origin. As the system under consideration has been derived by linearizing the nonlinear Saint Venant model, this stability property will only be valid for perturbations that are small enough.

From Theorem 6 in [12], we know that a sufficient stability condition is that

$$\rho(|A+K|) < 1$$

where $|A+K|$ denotes the matrix whose entries are the absolute values of the entries of $A+K$. We are thus interested in finding a set S such that $\rho(|A+K|) < 1 \ \forall K \in S$.

From [5], we know that if $A, B \geqslant 0$ then $\rho(A+B) \geqslant \rho(A)$. It implies that

$$\rho(|A+K|) \leqslant \rho(|A|+|K|)$$

and the set S may be found using the algorithm presented in Section 2.3.

If we apply the algorithm proposed in Section 2.3 with the following numerical values:

$$\alpha = 3.6$$

$$\beta = 2.6$$

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we obtain that there exists three blocks $\hat{M}_{ii}$ (one of dimension 2 and two of dimension 1). The different coefficients must be bounded as follows:

$$|k_0 k_1| < 1$$
$$|k_2| < 1.38$$
$$|k_3| < 1.38$$

to guarantee the stability of the system. This decomposition in three blocks is quite natural as only the two first parameters influence the stability of the first reach. If the first reach is stable, only the parameter k_2 has an influence on the stability of the second reach. Eventually, k_3 controls the stability of the third reach.

The decoupling of the problem in smaller subproblem allows to increase the possible value of some parameters, which may have a positive influence on the global behaviour of the system. In the example of Section 3, if we take the sufficient condition presented in [4] all the parameters must be bounded by 1. The decomposition in subproblems allows us to increase the value of k_3 and k_4 up to 1.38. It also allows to select $k_0 > 1$ provided k_1 is small enough and conversely.

4. CONCLUSIONS

In this paper, we have considered the problem of finding the largest set of perturbation such that a positive matrix remains stable. We have extended the results of [4] in the particular case where the perturbation matrix Δ is diagonal. In this case, the problem can be decoupled in smaller subproblems. For each of the subproblems, necessary and sufficient analytical conditions were derived to describe the starlike sets of admissible parameters. Outside of these sets, the perturbation destabilizes the system. These sets, which are not necessarily convex, contain the largest admissible ball described in [4].

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