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Network Controllability Is Determined by the Density of Low In-Degree and Out-Degree Nodes

Ginestra Bianconi

School of Mathematical Sciences, Queen Mary University of London, London, UK

Queen Mary
University of London

Reference and collaborators

G. Menichetti, L. Dall'Asta and G. Bianconi,
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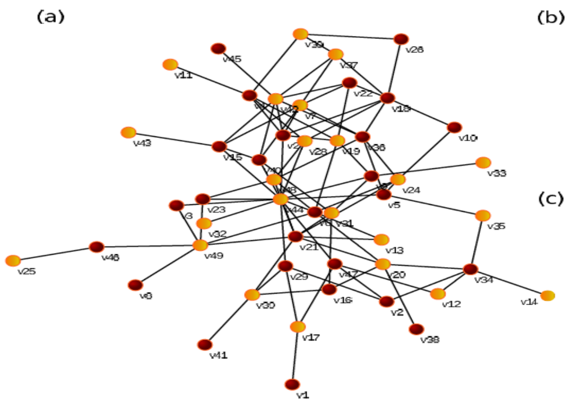
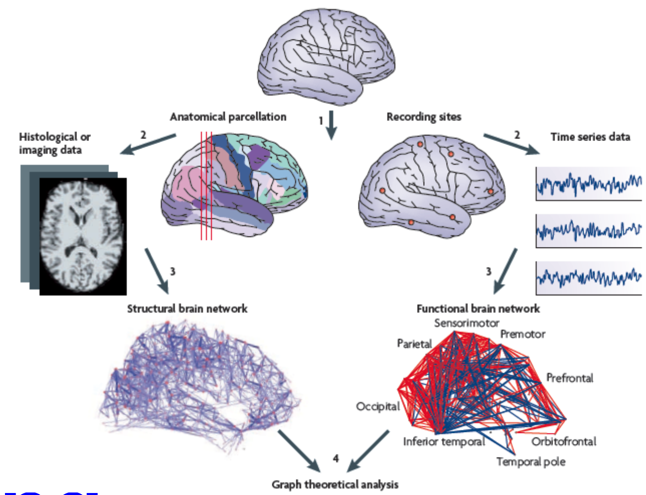
Giulia Menichetti

Luca Dall'Asta





Of relevance

for understanding

financial networks, and

the brain dynamics

and for network medicine

determining

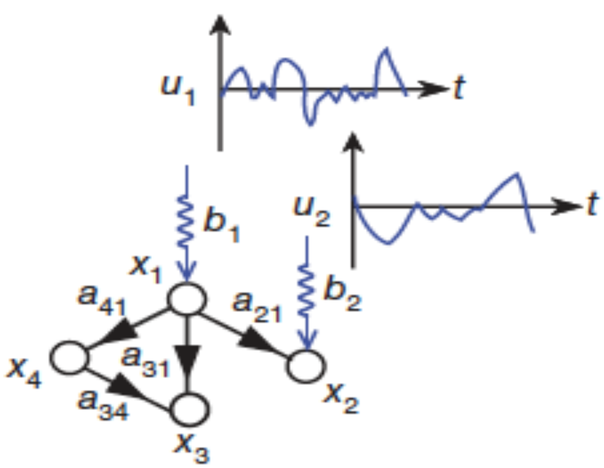
the controllability of networks

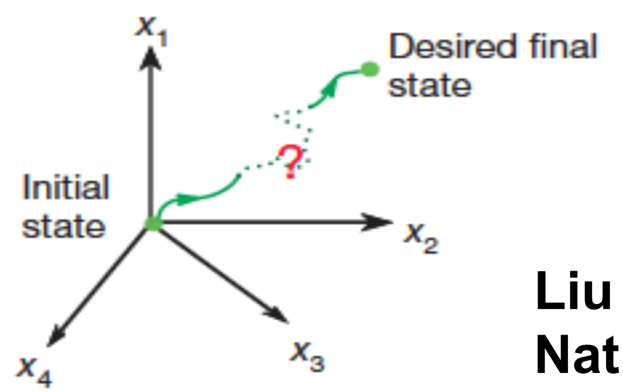
is a central theoretical problem

of network theory

Driver nodes

The driver nodes of a network
are the nodes that,
when stimulated by an external signal,
can drive
the dynamical state of a network
to any desired state.



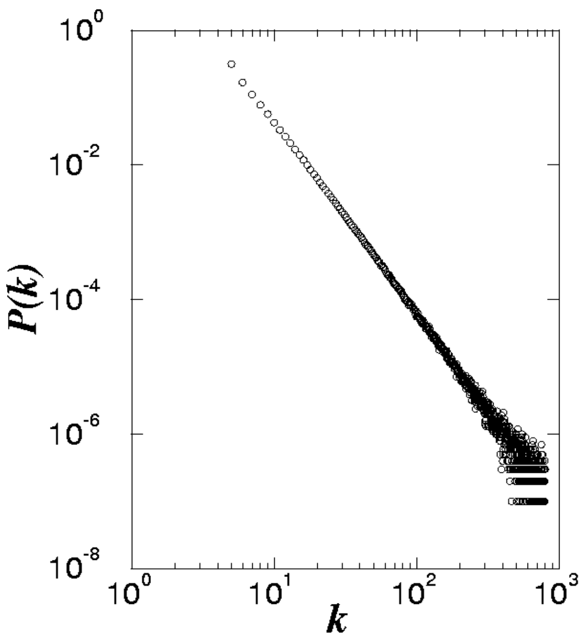


Liu et. al.
Nature (2011)

The importance of hubs for the dynamics on complex networks

Hubs and scale-free networks are essential for determining the

- The robustness of complex networks
- The stability of the ferromagnetic phase of the Ising model against thermal fluctuations
- Triggering and orchestrating the synchronization on neuronal networks
- Characterizing the epidemic spreading properties of networks



Which are the
key structural properties
of networks
that determine
their controllability?

The low In-degree and Out-degree nodes
(***hubs are irrelevant for determining the number of driver nodes***)

G. Menichetti, L.Dall'Asta and G. Bianconi
PRL 113, 078701 (2014)

Our framework:
Structural Controllability

The pivotal paper by Liu et al. Nature (2011) has shown that the structural controllability of networks can be mapped to the directed matching problem on these networks (that can be solved efficiently by Statistical Mechanics methods) and has open a new field in network theory.

ARTICLE

doi:10.1038/nature10011

Controllability of complex networks

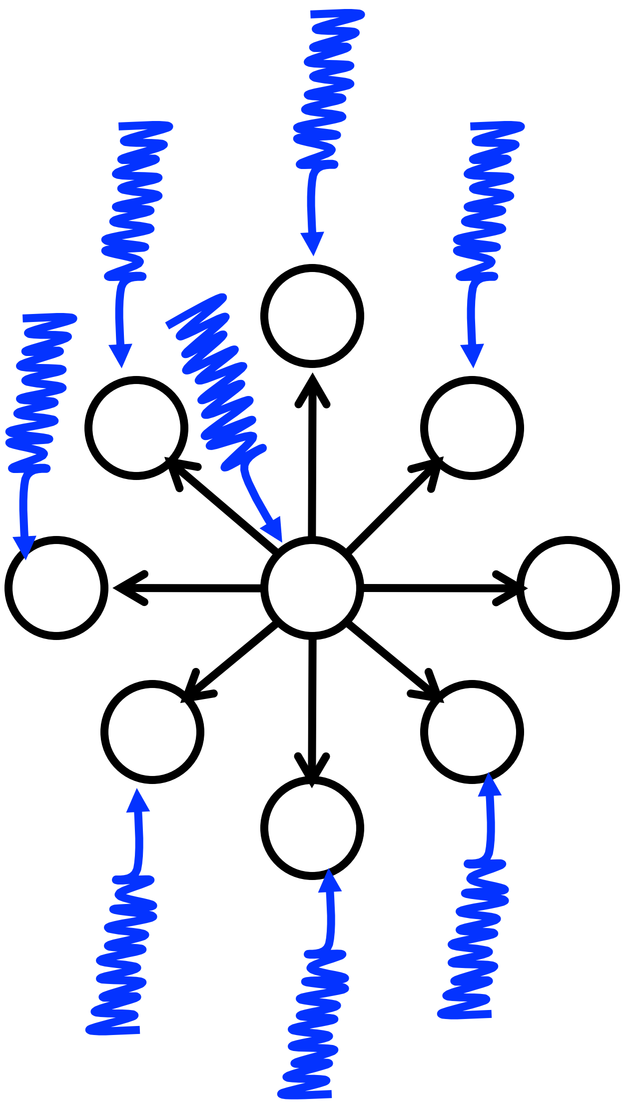
Yang-Yu Liu[1,2], Jean-Jacques Slotine[3,4] & Albert-László Barabási[1,2,5]

The ultimate proof of our understanding of natural or technological systems is reflected in our ability to control them. Although control theory offers mathematical tools for steering engineered and natural systems towards a desired state, a framework to control complex self-organized systems is lacking. Here we develop analytical tools to study the controllability of an arbitrary complex directed network, identifying the set of driver nodes with time-dependent control that can guide the system's entire dynamics. We apply these tools to several real networks, finding that the number of driver nodes is determined mainly by the network's degree distribution. We show that sparse inhomogeneous networks, which emerge in many real complex systems, are the most difficult to control, but that dense and homogeneous networks can be controlled using a few driver nodes. Counterintuitively, we find that in both model and real systems the driver nodes tend to avoid the high-degree nodes.

According to control theory, a dynamical system is controllable if, with a suitable choice of inputs, it can be driven from any initial state to any desired final state within finite time[1–3]. This definition agrees with our intuitive notion of control, capturing an ability to guide a system's

of traffic that passes through a node i in a communication network[24] or transcription factor concentration in a gene regulatory network[25]. The $N \times N$ matrix A describes the system's wiring diagram and the interaction strength between the components, for example the traffic

Controllability of a star network



- If we stimulate the **hub node** all the low in-degree nodes get the same signal we cannot explore any dynamical state of the network.
- **The driver nodes of the network are 8!**

The dynamical system

- Given a graph *G=(V,E)* of N nodes we consider the linear dynamical system

$$\boxed{\frac{d\,\mathbf{x}(\mathrm{t})}{dt} = A\mathbf{x} + B\mathbf{u}}$$

- In which
  - $\mathbf{x}$(t) is a vector of elements x_i(t) with i=1,2,…N and represents the dynamical state of the network,
  - A is an asymmetric $N \times N$ matrix describing directed weighted interactions between the nodes,
  - B is a $N \times M$ matrix describing the interaction between the nodes and M signals
  - The vector **u** of elements u_α(t) with α=1,2,…M describes the $M \leq N$ external signals.

Kalman's condition

- Given any realization of the matrices A and B, the dynamical system is controllable if it satisfy the Kalman's controllability condition, i.e. the $N \times MN$ matrix

$$C = (B, AB, A^2B, \ldots A^{N-1}B)$$

is full rank, i.e. $$rank(C) = N$$

The notion of exact controllability is computationally very demanding and often can turn out to be unusable since the entries of the nonzero matrix of A and B are not perfectly known it is then useful to introduce the notion of structural controllability.

Structural Controllability

- A system is structural controllable if, for every choice of the non-zero elements of the matrices A and B, except for a variety of zero Lebesgue measure in parameter space, C is full rank.

Minimum Input Theorem

Liu et al. Nature (2011) stated the

Minimum Input Theorem.

According to this theorem, the minimum set of driver nodes that guarantees the full structural controllability of a network is the set of unmatched nodes in a maximum matching of the same directed network.

The maximum directed matching problem

A matching M of a directed graph
is the set of directed edges
without common start and end vertices,
and it is maximum
when it contains
the maximum number of edges.

Statistical mechanics of the Maximum Matching Problem

- The variables $s_{ij}=0,1$ for every edge (i,j) define the matching and need to satisfy the following constraints

$$\boxed{\sum_{j\in\partial_+ i} s_{ij} \leq 1 \qquad \sum_{j\in\partial_- i} s_{ji} \leq 1}$$

- The maximum matching is the matching that minimize the energy equal to twice the number of unmatched nodes i.e. driver nodes of the network

$$\boxed{E = 2\sum_{i=1,\dots,N}\left(1-\sum_{j\in\partial_- i} s_{ji}\right)}$$

The statistical mechanics solution

- The Belief Propagation solution on a network in which the local tree-like assumption is valid, is given in terms of the messages sent along the links either in the same direction $h_{i\to j}$ or in the opposite direction $\hat{h}_{i\to j}$ of the links and can take three values, 1,-1,0 indicating respectively

$$h_{i\to j}=\hat{h}_{i\to j}=1$$ **Match me**

$$h_{i\to j}=\hat{h}_{i\to j}=-1$$ **Do not match me**

$$h_{i\to j}=\hat{h}_{i\to j}=0$$ **Do what you want**

Belief Propagation equations

The messages need to satisfy the following Belief Propagation (BP) equations

$$\boxed{\begin{aligned} h_{i\to j} &= -\max\left[-1, \max_{k\in\partial_+ i\setminus j} \hat{h}_{k\to i}\right] \\ \hat{h}_{i\to j} &= -\max\left[-1, \max_{k\in\partial_- i\setminus j} h_{k\to i}\right] \end{aligned}}$$

Finally the energy is given by

$$\boxed{E = -\sum_{i=1}^{N} \max\left[-1, \max_{k\in\partial_+ i} \hat{h}_{k\to i}\right] - \sum_{i=1}^{N} \max\left[-1, \max_{k\in\partial_- i} h_{k\to i}\right] + \sum_{<i,j>} \max\left[0, h_{i\to j} + \hat{h}_{j\to i}\right]}$$

First result: Sufficient condition for full controllability

For any sparse network
without a finite clustering coefficient,
(where the locally tree-like approximation is valid),

if the minimum in-degree
and the minimum out-degree of the network
are both greater than 2,

the network is fully controllable.

Sketch of the derivation

- Given the BP equation we observe that the solution in which all the **messages are zero correspond to a zero energy, i.e. full controllability.**
- From the BP equation it is easy to see that if all nodes have in-degree and out-degree greater than one this configuration of the messages if a BP solution

$$\boxed{\begin{aligned} h_{i\to j} &= -\max\left[-1, \max_{k\in\partial_+ i\setminus j} \hat{h}_{k\to i}\right] \\ \hat{h}_{i\to j} &= -\max\left[-1, \max_{k\in\partial_- i\setminus j} h_{k\to i}\right] \end{aligned}}$$

- The solution is also stable if all the in-degree and out-degree are greater than 2.

Second result: Necessary and sufficient condition for full controllability on a random network with given degree distribution

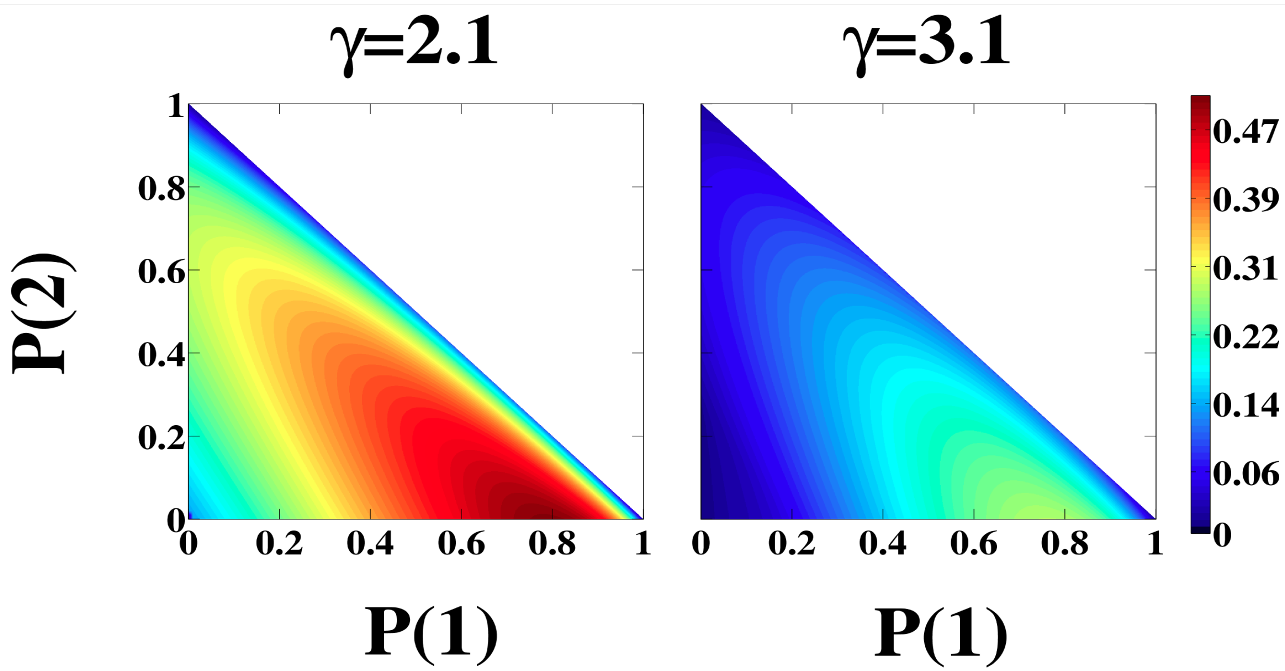
- A random network with given degree distribution is fully controllable iff

$$P^{out/in}(1) = P^{out/in}(2) = 0$$

$$P^{out}(2) < \frac{\langle k \rangle_{in}^{\;2}}{2\langle k(k-1) \rangle_{in}} \qquad P^{in}(2) < \frac{\langle k \rangle_{out}^{\;2}}{2\langle k(k-1) \rangle_{out}}$$

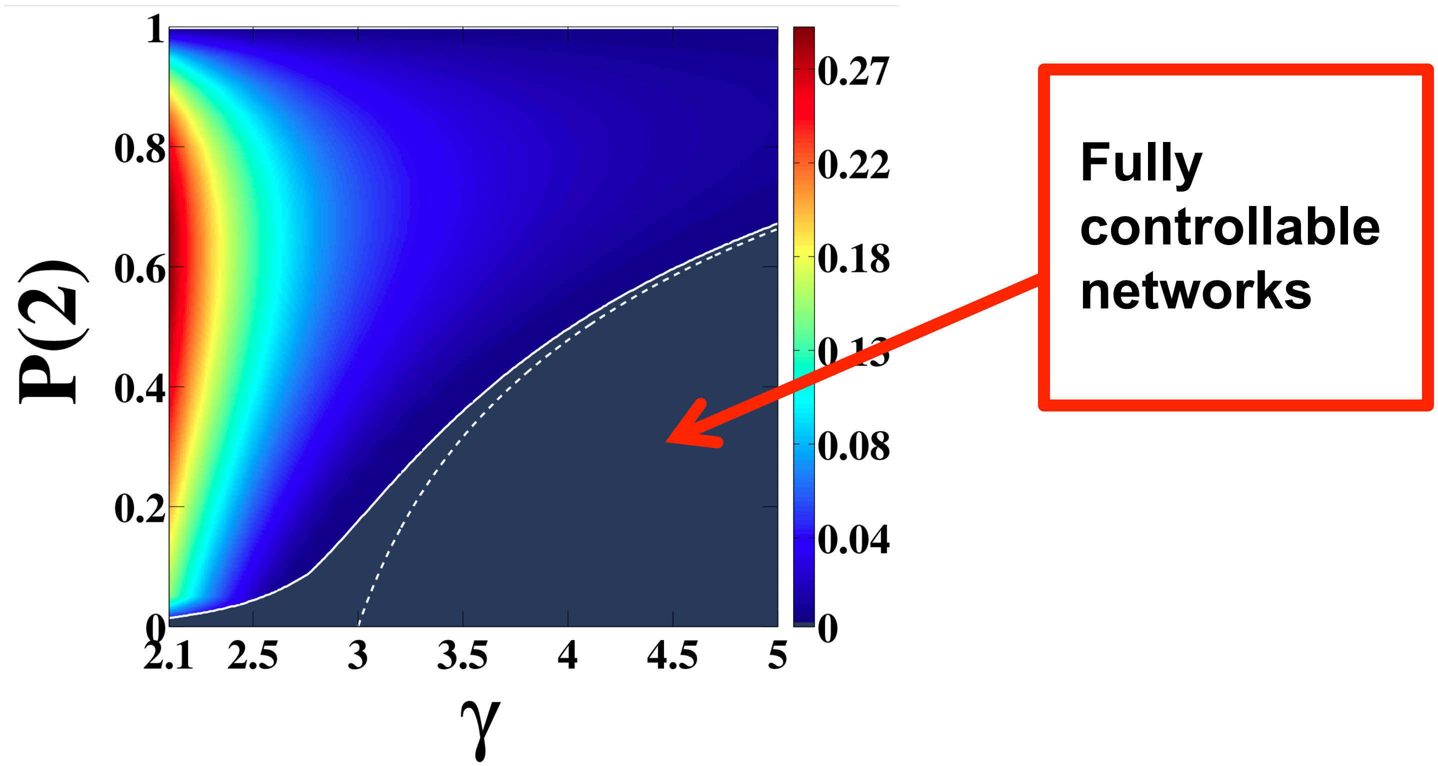
i.e. its minimum in and out-degree are 2 and the nodes with in/out degree 2 are less than a threshold.

Number of driver nodes as a function of the density of low in-degree and out-degree nodes changes smoothly



$$P^{in}(k) = P^{out}(k) = \begin{cases} P(1) & for \quad k = 1 \\ P(2) & for \quad k = 2 \\ Ck^{-\gamma} & for \quad k \in [3, K] \end{cases}$$

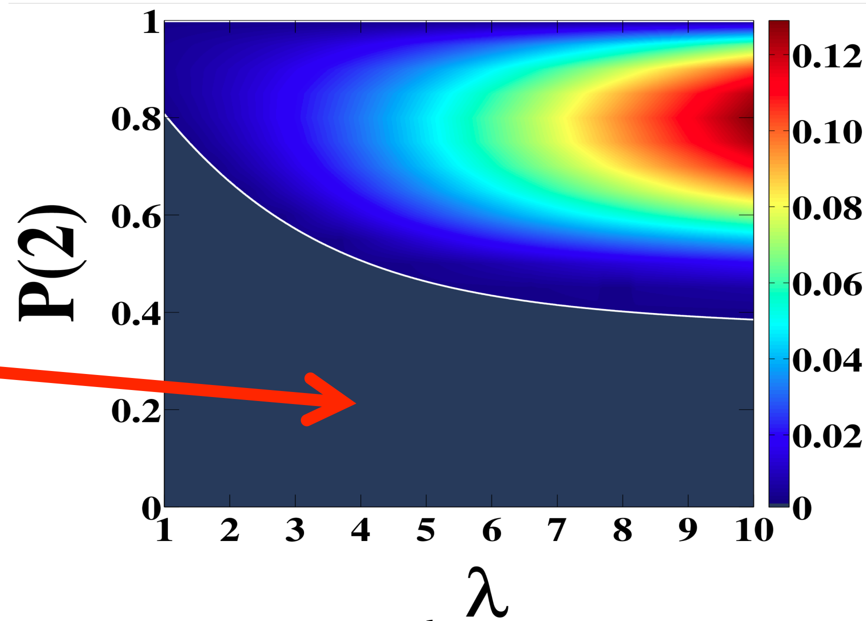
Phase diagram



$$P^{in}(k) = P^{out}(k) = \begin{cases} 0 & for \quad k = 0,1 \\ P(2) & for \quad k = 2 \\ Ck^{-\gamma} & for \quad k \in [3, K] \end{cases}$$

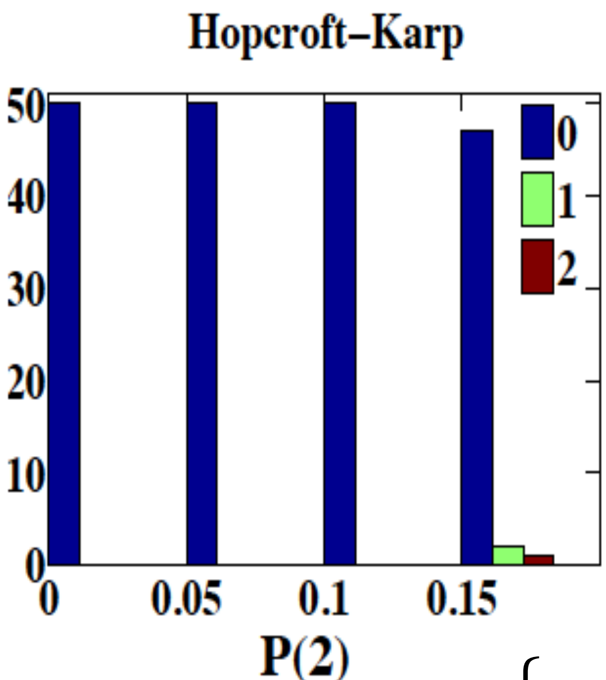
Phase diagram for a modified Poisson distribution

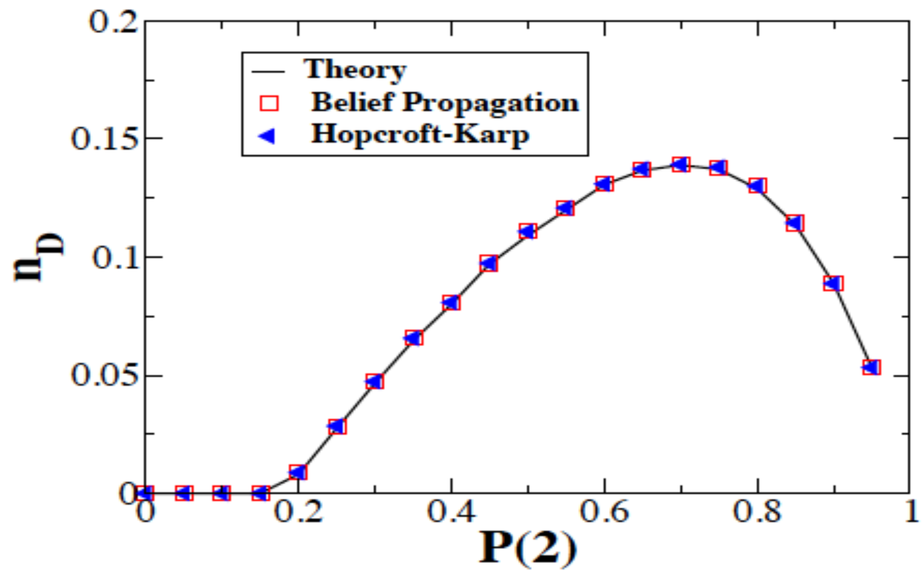
Fully controllable networks



$$P^{in}(k) = P^{out}(k) = \begin{cases} 0 \quad for \quad k = 0,1 \\ P(2) \quad for \quad k = 2 \\ C\dfrac{\lambda^k}{k!} \quad for \quad k \in [3,K] \end{cases}$$

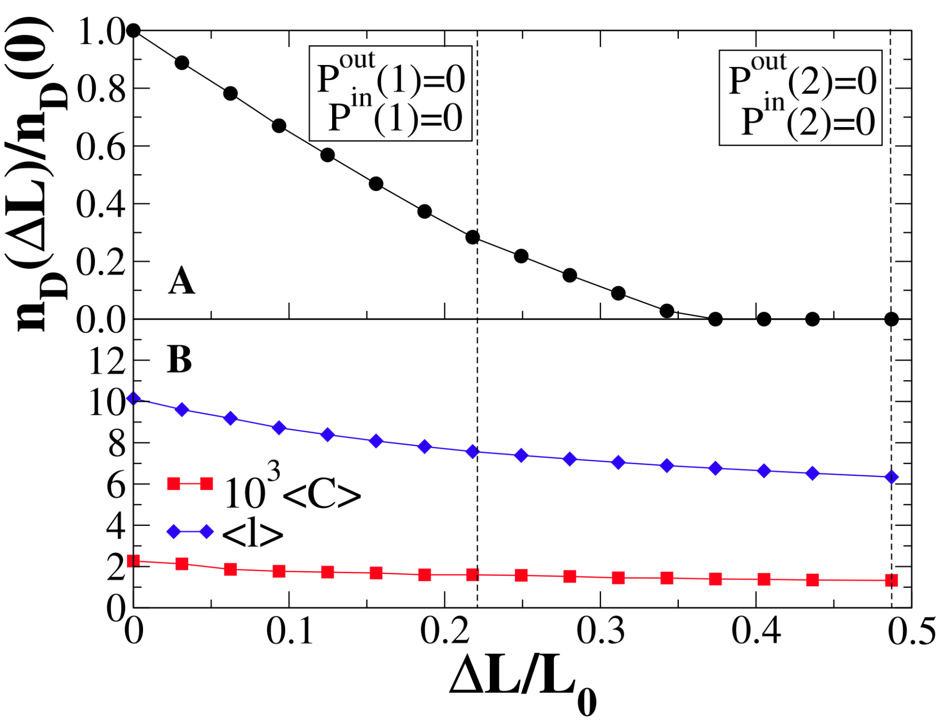
Comparison of the theoretical results with the BP and Hopcroft-Karp algorithms





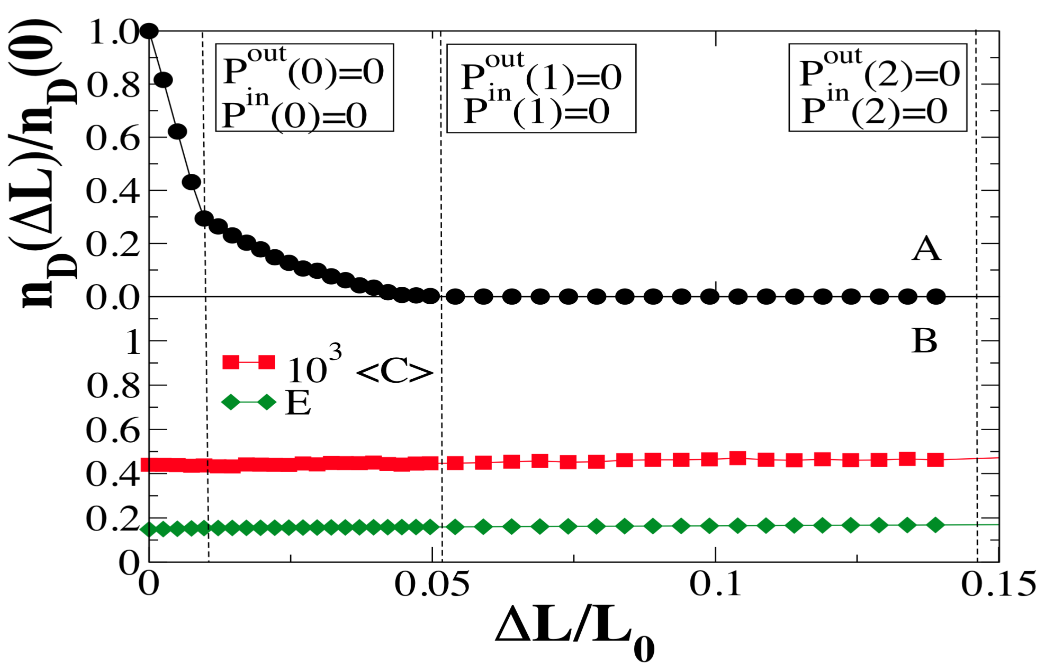
$$P^{in}(k) = P^{out}(k) = \begin{cases} 0 & for \quad k = 0,1 \\ P(2) & for \quad k = 2 \\ Ck^{-2.3} & for \quad k \in [3, K] \end{cases} \qquad N = 10^4, N_{run} = 50$$

Improving the controllability of networks by adding links to low in-degree and low out-degree nodes



Case of the pure scale-free distribution With γ=2.3

Improving the controllability of networks by adding links to low in-degree and low out-degree nodes



Case of the pure Poisson Distribution $\lambda=4$

Conclusions

- ***The controllability of a network is a fundamental problem with wide applications ranging from network medicine, to the characterization of the brain dynamics, and the evaluation of risk in financial markets.***

  Here we have shown that

  a) the structural controllability of networks is determined exclusively by the low in-degree and low out-degree nodes,

  i.e nodes with in/out degree equal or less than 2.

  b) all the networks with minimum in-degree and minimum out-degree nodes both greater than 2 are always fully controllable

  c) if a network is not fully controllable, it is possible to improve its controllability by adding links to low in/out degree nodes