A GENERAL AND FLEXIBLE FRAMEWORK FOR STUDYING DYNAMICS ON COMPLEX NETWORKS

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Dynamics & Networks

AME Framework

binary-state

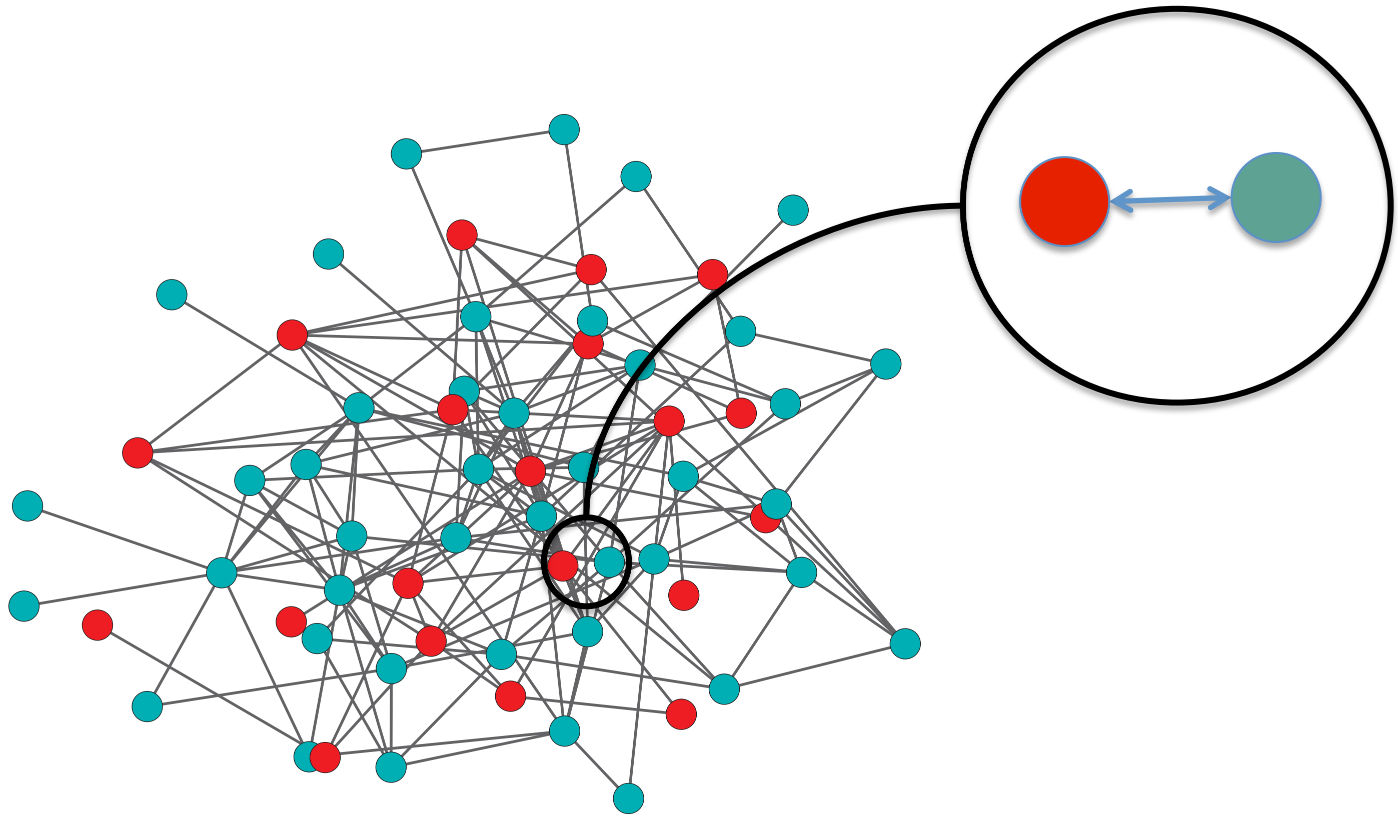
multi-state

Examples

Extensions

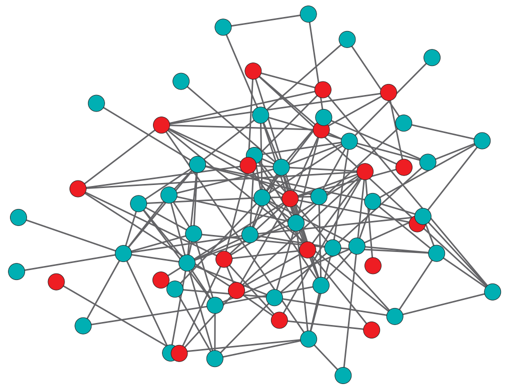
OUTLINE

DYNAMICS AND NETWORKS



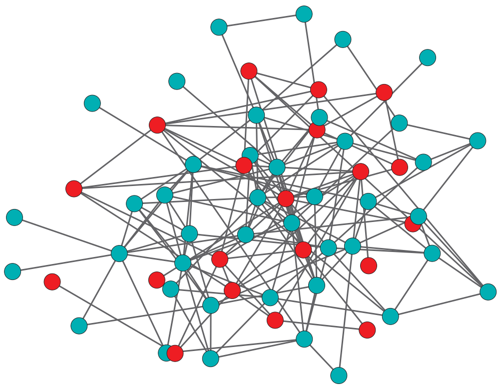
Specific Details

- Continuous-time Markov processes
  - $X_t^i \in \{0,1,\ldots,n\}$
  - rates



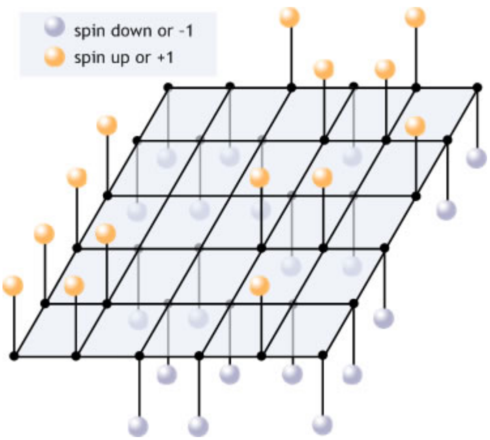
Specific Details

- Continuous-time Markov processes
  - $X_t^i \in \{0,1,\ldots,n\}$
  - rates
- Configuration model networks
  - infinitely large
  - p_k
  - tree-like

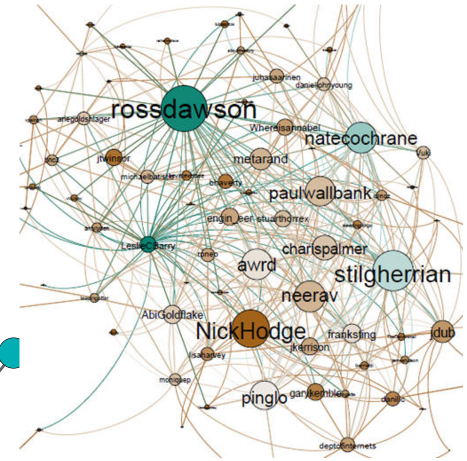


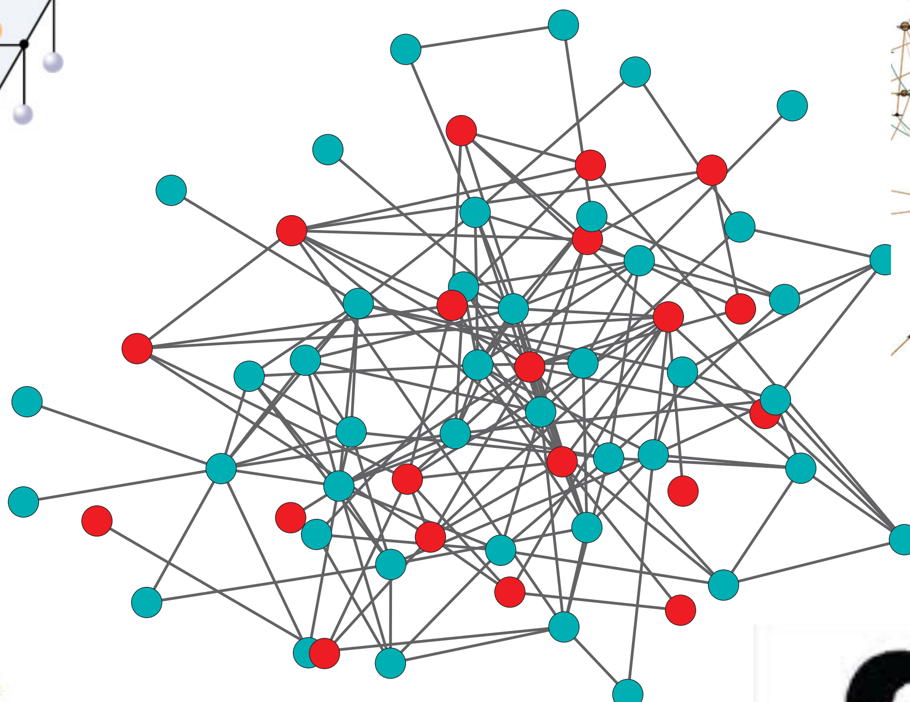
Examples

Physics

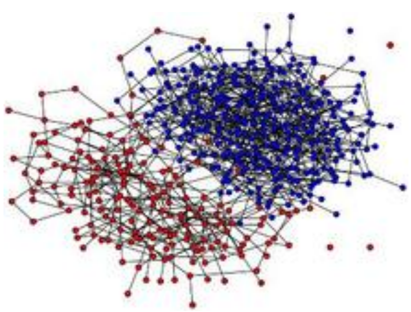


Social networks

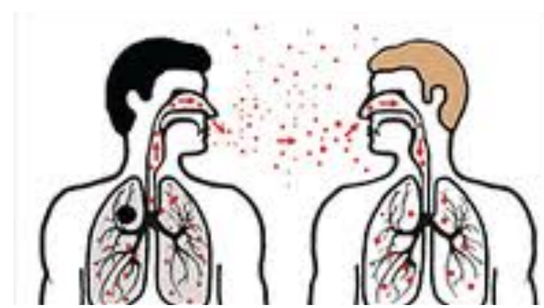




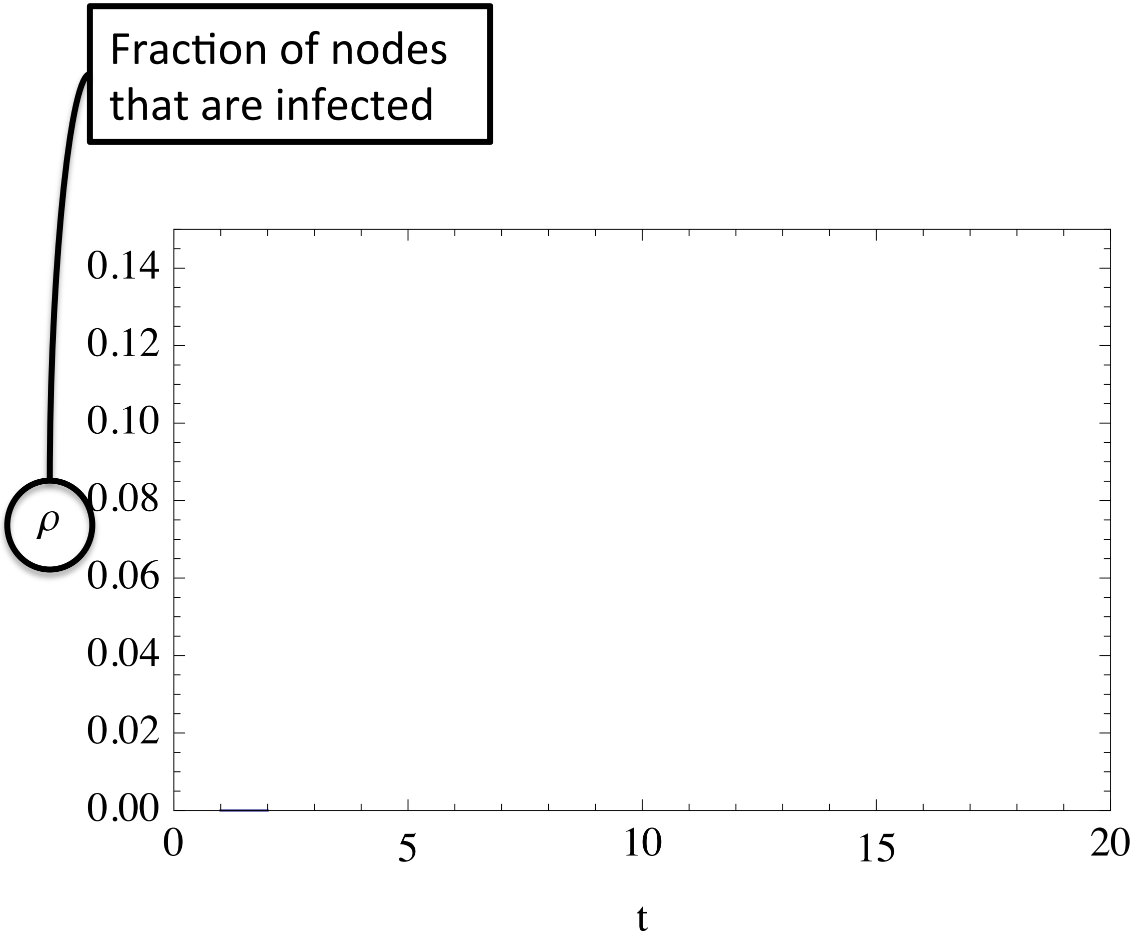
Opinions



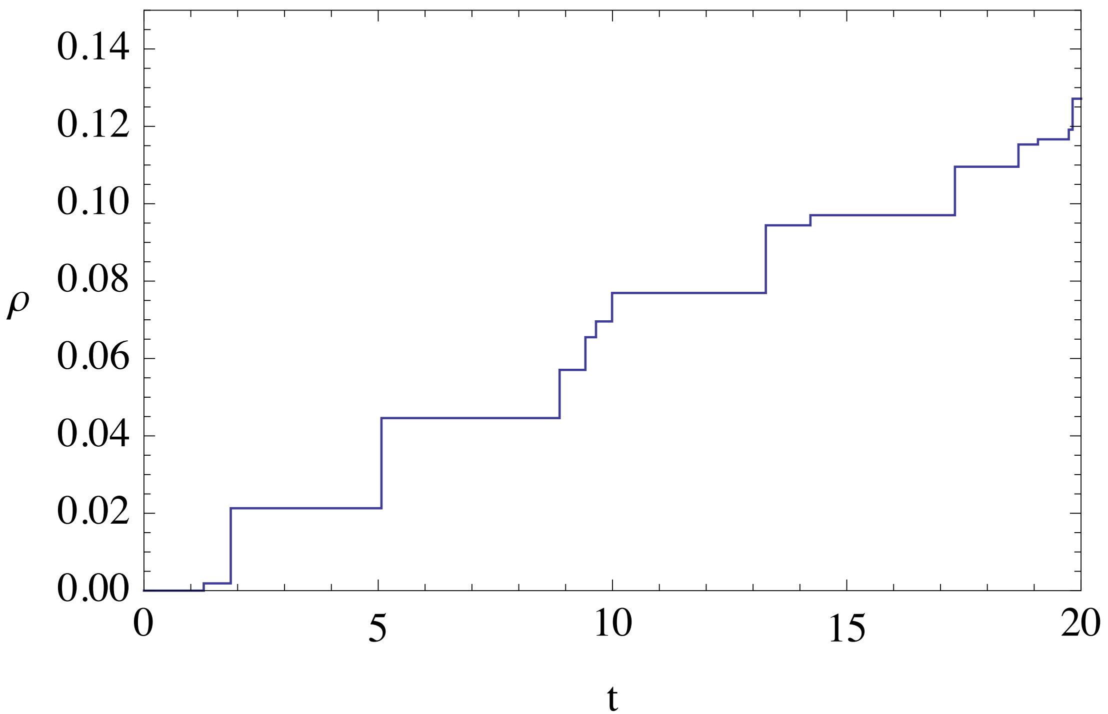
Epidemics



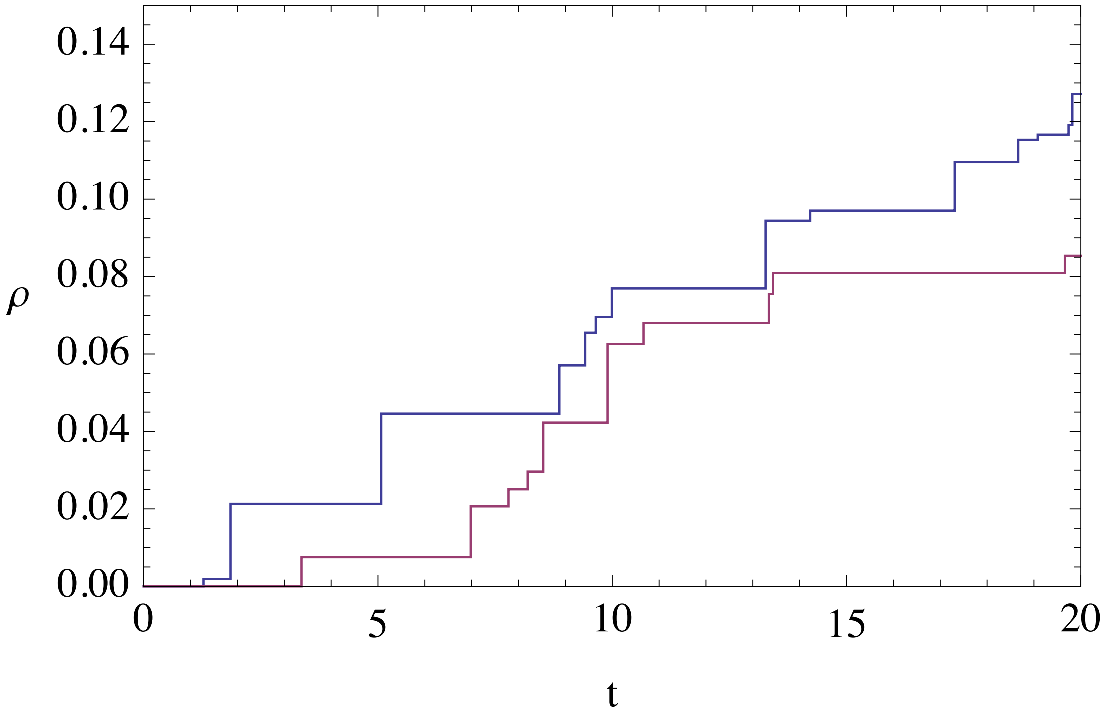
Stochastic Evolution



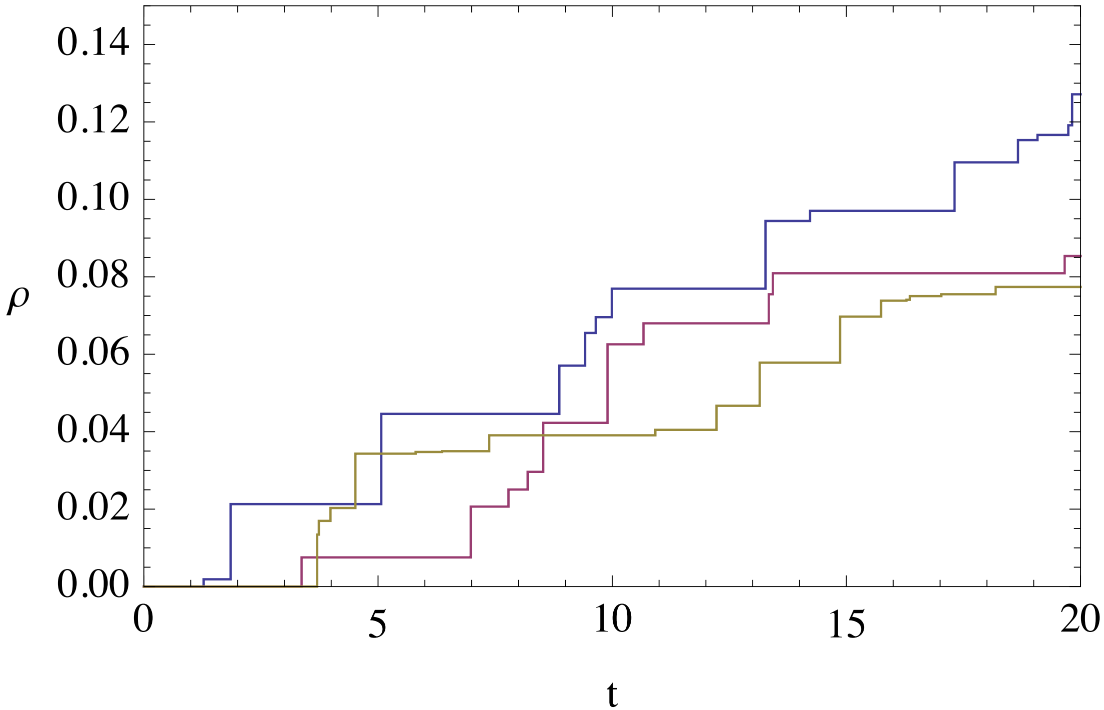
Stochastic Evolution



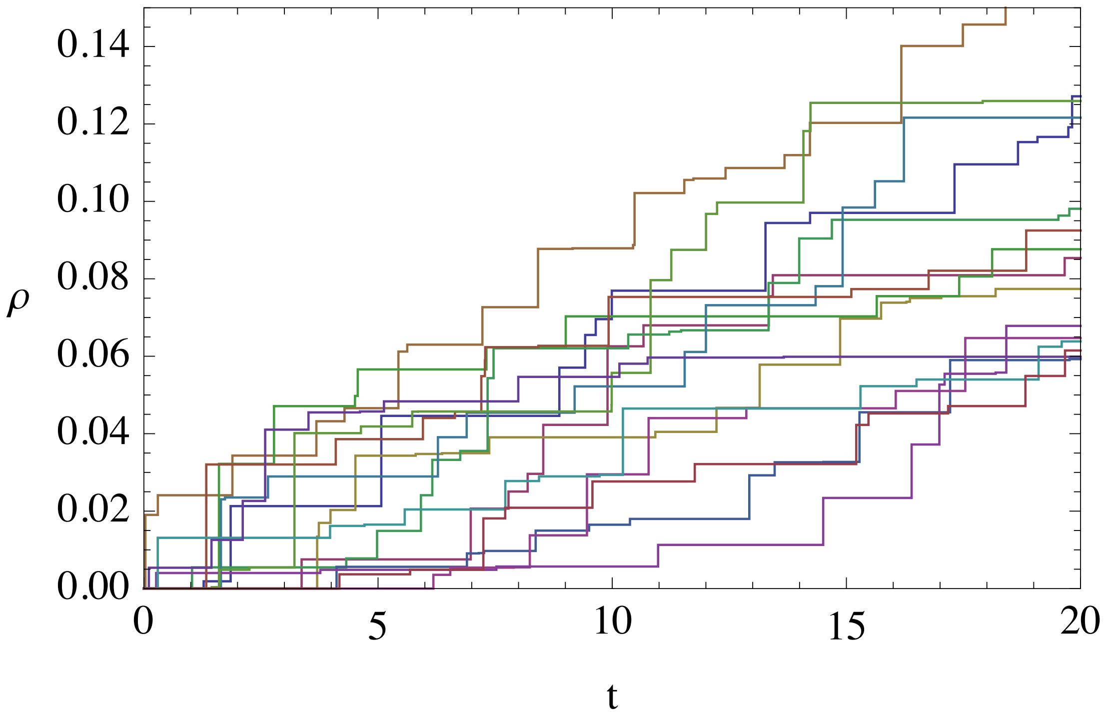
Stochastic Evolution



Stochastic Evolution

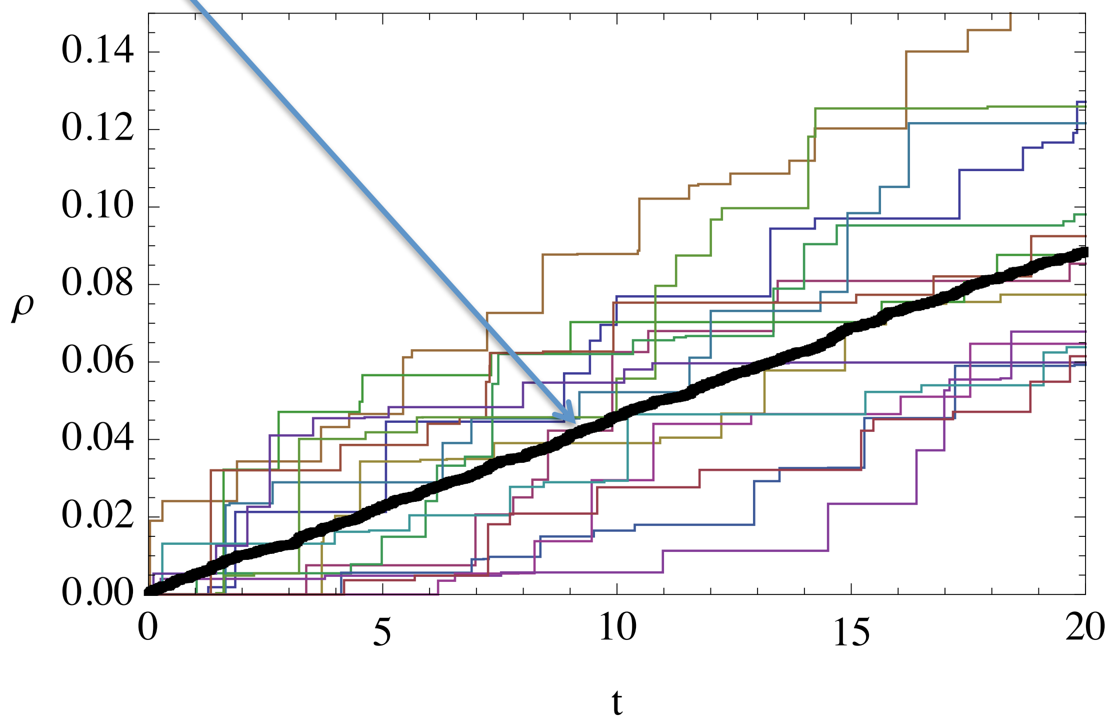


Stochastic Evolution



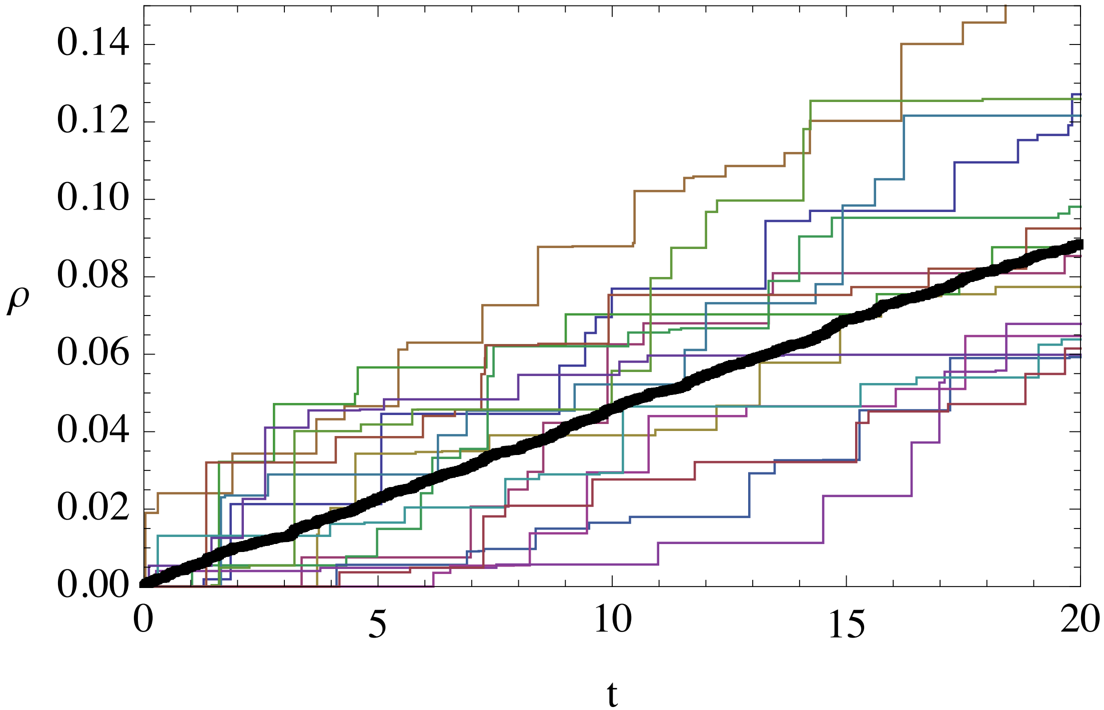
Stochastic Evolution

Expected Value



Stochastic Evolution

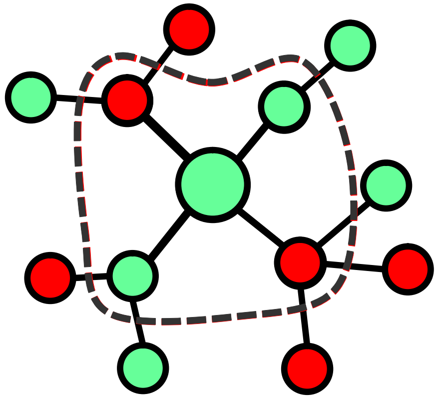
- Deterministic equations



APPROXIMATE MASTER EQUATION

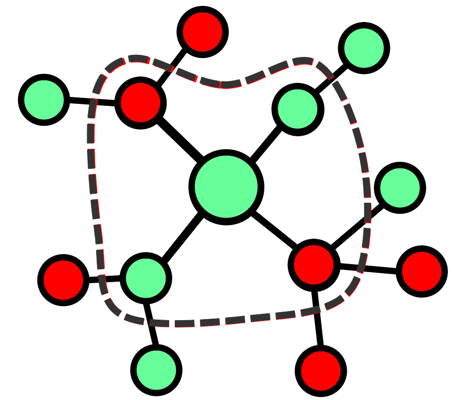
IDEA

- Consider nearest-neighbour network motifs



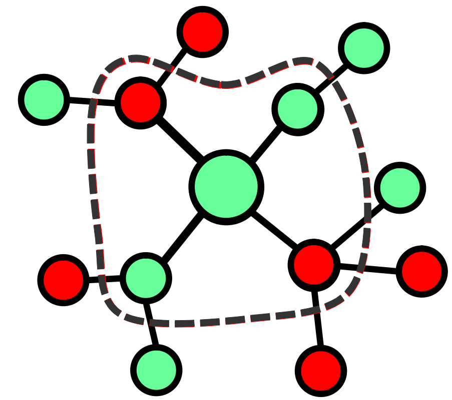
IDEA

- Consider nearest-neighbour network motifs



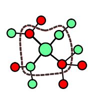
- variables $i_{k,m}$, $s_{k,m}$
  – Fraction of k-degree nodes in the network which
    - are infected (resp. susceptible) and
    - have m infected neighbours

IDEA



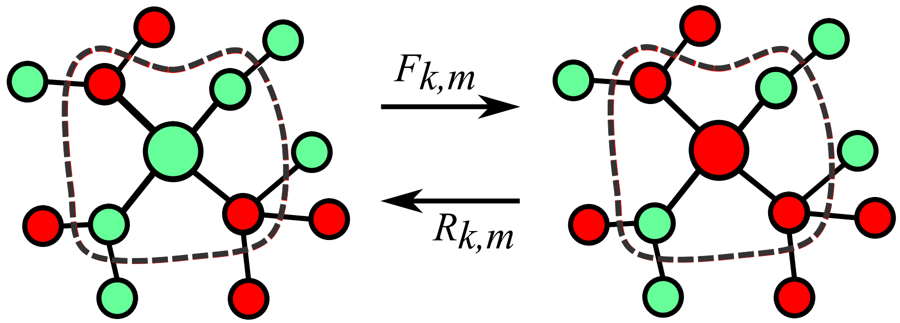
- Consider nearest-neighbour network motifs

- variables $i_{k,m}$, $s_{k,m}$
  - Fraction of k-degree nodes in the network which
    - are infected (resp. susceptible) and
    - have m infected neighbours

- Ex.  contributes to $s_{4,2}$ class

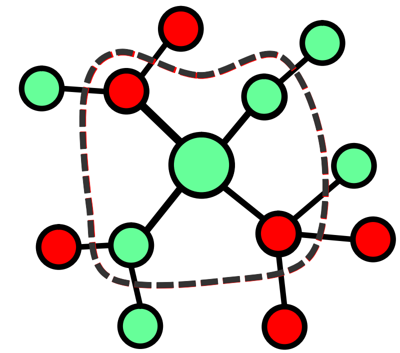
IDEA

- Nodes dynamically change state…



*ex SIS: $F_{k,m} = \lambda m, R_{k,m} = \mu$

IDEA



- Nodes dynamically change state…

  …and so variables $i_{k,m}$ and $s_{k,m}$ evolve over time

$$\frac{d}{dt}s_{k,m} = -F_{k,m}s_{k,m} + R_{k,m}i_{k,m}$$
$$-\beta^{s}(k-m)s_{k,m} + \beta^{s}(k-(m-1))s_{k,m} - \gamma^{s}ms_{k,m} + \gamma^{s}(m+1)s_{k,m+1}$$

$$\frac{d}{dt}i_{k,m} = -R_{k,m}i_{k,m} + F_{k,m}s_{k,m}$$
$$-\beta^{i}(k-m)i_{k,m} + \beta^{i}(k-(m-1))i_{k,m} - \gamma^{i}mi_{k,m} + \gamma^{i}(m+1)i_{k,m+1}$$

J.P. Gleeson, PRX (2013)

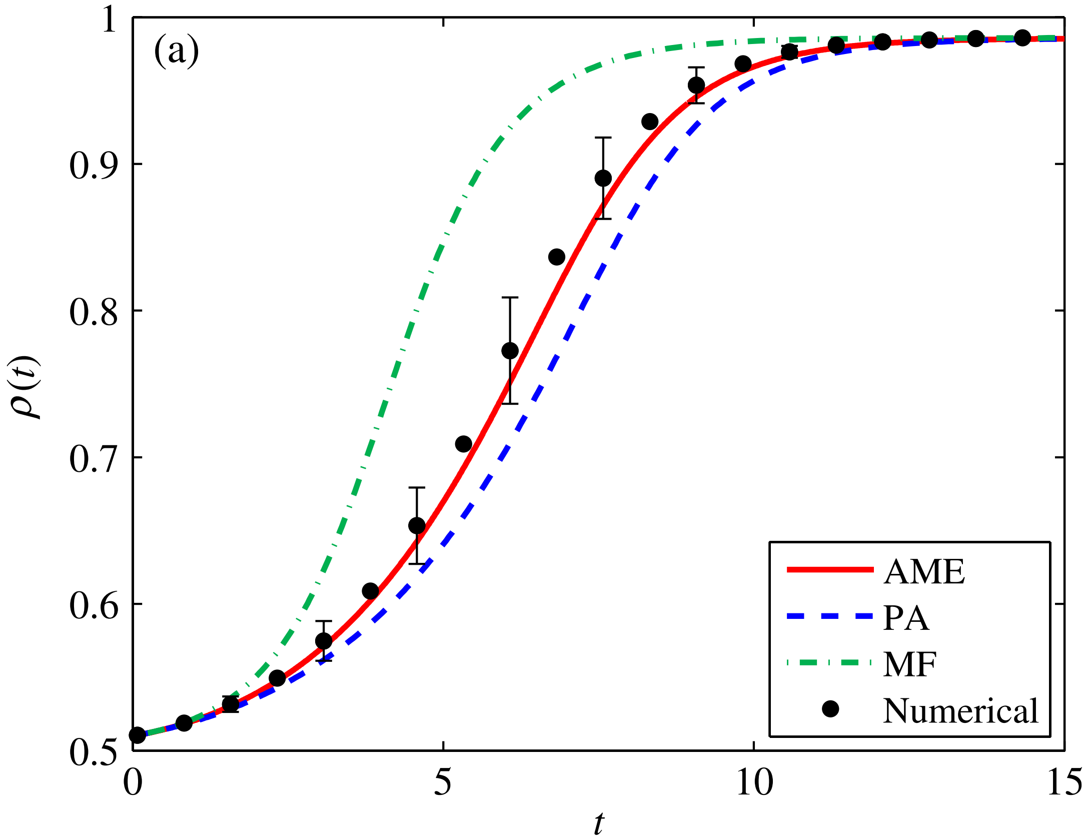
Macroscopic Variable

$$\bar{\rho} = \left\langle \sum_{m=0}^{k} i_{k,m} \right\rangle_k$$

BINARY STATE DYNAMICS

RESULTS

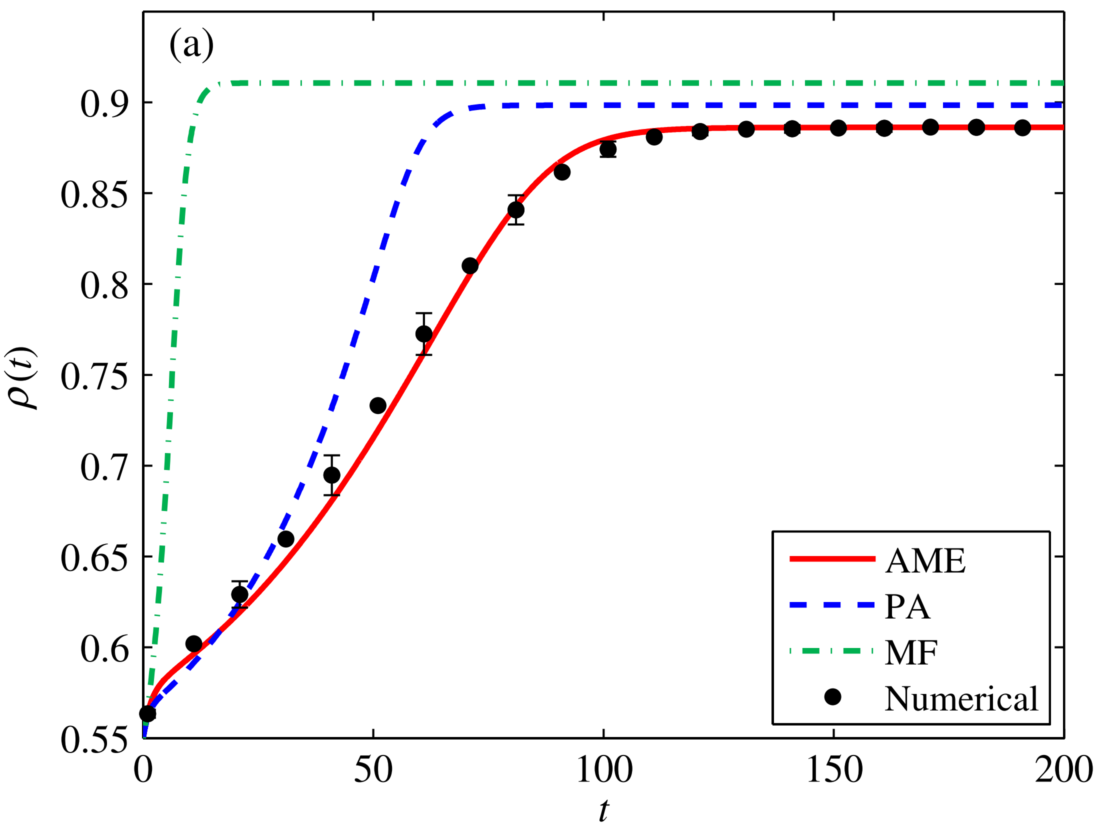
High Accuracy



Ising model dynamics

High Accuracy

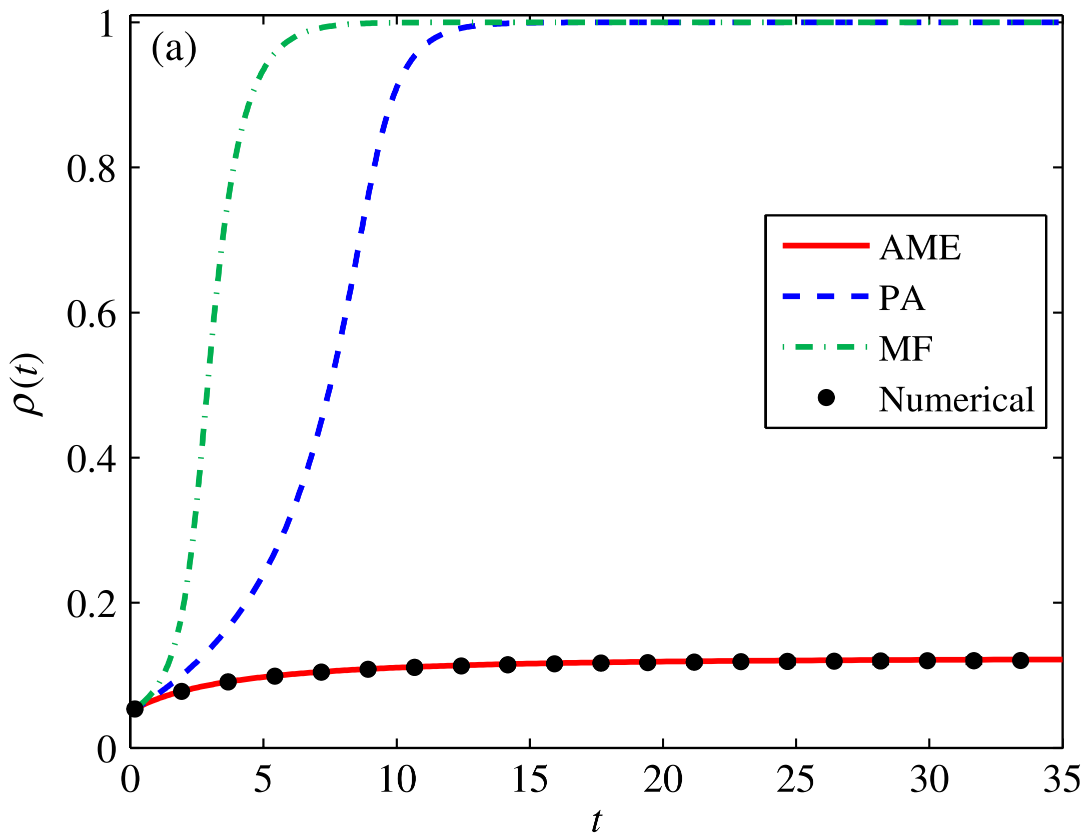
J.P. Gleeson, PRX (2013)



Majority voter model dynamics

High Accuracy

J.P. Gleeson, PRX (2013)

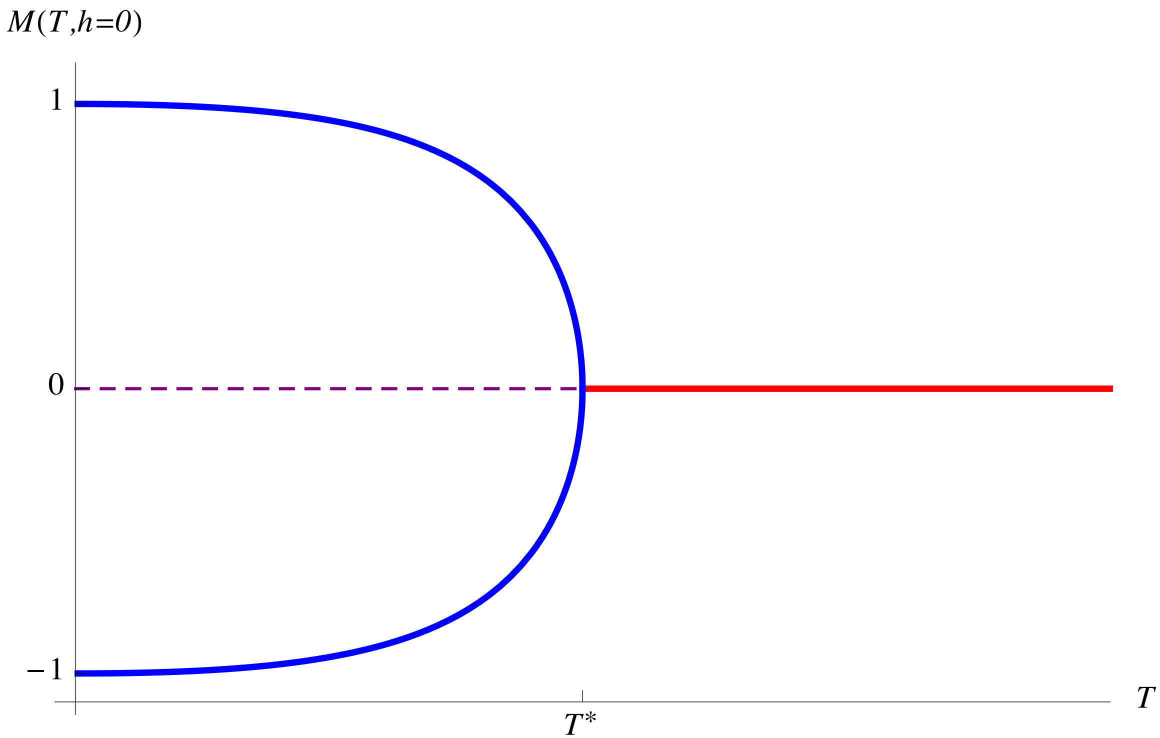


Threshold model dynamics

High Accuracy

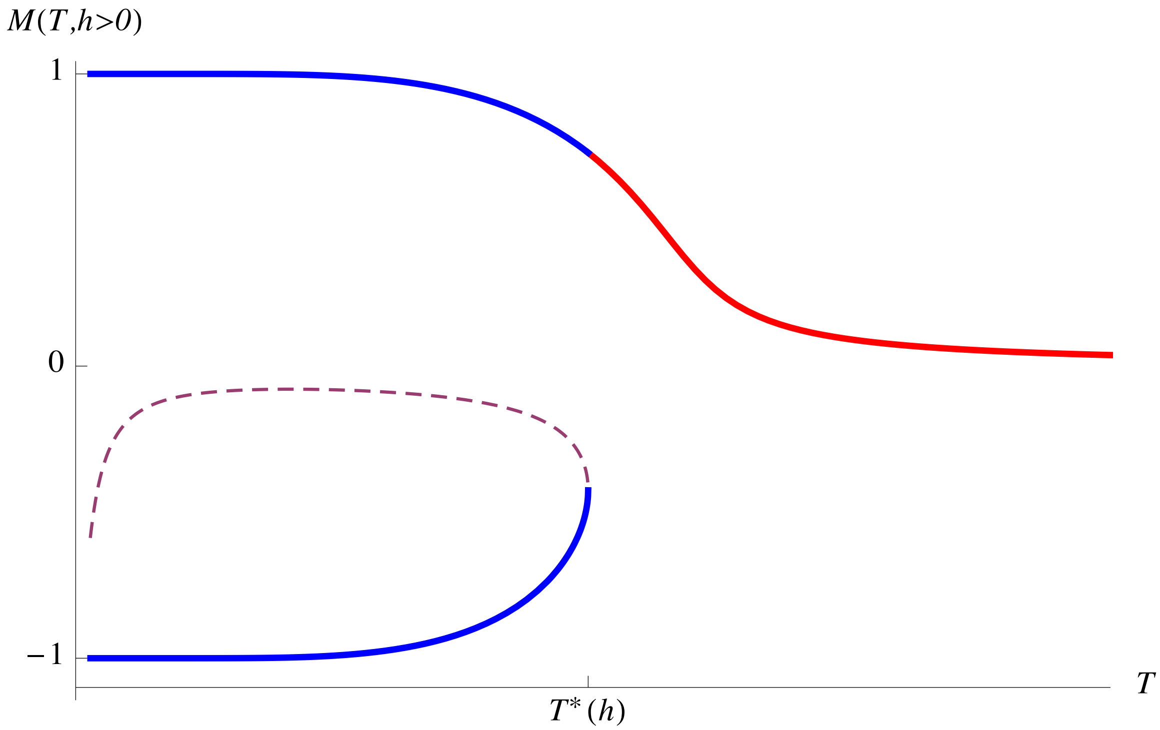
J.P. Gleeson, PRX (2013)

Analysis



Bifurcation analysis (phase diagram)
Ising model, $p_k = \delta_{k,4}$

Analysis



Bifurcation analysis (phase diagram)
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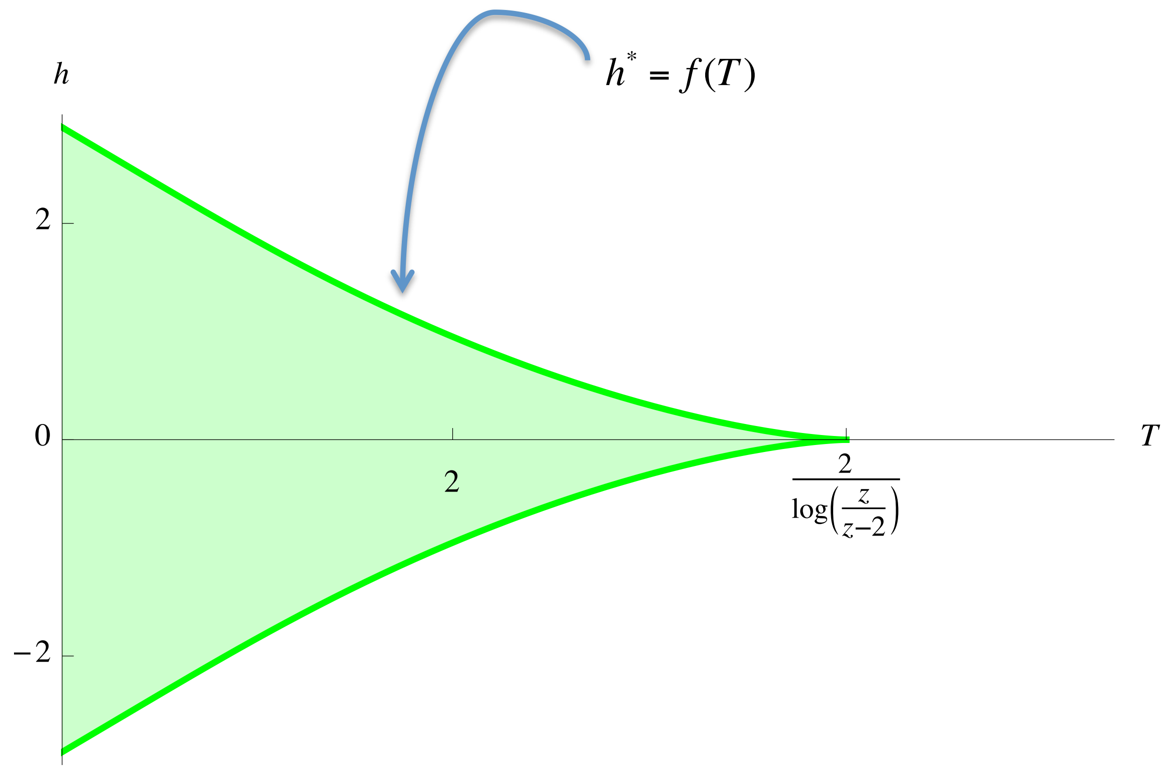
Analysis



Bifurcation analysis (phase diagram)
Ising model, $p_k = \delta_{k,4}$

Analysis

MULTI STATE DYNAMICS

MOTIVATION

FA MODEL OF GLASS TRANSITION

G.H. Fredrickson and H.C. Andersen, PRL (1984)

FA MODEL OF GLASS TRANSITION

- Nodes are spin-up (*+1*) or spin-down (*-1*)

G.H. Fredrickson and H.C. Andersen, PRL (1984)

FA MODEL OF GLASS TRANSITION

- Nodes are spin-up (*+1*) or spin-down (*-1*)
- Node can flip (change state) only if f or more of its neighbours are spin-down
  - spin-down nodes flip at rate 1
  - spin-up nodes flip at rate $e^{-1/T}$

G.H. Fredrickson and H.C. Andersen, PRL (1984)

FA MODEL OF GLASS TRANSITION

- $\phi(t)$
  - Fraction of unflipped nodes at time t
- $\Phi = \lim_{t\to\infty} \phi(t)$

G.H. Fredrickson and H.C. Andersen, PRL (1984)

FA MODEL OF GLASS TRANSITION

- $\phi(t)$
  - Fraction of unflipped nodes at time t
- $\Phi = \lim_{t\to\infty} \phi(t)$

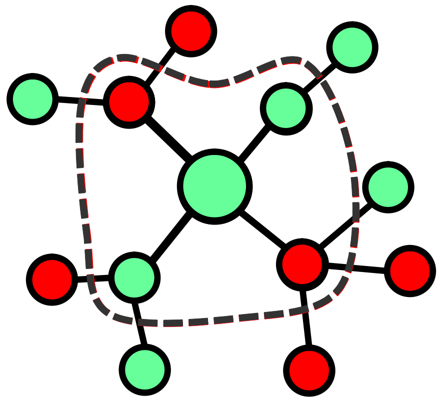
$\Phi = 0$ Liquid

$\Phi > 0$ Glass

G.H. Fredrickson and H.C. Andersen, PRL (1984)

Binary-state AME

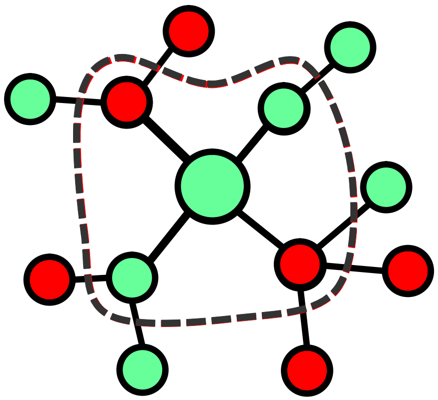
- Spin
  - -1 or +1



Binary-state AME

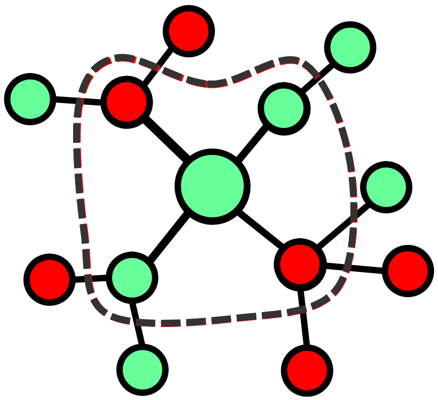
- Spin
  - -1 or +1

- Doesn't work



Binary-state AME

- Spin
  - -1 or +1

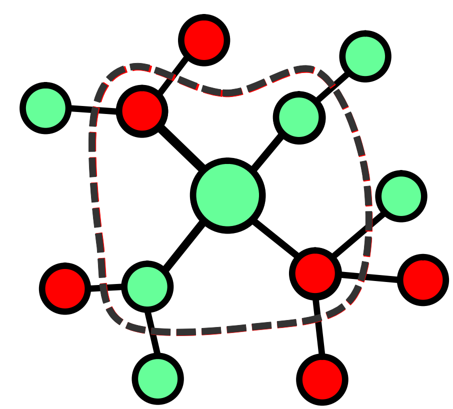


- Doesn't work
  - Binary approach fails to capture glass transition

$$\Phi \equiv 0$$

Binary-state AME

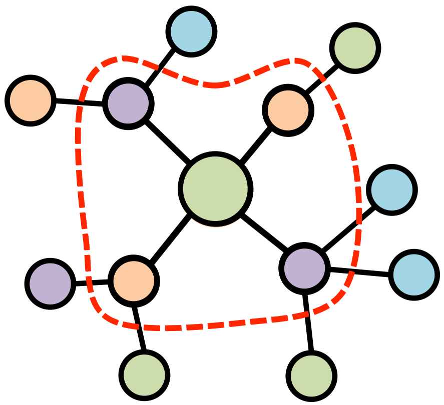
- Spin
  - -1 or +1



- Doesn't work
  - Binary approach fails to capture glass transition
  - Need to account for dynamical correlations between flipped and un-flipped nodes

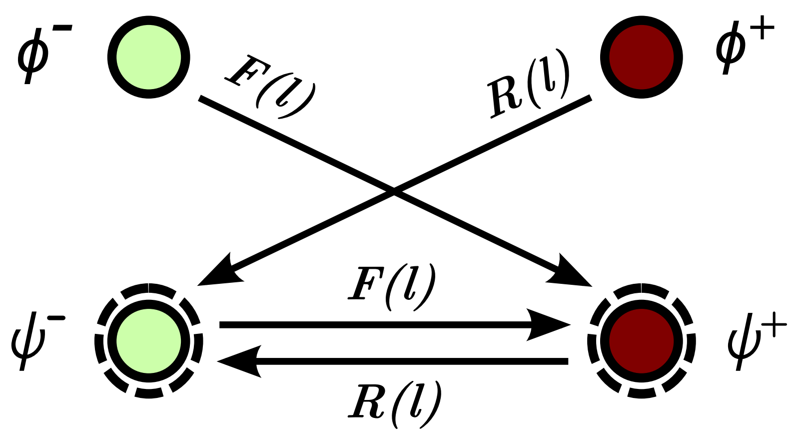
4-state AME

- Spin
  - -1 or +1
- Flipping history
  - unchanged (u) or changed (c)



4-state AME

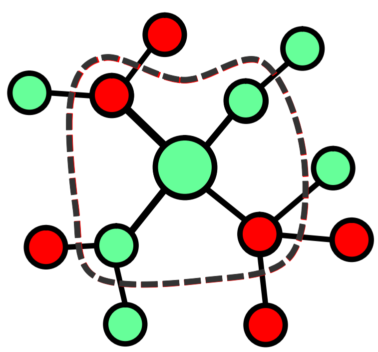
- Spin
  - -1 or +1
- Flipping history
  - unchanged (u) or changed (c)



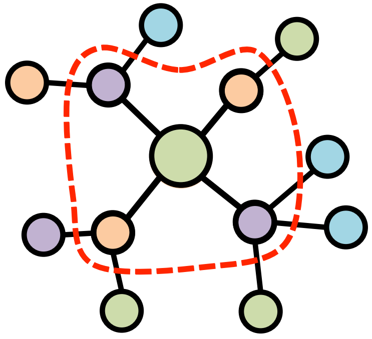
MULTI STATE DYNAMICS

FORMALISM

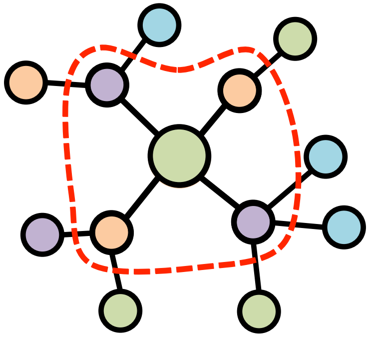
$$\vec{x}_{k,m} = \begin{pmatrix} s_{k,m} \\ i_{k,m} \end{pmatrix}$$



$$\vec{x}_{a_1,a_2,\dots,a_n} = \begin{pmatrix} x^1_{a_1,a_2,\dots,a_n} \\ x^2_{a_1,a_2,\dots,a_n} \\ \dots \\ x^n_{a_1,a_2,\dots,a_n} \end{pmatrix}$$



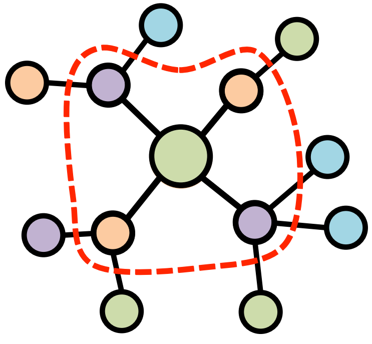
$$\frac{d}{dt}\boldsymbol{x}_{a_1,a_2,a_3,\ldots,a_n} = -\boldsymbol{R}_{a_1,a_2,a_3,\ldots,a_n}*\boldsymbol{x}_{a_1,a_2,a_3,\ldots,a_n}+\left(\boldsymbol{F}_{a_1,a_2,a_3,\ldots,a_n}{}^{T}\right)*\boldsymbol{x}_{a_1,a_2,a_3,\ldots,a_n}$$
$$-\sum_{l=1}^{n}\sum_{m\neq l}a_l*\boldsymbol{\beta}(l,m)*\boldsymbol{x}_{a_1,a_2,a_3,\ldots,a_n}+\sum_{l=1}^{n}\sum_{m\neq l}(a_l+1)*\boldsymbol{\beta}(l,m)*\boldsymbol{x}_{a_1,\ldots,a_l+1,\ldots,a_m-1,\ldots,a_n}$$



$$\frac{d}{dt}\boldsymbol{x}_{a_1,a_2,a_3,\ldots,a_n} = -\boldsymbol{R}_{a_1,a_2,a_3,\ldots,a_n} * \boldsymbol{x}_{a_1,a_2,a_3,\ldots,a_n} + \left(\boldsymbol{F}_{a_1,a_2,a_3,\ldots,a_n}{}^{T}\right) * \boldsymbol{x}_{a_1,a_2,a_3,\ldots,a_n}$$

$$-\sum_{l=1}^{n}\sum_{m\neq l} a_l * \boldsymbol{\beta}(l,m) * \boldsymbol{x}_{a_1,a_2,a_3,\ldots,a_n} + \sum_{l=1}^{n}\sum_{m\neq l}(a_l+1) * \boldsymbol{\beta}(l,m) * \boldsymbol{x}_{a_1,\ldots,a_l+1,\ldots,a_m-1,\ldots,a_n}$$

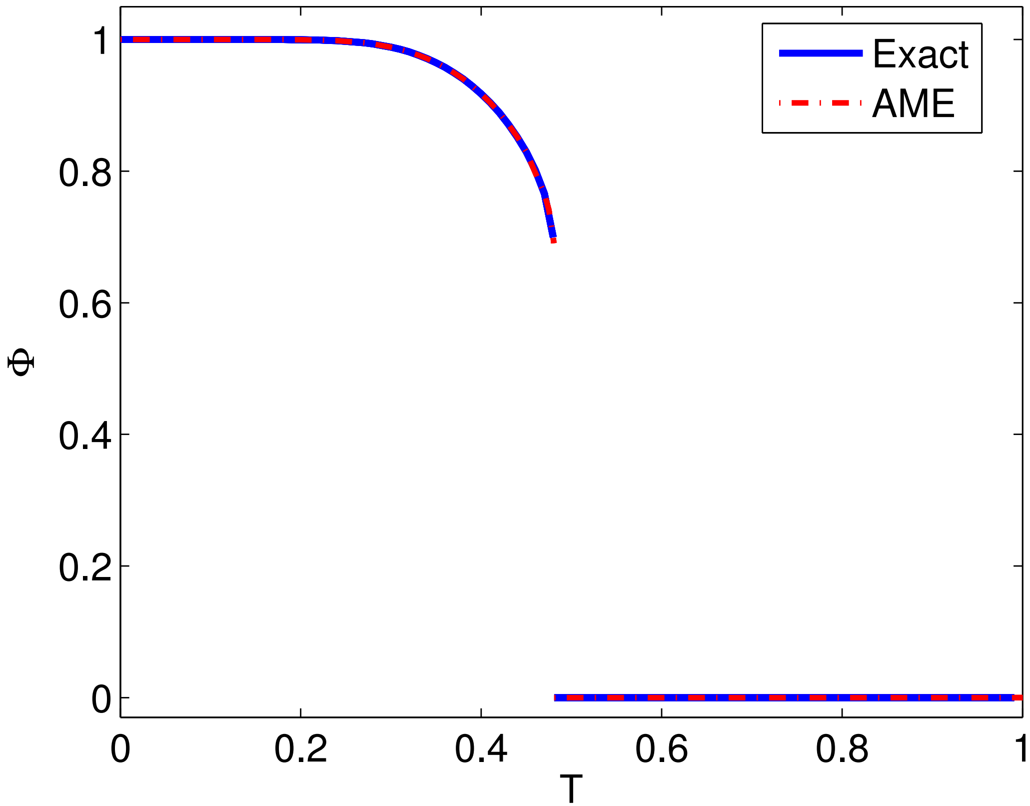
$$\phi(t), \Phi$$



MULTI STATE DYNAMICS

RESULTS

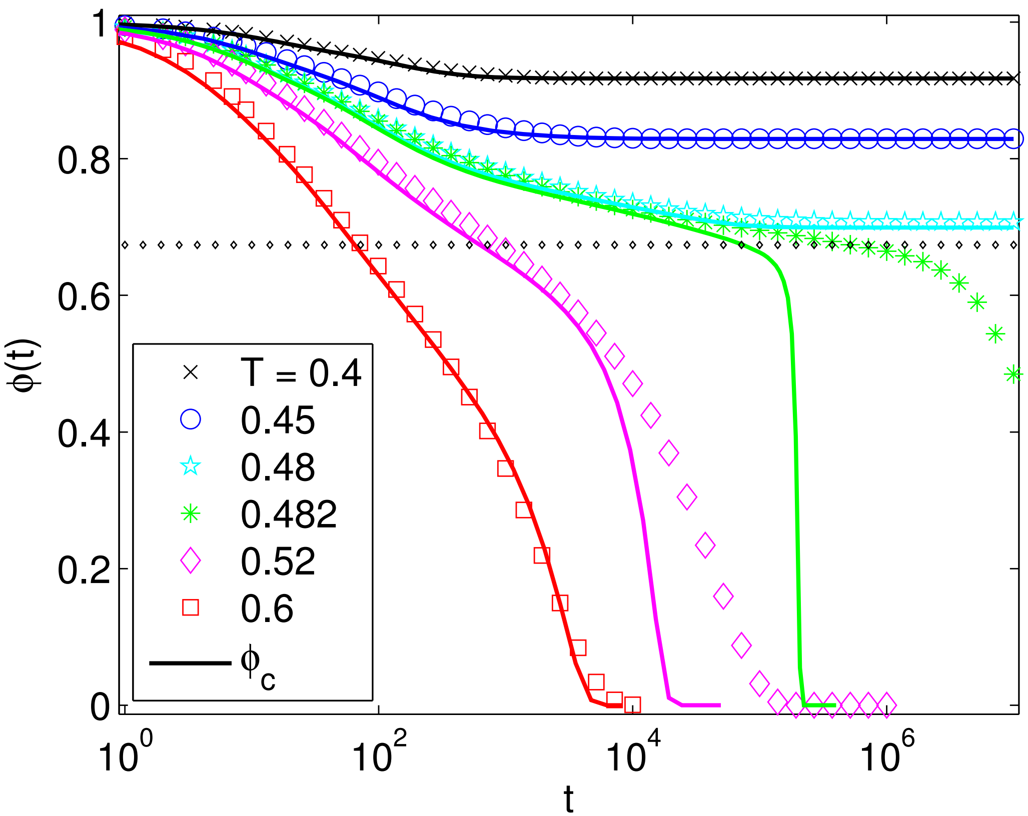
P. Fennell, J.P. Gleeson and D. Cellai, PRE, to appear



Correctly predicts phase diagram

High Accuracy

M. Sellitto, G. Biroli, and C. Toninelli, EPL (2005)



Correctly predicts transient regime
(completely novel)

High Accuracy

P. Fennell, J.P. Gleeson and D. Cellai, PRE, to appear

$$\frac{d}{dt}\phi^-_{m_1,m_2,m_3,m_4} = -F(m_1+m_3)\phi^-_{m_1,m_2,m_3,m_4}$$
$$-m_1\lambda^{\phi^-}_{1\to 4}\phi^-_{m_1,m_2,m_3,m_4} - m_2\lambda^{\phi^-}_{2\to 3}\phi^-_{m_1,m_2,m_3,m_4} - m_3\lambda^{\phi^-}_{3\to 4}\phi^-_{m_1,m_2,m_3,m_4} - m_4\lambda^{\phi^-}_{4\to 3}\phi^-_{m_1,m_2,m_3,m_4}$$
$$+(m_1+1)\lambda^{\phi^-}_{1\to 4}\phi^-_{m_1+1,m_2,m_3,m_4-1} + (m_2+1)\lambda^{\phi^-}_{2\to 3}\phi^-_{m_1,m_2+1,m_3-1,m_4}$$
$$+(m_3+1)\lambda^{\phi^-}_{3\to 4}\phi^-_{m_1,m_2,m_3+1,m_4-1} + (m_4+1)\lambda^{\phi^-}_{4\to 3}\phi^-_{m_1,m_2,m_3-1,m_4+1}$$

Behaviour in Glassy state

Analysis

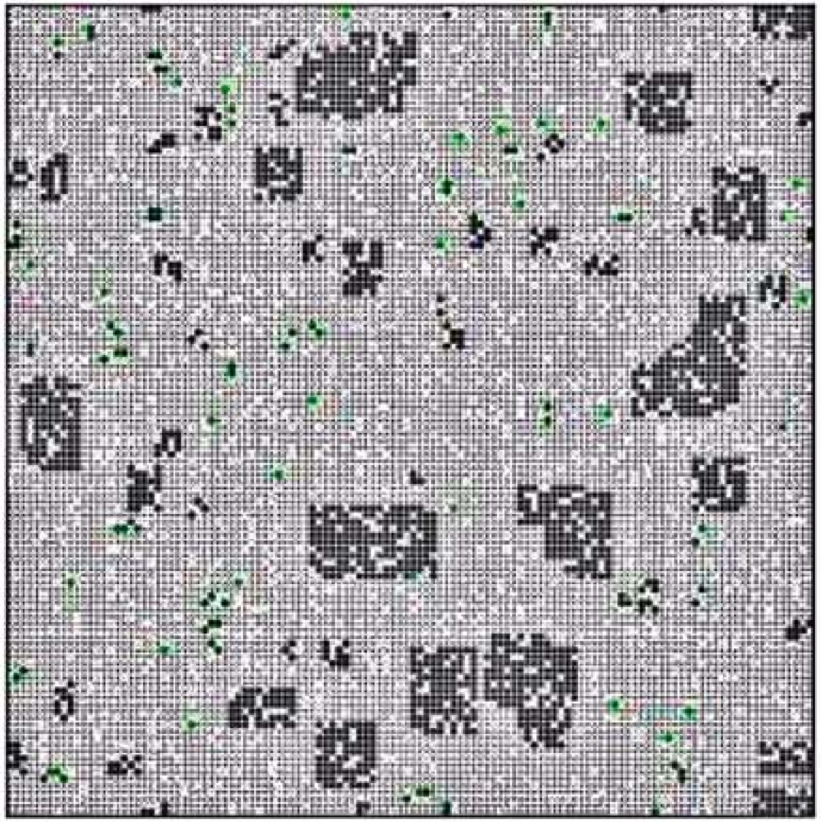
$$\frac{d}{dt}\phi_{m_1,m_2,m_3,m_4} = 0$$

$$\frac{d}{dt}\psi_{m_1,m_2,m_3,m_4} \neq 0$$

Behaviour in Glassy state

Analysis

P. Fennell, J.P. Gleeson and D. Cellai, PRE, to appear

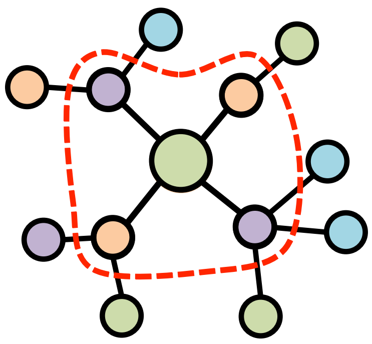


Dynamical Heterogeneity

Analysis

A. Lawlor et al., PRE (2005)

$$\frac{d}{dt}\boldsymbol{x}_{a_1,a_2,a_3,\ldots,a_n} = -\boldsymbol{R}_{a_1,a_2,a_3,\ldots,a_n} * \boldsymbol{x}_{a_1,a_2,a_3,\ldots,a_n} + \left(\boldsymbol{F}_{a_1,a_2,a_3,\ldots,a_n}{}^{T}\right) * \boldsymbol{x}_{a_1,a_2,a_3,\ldots,a_n}$$
$$-\sum_{l=1}^{n}\sum_{m\neq l} a_l * \boldsymbol{\beta}(l,m) * \boldsymbol{x}_{a_1,a_2,a_3,\ldots,a_n} + \sum_{l=1}^{n}\sum_{m\neq l}(a_l+1) * \boldsymbol{\beta}(l,m) * \boldsymbol{x}_{a_1,\ldots,a_l+1,\ldots,a_m-1,\ldots,a_n}$$



FURTHER POSSIBILITIES

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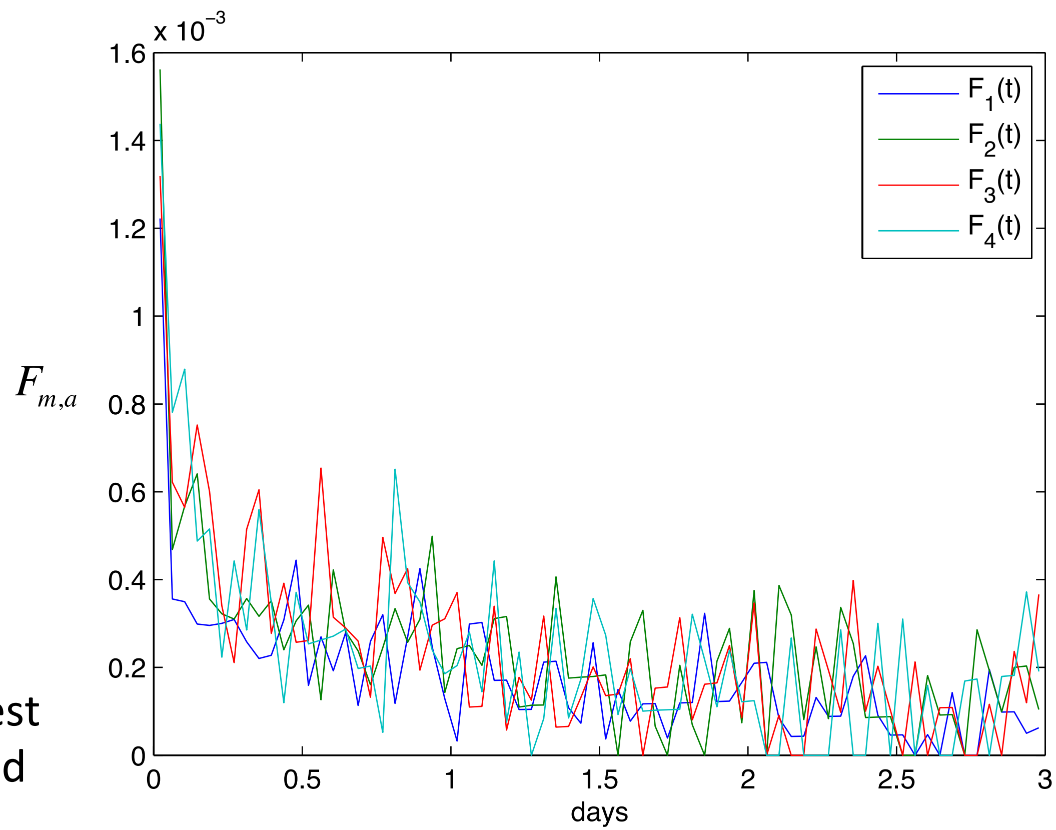
- Non-Markovian dynamics



time since latest friend tweeted hashtag

J. Borge-Holthoefer et al., PLoS One (2011)

FURTHER POSSIBILITIES

- Non-Markovian dynamics
- Directed Networks
- Multiplex networks
- Clustered Networks

Conclusions

- Deterministic approach to studying Stochastic processes
- AME, high-order approximation
  - beyond MF and PA
  - allows analysis
- Generalized to multi-state dynamics
  - FA model (novel insights)
  - Others (SIR etc.)

THANKS

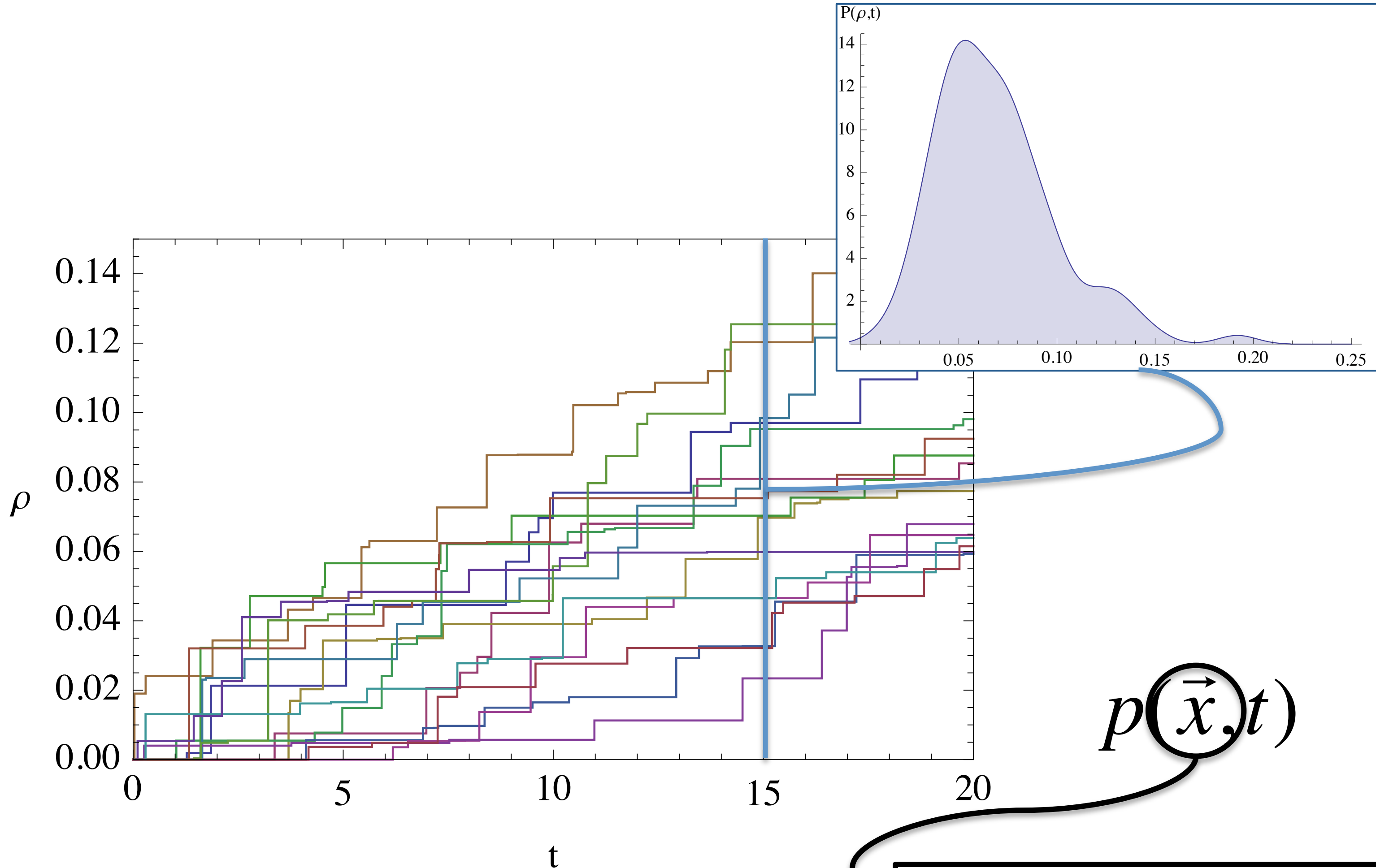
peter.fennell@ul.ie

@fennell_p

J. P. Gleeson*, Binary-State dynamics on Complex Networks: Pair Approximation and Beyond*, PRX (2013)

P. Fennell, J.P. Gleeson, and D. Cellai, *Analytical approach to the dynamics of facilitated spin models on random networks*, PRE, to appear

Stochastic Evolution



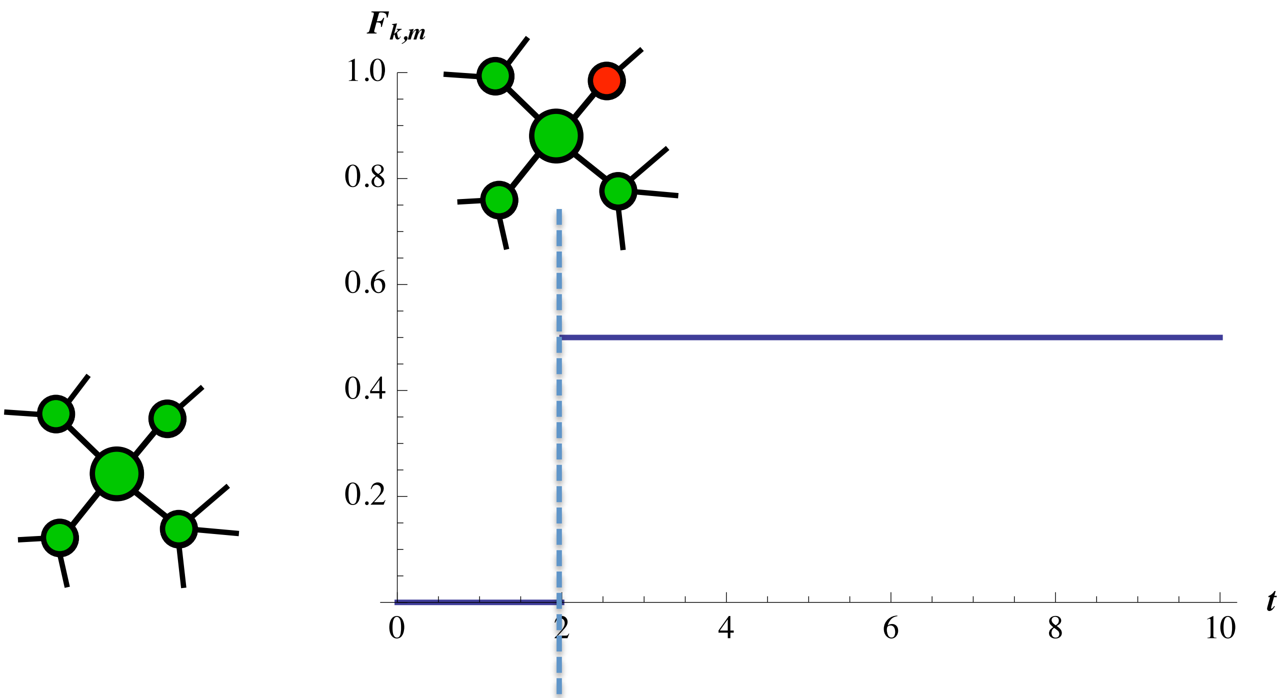
$$p(\vec{x}, t)$$

n^N possible configurations
e.g.. $2^{10} = 1024$

Rate Functions

$$F_{k,m},\ R_{k,m}$$

- Rates depend only on states of neighbours



Rate Functions

$F_{k,m}$, $R_{k,m}$

- Rates depend only on states of neighbours

$F_{k,m,a}$, $R_{k,m,a}$

- Rates also depend on "age"
  - amount of time node has been in its current neighbourhood configuration