Sobolev mappings into manifolds: nonlinear methods for approximation, extension and lifting problems

Lecture notes

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Contents

1 Towards Sobolev mappings

In order to introduce Sobolev maps, which are maps from a Riemannian manifolds into another manifold whose weak derivative satisfies an integrability condition or for which a fractional Gagliardo energy is finite, we first review the motivation, definition and properties of the more classical linear Sobolev spaces.

1.1 Linear Sobolev spaces

1.1.1 Motivation

Linear Sobolev spaces appear classically in the variational construction of solutions of elliptic boundary value problems. Let us consider the *Dirichlet problem* for the *Laplace equation*: given a set $\Omega \subset \mathbb{R}^d$ and a function $g : \partial\Omega \to \mathbb{R}$, we search for a function $u : \Omega \to \mathbb{R}$ that solves the problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \quad u = g & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

If $u \in C^2(\bar{\Omega})$ and if $v \in C^1(\bar{\Omega})$ is any function, then

$$\int_\Omega |\mathrm{D}v|^2 = \int_\Omega |\mathrm{D}u|^2 + 2\int_\Omega \mathrm{D}u \cdot \mathrm{D}(u - v) + \int_\Omega |\mathrm{D}(u - v)|^2.$$

By integration by parts (Gauß divergence theorem), we have

$$\int_{\partial\Omega} (u - v)\, \partial_\nu u = \int_\Omega \operatorname{div}\big((u - v)\nabla u\big) = \int_\Omega \nabla(u - v) \cdot \nabla u + \int_\Omega (u - v)\Delta u,$$

and therefore, if u is a solution of (1.1) and if $v = g$ on $\partial\Omega$, then

$$\int_\Omega \mathrm{D}u \cdot \mathrm{D}(u - v) = 0.$$

In particular, we have

$$\int_\Omega |\mathrm{D}v|^2 = \int_\Omega |\mathrm{D}u|^2 + \int_\Omega |\mathrm{D}(u - v)|^2 \geq \int_\Omega |\mathrm{D}u|^2,$$

with equality if and only if $u = v$ in Ω.

This suggests constructing a solution u to (1.1) by minimizing the *Dirichlet functional* defined for each $v \in C^1(\Omega)$ by

$$\mathcal{E}^{1,2}(v) \triangleq \int_\Omega |\mathrm{D}v|^2 \tag{1.2}$$

among the functions $v \in C^1(\Omega) \cap C(\bar{\Omega})$ such that $v = g$ on $\partial\Omega$. We consider a minimizing sequence, that is a sequence of functions $(u_j)_{j\in\mathbb{N}}$ in $C^1(\Omega) \cap C(\bar{\Omega})$ such that

$$\lim_{j\to\infty} \int_\Omega |\mathrm{D}u_j|^2 = c \triangleq \inf\left\{\int_\Omega |\mathrm{D}v|^2 \mid v \in C^1(\Omega) \cap C(\bar{\Omega})\right\} \geq 0. \tag{1.3}$$

We have for each $k, \ell \in \mathbb{N}$, the parallelogram identity

$$\int_\Omega |\mathrm{D}u_j - \mathrm{D}u_i|^2 + \int_\Omega |\mathrm{D}u_j + \mathrm{D}u_i|^2 = 2\int_\Omega |\mathrm{D}u_j|^2 + 2\int_\Omega |\mathrm{D}u_i|^2. \tag{1.4}$$

We observe that

$$\int_\Omega |\mathrm{D}u_j + \mathrm{D}u_i|^2 = 4\int_\Omega |\mathrm{D}w_{j,i}|^2 \geq 4c, \tag{1.5}$$

where $w_{j,i} \triangleq \frac{u_j+u_i}{2}$ on Ω, and hence $w_{j,i} = g$ on $\partial\Omega$. Therefore by (1.3), (1.4) and (1.5)

$$\lim_{j,i\to\infty} \int_\Omega |\mathrm{D}u_j - \mathrm{D}u_i|^2 = 0. \tag{1.6}$$

This is a *Cauchy condition* for the sequence of functions $(u_j)_{j\in\mathbb{N}}$ in a certain *seminorm*:

$$|v|_{W^{1,2}(\Omega)} \triangleq \|\mathrm{D}v\|_{L^2(\Omega)} = \left(\mathcal{E}^{1,2}(v)\right)^{\frac{1}{2}} = \left(\int_\Omega |\mathrm{D}v|^2\right)^{\frac{1}{2}}.$$

We still have a small issue now: the quantity $|\cdot|_{W^{1,2}}$ is not positive definite. We consider another norm, defined for $v \in C^1(\bar{\Omega})$ as

$$\|v\|_{W^{1,2}(\Omega)} = \left(\int_\Omega |v|^2 + |\mathrm{D}v|^2\right)^{\frac{1}{2}}. \tag{1.7}$$

The *Poincaré inequality*, ensures that if the set Ω is bounded in one direction (or has finite measure), then there exists a constant $C > 0$ such that for every $v \in C^1(\Omega) \cap C(\bar{\Omega})$ such that $v = 0$ on $\partial\Omega$, one has

$$\int_\Omega |v|^2 \leq C\int_\Omega |\mathrm{D}v|^2.$$

In particular

$$\int_\Omega |u_j - u_i|^2 + \int_\Omega |\mathrm{D}u_j - \mathrm{D}u_i|^2 \leq (C+1)\int_\Omega |\mathrm{D}u_j - \mathrm{D}u_i|^2,$$

and thus by (1.6),

$$\lim_{j,i\to\infty} \int_\Omega |u_j - u_i|^2 + \int_\Omega |\mathrm{D}u_j - \mathrm{D}u_i|^2 = 0.$$

It remains to see how to obtain a space that has suitable completeness properties.

1.1.2 Definitions linear Sobolev spaces

Let us now review the approaches that have been proposed and implemented to define complete spaces of functions on which (1.7) defines a norm for smooth functions.

Absolutely continuous functions

If $I \subset \mathbb{R}$ is an interval, a function $u : I \to \mathbb{R}^n$ is *absolutely continuous* whenever for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $\ell \in \mathbb{N}$, if $x_1, \dots, x_i, y_1, \dots, y_i \in I$, if $x_1 \le y_1 \le x_2 \le y_2 \le \dots \le x_i \le y_i$ and if

$$\sum_{i=1}^{\ell} |y_i - x_i| \le \delta,$$

then

$$\sum_{i=1}^{\ell} |u(y_i) - u(x_i)| \le \varepsilon.$$

In particular, any Lipschitz-continuous function is absolutely continuous and by taking $\ell = 1$, any absolutely continuous function is uniformly continuous and hence continuous.

If the function $u : I \to \mathbb{R}^m$ is absolutely continuous, then there exists a Lebesgue-integrable function $g \in L^1(I, \mathbb{R}^m)$ such that for every $x, y \in I$,

$$u(x) = u(y) + \int_x^y g(t)\,\mathrm{d}t. \tag{1.8}$$

That is, a weak version of the fundamental theorem of calculus holds for absolutely continuous functions. Conversely, by Lebesgue's dominated convergence theorem, any function u representable by (1.8) for some $g \in L^1(I, \mathbb{R}^m)$ is absolutely continuous. In such a case, we say that g is a *weak derivative* of u. By classical arguments in measure theory, two weak derivatives $g_0, g_1 \in L^1(I, \mathbb{R}^m)$ of a given function u coincide almost everywhere on the set I.

The characterization of absolutely continuous functions does not carry on to functions on higher-dimensional domains. For $\Omega \subseteq \mathbb{R}^m$ and $u : \Omega \to \mathbb{R}^n$, one can however consider the restrictions of the function u to straight lines that are parallel to a given direction axis $h \in \mathbb{R}^m$ and define the function u to be weakly differentiable in the direction h whenever the restriction of the function u to any straight line parallel to h is almost everywhere equal to a weakly differentiable function whose directional derivative is the restriction of some function $\partial_h u$. It can then be proved that the directional weak derivative $\partial_h u \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ depends linearly on the direction h and a weak derivative $\mathrm{D}u \in L^1_{\mathrm{loc}}(\Omega, \mathrm{Lin}(\mathbb{R}^m, \mathbb{R}^n)) \simeq \mathrm{Lin}(\mathbb{R}^m, L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n))$ can be defined in such a way that $\partial_h u = \mathrm{D}u[h]$ almost everywhere on Ω.

The Sobolev space is defined then as the space of measurable functions $u : \Omega \to \mathbb{R}^n$ such that $u \in L^p(\Omega, \mathbb{R}^n)$, u has a weak derivative Du and $\mathrm{D}u \in L^p(\Omega, \mathrm{Lin}(\mathbb{R}^m, \mathbb{R}^n))$.

This definition is appealing because it essentially relying on merely the fundamental theorem of calculus; its drawback is its dependence a priori on the linear structure of the domain $\Omega \subset \mathbb{R}^m$. A proof of the invariance under a smooth diffeomorphism of the domain is quite technical. On the other hand, manipulations on the target-side are very staightforward in the one-dimensional case, where Lipschitz functions are readily seen to preserve the absolute continuity.

Distributional derivatives

Given a set $\Omega \subset \mathbb{R}^m$, a function $g \in L^1_{\mathrm{loc}}(\Omega, \mathrm{Lin}(\mathbb{R}^m, \mathbb{R}^n))$ is defined to be a *weak derivative* of $u \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ whenever for every $\varphi \in C^1_c(\Omega, \mathbb{R}^m)$,

$$\int_\Omega g[\varphi] = -\int_\Omega u \operatorname{div} \varphi$$

(where for each $x \in \Omega$, $g[\varphi](x) = g(x)[\varphi(x)]$, that is, the linear map $g(x) \in \mathrm{Lin}(\mathbb{R}^m, \mathbb{R}^n)$ applied to the vector $\varphi(x) \in \mathbb{R}^m$). If $g_0, g_1 \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ are both weak derivatives, then $g_0 = g_1$ almost everywhere in Ω.

From the definition of weak derivative, one defines the *Sobolev space* $W^{1,p}(\Omega)$ to be the space of functions such that $u \in L^p(\Omega, \mathbb{R}^n)$, u has a weak derivative $\mathrm{D}u$ and $\mathrm{D}u \in L^p(\Omega, \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^m))$.

The distributional theory has the advantage of being independent on the dimension and to be quite stable under changes of variables in the domain.

Completion

For a given set $\Omega \subset \mathbb{R}^m$, the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^n)$ can be defined as the *completion* as a metric space of the set

$$\left\{ v \in C^1(\Omega, \mathbb{R}^n) \mid \int_\Omega |\mathrm{D}v|^p + |v|^p < +\infty \right\}, \tag{1.9}$$

under the Sobolev norm which is defined defined for each $vC^1(\Omega, \mathbb{R}^n)$ by

$$\|v\|_{W^{1,p}(\Omega)} \triangleq \int_\Omega |\mathrm{D}v|^p + |v|^p.$$

In (1.9) the behavior of v on the boundary $\partial\Omega$ is not constrained by any continuity or differentiabily assumption on the boundary but merely by the integrability condition. Under this assumption, the space is equivalent to the distributional space [DL54a, DL54b, MS64].

Fourier analysis

Given a function $u \in L^2(\mathbb{R}^m, \mathbb{R}^n)$, its *Fourier transform* $\widehat{u} : \mathbb{R}^m \to \mathbb{R}^n$ is defined for each $\xi \in \mathbb{R}^m$ by

$$\widehat{u}(\xi) = \int_{\mathbb{R}^m} u(x)\, e^{-2\pi i \xi \cdot x}\, \mathrm{d}x.$$

A function $u \in L^2(\mathbb{R}^m, \mathbb{R}^n)$ is in the Sobolev space $W^{1,2}(\mathbb{R}^m, \mathbb{R}^n)$ whenever

$$\int_{\mathbb{R}^m} \left(1 + |2\pi\xi|^2\right) \left|\widehat{u}(\xi)\right|^2 \,\mathrm{d}\xi < +\infty.$$

This approach has the advantage of relying on the standard machinery of the Fourier transform and generalizing straightforwardly to higher-order and fractional derivatives. The Fourier definition is more delicate to adapt to define $W^{1,p}(\Omega, \mathbb{R}^n)$ when Ω is not the Euclidean space or $p \neq 2$. The case where Ω is a rectangular parallelepiped or a torus can be treated through Fourier series; other domains by the analysis of the spectrum of the Laplacian operator $-\Delta$ on Ω. The case $p \neq 2$ can be treated by Littlwood–Paley theory which gives rise to the scale of Triebel–Lizorkin spaces $F^{s,p}_q(\mathbb{R}^m, \mathbb{R}^n)$ [RS96, Tri78].

Potential theory

When $p = 2$, it can be proved that if the *Bessel potential* is defined by

$$G_\alpha(x) = \frac{1}{(4\pi)^{\frac{\alpha}{2}}\Gamma(\frac{\alpha}{2})} \int_0^\infty \frac{e^{-\frac{\pi|x|^2}{\delta} - \frac{\delta}{4\pi}}}{\delta^{1+\frac{m-\alpha}{2}}} \,\mathrm{d}\delta.$$

and if $f \in L^2(\mathbb{R}^m, \mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} |D(G_1 * f)|^2 + |G_1 * f|^2 = \int_{\mathbb{R}^n} |f|^2$$

(see [Ste70, §V.3]). This suggest then the following definition of Sobolev spaces

$$W^{1,2}(\mathbb{R}^m, \mathbb{R}^n) = \left\{G_1 * f \mid f \in L^2(\mathbb{R}^m, \mathbb{R}^n)\right\}.$$

The advantage of this definition is that convolutions estimates can be readily applied to study the properties of the function $u \triangleq G_1 * f$. When $p \neq 2$, one can still consider the set

$$\left\{G_1 * f \mid f \in L^2(\mathbb{R}^m, \mathbb{R}^n)\right\}$$

which is then a *Newton potential space* which is slightly different from the Sobolev spaces and corresponds to the Triebel–Lizorkin space $F^{1,p}_2(\mathbb{R}^m, \mathbb{R}^n)$.

Metric definitions

Sobolev spaces can also be defined in a *metric space setting*. The starting observation is that given $u \in L^p(\mathbb{R}^m)$, one has $u \in W^{1,p}(\mathbb{R}^m, \mathbb{R}^m)$ if and only if there exists a constant $C > 0$ such that for every $h \in \mathbb{R}^n$,

$$\|u(x + h) - u(x)\|_{L^p(\mathbb{R}^m)} \le C|h|$$

(see for example [Bre11, proposition 9.3]). Various approaches lead to definitions of Sobolev functions and their energy (see [Chi07] for a synthesis of available approaches).

The interest of the metric space approach is that it just relies on the structure of the target as a metric space. The drawback is that the Sobolev energy depends on the precise definition being used and the Sobolev energy density $|\mathrm{D}u|^p$ cannot be in general written as powers of a given local derivative $|\mathrm{D}u|$. In other words, the derivative can depend on the exponent p.

Equivalence

Under wide and reasonable assumptions, the definitions above are equivalent up to redefinition of the function on null sets.

In order to avoid technical issues when we restrict functions to subsets, we shall assume everywhere in the sequel that all the measurable functions are defined everywhere and Borel-measurable. That is, we do not consider equivalence classes and we will be calling norms functions that are strictly speaking seminorms that vanish only on functions vanishing almost everywhere. This attitude that differs from a classical approach to L^p spaces and has been adopted by several authors, especially in the study of nonlinear Sobolev spaces, has the advantage of woking directly with functions without heavy circumlocutions or abuse of language and of removing the burden of checking that low-level statement and identities do not depend on the choice of a representative in an equivalence class[1].

1.1.3 Properties

We review the properties of Sobolev spaces, with emphasis on the facts that will be the most relevant in the nonlinear theory.

Functional analysis

For every $p \in [1, +\infty]$, the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^n)$ is a complete normed spaces. When $p = 2$, $W^{1,2}(\Omega, \mathbb{R}^n)$ is a Hilbert space. When $1 < p < +\infty$, the space $W^{1,p}(\Omega, \mathbb{R}^n)$ is reflexive.

The weak differentiation operator D is a bounded linear operator from $W^{1,p}(\Omega, \mathbb{R}^m)$ to the space $L^p(\Omega, \mathrm{Lin}(\mathbb{R}^m, \mathbb{R}^n))$. The space $W^{1,p}(\Omega, \mathbb{R}^n)$ can be identified to the closed subspace

$$\big\{(u, g) \mid g = \mathrm{D}u \text{ weakly in } \Omega\big\}$$

of the space $L^p(\Omega, \mathbb{R}^n) \oplus L^p(\Omega, \mathrm{Lin}(\mathbb{R}^m, \mathbb{R}^n))$. When $1 \le p < +\infty$, the space $W^{1,p}(\Omega, \mathbb{R}^n)$ is separable (as a closed subset of the separable space $L^p(\Omega, \mathbb{R}^n) \oplus L^p(\Omega, \mathrm{Lin}(\mathbb{R}^m, \mathbb{R}^n))$).

[1] At a logical level, it has the advantage of reducing multiple back-and-forth journey in equivalence classes through the axiom of choice. (*The axiom of choice is used to extract elements from equivalence classes where they should never have been put in the first place.* [BB85, p. 12])

Removable point singularities

Sobolev functions need not be smooth. The next proposition provides a criterion for functions that are smooth except at one point in a domain to be in a Sobolev space.

Proposition 1.1. *Let $\Omega \subset \mathbb{R}^m$, $a \in \Omega$ and $u : \Omega \to \mathbb{R}^m$. If $u \in W^{1,p}(\Omega \setminus \{a\})$ and if*

$$\liminf_{r \to 0} \frac{1}{r} \int_{B_r(a)} |u| = 0,$$

then $u \in W^{1,p}(\Omega)$.

In particular, if $m > 1$ and $u \in L^{\frac{m}{m-1}}(\Omega)$, we have by the Hölder inequality

$$\frac{1}{r} \int_{B_r(a)} |u| \le C_1 \left(\int_{\Omega \cap B_r(a)} |u|^{\frac{m}{m-1}} \right)^{1-\frac{1}{m}},$$

and thus the condition holds by Lebesgue's dominated convergence theorem.

As examples of applications of we get that the function $x \in \mathbb{B}^m \mapsto \frac{1}{|x|^\alpha}$ is in $W^{1,p}(\mathbb{B}^m, \mathbb{R})$ if and only if $\alpha < \frac{m}{p} - 1$.

Proof of proposition 1.1. Let $(r_j)_{j \in \mathbb{N}}$ be a sequence such that

$$\lim_{j \to \infty} \frac{1}{r_j} \int_{B_{r_j}(a_j)} |u| = 0.$$

We choose a function $\chi \in C^1(\mathbb{R}^m)$ such that $\chi = 1$ in $B_{1/2}(0)$ and $\chi = 0$ in $\mathbb{R}^m \setminus B_1(0)$. We define $\chi_j(x) = \chi(\frac{x-a}{r_j})$.

Given a test function $\varphi \in C^1_c(\Omega, \mathbb{R}^n)$, we have for every $j \in \mathbb{N}$, $\chi_j \varphi \in C^1_c(\Omega, \mathbb{R}^n)$ and since, $u|_{\Omega \setminus \{a\}}$ is weakly differentiable in $\Omega \setminus \{a\}$,

$$\begin{aligned} \int_\Omega \mathrm{D}u[\chi_j \varphi] = \int_{\Omega \setminus \{a\}} \mathrm{D}u[\chi_j \varphi] &= -\int_{\Omega \setminus \{a\}} u \operatorname{div}(\chi_j \varphi) \\ &= -\int_{\Omega \setminus \{a\}} \chi_j \, u \operatorname{div}(\varphi) - \int_{\Omega \setminus \{a\}} u \, \mathrm{D}\chi_j[\varphi]. \end{aligned}$$

We have for every $j \in \mathbb{N}$,

$$\left| \int_{\Omega \setminus \{a\}} u \mathrm{D}\chi_j[\varphi] \right| \le \frac{C_2}{r_j} \int_{\Omega \cap B_{r_j}(a)} |u|,$$

and we have thus by Lebesgue's dominated convergence theorem

$$\int_\Omega \mathrm{D}u[\varphi] = -\int_\Omega u \operatorname{div} \varphi,$$

and $\mathrm{D}u$ is a weak derivative of u on Ω. □

The Sobolev mappings will be bounded functions, and so we can look at a criterion for removing singularities for bounded Sobolev functions.

Proposition 1.2. *Let $\Omega \subset \mathbb{R}^m$, $F \subset \Omega$ be a closed set and assume that there exists a sequence $(\chi_j)_{j\in\mathbb{N}}$ in $C^1(\Omega,[0,1])$ such that for every $j \in \mathbb{N}$, $\chi_j = 0$ on a neighbourhood of F, $\chi_j \to 1$ almost everywhere in Ω and*

$$\lim_{n\to\infty} \int_\Omega |D\chi_j| = 0.$$

If $u \in L^\infty(\Omega,\mathbb{R}^n)$ and if $u \in W^{1,p}(\Omega \setminus F,\mathbb{R}^n)$, then $u \in W^{1,p}(\Omega,\mathbb{R}^n)$.

Proof of proposition 1.1. Given a test function $\varphi \in C^1_c(\Omega,\mathbb{R}^n)$, we have for every $j \in \mathbb{N}$, $\chi_j\varphi \in C^1_c(\Omega,\mathbb{R}^n)$ and since, $u|_{\Omega\setminus\{a\}}$ is weakly differentiable in $\Omega \setminus \{a\}$,

$$\begin{aligned}\int_\Omega \mathrm{D}u[\chi_j\varphi] = \int_{\Omega\setminus\{a\}} \mathrm{D}u[\chi_j\varphi] &= -\int_{\Omega\setminus\{a\}} u \operatorname{div}(\chi_j\varphi)\\ &= -\int_{\Omega\setminus\{a\}} \chi_j\, u \operatorname{div}(\varphi) - \int_{\Omega\setminus\{a\}} u\, \mathrm{D}\chi_j[\varphi].\end{aligned}$$

By assumption, we have

$$\left|\int_{\Omega\setminus\{a\}} u\mathrm{D}\chi_j[\varphi]\right| \le \frac{C_3}{r_j}\int_{\Omega\cap B_{r_j}(a)} |u|,$$

and we have thus by Lebesgue's dominated convergence theorem

$$\int_\Omega \mathrm{D}u[\varphi] = -\int_\Omega u \operatorname{div}\varphi,$$

and $\mathrm{D}u$ is a weak derivative of u on Ω. □

Restrictions to generic subsets

Sobolev functions can be restricted to lower dimensional linear subsets.

Proposition 1.3. *Let $\Omega \subset \mathbb{R}^m$. If $u \in W^{1,p}(\Omega,\mathbb{R}^n)$, then for almost every affine subspace $W \subset \mathbb{R}^m$, $u|_{\Omega\cap W} \in W^{1,p}(\Omega\cap W,\mathbb{R}^n)$, and $\mathrm{D}(u|_{\Omega\cap W}) = (\mathrm{D}u)|_{\Omega\cap W\times W}$.*

Here $\Omega \cap W$ is an open subset of $W \simeq \mathbb{R}^\ell$, with $\ell = \dim W$. The restriction $(\mathrm{D}u)|_{\Omega\cap W\times W}$ is applied to the map $\mathrm{D}u : \Omega\times\mathbb{R}^m \to \mathbb{R}^n$ defined for $(x,h) \in \Omega\times\mathbb{R}^m$ by $\mathrm{D}u(x,h) = \mathrm{D}u(x)[h]$.

We will need a slightly more refined version of proposition 1.3.

Proposition 1.4. *If $\Omega \subset \mathbb{R}^m$ is convex and bounded and $u \in W^{1,1}(\Omega)$, then for almost every $x,y \in \Omega$, the function $t \in [0,1] \mapsto \mathrm{D}u((1-t)x+ty))$ is summable and*

$$u(y) = u(x) + \int_0^1 \mathrm{D}u((1-t)x+ty)[y-x]\,\mathrm{d}t.$$

In comparison proposition 1.3 implies that for almost every x and y the function is equal almost everywhere on almost every line segment to an absolutely continuous function but there is no guarantee that the function behaves nicely at x and y on this segment.

In particular, we have for almost every $x, y \in \Omega$

$$|u(y) - u(x)| \le \int_0^1 |\mathrm{D}u((1-t)x + ty)||y - x| \,\mathrm{d}t.$$

The proof of proposition 1.4 relies on the next lemma.

Lemma 1.5. *If $\Omega \subset \mathbb{R}^m$ is convex and bounded and if $f : \Omega \to [0, +\infty)$ is measurable then for every α such that $\alpha + m > 1$,*

$$\int_\Omega \int_\Omega \int_{[0,1]} f((1-t)x + ty)|y - x|^\alpha \,\mathrm{d}y \,\mathrm{d}x \le C \int_\Omega f.$$

Proof. Since Ω is convex, by applying the substitution $y = x + (z - x)/t$, we obtain

$$\int_\Omega \int_\Omega \int_{[0,1]} f((1-t)x + ty) \,\mathrm{d}t|y - x|^\alpha \,\mathrm{d}y \,\mathrm{d}x = \int_\Omega \int_\Omega \int_{I_{x,z}} \frac{f(z)|z - x|^\alpha}{t^{\alpha+m}} \,\mathrm{d}x \,\mathrm{d}t \,\mathrm{d}z,$$

where

$$I_{x,z} = \Big\{ t \in [0,1] \mid \frac{z - x}{t} \in A - x \Big\}.$$

In particular we have

$$I_{x,z} \subset \Big[\frac{|z - x|}{\operatorname{diam}(\Omega)}, +\infty \Big),$$

and thus since $\alpha + m > 1$

$$\begin{aligned} &\int_\Omega \int_\Omega \int_{[0,1]} f((1-t)x + ty) \,\mathrm{d}t|y - x|^\alpha \,\mathrm{d}y \,\mathrm{d}x \\ &\qquad \le \int_\Omega \int_\Omega \int_{\frac{|z-x|}{\operatorname{diam}(\Omega)}}^{+\infty} \frac{f(z)|z - x|^\alpha}{t^{\alpha+m}} \,\mathrm{d}x \,\mathrm{d}t \,\mathrm{d}z \\ &\qquad \le C_4 \operatorname{diam}(\Omega)^{m+\alpha-1} \int_\Omega \int_\Omega \frac{f(z)}{|z - x|^{m-1}} \,\mathrm{d}x \,\mathrm{d}z \\ &\qquad \le C_5 \operatorname{diam}(\Omega)^{m+\alpha} \int_\Omega f, \end{aligned}$$

which ends the proof. □

Proof of proposition 1.4. Let $(u_j)_{j \in \mathbb{N}}$ be a sequence of smooth functions that converges to u in $W^{1,1}(\Omega, \mathbb{R}^n)$. Up to a subsequence, we can assume that there exists a set $E \subset \Omega$, such that $\mathcal{L}^m(E) = 0$ and for every $x \in \Omega \setminus E$, $(u_j(x))_{j \in \mathbb{N}}$ converges to $u(x)$.

Let $\alpha > 1 - m$. By lemma 1.5, we have

$$\lim_{j\to\infty} \int_\Omega \int_\Omega \int_{[0,1]} |Du_j((1-t)x+ty) - Du((1-t)x+ty)| \,\mathrm{d}t |y-x|^\alpha \,\mathrm{d}y \,\mathrm{d}x = 0.$$

Up to a subsequence, there exists a set $F \subset \Omega \times \Omega$ such that $\mathcal{L}^{2m}(E) = 0$, and for every $x, y \in (\Omega \times \Omega) \setminus E$,

$$\lim_{j\to\infty} \int_{[0,1]} |Du_j((1-t)x+ty) - Du((1-t)x+ty)| \,\mathrm{d}t = 0.$$

Since u_j is smooth, for every $(x, y) \in \Omega \times \Omega$

$$u_j(y) = u_j(x) + \int_0^1 \mathrm{D}u_j((1-t)x+ty)[y-x] \,\mathrm{d}t. \tag{1.10}$$

We conclude by observing that if $(x, y) \in \Omega \times \Omega \setminus (E \times \Omega \cup \Omega \times E \cup F)$, then all the terms in the identity (1.10) converge and bring us to the conclusion. □

Embeddings

The presence of a weak derivative gives improves several properties of Sobolev functions.

Poincaré inequality The Poincaré inequality relates controls mean oscillations of a function by an integral of the derivative:

Proposition 1.6. *For every $m \in \mathbb{N}$ and $p \in [1, +\infty)$, $C > 0$, such that if $A \subset \Omega \subset \mathbb{R}^m$ is convex and open, then*

$$\int_A \int_A |u(y) - u(x)|^p \,\mathrm{d}y \,\mathrm{d}x \le C_m \frac{\operatorname{diam}(A)^{p+m}}{p+m-1} \int_A |\mathrm{D}u(z)|^p \,\mathrm{d}z.$$

The Poincaré inequality is usually written as

$$\int_A \left| u(x) - \fint_A u(y) \,\mathrm{d}y \right|^p \mathrm{d}x \le C \frac{\operatorname{diam}(A)^{p+m}}{|A|} \fint_A |\mathrm{D}u|^p;$$

by Jensen's and Minkowski's inequalities, the latter inequality is equivalent to (1.11), with equivalent constants.

In particular, by proposition 1.6 for some $a \in \Omega$ and $r > 0$ and if $u \in W^{1,p}(\Omega, \mathbb{R}^n)$, then

$$\fint_{B_r(a)} \fint_{B_r(a)} |u(y) - u(x)|^p \,\mathrm{d}y \,\mathrm{d}x \le C\, r^p \fint_{B_r(a)} |\mathrm{D}u|^p. \tag{1.11}$$

Proof of proposition 1.6. By proposition 1.4, for almost every $x, y \in A$, we have since the set A is convex

$$|u(y) - u(x)| \le \int_0^1 |\mathrm{D}u((1-t)x + ty)||y - x| \,\mathrm{d}t,$$

and thus by Jensen's inequality, we deduce that

$$\int_A \int_A |u(y) - u(x)|^p \,\mathrm{d}y \,\mathrm{d}x \le \int_0^1 \int_A \int_A |\mathrm{D}u((1-t)x + ty)|^p |y - x|^p \,\mathrm{d}y \,\mathrm{d}x \,\mathrm{d}t.$$

Since A is convex, by applying the substitution $y = x + (z - x)/t$, we obtain

$$\int_A \int_A |u(y) - u(x)|^p \,\mathrm{d}y \,\mathrm{d}x \le \int_A \int_A \int_{I_{x,z}} \frac{|\mathrm{D}u(z)|^p |z - x|^p}{t^{p+m}} \,\mathrm{d}x \,\mathrm{d}t \,\mathrm{d}z,$$

where

$$I_{x,z} = \Big\{ t \in [0, 1] \mid \frac{z - x}{t} \in A - x \Big\}.$$

In particular we have

$$I_{x,z} \subset \Big[\frac{|z - x|}{\operatorname{diam} A}, +\infty \Big),$$

and thus

$$\begin{aligned}
\int_A \int_A |u(y) - u(x)|^p \,\mathrm{d}y \,\mathrm{d}x &\le \int_A \int_A \int_{\frac{|z-x|}{\operatorname{diam} A}}^{+\infty} \frac{|\mathrm{D}u(z)|^p |z - x|^p}{t^{p+m}} \,\mathrm{d}t \,\mathrm{d}x \,\mathrm{d}z \\
&\le \frac{\operatorname{diam}(A)^{p+m-1}}{p + m - 1} \int_A \int_A \frac{|\mathrm{D}u(z)|^p}{|z - x|^{m-1}} \,\mathrm{d}x \,\mathrm{d}z \\
&\le C_m \frac{\operatorname{diam}(A)^{p+m}}{p + m - 1} \int_A |\mathrm{D}u(z)|^p \,\mathrm{d}z.
\end{aligned}$$

□

Sobolev embedding When $p < m$, the Sobolev embedding states that there exists a constant $C \in (0, +\infty)$ such that for every $u \in W^{1,p}(\mathbb{R}^m, \mathbb{R}^n)$,

$$\Big(\int_{\mathbb{R}^m} |u|^{\frac{mp}{m-p}} \Big)^{\frac{1}{p} - \frac{1}{m}} \le C \int_{\mathbb{R}^m} |\mathrm{D}u|^p. \tag{1.12}$$

The exponent in (1.12) is chosen in such a way that both sides behave similary under scalings (homotheties) in the domain $\mathbb{R}^m$. When $\Omega \subset \mathbb{R}^m$ is a nice domain, then the Sobolev inequality (1.12) transports to the domain Ω with an additional zeroth-order term $\|u\|_{L^p(\Omega)}$.

Sobolev–Morrey embedding

Proposition 1.7. *Let $\Omega \subset \mathbb{R}^m$, $p > m$ and $u \in W^{1,p}(\Omega)$. For almost every $x \in \Omega$, for every convex set $A \subset \Omega$ such that $x \in A$ one has*

$$\int_A |u(y) - u(x)| \,\mathrm{d}y \le C \operatorname{diam}(A)^{1+m(1-\frac{1}{p})} \Big(\int_A |\mathrm{D}u|^p\Big)^{\frac{1}{p}}.$$

Proof. Since the set A is convex, for almost every $x, y \in A$, we have by proposition 1.4

$$|u(y) - u(x)| \le \int_0^1 |\mathrm{D}u((1-t)x + ty)||y - x| \,\mathrm{d}t,$$

and thus, we deduce that for almost every $x \in A$, we have

$$\int_A |u(y) - u(x)| \,\mathrm{d}y \le \int_0^1 \int_A |\mathrm{D}u((1-t)x + ty)||y - x| \,\mathrm{d}y \,\mathrm{d}t.$$

Since A is convex, by applying the substitution $y = x + (z - x)/t$, we obtain

$$\int_A |u(y) - u(x)| \,\mathrm{d}y \le \int_A \int_{I_{x,z}} \frac{|\mathrm{D}u(z)||z - x|}{t^{m+1}} \,\mathrm{d}t \,\mathrm{d}z,$$

where

$$I_{x,z} = \Big\{t \in [0,1] \mid \frac{z - x}{t} \in A - x\Big\}.$$

In particular we have

$$I_{x,z} \subset \Big[\frac{|z - x|}{\operatorname{diam} A}, +\infty\Big),$$

and thus

$$\begin{aligned}\int_A |u(y) - u(x)|^p \,\mathrm{d}y &\le \int_A \int_{\frac{|z-x|}{\operatorname{diam} A}}^{+\infty} \frac{|\mathrm{D}u(z)||z - x|^p}{t^{m+1}} \,\mathrm{d}t \,\mathrm{d}z\\ &\le \frac{\operatorname{diam}(A)^m}{m} \int_A \frac{|\mathrm{D}u(z)|}{|z - x|^{m-1}} \,\mathrm{d}z.\end{aligned}$$

By Hölder's inequality, we have, since $p > m$,

$$\begin{aligned}\int_A |u(y) - u(x)| \,\mathrm{d}y &\le C_6 \operatorname{diam}(A)^m \Big(\int_A |\mathrm{D}u|^p\Big)^{\frac{1}{p}} \bigg(\int_A \frac{1}{|x - z|^{\frac{(m-1)p}{p-1}}} \,\mathrm{d}z\bigg)^{1-\frac{1}{p}}\\ &\le C_7 \operatorname{diam}(A)^{1+m(1-\frac{1}{p})} \Big(\int_A |\mathrm{D}u|^p\Big)^{\frac{1}{p}},\end{aligned}$$

which is the conclusion. $\square$

In particular when $p > m$, proposition 1.7 implies the Sobolev–Morrey embedding states that there exists a constant such that for almost every $x, y \in B_r(a) \subset \mathbb{R}^m$ and $u \in W^{1,p}(B_r(a), \mathbb{R}^n)$, then

$$|u(y) - u(x)| \le C \left(\int_{B_r(a)} |\mathrm{D}u|^p \right)^{\frac{1}{p}} |y - x|^{1-\frac{m}{p}}. \tag{1.13}$$

The right hand side is a sublinear function of the distance between y and x; the inequality implies that the function u is in the space $C^{0,\alpha}(B_r(a), \mathbb{R}^m)$ of Hölder-continuous functions with exponent $\alpha \triangleq 1 - \frac{m}{p}$. Again the exponent in the Sobolev–Morrey inequality (1.13) is chosen in such a way that both sides scale similarly. In contrast with the Sobolev inequality (1.12), the Morrey–Sobolev inequality is local in nature on both sides of the inequality.

As a consequence we have the following

Proposition 1.8. *If $u \in \dot{W}^{1,p}(B_r(a), \mathbb{R}^\nu)$ with $p > m$ and if $x \in B_r(a)$,*

$$\operatorname*{ess\,sup}_{B_r(a)} |u(x)| \le \operatorname*{ess\,inf}_{B_r(a)} |u| + C\, r^{1-\frac{m}{p}} \left(\int_{B_r(a)} |\mathrm{D}u|^p \right)^{\frac{1}{p}}.$$

Proof. We observe that by proposition 1.7 for almost every $x, y \in B_r(a)$, we have by the triangle inequality

$$|u(x)| \le |u(y)| + |u(x) - u(y)| \le |u(y)| + C_8\, r^{1-\frac{m}{p}} \left(\int_{B_r(a)} |\mathrm{D}u|^p \right)^{\frac{1}{p}}. \qquad \square$$

The critical case Sobolev inequality (1.12) and (1.13) have left open the critical case $p = m$. When $I \subset \mathbb{R}$ is an interval, one has for almost every $x, y \in I$, the inequality

$$|u(y) - u(x)| \le C \int_{[y,x]} |\mathrm{D}u|,$$

which relates the modulus of continuity of u to the modulus of integrability of $\mathrm{D}u$.

When $m \ge 2$, it can be observed that if $\eta \in C^1(\mathbb{R}^m)$ and $\eta = 0$ in $\mathbb{R}^m \setminus B_{1/2}(0))$ and if the function $u_\gamma : \mathbb{R}^m \to \mathbb{R}$ is defined for $x \in \mathbb{R}^m \setminus \{0\}$ by

$$u_\gamma(x) = \eta(x) \left(\ln \frac{1}{|x|} \right)^\gamma,$$

then $u_\gamma \in W^{1,m}(\mathbb{R}^m, \mathbb{R})$ if and only if $\gamma < 1 - \frac{1}{m}$. In particular, when $m \ge 2$, u_γ can be chosen to be discontinuous. (When $m = 1$, the same example shows that can fail to be Hölder continuous.)

Compactness properties

If $(u_i)_{i \in \mathbb{N}}$ is a bounded sequence in $W^{1,p}(\Omega, \mathbb{R}^n)$, then there exists $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ is a subsequence $(u_{i_j})_{j \in \mathbb{N}}$ such that every compact set $K \subset \mathbb{R}^m$,

$$\lim_{j \to \infty} \int_K |u_j - u|^p = 0.$$

Chain rule

If $f \in C^1(\mathbb{R}^n, \mathbb{R}^\ell)$ and if Df is bounded, then for every $u \in W^{1,p}(\mathbb{R}^m, \mathbb{R}^n)$ we have $f \circ u \in W^{1,p}(\mathbb{R}^m, \mathbb{R}^\ell)$ and $D(f \circ u) = Df(u)[Du]$ almost everywhere on $\mathbb{R}^m$.

Fine properties

A measurable function $u : \Omega \to \mathbb{R}^n$ is approximately continuous at almost every point of its domain [EG92, theorem 1.7.3]: for every $a \in \Omega$ and for every $\varepsilon > 0$,

$$\lim_{\delta \to 0} \frac{\mathcal{L}^n\big(B_\delta(a) \setminus f^{-1}(B_\varepsilon(f(a)))\big)}{\mathcal{L}^n\big(B_\delta(a)\big)} = 0.$$

If moreover, $u \in L^p(\Omega, \mathbb{R}^n)$, then for almost every $a \in \Omega$ [EG92, corollary 1.7.1],

$$\lim_{\delta \to 0} \fint_{B_\delta(a)} |f - f(a)|^p = 0. \tag{1.14}$$

Sobolev functions have a better behavior: if $u \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$, then (1.14) holds outside a Borel-measurable set of vanishing p–capacity and is equal almost everywhere to a p–quasicontinuous function $u^* : \mathbb{R}^m \to \mathbb{R}^n$ [EG92, theorem 4.8.1].

Trace theory

In view of the solution of the Dirichlet problem for the Laplacian (1.1), it is natural to ask which boundary values $g : \partial\Omega \to \mathbb{R}$ extend to some Sobolev function $u : W^{1,p}(\Omega, \mathbb{R})$. This question is treated in trace theory.

For the half-space $\Omega = \mathbb{R}^m_+ \triangleq \mathbb{R}^{m-1} \times [0, +\infty)$, there exists a linear, continuous and bijective trace operator $\operatorname{tr} : W^{1,p}(\mathbb{R}^{m-1}_+, \mathbb{R}^n) \to W^{1-\frac{1}{p},p}(\mathbb{R}^m, \mathbb{R}^n)$ that coincides with the restriction operator on $\mathbb{R}^m \simeq \mathbb{R}^m \times \{0\}$ for smooth Sobolev functions. The codomain of the trace operator is a *fractional Sobolev space*, defined for by

$$W^{s,p}(\mathbb{R}^\ell) = \{u \in L^p(\mathbb{R}^\ell, \mathbb{R}^n) \mid \mathcal{E}^{s,p}(u) < +\infty\},$$

where the Gagliardo fractional Sobolev energy is defined by

$$\mathcal{E}^{s,p}(u) \triangleq \int_{\mathbb{R}^\ell} \int_{\mathbb{R}^\ell} \frac{|u(y) - u(x)|^p}{|y - x|^{\ell + sp}} \,\mathrm{d}y \,\mathrm{d}x. \tag{1.15}$$

By the construction, there exist constants $c, C > 0$ such that

$$c\,\mathcal{E}^{1-\frac{1}{p},p}(u) \le \inf\{\mathcal{E}^{1,p}(U) \mid \operatorname{tr} U = u\} \le C\,\mathcal{E}^{1-\frac{1}{p},p}(u).$$

When $\Omega \subset \mathbb{R}^m$ is a domain with a reasonably smooth boundary, the same theory holds with additional L^p norms appearing in the estimates.

Fractional Gagliardo–Nirenberg interpolation inequality

Proposition 1.9. *If the set $\Omega \subset \mathbb{R}^m$ is open and convex, then for every $s \in (0,1)$ and $p \in (1,+\infty)$ such that $sp > 1$, there exists a constant $C > 0$ such that for every $u \in W^{1,sp}(\Omega,\mathbb{R}^n) \cap L^\infty(\Omega,\mathbb{R}^n)$, one has*

$$\int_\Omega \int_\Omega \frac{|u(y) - u(x)|^p}{|y - x|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x \le C \|u\|_{L^\infty}^{(1-s)p} \int_\Omega |\mathrm{D}u|^{sp}.$$

Proof of proposition 1.9. For every $x, y \in \mathbb{R}^m$, we have by the triangle inequality and by the discrete Hölder inequality

$$|u(y) - u(x)|^p \le 2^{p-1}\Bigg(\bigg|u(y) - \fint_{B_{|y-x|/2}(\frac{x+y}{2})} u(z)\,\mathrm{d}z\bigg|^p + \bigg|\fint_{B_{|y-x|/2}(\frac{x+y}{2})} u(z)\,\mathrm{d}z - u(x)\bigg|^p\Bigg),$$

and thus by integrating over $\mathbb{R}^m \times \mathbb{R}^m$ and by symmetry

$$\begin{aligned}
&\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{|u(y) - u(x)|^p}{|y - x|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x \\
&\qquad\le 2^p \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{1}{|y - x|^{m+sp}} \bigg|\fint_{B_{|y-x|/2}(\frac{x+y}{2})} u(z)\,\mathrm{d}z - u(x)\bigg|^p \,\mathrm{d}y\,\mathrm{d}x \\
&\qquad\le C_9 \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{1}{|y - x|^{m+sp}} \bigg(\fint_{B_{|y-x|}(x)} |u(z) - u(x)|\,\mathrm{d}z\bigg)^p \,\mathrm{d}y\,\mathrm{d}x \\
&\qquad= C_{10} \int_{\mathbb{R}^m} \int_0^{+\infty} \frac{1}{r^{1+sp}} \bigg(\fint_{B_r(x)} |u(z) - u(x)|\bigg)^p \,\mathrm{d}r\,\mathrm{d}x.
\end{aligned}$$

Next, we have for every $x \in \mathbb{R}^m$ and $r \in (0,+\infty)$,

$$\begin{aligned}
\fint_{B_r(x)} |u(z) - u(x)|\,\mathrm{d}z &\le C_{11} \int_{B_r(x)} |\mathrm{D}u(z)| \bigg(\frac{1}{|z - x|^{m-1}} - \frac{1}{r^{m-1}}\bigg)\,\mathrm{d}z \\
&= C_{12} \int_0^r \frac{1}{\rho^m} \int_{B_\rho(x)} |\mathrm{D}u| \\
&\le C_{12}\, r \sup_{r>0} \frac{1}{\rho^m} \int_{B_\rho(x)} |\mathrm{D}u|\,\mathrm{d}\rho \\
&= C_{12}\, r\, \mathfrak{M}|\mathrm{D}u|(x),
\end{aligned}$$

where $\mathfrak{M}|\mathrm{D}u|(x)$ denotes the maximal function of $|\mathrm{D}u|$. It follows thus that for every

$\lambda > 0$ and $x \in \mathbb{R}^m$, we have

$$\begin{aligned}
&\int_0^{+\infty} \frac{1}{r^{1+sp}} \biggl(\fint_{B_r(x)} |u(z) - u(x)| \biggr)^p \,\mathrm{d}r \\
&\qquad \le C_{12} \int_0^{\lambda} \bigl|\mathfrak{M}|Du|(x)\bigr|^p r^{(1-s)p-1} \,\mathrm{d}r + 2^p \int_{\lambda}^{+\infty} \frac{\|u\|_{L^\infty}^p}{r^{1+sp}} \,\mathrm{d}r \\
&\qquad \le C_{13} \biggl(\bigl(\mathfrak{M}|Du|(x)\bigr)^p \lambda^{(1-s)p} + \frac{\|u\|_{L^\infty}^p}{\lambda^{sp}} \biggr).
\end{aligned}$$

If we set $\lambda = \|u\|_{L^\infty}/\mathfrak{M}|\mathrm{D}u|(x)$, we obtain

$$\int_0^{+\infty} \frac{1}{r^{1+sp}} \biggl(\fint_{B_r(x)} |u(z) - u(x)| \biggr)^p \,\mathrm{d}r \le C_{14} \|u\|_{L^\infty}^{(1-s)p} (\mathfrak{M}|\mathrm{D}u|(x))^{sp}.$$

We have thus proved that

$$\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{|u(y) - u(x)|^p}{|y - x|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x \le C_{15} \|u\|_{L^\infty}^{(1-s)p} \int_{\mathbb{R}^m} (\mathfrak{M}|\mathrm{D}u|)^{sp}.$$

By the maximal function theorem (see for example [Ste70, Theorem I.1; Duo01, Theorem 2.16]), we conclude since $sp > 1$ that

$$\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{|u(y) - u(x)|^p}{|y - x|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x \le C_{16} \, \|u\|_{L^\infty}^{(1-s)p} \int_{\mathbb{R}^m} |\mathrm{D}u|^{sp}. \qquad \square$$

The proof can be adapted to a ball $B_r(a)$ so that one obtains the estimate:

$$\int_{B_r(a)} \int_{B_r(a)} \frac{|u(y) - u(x)|^p}{|y - x|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x \le C \|u\|_{L^\infty(B_r(a))}^{(1-s)p} \int_{B_r(a)} |\mathrm{D}u|^{sp}. \tag{1.16}$$

Problem 1.1 (★★). Prove (1.16).

Fractional Sobolev–Morrey embedding

The next proposition states that $W^{1,sp}(\mathbb{R}^m, \mathbb{R}^n) \cap L^\infty(\mathbb{R}^m, \mathbb{R}^n) \subset W^{s,p}(\mathbb{R}^m, \mathbb{R}^n)$.

Proposition 1.10. *Let $s \in (0,1)$ and $p \in [1, +\infty)$. If $sp > m$, then there exists a constant $C > 0$ such that for every open set $\mathbb{R}^m \subseteq \mathbb{R}^m$, every $u \in W^{s,p}(\mathbb{R}^m)$ there exists a set $E \subset \mathbb{R}^m$ such that $\mathcal{L}^m(E) = 0$ and for every convex set $A \subset \mathbb{R}^m$ and every $x \in A \setminus E$, one has*

$$\fint_A |u(y) - u(x)| \,\mathrm{d}y \le C \frac{\operatorname{diam}(A)^{s+\frac{m}{p}}}{\mathcal{L}^m(A)^{\frac{2}{p}}} \Bigl(\int_A \int_A \frac{|u(y) - u(x)|^p}{|y - x|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x \Bigr)^{\frac{1}{p}}.$$

Lemma 1.11. *If $\Omega \subset \mathbb{R}^m$ and $u \in W^{s,p}(\Omega)$, then there exists $E \subset \Omega$ such that $\mathcal{L}^m(E) = 0$ and for each $x \in \Omega \setminus E$,*

$$\fint_A |u(y) - u(x)| \,\mathrm{d}y \le \sum_{j\in\mathbb{N}} \fint_{x+2^{-j}(A-x)} \fint_{x+2^{-j-1}(A-x)} |u(y) - u(z)| \,\mathrm{d}y \,\mathrm{d}z.$$

Proof. By a classical result in measure theory, since $u \in L^1_{\mathrm{loc}}(\Omega)$, there exists a measurable set $E \subset \Omega$ such that for every $x \in \Omega \setminus E$,

$$\lim_{r\to 0} \frac{1}{r^m} \int_{\Omega\cap B_r(x)} |u(y) - u(x)| \,\mathrm{d}y = 0.$$

The results comes then by letting $k \to \infty$ in the inequality

$$\fint_A |u(y) - u(x)| \,\mathrm{d}y \le \sum_{j=1}^{k} \fint_{x+2^{-j}(A-x)} \fint_{x+2^{-j-1}(A-x)} |u(y) - u(z)| \,\mathrm{d}y \,\mathrm{d}z + \fint_{x+2^{-k-1}(A-x)} |u(y) - u(x)| \,\mathrm{d}y. \qquad \square$$

Proof of proposition 1.10. We have by lemma 1.11 and by the Hölder inequality,

$$\begin{aligned}
&\int_A |u(y) - u(x)| \,\mathrm{d}y \\
&\quad\le \frac{C_{17}}{\mathcal{L}^m(A)} \sum_{j\in\mathbb{N}} 2^{j2m} \int_{x+2^{-j}(A-x)} \int_{x+2^{-j-1}(A-x)} |u(y) - u(z)| \,\mathrm{d}y \,\mathrm{d}z \\
&\quad\le \frac{C_{18}}{\mathcal{L}^m(A)^{\frac{2}{p}-1}} \Bigl(\sum_{j\in\mathbb{N}} 2^{-j\frac{\frac{m}{p}-s}{1-\frac{1}{p}}} \Bigr)^{1-\frac{1}{p}} \\
&\qquad\qquad \Bigl(\sum_{j\in\mathbb{N}} \int_{x+2^{-j}(A-x)} \int_{x+2^{-j-1}(A-x)} \frac{|u(y) - u(z)|^p}{2^{-j(m+sp)}} \,\mathrm{d}y \,\mathrm{d}z \Bigr)^{\frac{1}{p}} \\
&\quad\le C_{19} \frac{\operatorname{diam}(A)^{s+\frac{m}{p}}}{\mathcal{L}^m(A)^{\frac{2}{p}-1}} \Bigl(\int_A \int_A \frac{|u(y) - u(z)|^p}{|y - z|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}z \Bigr)^{\frac{1}{p}}.
\end{aligned}$$

$\square$

1.2 Sobolev mappings

1.2.1 Motivation and definition

Sobolev spaces from a manifold $\mathcal{M}$ into a manifold $\mathcal{N}$ appear in the counterpart of the Dirichlet problem (1.1). Whereas the Laplacian does not make sense as a linear operator acting on mappings, the minimization of the Dirichlet functional $\mathcal{E}^{1,2}$ defined by (1.2) still makes sense. Minimizers of this energy *harmonic maps* in Riemannian geometry and have been studied in geometric analysis (see for example [HL87, SU82, Jos11, HW08, ES64, Bre03]). When $\mathcal{M} = [0, 1]$ harmonic maps are geodesics with constant velocity parametrization.

Such problems appear in the mathematical analysis of *ordered media* in condensed matter physics: *superconductors* (see for example [HT01, Mer79]), *superfluids* (see for example [Mer79]), *ferromagnetism* (see for example [HS98, Mer79]) and *liquid crystals* (see for example [Mer79, BZ11, Bre91, Muc12, CGH91]), of Cosserat materials in *elasticity* ($N = \mathbb{R}^3 \times SO(3)$, see for example [Nef04]), of *gauge theories* in physics (see for example [Lie93])

First-order Sobolev spaces

For given $p \in [1, +\infty)$ and Riemannian manifolds $\mathcal{M}$ and $\mathcal{N}$, we define the Sobolev space

$$W^{1,p}(\mathcal{M}, \mathcal{N}) = \{u \in W^{1,p}(\mathcal{M}, \mathbb{R}^\nu) \mid u \in \mathcal{N} \text{ almost everywhere in } \mathcal{M}\}. \tag{1.17}$$

The Sobolev space $W^{1,p}(\mathcal{M}, \mathbb{R}^\nu)$ is defined by local charts and we have assumed that the manifold $\mathcal{N}$ was isometrically embedded into $\mathbb{R}^\nu$. (In view of Nash's isometric embedding theorem [Nas56], this is always possible.)

Proposition 1.12. *Let $\mathcal{N}$ and $\mathcal{L}$ be Riemmanian manifolds. If $\mathcal{N}$ is compact and $f \in C^1(\mathcal{N}, \mathcal{L})$ then the mapping*

$$u \in W^{1,p}(\mathcal{M}, \mathcal{N}) \mapsto f \circ u \in W^{1,p}(\mathcal{M}, \mathcal{L})$$

is well-defined and continuous.

In particular if $f \in C^1(\mathcal{N}, \mathcal{L})$ is a Riemannian isometry, then proposition 1.12 implies that the map $u \in W^{1,p}(\mathcal{M}, \mathcal{N}) \mapsto f \circ u \in W^{1,p}(\mathcal{M}, \mathcal{L})$ is a homeomorphism. This means in particular that the definition (1.17) does not depend on the embedding. This comes ultimately from the fact that Riemannian isometries relate tangent directions of embedded submanifolds and that the first-order chain rule formula only depends on such derivatives. This is not anymore the case for higher-order Sobolev spaces $W^{j,p}(\mathcal{M}, \mathcal{N})$ [CVS].

Proof of proposition 1.12. We assume that the manifold $\mathcal{N}$ is embedded isometrically in $\mathbb{R}^\nu$ and that $\mathcal{L}$ is embedded is embedded isometrically in $\mathbb{R}^\lambda$. Since $\mathcal{N}$ is compact, there exists map $\bar{f} \in C^1_c(\mathbb{R}^\nu, \mathbb{R}^\lambda)$ such that $\bar{f}|_{\mathcal{N}} = f$.

By the chain rule for Sobolev functions, for every $u \in W^{1,p}(\mathcal{M}, \mathbb{R}^\nu)$, one has $f \circ u \in W^{1,p}(\mathcal{M}, \mathbb{R}^\lambda)$. Assume now that the sequence $(u_n)_{n \in \mathbb{N}}$ converges to u in $W^{1,p}(\mathcal{M}, \mathbb{R}^\nu)$. Then $(u_n)_{n \in \mathbb{N}}$ converges to u and $(\mathrm{D}u_n)_{n \in \mathbb{N}}$ converges to $\mathrm{D}u$ in measure. Moreover,

$$|\mathrm{D}(f \circ u_n) - \mathrm{D}(f \circ u)| = |\mathrm{D}f(u_n)[\mathrm{D}u_n] - \mathrm{D}f(u)[\mathrm{D}u]| \le |\mathrm{D}f(u_n)||\mathrm{D}u_n| + |\mathrm{D}f(u)||\mathrm{D}u|.$$

By (a variant of) Lebesgue's dominated convergence theorem, the sequence $(\mathrm{D}(f \circ u_n))_{n\in\mathbb{N}}$ converges to $\mathrm{D}(f \circ u)$ in $L^p(\mathcal{N})$, and it follows then that the sequence $(f \circ u_n)_{n\in\mathbb{N}}$ converges to $f \circ u$ in $W^{1,p}(\mathcal{M}, \mathcal{L})$. □

The proof of proposition 1.12 relies on Lebesgue's dominated convergence theorem and thus does not relate the rate of convergences. If fact the map f is not uniformly continuous in general [CVS16, proposition 4.15].

Problem 1.2 (★★). Construct two $\mathcal{N}$ and $\mathcal{L}$ that are isometrically embedded into Euclidean spaces and a global Riemannian isometry $i : \mathcal{N} \to \mathcal{L}$ such that the map $u \in W^{1,p}(\mathcal{M},\mathcal{N}) \mapsto i \circ u \in W^{1,p}(\mathcal{M},\mathcal{N})$ is not uniformly continuous.

Problem 1.3 (★★★★). Construct two $\mathcal{N}$ and $\mathcal{L}$ that are isometrically emedded respectively in the Euclidean spaces $\mathbb{R}^\nu$ and $\mathbb{R}^\lambda$ and a global Riemannian isometry $i : \mathcal{N} \to \mathcal{L}$ and a global Riemannian isometry $i : \mathcal{N} \to \mathcal{L}$ such that the map $u \in W^{1,p}(\mathcal{M},\mathcal{N}) \mapsto i \circ u \in W^{1,p}(\mathcal{M},\mathcal{N})$ is not continous for the topologies induced respectively on $W^{1,p}(\mathcal{M},\mathcal{N})$ and $W^{1,p}(\mathcal{M},\mathcal{L})$ by the weak topologies on $W^{1,p}(\mathcal{M},\mathbb{R}^\nu)$ and $W^{1,p}(\mathcal{M},\mathbb{R}^\lambda)$.

This property of continuity relies on the continuity of the derivative. It is known that the composition with Lipschitz-continuous mapping is not continuous [Haj07, theorem 1.2].

Fractional spaces

The fractional spaces can be defined intrinsically. Indeed, the Riemannian manifolds $\mathcal{M}$ and $\mathcal{N}$ are naturally endowed with their geodesic distances $d_\mathcal{N}$ and $d_\mathcal{N}$. We define for $s \in (0,1)$ and $p \in [1,+\infty)$, for every measurable map $u : \mathcal{M} \to \mathcal{N}$, the fractional energy

$$\mathcal{E}^{s,p}(u) = \int_\mathcal{M} \int_\mathcal{M} \frac{d_\mathcal{N}(u(y), u(x))^p}{d_\mathcal{M}(y,x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x.$$

We define a sequence $(u_n)_{n\in\mathbb{N}}$ to be strongly converging to u whenever $(u_i)_{i\in\mathbb{N}}$ converges locally in measure to u and

$$\lim_{i\to\infty} \mathcal{E}^{s,p}(u_i) = \mathcal{E}^{s,p}(u).$$

By Fatou's lemma, this is equivalent to require

$$\limsup_{i\to\infty} \mathcal{E}^{s,p}(u_i) = \mathcal{E}^{s,p}(u).$$

This is still equivalent to require that

$$\lim_{i\to\infty} \int_\mathcal{M} \int_\mathcal{M} \frac{\left|d_\mathcal{N}(u(y), u(x)) - d_\mathcal{N}(u_i(y), u_i(x))\right|^p}{d_\mathcal{M}(y,x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x$$

If we assume now that the manifold $\mathcal{N}$ is isometrically embedded into some Euclidean space $\mathbb{R}^\nu$ and if the sequence $(u_i)_{i\in\mathbb{N}}$ converges strongly in $W^{s,p}(\mathcal{M},\mathcal{N})$ to u, then since for every $x,y\in\mathcal{M}$,

$$\begin{aligned}\frac{|(u_i(y)-u(y))-(u_i(x)-u(x))|^p}{d_{\mathcal{M}}(y,x)^{m+sp}} &\le 2^{p-1}\frac{|u_i(y)-u_i(x)|^p+|u(y)-u(x)|^p}{d_{\mathcal{M}}(y,x)^{m+sp}}\\ &\le 2^{p-1}\frac{d_{\mathcal{N}}(u_i(y),u_i(x))^p+d_{\mathcal{N}}(u(y),u(x))^p}{d_{\mathcal{M}}(y,x)^{m+sp}},\end{aligned}$$

so that $(u_i)_{i\in\mathbb{N}}$ converges strongly to u in $W^{s,p}(\mathcal{M},\mathbb{R}^\nu)$.

Conversely, if $(u_i)_{i\in\mathbb{N}}$ converges strongly to u in $W^{s,p}(\mathcal{M},\mathbb{R}^\nu)$, then since

$$\frac{d_{\mathcal{N}}(u_i(y),u_i(x))^p}{d_{\mathcal{N}}(y,x)^{m+sp}}\le\frac{|u_i(y)-u_i(x)|^p}{d_{\mathcal{N}}(y,x)^{m+sp}},$$

and thus by Lebesgue's dominated convergence theorem, we have that $(u_i)_{i\in\mathbb{N}}$ converges strongly to u in $W^{s,p}(\mathcal{M},\mathcal{N})$.

We could also have defined the strong convergence by embedding. It follows from [BPVS14, lemma 2.5; BBM04, claim (5.43)], that changes of the embedding induce uniformly continuous maps for the induced distances on $W^{s,p}(\mathcal{M},\mathcal{N})$.

Localizing the target

One of the basic facts in differential topology, is that smooth maps are continuous and thus when restricted to sets that are small enough in the domain result have an image which is in a single local chart domain. By the Morrey–Sobolev embedding (1.13), this remains the case when $sp>m$; this is not anymore the case when $sp\le m$.

Indeed, if $sp<1$ and one does not have $s=p=m=1$, then for every $\varepsilon>0$, there exists a map $u\in W^{s,p}(\mathbb{R}^m,\mathbb{R})$ such that $u\in C^\infty(\mathbb{R}^m\setminus\{0\})$, $\lim_{x\to 0}u(x)=+\infty$. If $\mathcal{M}$ is connected, then there exist a function $g\in C^1(\mathbb{R},\mathcal{M})$ be a function such that for every $T\in\mathbb{R}$, the set $g([T,+\infty))$ is dense in $\mathcal{M}$. The image of the function $g\circ u$ is then dense in $\mathcal{M}$ in every neighbourhood of 0.

Problem 1.4 (★). If $\mathcal{M}$ is connected, a function $\gamma\in C^\infty(\mathbb{R},\mathcal{M})$ such that for every $T>0$, the set $\gamma([T,+\infty))$ is dense in $\mathcal{M}$.

Problem 1.5 (★★). If $\mathcal{N}$ is connected and $sp<m=\dim\mathcal{M}$, construct a function in $u\in W^{s,p}$ such that for every open set $G\subset\mathcal{M}$, the set $u(G)$ is dense in $\mathcal{M}$.

1.2.2 Questions

In order to study problems in calculus of variations and partial differential equations involving mappings, it is useful to understand the structure of Sobolev mappings.

By nature, Sobolev spaces into manifolds do not have any linear structure. This means that classical linear functional analysis theorems (Hahn–Banach theorem, uniform boundedness theorem) do not apply anymore and also that classical linear tools such as the regularization by convolution do not work.

Several questions arise arise naturally for Sobolev mappings.

The question of *approximation* asks whether and how a Sobolev mapping can be approximated by a class of smoother maps. If it is possible, then this means that the theory of Sobolev maps is the study of the properties of this class of smoother maps that are invariant under the notion of approximation. When this is not possible, this indicates that the Sobolev space is much larger than the class of smoother maps. Hence, the function space becomes part of the problem (or of the model in physical settings), one can expect Lavrentiev phenomena (gap between infima of energies defined on different spaces) and associated loss of regularity (one should expect in general that maps having the regularity of minimizers in calculus of variations should be dense in the domain; however the regularity can be worse than that).

The question of *traces* asks about the possible boundary values of Sobolev mappings. This question is fundamental to understand admissible boundary data for problems in calculus of variations.

Another question is about *connected components* of the Sobolev spaces of mappings. One expects the multiplicity of connected components to be related to multiplicity of minimizers (although it is known in general that each connected component carries a minimizer). The probably simplest setting is the study of maps from $W^{s,p}(\mathbb{S}^m, \mathbb{S}^m)$ when $sp \geq m$; maps in this space are connected to each other if and only if their Brouwer topological degree coincide.

A last question is the question of *lifting*. Given a fibration $\pi : \mathcal{E} \to \mathcal{N}$, one asks whether for $u \in W^{s,p}(\mathcal{M}, \mathcal{N})$ there exists a mapping $\varphi : \mathcal{M} \to \mathcal{E}$ such that $u = \pi \circ \varphi$. The goal is to have space $\mathcal{E}$ which has a simpler structure than $\mathcal{N}$ so that studying the lifting simplifies the analysis of the problem. An important class of fibrations are *coverings*, where π is a local homeomorphism that covers evenly $\mathcal{N}$ and in particular universal coverings (when $\mathcal{E}$ is moreover simply connected). For example the map $t \in \mathbb{R} \mapsto (\cos t, \sin t)$ is the universal covering of the circle $\mathbb{S}^1$ by $\mathbb{R}$. Another example is the *Hopf fibration* from $\mathbb{S}^3$ to $\mathbb{S}^2$.

1.3 Comments

Proposition 1.9 appears in [Run86, lemma 2.1; BM01, corollary 3.2]. The proof of proposition 1.9 is inspired by [MS02].

2 Supercritical and critical Sobolev mappings

In this lecture, we consider problems for supercritical and critical Sobolev spaces. The common feature of these spaces is that the integrability of the gradient guarantees that these maps are continuous or close enough to continous and thus they inherit their properties from Sobolev and continuous maps.

2.1 Approximation

We first consider the question of strong approximation in supercritical Sobolev spaces.

Theorem 2.1. *If $u \in W^{s,p}(\mathcal{M},\mathcal{N})$ and if $sp \ge m$, then there exists a sequence $(u_j)_{j\in\mathbb{N}}$ in $C^\infty(\mathcal{M},\mathcal{N})$ that converges to u in $W^{s,p}(\mathcal{M},\mathcal{N})$.*

A first tool is the existence of a retraction from a neighbourhood of the embedded manifold $\mathcal{N} \subset \mathbb{R}^\nu$.

Lemma 2.2. *If $\mathcal{N}$ is a smooth compact submanifold of $\mathbb{R}^\nu$, then there exists $\delta_{\mathcal{N}} > 0$, and a map $\Pi \in C^\infty(\overline{\mathcal{N} + B_{\delta_{\mathcal{N}}}},\mathcal{N})$ such that $\Pi|_{\mathcal{N}} = \mathrm{id}$.*

Proof. We observe that if $\delta_{\mathcal{N}}$ is small enough, then for each $y \in \overline{\mathcal{N} + B_{\delta_{\mathcal{N}}}}$ we can define $\Pi(y) \in \mathcal{N}$ to be the nearest point projection of y on $\mathcal{N}$. By the implicit function theorem, the map Π is smooth. □

Problem 2.1 (★). Show the existence of $\delta_{\mathcal{N}}$ and the smoothness of Π by the implicit function theorem.

We first prove theorem 2.1 in the supercritical case $sp > m$.

Proof of theorem 2.1 when $sp > m$. Since $u \in W^{s,p}(\mathcal{M},\mathbb{R}^\nu)$, there exists a sequence $(v_j)_{j\in\mathbb{N}}$ that converges to u in $W^{s,p}(\mathcal{M},\mathbb{R}^\nu)$. By proposition 1.8, we have for almost every $x \in \mathcal{M}$

$$\operatorname{dist}\bigl(v_j(x),\mathcal{N}\bigr) \le |v_j(x) - v(x)| \le C_1\left(\operatorname*{ess\,inf}_{\mathcal{M}}|v_j - v| + \left(\int_{\mathcal{M}} |\mathrm{D}(v_j - v)|^p\right)^{\frac{1}{p}}\right).$$

Thus when $j \in \mathbb{N}$ is large enough, we have by continuity of v_j for every $x \in \mathcal{M}$, $\operatorname{dist}(v_j(x),\mathcal{N}) \le \delta_{\mathcal{N}}$ and we can set for such $j \in \mathbb{N}$, $u_j = \Pi \circ v_j$. Since the map Π is smooth, the sequence $(u_j)_{j\in\mathbb{N}}$ converges to $u = \Pi \circ u$. □

The proof of theorem 2.1 in the critical case $sp = m$ is more subtle since the Morrey embbedding proposition 1.8 does not hold in the limiting case $sp = m$.

The main tool is the following estimate on the distance to the target.

Lemma 2.3. *If $\varphi \in (L^1 \cap L^\infty)(\mathcal{M}, \mathbb{R})$, if $\varphi \ge 0$ and if $\int_{\mathcal{M}} \varphi = 1$, then for every $u \in L^1(\mathcal{M}, \mathbb{R}^\nu)$, one has*

$$\inf_{z \in u(\mathcal{M})} \left| \int_{\mathcal{M}} u\, \varphi - z \right| \le \int_{\mathcal{M}} \int_{\mathcal{M}} |u(y) - u(x)| \varphi(y) \varphi(x)\, \mathrm{d}y\, \mathrm{d}x.$$

Proof. For every $y \in \mathcal{M}$, we have $u(y) \in u(\mathcal{M})$ and thus

$$\begin{aligned}\inf_{z \in u(\mathcal{M})} \left| \int_{\mathcal{M}} \varphi u - z \right| &\le \left| \int_{\mathcal{M}} u(x) \varphi(x)\, \mathrm{d}x - u(y) \right| \\ &= \left| \int_{\mathcal{M}} (u(y) - u(x)) \varphi(x)\, \mathrm{d}x \right| \le \int_{\mathcal{M}} |u(y) - u(x)| \varphi(x)\, \mathrm{d}x.\end{aligned}$$

We reach the conclusion by multiplying by $\varphi(y)$ and by integrating with respect to y over the domain $\mathcal{M}$. □

In fact in the statement of lemma 2.3, we could neglect the values taken on a Lebesgue null set of $\mathcal{M}$. This can be stated and proved through the notion of *essential range* of a mapping [BN95].

Problem 2.2 (★★). State and prove lemma 2.3 where $u(\mathcal{M})$ is replaced by the essential range of u.

Lemma 2.4. *If $sp = m$ and $u \in W^{s,p}(\mathcal{M}, \mathbb{R}^\nu)$, then*

$$\lim_{r \to 0} \fint_{B_r(a)} \fint_{B_r(a)} |u(y) - u(x)|\, \mathrm{d}y\, \mathrm{d}x = 0.$$

Lemma 2.4 states that functions in the space $W^{s,p}(\mathcal{M}, \mathbb{R}^\nu)$ have vanishing mean oscillation.

Proof of lemma 2.4. If $s = 1$ we have by the Poincaré inequality (1.11) for every $a \in \mathcal{M}$ if $r > 0$ is small enough (because we are working on a manifold $\mathcal{M}$)

$$\begin{aligned}\fint_{B_r(a)} \fint_{B_r(a)} |u(y) - u(x)|\, \mathrm{d}y\, \mathrm{d}x &\le \left(\fint_{B_r(a)} \fint_{B_r(a)} |u(y) - u(x)|^m\, \mathrm{d}y\, \mathrm{d}x \right)^{\frac{1}{m}} \\ &\le C_2 \left(\int_{B_r(a)} |\mathrm{D}u|^m \right)^{\frac{1}{m}}.\end{aligned} \tag{2.1}$$

Since

$$\int_{\mathcal{M}} |\mathrm{D}u|^m < +\infty,$$

by Lebesgue's dominated convergence theorem, we have

$$\lim_{r\to 0} \sup_{a\in\mathbb{R}^m} \int_{B_r(a)} |\mathrm{D}u|^m = 0,$$

and thus the conclusion holds.

When $s \in (0,1)$, we have by Hölder's inequality, if $a \in \mathcal{M}$ and if $r > 0$ is small enough

$$\begin{aligned}
&\fint_{B_r(a)} \fint_{B_r(a)} |u(y) - u(x)| \,\mathrm{d}y \,\mathrm{d}x \\
&\quad\le \left(\fint_{B_r(a)} \fint_{B_r(a)} |y-x|^{\frac{2m}{p-1}} \,\mathrm{d}y\,\mathrm{d}x\right)^{1-\frac{1}{p}} \left(\fint_{B_r(a)} \fint_{B_r(a)} \frac{|u(y)-u(x)|^p}{|y-x|^{2m}} \,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{1}{p}} \\
&\quad\le C_3 \sup_{a\in\mathbb{R}^m} \left(\int_{B_r(a)} \int_{B_r(a)} \frac{|u(y)-u(x)|^p}{|y-x|^{2m}} \,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned}$$

We conclude again by Lebesgue's dominated convergence theorem that

$$\lim_{r\to 0} \sup_{a\in\mathbb{R}^m} \left(\int_{B_r(a)} \int_{B_r(a)} \frac{|u(y)-u(x)|^p}{|y-x|^{2m}} \,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{1}{p}} = 0. \qquad \square$$

Remark 2.5. Whereas the proof of lemma 2.4 seems to cover the range $p \in [1, +\infty)$ with $sp = m$, it turns out that when $s > 1$ the Gagliardo seminorm defined in (1.15) is finite only on mappings that are almost everywhere equal to a constant on each connected component of $\mathcal{M}$.

Proof of theorem 2.1 when $\mathcal{M} = \mathbb{R}^m$ and $sp = m$. We define for each $\varepsilon > 0$, the function $u_\varepsilon \in C^\infty(\mathbb{R}^m, \mathbb{R}^\nu)$ by setting for each $x \in \mathbb{R}^m$,

$$u_\varepsilon(x) \triangleq \int_{\mathbb{R}^m} \varphi(y)\, u(x - \varepsilon y) \,\mathrm{d}y,$$

where $\varphi \in C^\infty_c(\mathbb{R}^m)$, $\varphi \ge 0$ and $\operatorname{supp} \varphi \subset B_1(0)$. By classical properties of Sobolev spaces, $u_\varepsilon \to u$ in $W^{s,p}(\mathbb{R}^m, \mathbb{R}^n)$ as $\varepsilon \to 0$ to u.

By lemma 2.3, we have for every $\varepsilon > 0$

$$\operatorname{dist}(u_\varepsilon, \mathcal{N}) \le C_1 \sup_{a\in\mathbb{R}^m} \fint_{B_\varepsilon(a)} \fint_{B_\varepsilon(a)} |u(y) - u(x)| \,\mathrm{d}y\,\mathrm{d}x, \tag{2.2}$$

and thus by lemma 2.4, we deduce that

$$\lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{x\in\mathbb{R}^m} \operatorname{dist}(u_\varepsilon(x), \mathcal{N}) = 0. \tag{2.3}$$

We conclude by lemma 2.3 as in the proof of theorem 2.1. $\square$

Problem 2.3 (★★). Prove theorem 2.1 when $sp = m$ on $\mathcal{M} = \mathbb{R}^m$ by taking φ to be the *Poisson kernel*.

Problem 2.4. Give an example of map $u \in W^{1,m}(\mathcal{M},\mathcal{N})$, which cannot be strongly approximated by smooth maps.

Problem 2.5. Give an example of map $u \in W^{s,p}(\mathcal{M},\mathcal{N})$ with $sp = m$, which cannot be strongly approximated by smooth maps.

2.2 Homotopy

In the classical setting of continuous map, to maps $u, v \in C(\mathcal{M},\mathcal{N})$ are homotopic if there exists a map $H \in C([0,1] \times \mathcal{M},\mathcal{N})$ such that $H(0,\cdot) = u$ and $H(1,\cdot) = v$. The homotopy is an equivalence relation. Equivalently when $\mathcal{M}$ is compact, there exists a continuous map $H \in C([0,1], C(\mathcal{M},\mathcal{N}))$ (where $C(\mathcal{M},\mathcal{N}))$ is endowed with the norm of the uniform convergence) such that $H(0) = u$ and $H(1) = v$. If $\mathcal{N}$ is compact, there exists $\delta > 0$ such that if $u, v \in C(\mathcal{M},\mathcal{N})$ and if $d_{\mathcal{N}}(u,v) \le \delta$, then u and v are homotopic.

We consider the question about homotopy in Sobolev spaces, that is whether two given maps $u, v \in W^{s,p}(\mathcal{M},\mathcal{N})$ can be connected by a continuous path in $W^{s,p}(\mathcal{M},\mathcal{N})$.

The first result states that any supercritical or critical Sobolev map can be connected to a continuous map. That is there are not more Sobolev connected than there were continuous connected components.

Proposition 2.6. *If $sp \ge m$, then every $u \in W^{s,p}(\mathcal{M},\mathcal{N})$, is connected continuously in $W^{s,p}(\mathcal{M},\mathcal{N})$ to a map in $(C^\infty \cap W^{s,p})(\mathcal{M},\mathcal{N})$.*

Proof. The proof goes as the proof of theorem 2.1, by letting $\varepsilon > 0$ be the homotopy parameter. □

A second fact is that any supercritical or critical Sobolev map is surrounded by a neighbourhood of Sobolev maps belonging to the same connected component.

Proposition 2.7. *Every path-connected connected component of $W^{s,p}(\mathcal{M},\mathcal{N})$ is open.*

In contrast with continuous maps between compact manifolds, we have no uniform quantitative information on the size of the neighbourhood when $sp = m$.

Proof of proposition 2.7. We follow the proof of theorem 2.1. Given $u \in W^{s,p}(\mathcal{M},\mathcal{N})$, we observe that there exists $\varepsilon > 0$ and a neighbourhood $\mathcal{U}$ of u such that any $v \in \mathcal{U}$ satisfies

$$\operatorname*{ess\,sup}_{x \in \mathcal{M}} \operatorname{dist}(v_\varepsilon(x), \mathcal{N}) \le \frac{\delta_{\mathcal{N}}}{2}.$$

We have then for every $x \in \mathcal{M}$,

$$|v_\varepsilon(x) - u_\varepsilon(x)| \le C_2 \fint_{B_\varepsilon(x)} |v(x - \varepsilon y) - u(x - \varepsilon y)| \,\mathrm{d}y \le \frac{C_3}{\varepsilon^{m(1-\frac{1}{p})}} \left(\int_{\mathcal{M}} |v - u|^p \right)^{\frac{1}{p}}.$$

Thus for fixed $\varepsilon > 0$, if we assume that u and v are close enough in $L^p(\mathcal{M},\mathcal{N})$, then we have everywhere on $\mathcal{M}$,

$$|v_\varepsilon - u_\varepsilon| \le \frac{\delta_{\mathcal{N}}}{2}.$$

We conclude by combining a homotopy from u to u_ε, a homotopy from u_ε to v_ε and a homotopy from v_ε to v. □

Proposition 2.8. *If $sp \ge m$, then $u, v \in (C \cap W^{s,p})(\mathcal{M},\mathcal{N})$, are connected to each other in $W^{s,p}(\mathcal{M},\mathcal{N})$ if and only if they are connected to each other in $C(\mathcal{M},\mathcal{N})$.*

Proof. This is done by applying the construction of theorem 2.1 to smoothen the continuous homotopy. □

2.3 Extension from the boundary

2.3.1 Local extension

The application of the classical trace theory shows that traces of mappings in $W^{1,p}(\mathcal{M} \times [0,1],\mathcal{N})$ on $\mathcal{M} \times \{0\}$ lie in the fractional space $W^{1-\frac{1}{p},p}(\mathcal{M},\mathcal{N})$.

Proposition 2.9. *If $p > m = \dim \mathcal{M} + 1$, then for every $u \in W^{1-\frac{1}{p},p}(\mathcal{M},\mathcal{N})$, there exists a map $U \in C^\infty(\mathcal{M} \times (0,1],\mathcal{N}) \cap W^{1,p}(\mathcal{M},\mathcal{N})$, such that $\operatorname{tr}_{\mathcal{M}\times\{0\}} U = u$ on $\mathcal{M} \times \{0\} \simeq \mathcal{M}$. Moreover, there exists a constant $C > 0$ such that*

$$\mathcal{E}^{1,p}(U) \le C\,\mathcal{E}^{1-\frac{1}{p},p}(u)\bigl(1 + \mathcal{E}^{1-\frac{1}{p},p}(u)^{\frac{p}{p-m}}\bigr).$$

Proof. By the classical linear theory of extensions, there exists a function $V \in C^\infty(\mathcal{M} \times (0,1],\mathbb{R}^\nu) \cap W^{1,p}(\mathcal{M},\mathbb{R}^\nu)$ such that $\operatorname{tr}_{\mathcal{M}\times\{0\}} V = u$ on $\mathcal{M} \times \{0\} \simeq \mathcal{M}$. Moreover

$$\mathcal{E}^{1,p}(V) \le C_1 \mathcal{E}^{1-\frac{1}{p},p}(u).$$

By the Sobolev–Morrey embedding, we have for almost every $(x,t) \in \mathcal{M} \times [0,1]$

$$\operatorname{dist}(V(t,x),\mathcal{N}) \le |V(t,x) - u(x)| \le C_2\, t^{1-\frac{m}{p}}\, \bigl(\mathcal{E}^{1,p}(V)\bigr)^{\frac{1}{p}} \le C_3\, t^{1-\frac{m}{p}}\, \bigl(\mathcal{E}^{1-\frac{1}{p},p}(u)\bigr)^{\frac{1}{p}}.$$

If we define

$$\tau \triangleq \min\left(1, \left(\frac{\delta_{\mathcal{N}}}{C_3 \mathcal{E}^{1-\frac{1}{p},p}(u)^{\frac{1}{p}}}\right)^{\frac{p}{p-m}}\right)$$

then $V([0,\tau]) \subseteq \mathcal{N} + B_{\delta_{\mathcal{N}}}$. We conclude by defining $U : \mathcal{M} \times [0,1] \to \mathcal{N}$ for almost every $(x,t) \in \mathcal{M} \times [0,1]$ by

$$U(x,t) \triangleq \Pi\bigl(V(x,\tau t)\bigr)$$

and by observing that

$$\mathcal{E}^{1,p}(U) \le \frac{\mathcal{E}^{1,p}(V)}{\tau^p} \le C_4 \mathcal{E}^{1-\frac{1}{p},p}(u) \max\bigl(1, \mathcal{E}^{1-\frac{1}{p},p}(u)^{\frac{p}{p-m}}\bigr).$$ □

Problem 2.6 (★★). Prove proposition 2.9 on a compact manifold.

Proposition 2.10. *Assume that* $\dim \mathcal{M} = m - 1$*. For every* $u \in W^{1-\frac{1}{m},m}(\mathcal{M},\mathcal{N})$*, there exists a map* $U \in C^\infty(\mathcal{M} \times (0,1],\mathcal{N}) \cap W^{1,m}(\mathcal{M},\mathcal{N})$*, such that* $\operatorname{tr}_{\mathcal{M}\times\{0\}} U = u$ *on* $\mathcal{M} \times \{0\} \simeq \mathcal{M}$*.*

Proof. We assume that $\mathcal{M} = \mathbb{R}^m$. By the classical linear theory of extensions we can define an extension

$$V = \int_{\mathbb{R}^m} u(x - ty)\varphi(x)\,\mathrm{d}y,$$

where $\varphi \in C^1_c(\mathbb{R}^m)$, $\operatorname{supp}\varphi \subset B_1(0)$, $\varphi \geq 0$ on $\mathbb{R}^m$ and $\int_{\mathbb{R}^m} \varphi = 1$. By the classical linear trace theory, we have $V \in C^\infty(\mathcal{M} \times (0,1],\mathbb{R}^\nu) \cap W^{1,p}(\mathcal{M},\mathbb{R}^\nu)$ and $\operatorname{tr}_{\mathcal{M}\times\{0\}} U = u$ on $\mathcal{M} \times \{0\} \simeq \mathcal{M}$. By lemma 2.3 and by lemma 2.4, there exists $\tau \in [0,1]$ such that $V([0,\tau]) \subseteq \mathcal{N} + B_{\delta_\mathcal{N}}$. We conclude by taking

$$U(x,t) \triangleq \Pi\big(V(x,\tau t)\big). \qquad \square$$

Problem 2.7 (★★★). Prove proposition 2.9 on a compact manifold.

In contrast with proposition 2.9, proposition 2.10 does not come with any estimate. This comes from the fact that τ is defined in the proof through lemma 2.4 which is based on Lebesgue's dominated convergence theorem. The control given by this proof would be in terms of the modulus of integrability of the Sobolev energy density; this would imply uniform control on *compact subsets* of $W^{s,p}(\mathcal{M},\mathcal{N})$. We will see later on that there is no quantitative control when the homotopy groups $\pi_1(\mathcal{N}), \pi_2(\mathcal{N}), \dots, \pi_m(\mathcal{N})$ are not all finite. One way to bypass this lack of control would be to lower the requirement on the integrability of the gradient to a Marcinkiewicz weak L^{m+1} condition [PR14, PVS17].

2.3.2 Global extension

We now consider the extension problem from the boundary of a manifold.

Proposition 2.11. *If* $p \geq m = \dim \mathcal{M}$*, then for every* $u \in W^{1-\frac{1}{p},p}(\partial\mathcal{M},\mathcal{N})$*, the following are equivalent*

(i) there exists a map $U \in (C^\infty \cap W^{1,m})(\mathcal{M},\mathcal{N})$*, such that* $\operatorname{tr}_{\partial\mathcal{M}} U = u$ *on* $\partial\mathcal{M}$*,*

(ii) u *is homotopic in* $W^{1-\frac{1}{p},p}(\partial\mathcal{M},\mathcal{N})$ *to the restriction of a smooth map.*

In the continuous case, this corresponds to the homotopy extension property: if $u \in C(\partial\mathcal{M},\mathcal{N})$ is homotopic to $V|_{\partial\mathcal{M}}$ for some $V \in C(\partial\mathcal{M},\mathcal{N})$, then $u = U|_{\partial\mathcal{M}}$ for some $U \in C(\mathcal{M},\mathcal{N})$ which is homotopic to V on the manifold $\mathcal{M}$.

Proof of proposition 2.11. We first apply proposition 2.9 or proposition 2.10 to extend u to a neighbourhood of $\partial\mathcal{M}$. Since this extension is smooth far from the boundary, we apply classical differential topology techniques to extend this map inside $\mathcal{M}$.

Conversely, if $u = \operatorname{tr}_{\partial M} U$, then there exists a sequence of maps $(U_\ell)_{\ell\in\mathbb{N}}$ in $C^\infty(\mathcal{M},\mathcal{N})$ converging to U in $W^{1,m}(\mathcal{M},\mathcal{N})$. By the continuity of the traces, the sequence $(\operatorname{tr}_{\partial\mathcal{M}} U_\ell)_{\ell\in\mathbb{N}}$ converges to u in $W^{1-\frac{1}{p},p}(\partial\mathcal{M},\mathcal{N})$ and thus by proposition 2.7, u is connected in $W^{1-\frac{1}{p},p}(\partial\mathcal{M},\mathcal{N})$ to the restriction of a smooth map $U_\ell \in C^\infty(\mathcal{M},\mathcal{N})$. □

Problem 2.8 (★★★). Prove proposition 2.9 on a compact manifold.

When $p > m$, it is possible to prove by a compactness argument that the extension of proposition 2.11 can be taken to be bounded on bounded sets [PVS17]. More explicit estimates should be connected to quantitative homotopy theory [?Gromov_1996, FW13].

2.4 Lifting

Definition 2.12. Let $\widetilde{\mathcal{N}}$ and $\mathcal{N}$ be Riemannian manifolds. The map $\pi : \widetilde{N} \to \mathcal{N}$ is a Riemannian covering whenever π is a local isometry.

In particular, if $\rho \triangleq \operatorname{inj}(\mathcal{N})$ denotes the injectivity radius of the manifold $\mathcal{N}$, then for every $\widetilde{y} \in \widetilde{\mathcal{N}}$ is an isometry on the geodesic ball $B_\rho(\widetilde{y})$. This means that for every $y \in \mathcal{N}$,

$$\pi^{-1}(B_\rho(y)) = \bigcup_{\widetilde{y}\in\pi^{-1}(y)} B_\rho(\widetilde{y}),$$

that is $\pi^{-1}(B_\rho(y))$ can be written as a union of open sets on which π is a homeomorphism.

It is a classical fact from homotopy theory that if the manifold $\mathcal{M}$ is simply connected then for every mapping $u \in C(\mathcal{M},\mathcal{N})$ there exists a mapping $\widetilde{u} \in C(\widetilde{\mathcal{M}},\mathcal{N})$ such that $u = \pi \circ \widetilde{u}$ [Hat02, proposition 1.33].

Proposition 2.13. *Let $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ be a covering map. If $\widetilde{\mathcal{N}}$ is either compact or a Euclidean space, if $sp \geq m$ and $\mathcal{M}$ is compact, if $\mathcal{M}$ simply connected and either $\mathcal{M}$ is compact or $s = 1$, then for every $u \in W^{s,p}(\mathcal{M},\mathcal{N})$, there exists $\widetilde{u} \in W^{s,p}(\mathcal{M},\mathcal{N})$ such that $u = \pi \circ \widetilde{u}$.*

The role of the assumption that $\widetilde{\mathcal{N}}$ is either compact or a Euclidean space is to have simply defined Sobolev spaces.

Proof of proposition 2.13 when $s = 1$ and $p > m$. By the Morrey–Sobolev embedding, we can assume that u is continuous. By the classical lifting theorem, there exists a map $\widetilde{u} \in C(\mathcal{M},\widetilde{\mathcal{N}})$ such that $\pi \circ \widetilde{u} = u$. Since $\widetilde{u}$ is continuous and since by definition π is local isometry, it follows that locally we have that $\widetilde{u}$ is weakly differentiable and

$$\mathcal{E}^{1,p}(\widetilde{u}) = \mathcal{E}^{1,p}(u).$$ □

Proof of proposition 2.13 when $s = 1$ and $sp > m$. By the fractional Morrey–Sobolev embedding, we can assume that the map u is continuous. Since $\mathcal{M}$ is simply-connected, by the classical lifting theorem, there exists a map $\widetilde{u} \in C(\mathcal{M},\widetilde{\mathcal{N}})$ such that $\pi \circ \widetilde{u} = u$.

By the Morrey–Sobolev embedding, there exists a constant $C_1 > 0$ such that if

$$r^{sp-m}\mathcal{E}^{s,p}(u) \leq C_1,$$

then $\operatorname{diam}(u(B_r(a))) \leq \operatorname{inj}(\mathcal{N})$ for every $a \in \mathcal{M}$. Hence, $\operatorname{diam}(\widetilde{u}(B_r(a))) \leq \operatorname{inj}(\mathcal{N})$ for every $a \in \mathcal{M}$ and for every $x, y \in \mathcal{M}$ such that $d_{\mathcal{M}}(y,x) \leq r$, we have $d_{\widetilde{\mathcal{N}}}(\widetilde{u}(y), \widetilde{u}(x)) = d_{\mathcal{N}}(u(y), u(x))$. We have thus

$$\iint\limits_{\substack{(x,y)\in\mathcal{M}^2\\ d_{\mathcal{M}}(y,x)\leq r}} \frac{d_{\mathcal{N}}(\widetilde{u}(y), \widetilde{u}(x))^p}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x = \iint\limits_{\substack{(x,y)\in\mathcal{M}^2\\ d_{\mathcal{M}}(x,y)\leq r}} \frac{d_{\mathcal{N}}(u(y), u(x))^p}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x.$$

It follows then that $u \in W^{s,p}(\mathcal{M}, \mathcal{N})$. □

Proof of proposition 2.13 when $s = 1$ and $p = m$. By theorem 2.1, there exists a sequence $(u_j)_{j\in\mathbb{N}}$ that approximates u in $W^{1,m}(\mathcal{M}, \mathcal{N})$. Since u_j is smooth, there exists $\widetilde{u}_j$ such that $\pi \circ \widetilde{u}_j = u_j$. If $\widetilde{\mathcal{N}}$ is a Euclidean space, then π is the universal covering of the compact manifold $\mathcal{N}$, and we can choose the u_j in such a way that

$$\left|\int_{\mathcal{M}} \widetilde{u}_j\right| \leq C_2.$$

As in the proof of the case $s = 1$ and $p > m$, we have

$$\mathcal{E}^{1,m}(\widetilde{u}_j) = \mathcal{E}^{1,m}(u_j).$$

Since $\mathcal{M}$ is compact, the sequence $\widetilde{u}$ converges then in measure to some $\widetilde{u}$. □

Proof of proposition 2.13 when $s \in (0,1)$ and $sp = m$. Since $u \in W^{s,p}(\mathcal{M}, \mathcal{N})$, there exists a map $V \in C^\infty(\mathcal{M} \times (0,1), \mathbb{R}^\nu)$ such that

$$\iint\limits_{\mathcal{M}\times[0,1]} \frac{|\mathrm{D}V(t,x)|^p}{t^{1-(1-s)p}} \,\mathrm{d}t\,\mathrm{d}x \leq \mathcal{E}^{s,p}(u),$$

$\operatorname{tr}_{\mathcal{M}\times\{0\}} V = u$ and $U(t,x)$ is obtained by integration of u on $B_t(x)$ [MR15; Maz11, theorem 10.1.1.1]. It follows from lemma 2.3 and lemma 2.4, that when $\tau > 0$ is small enough we have $V(\mathcal{M} \times [0,\tau]) \subset \mathcal{N} + B_{\delta_{\mathcal{N}}}(0)$. We define then $U : \mathcal{M} \times [0,\tau]$ by $V \triangleq \pi \circ V|_{\mathcal{M}\times[0,\tau]}$. We have

$$\iint\limits_{\mathcal{M}\times[0,\tau]} \frac{|\mathrm{D}U(t,x)|^p}{t^{1-(1-s)p}} \,\mathrm{d}t\,\mathrm{d}x \leq C_3 \iint\limits_{\mathcal{M}\times[0,\tau]} \frac{|\mathrm{D}V(t,x)|^p}{t^{1-(1-s)p}} \,\mathrm{d}t\,\mathrm{d}x$$

Since U is smooth on $\mathcal{M} \times (0,1)$ and $\mathcal{M} \times (0,1)$ is simply connected by the simple-connectedness of $\mathcal{M}$, there exists $\widetilde{U} \in C^\infty_c(\mathcal{M} \times (0,1), \widetilde{\mathcal{N}})$ such that $\pi \circ \widetilde{U} = U$ and we have

$$\iint\limits_{\mathcal{M}\times[0,\tau]} \frac{|\mathrm{D}\widetilde{U}(t,x)|^p}{t^{1-(1-s)p}} \,\mathrm{d}t\,\mathrm{d}x = \iint\limits_{\mathcal{M}\times[0,\tau]} \frac{|\mathrm{D}U(t,x)|^p}{t^{1-(1-s)p}} \,\mathrm{d}t\,\mathrm{d}x.$$

Moreover, by the Hölder inequality, we have for every compact subset $K \subset \mathcal{M}$,

$$\begin{aligned}\iint\limits_{K\times[0,\tau]} |\mathrm{D}\widetilde{U}(t,x)| \,\mathrm{d}t\,\mathrm{d}x \\ \le \Bigg(\iint\limits_{K\times[0,\tau]} \frac{|\mathrm{D}\widetilde{U}(t,x)|^p}{t^{1-(1-s)p}}\,\mathrm{d}t\,\mathrm{d}x\Bigg)^{\frac{1}{p}}\Bigg(\iint\limits_{K\times[0,\tau]} \frac{1}{t^{1-\frac{sp}{p-1}}}\,\mathrm{d}t\,\mathrm{d}x\Bigg)^{1-\frac{1}{p}} < +\infty.\end{aligned}$$

Hence for almost every $x \in \mathcal{M}$, we set $\widetilde{u}(x) = \lim_{t\to 0} \widetilde{U}(x,t)$. By classical weighted trace theorems, we have $\widetilde{u} = \operatorname{tr}_{\mathcal{M}\times\{0\}} \widetilde{U} \in W^{s,p}(\mathcal{M},\mathcal{N})$. □

Proposition 2.14. *Under the assumptions of proposition 2.13, there exists a countable family of liftings $(\widetilde{u}_{\widetilde{b}})_{\widetilde{b}\in\pi^{-1}(b)}$ in $W^{s,p}(\mathcal{M},\mathcal{N})$ such that for almost every $x \in \Omega$, $\pi^{-1}(\{u(x)\})$ is the disjoint union $\bigcup_{\widetilde{b}\in\pi^{-1}(b)}\{\widetilde{u}_{\widetilde{b}}(x)\}$.*

Proof. Let $U \in C^\infty(\mathcal{M}\times(0,1),\mathcal{N})$ be given as in the proof of proposition 2.13. Since $\mathcal{M}$ is connected and simply-connected, there exists a family $(\widetilde{U}_{\widetilde{b}})_{\widetilde{b}\in\pi^{-1}(b)}$ of smooth liftings such that for every $(x,t) \in \mathcal{M}\times(0,\tau)$, $\pi^{-1}(U(x,t))$ is the disjoint union $\{\widetilde{U}_{\widetilde{b}}(x,t) \mid \widetilde{b} \in \pi^{-1}(\{b\})\}$. We then set $\widetilde{u}_{\widetilde{b}}(x) = \lim_{t\to 0}\widetilde{U}_{\widetilde{b}}$. □

Proposition 2.15. *Assume that $sp \ge m$ and that $\mathcal{M}$ is connected. If $\widetilde{u},\widetilde{v} \in W^{s,p}(\mathcal{M},\mathcal{N})$, then either $\widetilde{u} = \widetilde{v}$ almost everywhere or $\widetilde{u} \ne \widetilde{v}$ almost everywhere.*

Proposition 2.15 is a particular case of proposition 3.9 below, that covers the case $sp \ge 1$. We give a direct proof in the spirit of the present section.

Proof of proposition 2.15. Since the covering map π is a local isometry and since the manifold $\mathcal{N}$ is compact, there exists $\delta > 0$ such that if $\pi(y) = \pi(z)$ and $d_{\mathcal{N}}(y,z) \le \delta$ implies $y = z$. We choose a function $\theta \in C^\infty([0,+\infty))$ such that $\theta(0) = 0$ and $\theta = 1$ on $[\delta,+\infty)$. We define $f = \theta(d_{\mathcal{N}}(u,v))$. By the chain rule, $f \in W^{s,p}(\mathcal{M},\{0,1\})$. By theorem 2.1 applied to the 0–dimensional Riemannian manifold $\{0,1\}$, f is the limit of constant functions. Since $\mathcal{M}$ is connected, we either have $f = 0$ almost everywhere on $\mathcal{M}$ or $f = 1$ almost everywhere on $\mathcal{M}$. □

Proposition 2.16. *Assume that $\Gamma \subset \mathcal{M}$ is a connected submanifold and* $\dim\Gamma = \ell$. *If $u \in W^{1,m}(\mathcal{M},\mathcal{N})$ and $\widetilde{v} \in W^{1-\frac{\ell}{m},m}(\Gamma,\widetilde{\mathcal{N}})$, and* $\operatorname{tr}_\Gamma u = \pi\circ\widetilde{v}$, *then there exists $\widetilde{u} \in W^{1,m}(\mathcal{M},\widetilde{\mathcal{N}})$ such that $u = \pi\circ\widetilde{u}$ and* $\operatorname{tr}_\Gamma \widetilde{u} = \widetilde{v}$ *on Γ.*

Proof. We consider the liftings given by proposition 2.14. We claim that there exists $\widetilde{u}_{\widetilde{b}}$ such that $\operatorname{tr}_\Gamma \widetilde{u}_{\widetilde{b}} = \widetilde{v}$.

Indeed, assume that $K \subset \mathcal{N}$ is compact and that π is a homeomorphism on every connected component of $\pi^{-1}(K)$. Then there exists an open set G such that $K \subset G \subset \mathcal{N}$ such that π is a homeomorphism on every connected component of $\pi^{-1}(K)$. Let $\varphi \in C^1_c$

be such that $\varphi > 0$ on G and $\varphi = 0$ on $\mathcal{M} \setminus G$. There exists a family $\varphi_{\widetilde{b}}$ in $C^1_c(\widetilde{\mathcal{N}})$ such that $\sum_{\widetilde{b} \in \pi^{-1}(\{b\})} \chi_{\widetilde{b}} = \chi \circ \pi$. We have then for every $\widetilde{d} \in \pi^{-1}(\{b\})$,

$$\sum_{\widetilde{b} \in \pi^{-1}(\{b\})} \chi_{\widetilde{d}} \circ \widetilde{u}_{\widetilde{b}} = \chi \circ u.$$

This convergence holds in $W^{s,p}(\mathcal{M},\mathcal{N})$. Indeed, if $s = 1$, we have for every subset $F \subseteq \pi^{-1}(\{b\})$ $|\mathrm{D} f_F| \le C_4 |\mathrm{D}\widetilde{u}_{\widetilde{d}}|$ almost everywhere, where $f_F = \sum_{\widetilde{b} \in F} \chi_{\widetilde{d}} \circ \widetilde{u}_{\widetilde{b}}$; if $0 < s < 1$, we have for every $x, y \in \mathcal{M}$, $|f_F(y) - f_F(x)| \le d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{\widetilde{d}}(y), \widetilde{u}_{\widetilde{d}}(y))$.

Thus we have

$$\sum_{\widetilde{b} \in \pi^{-1}(\{b\})} \chi_{\widetilde{d}} \circ \operatorname{tr}_\Gamma \widetilde{u}_{\widetilde{b}} = \chi \circ \operatorname{tr}_\Gamma u,$$

from which it follows that for almost every $x \in (\operatorname{tr}_\Gamma u)^{-1}(G)$, $\pi^{-1}(\operatorname{tr}_\Gamma u(x)) = \{\operatorname{tr}_\Gamma \widetilde{u}_{\widetilde{b}} \mid \widetilde{b} \in \pi^{-1}(\{b\})\}$. □

Problem 2.9. ★★ For $\mathcal{M} = \mathbb{B}^m$ and $\pi : \mathbb{R} \to \mathbb{S}^1$ defined by $\pi(t) = (\cos t, \sin t)$, show that when $sp = 0$ there exists a sequence of smooth maps $(\widetilde{u}_j)_{j \in \mathbb{N}}$ in $C^\infty(\mathbb{S}^m, \mathbb{R})$, such that

$$\mathcal{E}^{s,p}(\widetilde{u}_j) \to \infty$$

and

$$\limsup_{j \to \infty} \frac{\mathcal{E}^{s,p}(\widetilde{u}_j)}{\mathcal{E}^{s,p}(\pi \circ \widetilde{u}_j)^{\frac{1}{s}}} > 0.$$

2.5 Comments

Approximation

The fact that when $sp = m$, the observation that approximations by convolution are close to the target manifold goes back to the seminal work of Schoen and Uhlenbeck [SU82, §3; SU83, §4].

For the approximation problem, when the target manifold $\mathcal{N}$ is not assumed to be compact, then additional restrictions on the target manifold appear in theorem 2.1 in the critical case $sp = m$ [HS14, BPVS17].

Homotopy

The homotopy classes of Sobolev mappings have been well studied for $W^{s,p}(\mathbb{S}^m, \mathbb{S}^m)$, where they are related to the topological Brouwer degree. The degree of maps in $W^{1,2}(\mathbb{S}^2, \mathbb{S}^2)$ was studied by Brezis and Coron [BC83] (see also [Bre85]) in the context of harmonic maps. The degree was defined for maps in $W^{\frac{1}{2},2}(\mathbb{S}^1, \mathbb{S}^1)$ in the context of Ginzburg–Landau equations [BdMBGP91] (see also [BBH94]). The general theory of Brezis and Nirenberg treats all the critical and supercritical Sobolev maps as maps of vanishing mean oscillation (VMO), for which a degree can be defined [BN95, BN96].

Extension

The extensions in the supercritical case where obtained by Bethuel and Demengel [BD95, theorems 1 and 2].

Lifting

Bethuel and Zheng have proved the existence and uniqueness of liftings for maps $W^{1,2}(\mathbb{B}^2, \mathbb{R})$ [BZ88, lemma 4]. When $sp \ge m$, liftings have been constructed for maps in $W^{s,p}(\mathbb{B}^m, \mathbb{S}^1)$ [BBM00, §2] by relying on iterated extensions in Sobolev spaces; the same method was used to construct liftings for maps in $W^{s,p}(\mathbb{B}^m, \mathcal{N})$ when $sp \ge m$ by Bethuel and Chiron [BC07, theorem 3 ii)]. We present here the method developped by Petru Mironescu, Emmanuel Russ and Yannick Sire to cover the Besov spaces for subcritical dimensions [MRS].

3 Topological obstructions

3.1 Density of smooth maps

We consider the question of the density of the set $C^\infty(\mathcal{M},\mathcal{N})$ of smooth maps in the Sobolev space $W^{1,p}(\mathcal{M},\mathcal{N})$.

Proposition 3.1. *If $p < m$ and if the set $C^\infty(\mathcal{M},\mathcal{N})$ is dense in the space $W^{1,p}(\mathcal{M},\mathcal{N})$, then the restriction operator $F \in C(\mathbb{B}^{\lfloor p+1\rfloor},\mathcal{N}) \mapsto F|_{\mathbb{S}^{\lfloor p\rfloor}} \in C(\mathbb{S}^{\lfloor p\rfloor},\mathcal{N})$ is surjective.*

Here and in the sequel $\lfloor p\rfloor$ denotes the *integer part* of the real number $p \in \mathbb{R}$:

$$\lfloor p\rfloor = \sup\{k \in \mathbb{Z} \mid k \le p\};$$

it is characterized by the conditions that $\lfloor p\rfloor \in \mathbb{Z}$ and $\lfloor p\rfloor \le p < \lfloor p+1\rfloor$.

In the language of algebraic topology, the necessary condition for the density of smooth maps is that the $\lfloor p\rfloor$-th homotopy group $\pi_{\lfloor p\rfloor}(\mathcal{N})$ of the target manifold $\mathcal{N}$ is trivial: $\pi_{\lfloor p\rfloor}(\mathcal{N}) \simeq \{0\}$. This way of expressing the condition is a handy shortcut, but might suggest the interaction with the group structure which does not appear in the long run and obfuscate the more fundamental role of obstruction theory.

The necessary condition for density of proposition 3.1 only involves the target manifold $\mathcal{N}$. It will appear later that in general the condition does involve extensions from general $\lfloor p\rfloor$–dimensional skeletons contained in the domain manifold $\mathcal{M}$ (see **??** below); when the topology domain manifold $\mathcal{M}$ is simple enough, it turns out that there is no additional obstruction to the density of smooth maps.

Lemma 3.2. *Let $m \in \mathbb{N}$, $p \in [1,+\infty)$ and $u : \mathbb{B}^m \to \mathbb{S}^{m-1}$ be defined for each $x \in \mathbb{B}^m \setminus \{0\}$ by*

$$u(x) = \frac{x}{|x|}.$$

Then $u \in W^{1,p}(\mathbb{B}^m,\mathbb{S}^{m-1})$ if and only if $p < m$.

Proof. Since when $m = 1$, the map u is discontinuous and cannot be weakly differentiable, we assume henceforth that $m \ge 2$.

Since the map u is smooth in $\mathbb{B}^m \setminus \{0\}$, u is weakly differentiable on $\mathbb{B}^m \setminus \{0\}$ and for every $x \in \mathbb{B}^m \setminus \{0\}$ and $h \in \mathbb{R}^m$,

$$\mathrm{D}u(x)[h] = \frac{h}{|x|} - \frac{x(x\cdot h)}{|x|^3}.$$

In particular, we have for every $x \in \mathbb{B}^m \setminus \{0\}$,

$$|\mathrm{D}u(x)| = \sqrt{\operatorname{tr}(\mathrm{D}u(x)^* \circ \mathrm{D}u(x))} = \frac{\sqrt{m-1}}{|x|}.$$

(we are using the canonical Euclidean norm on linear mappings, also known as Frobenius norm or as Hilbert–Schmidt norm). We then have

$$\int_{\mathbb{B}^m \setminus \{0\}} |\mathrm{D}u|^p = \int_{\mathbb{B}^m} \frac{(m-1)^{\frac{p}{2}}}{|x|^p} \,\mathrm{d}x,$$

which is finite if and only $p < m$. This condition is then necessary and sufficient to have $u|_{\mathbb{B}^m \setminus \{0\}} \in W^{1,p}(\mathbb{B}^m \setminus \{0\}, \mathbb{S}^{m-1})$. By proposition 1.1, this is equivalent to have $u \in W^{1,p}(\mathbb{B}^m, \mathbb{S}^{m-1})$. □

Proof of proposition 3.1 when $\mathcal{M} = \mathbb{B}^m$ and $m-1 \le p < m$. Let $f \in C(\mathbb{S}^{m-1}, \mathcal{N})$. By standard approximation of continuous maps by smooth maps, the map f is homotopic to some smooth map $\breve{f} \in C^\infty(\mathbb{S}^{m-1}, \mathcal{N})$.

We define the map $u : \mathbb{B}^m \to \mathcal{N}$ for each $x \in \mathbb{B}^m \setminus \{0\}$ by

$$u(x) \triangleq \breve{f}\Big(\frac{x}{|x|}\Big). \tag{3.1}$$

By lemma 3.2, the smoothness of $\breve{f}$ and by the chain rule for Sobolev mappings, we have $u \in W^{1,p}(\mathbb{B}^m, \mathcal{N})$ for every $p \in [1, m)$.

We assume now that the sequence $(u_j)_{j \in \mathbb{N}}$ of maps in $C^\infty(\mathbb{B}^m, \mathcal{N})$ converges strongly to the map u in the Sobolev space $W^{1,p}(\mathbb{B}^m, \mathcal{N})$. By integration in spherical coordinates and Fubini's theorem (or by the coarea formula), we have for every $j \in \mathbb{N}$

$$\int_{\mathbb{B}^m} |\mathrm{D}u_j - \mathrm{D}u|^p + |u_j - u|^p \,\mathrm{d}\mathcal{L}^m = \int_0^1 \int_{\mathbb{S}^{m-1}_r} |\mathrm{D}u_j - \mathrm{D}u|^p + |u_j - u|^p \,\mathrm{d}\mathcal{H}^{m-1} \,\mathrm{d}r.$$

In particular, up to a subsequence, for almost every $r \in (0,1)$, we have

$$\lim_{j \to \infty} \int_{\mathbb{S}^{m-1}_r} |\mathrm{D}u_j - \mathrm{D}u|^p + |u_j - u|^p \,\mathrm{d}\mathcal{H}^{m-1} = 0.$$

If $p \ge m-1$, this implies by propositions 2.7 and 2.8 that when $j \in \mathbb{N}$ is large enough the maps $u|_{\mathbb{S}^{m-1}_r}$ and $u_j|_{\mathbb{S}^{m-1}_r}$ are homotopic as continous maps from $\mathbb{S}^{m-1}_r$ to $\mathcal{N}$. By assumption, the map $u_j|_{\mathbb{S}^{m-1}_r}$ has an extension to $\mathbb{B}^{m-1}_r$. By construction, this means that $\breve{f}$ is homotopic to the restriction of a continous map from $\mathbb{B}^m$ to $\mathcal{N}$. By transitivity of homotopies, the map f is also homotopic to this restriction. In view of the homotopy extension property, the map f is the restriction of a continuous map from $\mathbb{B}^m$ to $\mathcal{N}$. □

Remark 3.3. When $\mathcal{M} = \mathbb{B}^m$ and $m-1 \le p < m$, the topological obstruction already arises for the approximation of maps in $C^\infty(\mathbb{B}^m \setminus \{0\}, \mathcal{N}) \cap W^{1,p}(\mathbb{B}^m, \mathcal{N})$.

The proof of proposition 3.1 gives a formula (3.1) to construct a Sobolev map that cannot be approximated by smooth maps.

Proof of proposition 3.1 when $\mathcal{M} = \mathbb{S}^m$ and $m - 1 \le p < m$. We proceed as in the case $\mathcal{M} = \mathbb{S}^m$ and $m - 1 \le p < m$, except that we define the map $u : \mathbb{S}^m \to \mathcal{N}$ for each $(y, z) \in \mathbb{S}^m \subset \mathbb{R}^m \times \mathbb{R}$ by

$$u(y, z) = \check{f}\Big(\frac{y}{|y|}\Big).$$

The proof continues as previously. □

Remark 3.4. When $\mathcal{M} = \mathbb{B}^m$ and $m - 1 \le p < m$, the topological obstruction already arises for the approximation of maps in $C^\infty(\mathbb{S}^m \setminus \{e_{m+1}, -e_{m+1}\}, \mathcal{N})$, where $e_{m+1} = (0, \dots, 0, 1) \in \mathbb{S}^m \subset \mathbb{R}^{m+1}$ is the $(m+1)$–th vector in the canonical basis of $\mathbb{R}^{m+1}$. Equivalently, the map is smooth except at the south and north poles; such a map is called a *dipole*.

Proof of proposition 3.1 in the general case. We choose a point $b \in \mathbb{S}^{\lfloor p+1 \rfloor}$ and let $\hat{u}$ be the function constructed in the proof on the sphere $\mathbb{S}^{\lfloor p+1 \rfloor}$. We consider a mapping $\Psi : \mathcal{M} \to \mathbb{S}^{\lfloor p+1 \rfloor} \times \mathbb{S}^{m - \lfloor p+1 \rfloor}$ such that Ψ is a diffeomorphism on $\Psi^{-1}((\mathbb{S}^{\lfloor p+1 \rfloor} \setminus \{b\}) \times \mathbb{S}^{m - \lfloor p+1 \rfloor})$. We then define $u = \hat{u} \circ P \circ \Psi$, where $P : \mathbb{S}^{\lfloor p+1 \rfloor} \times \mathbb{S}^{m - \lfloor p+1 \rfloor} \to \mathbb{S}^{\lfloor p+1 \rfloor}$ is the canonical projection defined for each $(y, z) \in \mathbb{S}^{\lfloor p \rfloor} \times \mathbb{S}^{m - \lfloor p \rfloor}$ by $P(y, z) = y$. Since the map $\hat{u}$ is smooth in a neighbourhood of b, it follows that $u \in W^{1,p}(\mathcal{M}, \mathcal{N})$. By Fubini's theorem, if u can be approximated by smooth maps, then . □

Remark 3.5. The proof shows that the map is constant outside a solid torus $\mathbb{B}^{\lfloor p+1 \rfloor} \times \mathbb{S}^{m - \lfloor p+1 \rfloor}$; the map constructed when the topological condition is not satisfied is smooth outside two sets diffeomorphic to two spheres $\mathbb{S}^{m - \lfloor p+1 \rfloor}$. When $\lfloor p + 1 \rfloor = m$, the two corresponding points form a *dipole*.

Problem 3.1 (★★). Write a detailed proof of proposition 3.1 when $\mathcal{M}$ is a general manifold.

Problem 3.2 (★). Explain where it is used in the proof of proposition 3.1 that $p < m$.

Problem 3.3 (★★★). State and write proposition 3.1 for $W^{s,p}(\mathcal{M}, \mathcal{N})$ when $s \in (0, 1)$, $p \in [1, +\infty)$ and $sp < m$.

3.2 Extension from the boundary

We consider now the question of the extension of Sobolev mappings, for which we have a similar restriction.

Theorem 3.6. *Let $1 \le p < m$. If every map $u \in W^{1-1/p,p}(\partial\mathcal{M}, \mathcal{N})$ is the trace of some map $U \in W^{1-1/p,p}(\mathcal{M}, \mathcal{N})$, then the restriction operator $F \in C(\mathbb{B}^{\lfloor p \rfloor}, \mathcal{N}) \mapsto F|_{\mathbb{S}^{\lfloor p-1 \rfloor}} \in C(\mathbb{S}^{\lfloor p-1 \rfloor}, \mathcal{N})$.*

Equivalently, if the trace operator is onto then the homotopy group $\pi_{\lfloor p-1 \rfloor}(\mathcal{N})$ is trivial. When $p \in [1,2)$, the necessary condition is that the target manifold $\mathcal{N}$ should be connected.

We will see later that other obstructions appear on lower-dimensional spheres as well (see chapters 5 and 7 below); this other obstructions are appear for quantitative reasons rather than qualitative reasons presented here.

The proof of theorem 3.6 relies on the fractional counterpart of lemma 3.2.

Lemma 3.7. *Let $m \in \mathbb{N}$, $s \in (0,1)$, $p \in [1,+\infty)$ and $u : \mathbb{B}^m \to \mathbb{S}^{m-1}$ be defined for each $x \in \mathbb{B}^m \setminus \{0\}$ by*

$$u(x) = \frac{x}{|x|}.$$

Then $u \in W^{s,p}(\mathbb{B}^m, \mathbb{S}^{m-1})$ if and only if $sp < m$.

In view of the fractional Gagliardo–Nirenberg interpolation inequality proposition 1.9, lemma 3.2 implies the sufficiency part of lemma 3.7 when $sp \in (1,m)$ and hence, since the domain $\mathbb{B}^m$ is bounded, for $sp \in (0,m)$. We give a direct proof of lemma 3.7.

Proof of lemma 3.2. We have

$$\mathcal{E}^{s,p}(u) = \iint\limits_{\mathbb{B}^m \times \mathbb{B}^m} \frac{|u(y) - u(x)|^p}{|y-x|^{m+sp}} \,\mathrm{d}x \,\mathrm{d}y = 2 \iint\limits_{\substack{(x,y) \in \mathbb{B}^m \times \mathbb{B}^m \\ |x| \le |y|}} \frac{|u(y) - u(x)|^p}{|y-x|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x. \tag{3.2}$$

For every $x, y \in \mathbb{B}^m \setminus \{0\}$, we have by the triangle inequality

$$|u(y) - u(x)| \le |u(y)| + |u(x)| = 2; \tag{3.3}$$

if moreover $|x| \le |y|$, then we have

$$\begin{aligned} |u(y) - u(x)| &= \left| \frac{y}{|y|} - \frac{x}{|x|} \right| = \frac{\big| y|x| - x|y| \big|}{|y||x|} \\ &\le \frac{\big| y|x| - x|x| \big|}{|y||x|} + \frac{\big| x|x| - x|y| \big|}{|y||x|} = \frac{|y-x|}{|x|} + \frac{|y-x|}{|y|} \le 2 \frac{|y-x|}{|x|}. \end{aligned} \tag{3.4}$$

We estimate then by (3.3)

$$\begin{aligned} \iint\limits_{\substack{(x,y) \in \mathbb{B}^m \times \mathbb{B}^m \\ |x| \le |y| \\ |y-x| \ge |x|}} \frac{|u(y) - u(x)|^p}{|y-x|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x &\le \iint\limits_{\substack{(x,y) \in \mathbb{B}^m \times \mathbb{B}^m \\ |x| \le |y| \\ |y-x| \ge |x|}} \frac{2^p}{|y-x|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x \\ &\le \iint\limits_{\substack{(x,z) \in \mathbb{B}^m \times \mathbb{R}^m \\ |z| \ge |x|}} \frac{2^p}{|z|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x \\ &\le C_1 \int_{\mathbb{B}^m} \frac{1}{|x|^{sp}} \,\mathrm{d}x < +\infty, \end{aligned} \tag{3.5}$$

since we have assumed $sp < m$. On the other hand, we have by (3.4)

$$\begin{aligned}\iint\limits_{\substack{(x,y)\in\mathbb{B}^m\times\mathbb{B}^m\\|x|\le|y|\\|y-x|\le|x|}} \frac{|u(y)-u(x)|^p}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x &\le \iint\limits_{\substack{(x,y)\in\mathbb{B}^m\times\mathbb{B}^m\\|y-x|\le|x|}} \frac{2^p}{|x|^p|y-x|^{m-(1-s)p}}\,\mathrm{d}y\,\mathrm{d}x\\ &\le \iint\limits_{\substack{(x,z)\in\mathbb{B}^m\times\mathbb{R}^m\\|z|\le|x|}} \frac{2^p}{|x|^p|z|^{m-(1-s)p}}\,\mathrm{d}z\,\mathrm{d}x\\ &\le C_2\int_{\mathbb{B}^m}\frac{1}{|x|^{sp}}\,\mathrm{d}x < +\infty,\end{aligned} \tag{3.6}$$

since $s < 1$ and $sp < m$. By (3.2), (3.5) and (3.6) we conclude that $u \in W^{s,p}(\mathbb{B}^m, \mathbb{R}^m)$.

For the other direction, we have for every $r \in (0,1)$, since u is homogeneous of degree 0 and since $u(x) \neq u(y)$ for almost every $x, y \in \mathbb{B}^m \times \mathbb{B}^m$,

$$\begin{aligned}\iint\limits_{\mathbb{B}^m\times\mathbb{B}^m} \frac{|u(y)-u(x)|^p}{|y-x|^{m+sp}}\,\mathrm{d}x\,\mathrm{d}y &> \iint\limits_{\mathbb{B}^m_r\times\mathbb{B}^m_r} \frac{|u(y)-u(x)|^p}{|y-x|^{m+sp}}\,\mathrm{d}x\,\mathrm{d}y\\ &= r^{m-sp}\iint\limits_{\mathbb{B}^m\times\mathbb{B}^m} \frac{|u(ry)-u(rx)|^p}{|y-x|^{m+sp}}\,\mathrm{d}x\,\mathrm{d}y\\ &= r^{m-sp}\iint\limits_{\mathbb{B}^m\times\mathbb{B}^m} \frac{|u(y)-u(x)|^p}{|y-x|^{m+sp}}\,\mathrm{d}x\,\mathrm{d}y,\end{aligned} \tag{3.7}$$

This implies that $1 > r^{m-sp}$ for every $r \in (0,1)$ and thus $sp < m$. □

Proof of theorem 3.6 when $\mathcal{M} = \mathbb{B}^{m-1} \times [0,1)$ and $m-1 \le p < m$. Let $f \in C(\mathbb{S}^{m-2}, \mathcal{N})$. By classical approximation, f is homotopic to some map $\check{f} \in C^\infty(\mathbb{S}^{m-2}, \mathcal{N})$. We define the map $u : \mathbb{B}^{m-1} \to \mathcal{N}$ for $x \in \mathbb{B}^{m-1} \setminus \{0\}$ by

$$u(x) = \check{f}\Big(\frac{x}{|x|}\Big). \tag{3.8}$$

Since $p < m$, by lemma 3.7 and the composition of fractional Sobolev maps by Lipschitz continuous maps, we have $u \in W^{1-\frac{1}{p},p}(\mathbb{B}^{m-1}, \mathcal{N})$.

We assume now that $u = \operatorname{tr}_{\mathbb{B}^{m-1}\times\{0\}} U$ on $\mathbb{B}^{m-1} \times \{0\}$. By density of smooth maps in $W^{1,p}(\mathbb{B}^{m-1} \times [0,1), \mathbb{R}^\nu)$, there exists a sequence of smooth maps $(V_j)_{j\in\mathbb{N}}$ in $C^\infty(\bar{\mathbb{B}}^{m-1} \times [0,1], \mathbb{R}^\nu)$ such that

$$\int_{\mathbb{B}^{m-1}\times[0,1)} |\mathrm{D}V_j - \mathrm{D}U|^p + |V_j - U|^p \le 2^{-j}.$$

By spherical integration and by Fubini's theorem, we have for every $j \in \mathbb{N}$,

$$\int_0^1 \int_{\mathbb{S}_r^{m-1} \cap \mathbb{R}_+^m} |DV_j - DU|^p + |V_j - U|^p \, d\mathcal{H}^{m-1} \\ \le \int_{\mathbb{B}^{m-1} \times [0,1)} |DV_j - DU|^p + |V_j - U|^p \le 2^{-j}.$$

By Lebesgue's monotone convergence theorem, it follows thus that for almost every $r \in (0,1)$,

$$\lim_{j \to 0} \int_{[0,1]} \int_{\mathbb{S}_r^{m-1} \cap \mathbb{R}_+^m} |DV_j - DU|^p + |V_j - U|^p \, d\mathcal{H}^{m-1} = 0.$$

Since by assumption $\operatorname{tr}_{\mathbb{B}^{m-1} \times \{0\}} U = u$, we have by the classical trace theory that $(V_j|_{\mathbb{B}^{m-1} \times \{0\}})_{j \in \mathbb{N}}$ converges to u in $L^p(\mathbb{B}^{m-1}, \mathbb{R}^\nu)$ and thus for almost every $r \in (0,1)$, $\operatorname{tr}_{\mathbb{S}_r^{m-1} \times 0} U|_{\mathbb{S}_r^m \cap \mathbb{R}_+^m} = u|_{\mathbb{S}_r^{m-1} \times 0}$.

By proposition 2.11, it follows then that $u|_{\mathbb{S}_r^{m-2} \times \{0\}}$ is homotopic in $W^{1-1/p,p}(\mathbb{S}_r^{m-2} \times \{0\}, \mathcal{N})$ to the restriction of a smooth map. By proposition 2.8, this implies that $\check{f}$ is homotopic in $C(\mathbb{S}^{m-2}, \mathcal{N})$ to the restriction of a smooth function from $\bar{\mathbb{B}}^{m-1}$ to $\mathcal{N}$ and thus by transitivity of homotopies and by the homotopy extension property the map $f \in C(\mathbb{S}^{m-2}, \mathcal{N})$ can be written as $F|_{\mathbb{S}^{m-2}}$ for some $F \in C(\mathbb{B}^{m-1}, \mathcal{N})$. □

3.3 Lifting

We now describe local obstructions to lifting.

Proposition 3.8. *Let $\pi : \widetilde{\mathcal{M}} \to \mathcal{M}$ be a Riemannian covering. If $1 \le sp < m = \dim \mathcal{M}$ and every map $u \in W^{s,p}(\mathcal{M}, \mathcal{N})$ has a lifting, then every map $f \in C(\mathbb{S}^1, \mathcal{N})$ has a lifting.*

In many cases, is known that there exists maps $f \in C(\mathbb{S}^1, \mathcal{N})$ that have no lifting: $\pi : \mathbb{R}^1 \to \mathbb{S}^1$, $\pi : \mathbb{R}P^n \to \mathbb{S}^n$, $\pi : U(2) \to SO(3)$.

In order to establish an obstruction it is useful to know that the lifting is essentially unique.

Proposition 3.9. *Assume that $\mathcal{M}$ is connected, that $s \in (0,1]$, $p \in [1, +\infty)$ and that $sp \ge 1$. For every $\widetilde{u}, \widetilde{v} \in W^{s,p}(\mathcal{M}, \widetilde{\mathcal{N}})$, if $\pi \circ \widetilde{u} = \pi \circ \widetilde{v}$ almost everywhere in $\mathcal{M}$, then either $\widetilde{u} = \widetilde{v}$ almost everywhere in $\mathcal{M}$ or $\widetilde{u} \ne \widetilde{v}$ almost everywhere in $\mathcal{M}$.*

Proof of proposition 3.9 for $s = 1$. Since π is a local isometry and $\mathcal{N}$ is compact, there exists $\delta > 0$ such that if $\pi(y) = \pi(z)$ and $d_{\mathcal{N}}(y,z) \le \delta$ implies $y = z$.

We define the function $f : \mathcal{M} \to \mathbb{R}$ for $x \in \mathcal{M}$ by $f(x) = d_{\mathcal{N}}(u(x), v(x))$. By composition of Lipschitz maps with Sobolev mappings, $f \in W^{1,p}(\mathcal{M})$. We observe that for every $x \in \mathcal{M}$, $f(x) \in \{0\} \cap [\delta, +\infty)$. We choose a function $\theta \in C^\infty([0, +\infty))$ such that

$\theta(0) = 0$ on $[0, \delta/3]$ and $\theta = 1$ on $[2\delta/3, +\infty)$. By the chain rule for Sobolev functions, $\theta \circ f \in W^{1,p}(\mathcal{M}, \{0,1\})$ and for almost every $x \in \mathcal{M}$,

$$|\mathrm{D}(\theta \circ f)(x)| = |\theta'(f(x))||\mathrm{D}f(x)| = 0.$$

Since the manifold $\mathcal{M}$ is connected, the function $\theta \circ f$ is almost everywhere equal to a constant, and thus either $\theta \circ f = 0$ almost everywhere on $\mathcal{M}$ and then $\widetilde{u} = \widetilde{v}$ almost everywhere on $\mathcal{M}$, or $\theta \circ f = 1$ almost everywhere on $\mathcal{M}$ and then $\widetilde{u} \neq \widetilde{v}$ almost everywhere on $\mathcal{M}$. □

The proof of proposition 3.9 in the fractional case relies on the following property:

Lemma 3.10. *Let $\Omega \subset \mathbb{R}^m$ be convex and $A \subset \Omega$ be measurable. If*

$$\int_A \int_{\Omega \setminus A} \frac{1}{|y - x|^{m+1}} \,\mathrm{d}y \,\mathrm{d}x < +\infty,$$

then either $\mathcal{L}^m(A) = 0$ or $\mathcal{L}^m(\Omega \setminus A) = 0$.

Proof. We have by addidivity of the integral and by the change of variables $z = \frac{x+y}{2}$.

$$\begin{aligned}
&\iint\limits_{(x,y) \in A \times \Omega \setminus A} \frac{1}{|y - x|^{m+1}} \,\mathrm{d}y \,\mathrm{d}x \\
&\quad = \iint\limits_{\substack{(x,y) \in A \times (\Omega \setminus A) \\ \frac{x+y}{2} \in \Omega \setminus A}} \frac{1}{|y - x|^{m+1}} \,\mathrm{d}y \,\mathrm{d}x + \iint\limits_{\substack{(x,y) \in \Omega \times (\Omega \setminus A) \\ \frac{x+y}{2} \in A}} \frac{1}{|y - x|^{m+1}} \,\mathrm{d}y \,\mathrm{d}x \\
&\quad = \frac{1}{2} \iint\limits_{\substack{(x,z) \in A \times (\Omega \setminus A) \\ 2z - x \in \Omega \setminus A}} \frac{1}{|z - x|^{m+1}} \,\mathrm{d}z \,\mathrm{d}x + \frac{1}{2} \iint\limits_{\substack{(z,y) \in \Omega \times (\Omega \setminus A) \\ 2z - y \in A}} \frac{1}{|y - z|^{m+1}} \,\mathrm{d}y \,\mathrm{d}z \\
&\quad = \frac{1}{2} \iint\limits_{\substack{(x,y) \in A \times \Omega \setminus A \\ 2y - x \in \Omega \setminus A}} \frac{1}{|y - x|^{m+1}} \,\mathrm{d}y \,\mathrm{d}x + \frac{1}{2} \iint\limits_{\substack{(x,y) \in A \times (\Omega \setminus A) \\ 2x - y \in A}} \frac{1}{|y - x|^{m+1}} \,\mathrm{d}y \,\mathrm{d}x,
\end{aligned}$$

Since the integrals are finite, this implies that for almost every $(x, y) \in A \times \Omega \setminus A$, we have $2y - x \in \Omega \setminus A$ and $2x - y \in A$. Assume now that $\mathcal{L}^m(A) > 0$. Then, one has for almost every $y \in \Omega \setminus A$, $2y - x \in \Omega \setminus A$ and hence $\mathcal{L}^m(\Omega \setminus A \setminus (\frac{1}{2}((\Omega \setminus A) + x)))$, which implies that

$$\mathcal{L}^m(\Omega \setminus A) \leq 2^{-m} \mathcal{L}^m(\Omega \setminus A)$$

and thus $\mathcal{L}^m(\Omega \setminus A) = 0$. □

Remark 3.11. If we assume for some $\alpha \in (0, +\infty)$ that

$$\int_A \int_{\Omega \setminus A} \frac{1}{|y - x|^{m+\alpha}} \,\mathrm{d}y \,\mathrm{d}x < +\infty \tag{3.9}$$

and if $\alpha \geq 1$ we reach the same conclusion. When $\alpha > 1$, this can be proved by noting that by boundedness of the set Ω, $|y - x|^{m+\alpha} \leq \operatorname{diam}(\Omega)^{\alpha-1}|y - x|$ or by noting that the same change of variables, results in the inequality

$$\iint\limits_{(x,y)\in A\times\Omega\setminus A} \frac{1}{|y - x|^{m+1}} \,\mathrm{d}y \,\mathrm{d}x \leq \frac{1}{2^{\alpha-1}} \iint\limits_{(x,y)\in A\times\Omega\setminus A} \frac{1}{|y - x|^{m+1}} \,\mathrm{d}y \,\mathrm{d}x,$$

which implies immediately that the integral should be 0. When $\alpha < 1$, (3.9) holds for any set $A \subset \Omega$ that has a smooth boundary.

Proof of proposition 3.9 for $s \in (0, 1)$. We define the set

$$A \triangleq \{x \in \mathcal{M} \mid \widetilde{u}(x) = \widetilde{v}(x)\}.$$

We observe that for every $x, y \in \mathcal{M}$, we have by the triangle inequality,

$$d_{\widetilde{\mathcal{N}}}(\widetilde{u}(y), \widetilde{v}(y)) \leq d_{\widetilde{\mathcal{N}}}(\widetilde{u}(y), \widetilde{u}(x)) + d_{\widetilde{\mathcal{N}}}(\widetilde{u}(x), \widetilde{v}(x)) + d_{\widetilde{\mathcal{N}}}(\widetilde{v}(x), \widetilde{v}(y))$$

and thus, if if $x \in A$ and $y \in \mathcal{M} \setminus A$ then $d_{\widetilde{\mathcal{N}}}(\widetilde{u}(y), \widetilde{v}(y)) \geq \rho$ and $\widetilde{u}(x) = \widetilde{v}(x)$ and thus

$$\rho^p \leq 2^{p-1}\big(d_{\widetilde{\mathcal{N}}}(\widetilde{u}(y), \widetilde{u}(x))^p + d_{\widetilde{\mathcal{N}}}(\widetilde{v}(x), \widetilde{v}(y))^p\big).$$

If $s \in (0, 1)$, we have then by definition of the Gagliardo energy

$$\begin{aligned}\int_A \int_{\mathcal{M}\setminus A} &\frac{1}{d_{\mathcal{M}}(y, x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x \\ &\leq \frac{2^{p-1}}{\rho^p}\left(\iint\limits_{\mathcal{M}\times\mathcal{M}} \frac{d_{\widetilde{\mathcal{N}}}(\widetilde{u}(y), \widetilde{u}(x))^p}{d_{\mathcal{M}}(y, x)^{m+sp}} \,\mathrm{d}x \,\mathrm{d}y + \iint\limits_{\mathcal{M}\times\mathcal{M}} \frac{d_{\widetilde{\mathcal{N}}}(\widetilde{v}(y), \widetilde{v}(x))^p}{d_{\mathcal{M}}(y, x)^{m+sp}} \,\mathrm{d}x \,\mathrm{d}y\right) \\ &< +\infty.\end{aligned}$$

Therefore by lemma 3.10, since $sp \geq 1$, for every $B_r(a) \subset \mathcal{M}$, either $\widetilde{u} = \widetilde{v}$ almost everywhere in $B_r(a)$ or $\widetilde{u} \neq \widetilde{v}$ almost everywhere in $B_r(a)$. □

When $sp < 1$, the uniqueness of the lifting fails.

Proposition 3.12. *If $sp < 1$ and $\widetilde{M}$ is connected and $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ is not injective, there exists a map $\widetilde{u} \in W^{s,p}(\mathcal{M}, \widetilde{\mathcal{N}})$ which is not constant and such that $\pi \circ \widetilde{u}$ is not constant.*

Proof of proposition 3.12 when $M = \mathbb{B}^m$. By assumption, there exist $\widetilde{a}, \widetilde{b} \in \widetilde{\mathcal{N}}$ such that $\pi(\widetilde{a}) = \pi(\widetilde{b})$. We define

$$\widetilde{u}(x) = \begin{cases} \widetilde{a} & \text{if } |x| \leq \frac{1}{2}, \\ \widetilde{b} & \text{if } \frac{1}{2} < |x| < 1. \end{cases}$$

We have

$$\iint_{\mathbb{B}^m\times\mathbb{B}^m} \frac{d_{\widetilde{\mathcal{N}}}(\widetilde{u}(y), \widetilde{u}(x))^p}{|y - x|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x \leq 2d_{\widetilde{\mathcal{N}}}(\widetilde{b}, \widetilde{b})^p \int_{\mathbb{B}^m_{1/2}} \int_{\mathbb{B}^m\setminus\mathbb{B}^m_{1/2}} \frac{1}{|y - x|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x.$$

We estimate

$$\int_{\mathbb{B}^m_{1/2}} \int_{\mathbb{B}^m \setminus \mathbb{B}^m_{1/2}} \frac{1}{|y-x|^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x \le \int_{\mathbb{B}^m_{1/2}} \int_{\mathbb{R}^m \setminus \mathbb{B}^m_{1/2-|x|}(x)} \frac{1}{|y-x|^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x$$
$$\le C_3 \int_{\mathbb{B}^m_{1/2}} \frac{1}{(1-2|x|)^{sp}} \,\mathrm{d}x < +\infty,$$

since $0 < sp < 1$. □

Lemma 3.13. *If $\widetilde{u} \in W^{s,p}(\mathcal{M}, \widetilde{\mathcal{N}})$ and if $\pi \circ u \in C(\mathcal{M}, \mathcal{N})$, then $\widetilde{u}$ is almost everywhere equal to a continuous function $\widetilde{u}' \in C(\mathcal{M}, \widetilde{\mathcal{N}})$.*

Proof of lemma 3.13. Since continuity is local, it is sufficient to prove that $\widetilde{u}$ is almost everywhere equal to a continuous map at a neighbourhood of any point.

Let $a \in \mathcal{M}$. Since $\pi \circ \widetilde{u}$ is continuous, there exists $r > 0$ such that $\pi(\widetilde{u}(B_r(a))) \subset B_\rho(\pi(\widetilde{u}(a)))$. Since $B_r(a)$ is connected, there exists a countable family of continuous maps $(\widetilde{u}_{\widetilde{b}})_{\widetilde{b} \in \pi^{-1}(\{b\})}$ such that $\pi \circ \widetilde{u}_{\widetilde{b}} = \pi \circ \widetilde{u}$ and for every $x \in B_r(a)$, $\pi^{-1}(\pi(\widetilde{u}(x)) = \{\widetilde{u}_{\widetilde{b}}(x) \mid \widetilde{b} \in \pi^{-1}(\{b\})\}$. Since π is a local isometry, $\widetilde{u}_{\widetilde{b}}|_{B_r(a)} \in W^{s,p}(B_r(a), \widetilde{N})$ for every $\widetilde{b} \in \pi^{-1}(\{b\})$. Hence by proposition 3.9, there exists $\widetilde{b} \in \pi^{-1}(\{b\})$ such that $\widetilde{u} = \widetilde{u}_{\widetilde{b}}$ almost everywhere in the ball $B_r(a)$ and the conclusion follows. □

Proof of proposition 3.8 when $\mathcal{M} = \mathbb{B}^2$. Let $f \in C(\mathbb{S}^1, \mathcal{N})$. By standard approximation, f is homotopic to some $\check{f} \in C^\infty(\mathbb{S}^1, \mathcal{N})$. We define for each $x \in \mathbb{B}^2 \setminus \{0\}$,

$$u(x) = \check{f}\big(x/|x|\big).$$

Since $sp < 2$, we have $u \in W^{s,p}(\mathbb{B}^2, \mathcal{N})$ (lemma 3.2 if $s = 1$ and lemma 3.7 if $0 < s < 1$). If $u = \pi \circ \widetilde{u}$, then by lemma 3.13, we have $\widetilde{u} \in C(\mathbb{B}^2 \setminus \{0\}, \mathcal{N})$ and hence the map $\check{f}$ has a continuous lifting and thus by the homotopy lifting property, the map f also has a continuous lifting. □

3.4 What abouth the homotopy problem?

We end this chapter, with a brief explanation about why we do not consider the homotopy problem.

Proposition 3.14. *For every $s \in (0,1]$, $p \in [1,+\infty)$ and $\mathcal{N}$, every map $u \in W^{s,p}(\mathbb{B}, \mathcal{N})$ is homotopic in $W^{s,p}(\mathbb{B}, \mathcal{N})$ to a constant map.*

Proof. Let $a \in \mathbb{B}^m$ be a Lebesgue point of the map u. We define then for every $t \in [0,1]$ and $x \in \mathbb{B}^m$,

$$H(t)(x) = u((1-t)x + ta).$$

We have for every $x \in \mathbb{B}^m$, $H(0)(x) = u(x)$ and $H(1)(x) = u(a)$. Moreover, $H \in C([0,1], W^{s,p}(\mathbb{B}^m, \mathcal{N}))$. □

3.5 Comments

3.5.1 Approximation

The necessary condition of proposition 3.1 was observed by Schoen and Uhlenbeck on a ball [SU83, §4] (see also [BZ88, Theorem 2]) and extended by Bethuel to a general domain [Bet91, Theorem A.0].

The fractional counterpart of proposition 3.1 was proved by Escobedo when $\mathcal{M} = \mathbb{B}^m$ [Esc88, Theorem 3].

3.5.2 Extension

Theorem 3.6 is due to Robert Hardt and Lin Fanghua [HL87, §6.3] with a proof relying on singularities of p–harmonic extensions (see also [BD95, theorem 4] for a proof using the density of maps that are smooth outside a small-dimensional set).

3.5.3 Lifting

Proposition 3.8 when $\pi : \mathbb{R} \to \mathbb{S}^1$ is due to Jean Bourgain, Haïm Brezis and Petru Mironescu [BBM01, (4.2)]. The case where $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ is a universal covering or $\pi : \mathbb{S}^1_k \mapsto \mathbb{S}^1$ is k–tuple covering of the circle due to Fabrice Bethuel and David Chiron [BC07, lemma 1, theorem 3 i) and proposition 3 i)]. In the case of liquid crystals $\pi : \mathbb{R}P^n \to \mathbb{S}^n$, proposition 3.8 has been proved by John Ball and Argir Zarnescu [BZ11, theorem 2].

The uniqueness of the lifting when $\pi : \mathbb{R} \to \mathbb{S}^1$ is due to Jean Bourgain, Haïm Brezis and Petru Mironescu [BBM01, Appendix B]. For the projective space, in connection with liquid crystals, proposition 3.8 has been proved when $s = 1$ by Ball and Zarnescu [BZ11, proposition 2] and by Mucci [Muc12, theorem 2.5].

When $\alpha \in (0, 1)$, the double integral in eq. (3.9) defines an α–fractional perimeter (see for example [PS17; CSV15; FMM11; FS08, (4.2)]).

Lemma 3.10 is originally due to Jean Bourgain, Haïm Brezis and Petru Mironescu [Bre02; BBM01, Appendix B]. Other proofs have been given since [DMMS08]. We present a proof inspired from Alireza Ranjbar-Motlagh.

4 Singular retractions methods

4.1 Homotopic preliminaries

4.1.1 ℓ–connected sets

We will assume that the target manifold $\mathcal{N}$ is ℓ–connected.

Definition 4.1. The manifold $\mathcal{N}$ is ℓ–connected whenever for every $j \in \{0, \dots, \ell\}$, the restriction operator $F \in C(\bar{\mathbb{B}}^{j+1}, \mathcal{N}) \mapsto F|_{\mathbb{S}^j} \in C(\mathbb{S}^j, \mathcal{N})$ is surjective.

A practical consequence is that if $A \subset [0,1]^{\ell+1}$ is a union of lower-dimensional spaces, the restriction operator $F \in C([0,1]^{\ell+1}, \mathcal{N}) \mapsto F|_A \in C(A, \mathcal{N})$ is surjective.

For example, the sphere $\mathbb{S}^n$ is ℓ–connected if and only if $\ell < n$.

In the language of homotopy groups, a manifold $\mathcal{N}$ is ℓ–connected if and only if $\pi_0(\mathcal{N}) \simeq \dots \simeq \pi_\ell(\mathcal{N}) \simeq \{0\}$. In particular, $\mathcal{N}$ is 0–connected if and only if it is path-connected; $\mathcal{N}$ is 1–connected if and only if it is path-connected and simply-connected.

By the Hurewicz theorem, if $\ell \geq 1$, the manifold $\mathcal{N}$ is ℓ–connected if and only if $\mathcal{N}$ is connected and simply connected and for every $j \in \{2, \dots, \ell\}$, the j–th homology group with integer coefficients is trivial $H_j(\mathcal{N}, \mathbb{Z}) \simeq \{0\}$ [Hu59, corollary II.9.2]. A compact manifold $\mathcal{N}$ of dimension n is not n–connected: If the manifold $\mathcal{N}$ is connected and orientiable, and if $n = \dim \mathcal{N}$, then $H_n(\mathcal{N}, \mathbb{Z}) = \mathbb{Z}$ and $\mathcal{N}$ is not n–connected [Spa66, Exercise 4.E]; in general if $\mathcal{N}$ is connected, then $H_n(\mathcal{N}, \mathbb{Z}/2) \simeq H_n(\mathcal{N}, \mathbb{Z}) \otimes \mathbb{Z}/2 \simeq \mathbb{Z}/2$.

4.1.2 Euclidean space as a cubical complex

In order to exploit the ℓ–connectedness of the target manifold $\mathcal{N}$, we will endow the space $\mathbb{R}^\nu$ into which $\mathcal{N}$ is embedded with a cubical complex structure.

We consider a decomposition of the Euclidean space $\mathbb{R}^\nu$ into cubes of edge-length $\eta \in (0, +\infty)$

$$\mathcal{L}^\nu_\eta \triangleq \left\{ \eta k + [-\tfrac{\eta}{2}, \tfrac{\eta}{2}]^\nu \mid k \in \mathbb{Z}^\nu \right\}$$

and we consider the set $\mathcal{K}^\ell_\eta$ of ℓ–dimensional faces of cubes of $\mathcal{Q}_\eta$. The realization of the complex is

$$\bigcup \mathcal{K}^\ell_\eta \triangleq \bigcup_{Q \in \mathcal{K}^\ell_\eta} Q.$$

We have immediately the inclusions

$$\eta\big(\mathbb{Z} + \tfrac{1}{2}\big)^\nu = \bigcup \mathcal{K}^0_\eta \subset \bigcup \mathcal{K}^1_\eta \subset \dots \subset \bigcup \mathcal{K}^{\nu-1}_\eta \subset \bigcup \mathcal{K}^\nu_\eta = \mathbb{R}^\nu.$$

We also consider the *dual decomposition*

$$\mathcal{L}^\ell_\eta \triangleq \{Q + \eta(\tfrac{1}{2}, \dots, \tfrac{1}{2}) \mid Q \in \mathcal{K}^\ell_\eta\}.$$

We observe that for every $\ell \in \{0, \dots, \nu - 1\}$, $\bigcup \mathcal{K}^\ell_\eta \cap \bigcup \mathcal{L}^{\nu-\ell-1}_\eta = \varnothing$ and that the set $\bigcup \mathcal{K}^\ell_\eta$ is a homotopy retract of $\mathbb{R}^\nu \setminus \bigcup \mathcal{L}^{\nu-\ell-1}_\eta$. We will use a quantitative version of this fact.

Lemma 4.2. *For every $\ell \in \{0, \dots, \nu - 1\}$, there exists a map $\Theta^\ell_\eta \in C(\bigcup \mathcal{K}^{\ell+1}_\eta \setminus \bigcup \mathcal{L}^{\nu-\ell-1}_\eta, \bigcup \mathcal{K}^\ell_\eta)$ such that*

(i) for every $Q \in \mathcal{K}^{\ell+1}_\eta$, one has $\Theta^\ell_\eta(Q \setminus \bigcup \mathcal{L}^{\nu-\ell-1}_\eta) \subset Q$,

(ii) for every $y \in \bigcup \mathcal{K}^\ell_\eta$, one has $\Theta^\ell_\eta(y) = y$,

(iii) for every $y \in \bigcup \mathcal{K}^{\ell+1}_\eta \setminus \bigcup \mathcal{L}^{\nu-\ell-1}_\eta$, one has $[y, \Theta^\ell_\eta(y)] \subset \bigcup \mathcal{K}^{\ell+1}_\eta \setminus \bigcup \mathcal{L}^{\nu-\ell-1}_\eta$,

(iv) the map Θ^ℓ_η is locally Lipschitz-continuous, and for every $y \in \bigcup \mathcal{K}^{\ell+1}_\eta \setminus \bigcup \mathcal{L}^{\nu-\ell-1}_\eta$,

$$|D\Theta^\ell_\eta(y)| \le \frac{C\eta}{\operatorname{dist}_\infty(y, \bigcup \mathcal{L}^{\nu-\ell-1}_\eta)},$$

(v) for every $j \in \{\nu - \ell, \dots, \nu - 1\}$ and every $y \in \bigcup \mathcal{K}^{\ell+1}_\eta \setminus \bigcup \mathcal{L}^{\nu-\ell-1}_\eta$,

$$\operatorname{dist}_\infty\big(\Theta^\ell_\eta(y), \bigcup \mathcal{L}^j_\eta\big) = \frac{\eta}{2\operatorname{dist}_\infty\big(y, \bigcup \mathcal{L}^{\nu-\ell-1}_\eta\big)} \operatorname{dist}_\infty\big(y, \bigcup \mathcal{L}^j_\eta\big).$$

In particular for every $Q \in \mathcal{K}^{\ell+1}_\eta$ and for every $j \in \{\nu - \ell, \dots, \nu - 1\}$

$$\Theta^\ell_\eta(Q \setminus \bigcup \mathcal{L}^j_\eta) \subset Q \setminus \bigcup \mathcal{L}^j_\eta.$$

We have defined here for every $y \in \mathbb{R}^\nu$ and $A \subset \mathbb{R}^\nu$,

$$\operatorname{dist}_\infty(y, A) = \inf\big\{|y - z|_\infty \mid z \in A\big\},$$

where for every $v = (v_1, \dots, v_\nu) \in \mathbb{R}^\nu$, $|v|_\infty = \max\big\{|v_1|, \dots, |v_\nu|\big\}$.

Proof of lemma 4.2. We need to define Θ^ℓ_η on $Q \setminus \mathcal{L}^{\nu-\ell-1}_\eta$ for $Q \in \mathcal{K}^{\ell+1}_\eta$. We assume without loss of generality that $\eta = 1$ and that $Q = [-\frac{\eta}{2}, \frac{\eta}{2}]^{\ell+1} \times \{0\}^{\nu-\ell-1}$. We have then

$$Q \cap \mathcal{L}^{\nu-\ell-1}_\eta = \{0\},$$

We define for every $y \in Q \setminus \bigcup \mathcal{L}^{\nu-\ell-1}_\eta$

$$\Theta^\ell_\eta(y) \triangleq \frac{\eta}{2|y|_\infty} y.$$

We conclude by observing that for every $y \in Q$,

$$\operatorname{dist}\big(y, \bigcup \mathcal{L}^{\nu-\ell-1}_\eta\big) = |y|_\infty,$$

and that, more generally, for every $j \in \{\nu - \ell - 1, \dots, \nu - 1\}$ and every $y \in Q$,

$$\operatorname{dist}\big(y, \bigcup \mathcal{L}^j_\eta\big) = \min_{\substack{I \subset \{1, \dots, \ell+1\} \\ \# I = \nu - j}} \max_{i \in I} |y_i|. \qquad \square$$

4.2 Extension for $\lfloor p-1 \rfloor$–connected manifolds

As a first result for ℓ–connected manifolds, we have the existence of an extension from the trace space for $\lfloor p-1 \rfloor$–connected manifolds.

Theorem 4.3. *If $p < m = \dim \mathcal{M}$ and if the manifold $\mathcal{N}$ is $\lfloor p-1 \rfloor$–connected, then for each map $u \in W^{1-\frac{1}{p},p}(\partial\mathcal{M}, \mathcal{N})$, there exists a map $U \in W^{1,p}(\mathcal{M}, \mathcal{N})$ such that* $\operatorname{tr}_{\partial\mathcal{M}} = u$. *Moreover,*

$$\mathcal{E}^{1,p}(U) \le C\,\mathcal{E}^{1-\frac{1}{p},p}(u),$$

for some constant C depending on p, $\mathcal{M}$ and $\mathcal{N}$.

In view of theorem 5.1, the condition is necessary and sufficient when $\mathcal{N}$ is connected and $p < 3$. For $p \ge 3$, the weaker condition $\pi_{\lfloor p-1 \rfloor}(\mathcal{N}) \simeq \{0\}1$ is necessary (theorem 3.6); we will see further that it is also necessary that the homotopy groups $\pi_1(\mathcal{N}), \dots, \pi_{\lfloor p-2 \rfloor}(\mathcal{N})$ are all finite (see theorem 3.6).

In the supercritical and critical cases $p \ge m$ (proposition 2.9), lemma 2.2 provided a smooth retraction on a tubular neighbourhood of the manifold $\mathcal{N}$ in the ambient space $\mathbb{R}^\nu$. In the subcritical case $p < m$, there is no reason for which an approximation by convolution, a homotopy or an extension should stay in this neighbourhood. One way to overcome this difficulty is to define a retraction on the whole on $\mathbb{R}^m$. This can still be done when the manifold $\mathcal{N}$ is ℓ–connected if one is ready to treat singularities of codimension $\ell + 2$ in the retraction map.

Proposition 4.4. *Let $\ell \in \{0, \dots, \nu - 2\}$. If the manifold $\mathcal{N}$ is ℓ–connected, then for every $\rho \in (0, \delta_{\mathcal{N}}/2)$ and every $\eta \in (0, +\infty)$ small enough, there exists a compact set $\Sigma \subset \bigcup \mathcal{L}_\eta^{\nu-\ell-2} \cap (\mathcal{N} + \mathbb{B}^\nu_{\delta_{\mathcal{N}}}) \setminus (\mathcal{N} + \mathbb{B}^\nu_\rho)$ and a map $\Phi \in C(\mathbb{R}^\nu \setminus \Sigma, \mathcal{N} + \mathbb{B}^\nu_{\delta_{\mathcal{N}/2}})$ such that*

(i) for every $y \in \mathcal{N} + \mathbb{B}^\nu_\rho$, one has $\Phi(y) = y$,

(ii) the map Φ is locally Lipschitz-continuous and for almost every $y \in \mathbb{R}^\nu \setminus \Sigma$, one has

$$|D\Phi(y)| \le \frac{C}{\operatorname{dist}(y, \Sigma)},$$

for some constant $C > 0$ depending on $\mathcal{N}$,

(iii) the map Φ is constant outside a compact subset of $\mathbb{R}^\nu$.

Here $\delta_{\mathcal{N}}$ is given by lemma 2.2 so that there exists a smooth retraction $\Pi : \mathcal{N} + \mathbb{B}^\nu_{\delta_{\mathcal{N}}} \to \mathcal{N}$.

When $\mathcal{N} = \mathbb{S}^{\ell+1} \subset \mathbb{R}^{\ell+2}$, a mapping Φ satisfying the conclusions of proposition 4.4 (except the constancy outside a compact set) is given for each $y \in \mathbb{R}^{\ell+2} \setminus \Sigma$ with $\Sigma = \{0\}$ by

$$\Phi(y) = \begin{cases} \frac{1-\rho}{|y|} y & \text{if } |y| \le 1 - \rho, \\ y & \text{if } 1 - \rho \le |y| \le 1 + \rho, \\ \frac{1+\rho}{|y|} y & \text{if } |y| \ge 1 + \rho. \end{cases}$$

Proof of proposition 4.4. We define $\mathcal{U} = \{Q \in \mathcal{K}^\nu_\eta \mid Q \subset \mathcal{N} + \mathbb{B}^\nu_{\delta_\mathcal{N}/2}(0)\}$ and we observe that if η is small enough, then

$$\mathcal{N} + \mathbb{B}^\nu_\rho \subset \bigcup \mathcal{U} \triangleq \bigcup_{Q \in \mathcal{U}} Q.$$

We define Φ for each $y \in U$ by $\Phi(y) = y$.

We set

$$\mathcal{V} \triangleq \{Q \in \mathcal{Q} \mid Q \cap \bigcup U = \varnothing\},$$

we choose $b \in \mathcal{N}$ and we define Φ for $y \in \bigcup V$ by $\Phi(y) = b$.

We set now for every $j \in \{0, \dots, \nu\}$,

$$\mathcal{W}^j = \{Q \in \mathcal{K}^\ell_\eta \mid Q \notin \bigcup \mathcal{U} \cup \bigcup \mathcal{V}\}.$$

We define for each $y \in \bigcup \mathcal{W}^0$, $\Psi(y) = b$.

We assume now that Ψ has been defined on $\bigcup \mathcal{W}^j$ for some $j \in \{0, \dots, \nu - 1\}$. By definition of $\mathcal{U}$ and $\mathcal{V}$, the map Ψ is already defined on $\bigcup \mathcal{K}^j_\eta$. If $j \le \ell$, we observe that since the manifold $\mathcal{N}$ is a deformation retract of its tubular neighbourhood $\mathcal{N} + \mathbb{B}^\nu_{\delta_\mathcal{N}}$, the set $\mathcal{N} + \mathbb{B}^\nu_{\delta_\mathcal{N}}$ is also ℓ–connected. Thus by definition of ℓ–connectedness and classical regularization arguments, for every $Q \in \mathcal{W}^{j+1}$, we define Ψ to be a Lipschitz-continuous extension of the boundary values. If $j \ge \ell + 1$, we define for every $Q \in \mathcal{W}^{j+1}$, $\Psi|_{Q \setminus \bigcup \mathcal{L}^{\nu-\ell-2}_\eta} = \Psi \circ \Theta^j_\eta|_{Q \setminus \bigcup \mathcal{L}^{\nu-\ell-2}_\eta}$, where Θ^j_η was defined in lemma 4.2.

The conclusion holds with $\Sigma \triangleq \bigcup \mathcal{L}^{\nu-\ell-2} \setminus (\bigcup \mathcal{U} \cup \bigcup \mathcal{V})$. □

Problem 4.1 (★★). Show that one can take $\Phi \in C^\infty(\mathbb{R}^\nu \setminus \Sigma, \mathcal{N})$ such that for every $k \in \mathbb{N}$, there exists a constant C_k such that for every $y \in \mathbb{R}^\nu \setminus \Sigma$,

$$|D^k \Phi(y)| \le \frac{C_k}{\operatorname{dist}(y, \Sigma)^k}.$$

Proof of theorem 4.3. Let $V \in W^{1,p}(\mathcal{M}, \mathbb{R}^\nu)$ be an extension given by the classical linear trace theory. Let $\Pi : \mathcal{N} + \mathbb{B}^\nu_\rho \to \mathcal{N}$ be the retraction given by lemma 2.2. Let $\Psi : \mathbb{R}^\nu \setminus \bigcup \mathcal{L}^{\nu - \lfloor p+1 \rfloor}_\nu \to \mathcal{N} + \mathbb{B}^\nu_\rho$ be given by proposition 4.4 with $\ell = \lfloor p - 1 \rfloor$.

Since the function V is smooth in the interior of $\mathcal{M}$, by Sard's theorem and by the implicit function theorem, for almost every $h \in \mathbb{R}^\nu$, the set $V^{-1}(K - h)$ is a countable union of $(m - \lfloor p+1 \rfloor)$–dimensional submanifolds of $\mathcal{M}$ and V is transversal to these manifolds and thus the map $\Phi \circ (V + h)$ is weakly differentiable.

We set $\rho = \delta_{\mathcal{N}}/2$ and we compute now the quantity

$$\begin{aligned}\int_{\mathbb{B}^\nu_\rho}\int_{\mathcal{M}} |\mathrm{D}(\Phi \circ (V+h) - h)|^p &\le \int_{\mathcal{M}} |\mathrm{D}V|^p \int_{\mathbb{B}^\nu_\rho} |\mathrm{D}\Pi(V(x)+h)|^p \,\mathrm{d}h\,\mathrm{d}x\\ &\le C_1 \int_{\mathcal{M}} |\mathrm{D}V|^p \int_{\mathbb{B}^\nu_\rho} \frac{1}{\operatorname{dist}(V(x)+h,\Sigma)^p}\,\mathrm{d}h\,\mathrm{d}x\\ &\le C_2 \int_{\mathcal{M}} |\mathrm{D}V|^p \int_0^1 \frac{r^{\lfloor p \rfloor}}{r^p}\,\mathrm{d}r\\ &\le C_3 \int_{\mathcal{M}} |\mathrm{D}V|^p,\end{aligned}$$

since the set $\Sigma \subset \bigcup T^{\nu-\lfloor p+1 \rfloor}_\rho$ is a union of convex sets of codimension $\lfloor p+1 \rfloor$. There exists thus a vector $h \in \mathbb{B}^\nu_\rho$ such that $\Phi \circ (V+h) - h \in W^{1,p}(\mathcal{M}, \mathbb{R}^\nu)$.

Moreover, since $h \in \mathbb{B}^\nu_{\delta_{\mathcal{N}}/2}$, then $\Phi \circ (V+h) - h \in \mathcal{N} + \mathbb{B}^\nu_{\delta_{\mathcal{N}}}$ in $\mathcal{M}$ and $\operatorname{tr}_{\partial\mathcal{M}} \Phi \circ (V + h) - h = u$ on $\partial\mathcal{M}$. We conclude by setting $U \triangleq \Pi \circ (\Phi \circ (V+h) - h)$. □

Remark 4.5. As a consequence, of the proof, there exists a set $T \subset \mathcal{M}$ which is a countable union of $(m - \lfloor p+1 \rfloor)$–dimensional manifolds such that $U \in C^\infty(\mathcal{M} \setminus T)$. Moreover, for every $x \in T$, $\limsup_{y \to x} \operatorname{dist}_{\mathcal{M}}(y,x)|DU(y)| < +\infty$.

Problem 4.2 (★). Prove that when the manifold $\mathcal{N}$ is $m-1$–connected, then every map $u \in W^{1-\frac{1}{m},m}(\partial\mathcal{M}, \mathcal{N})$ can be extended to a map $U \in C^\infty(\mathcal{M}, \mathcal{N})$ and such that

$$\mathcal{E}^{1,m}(U) \le C\,\mathcal{E}^{1-\frac{1}{m},m}(u).$$

Compare the result with proposition 2.10.

4.3 Approximation for $\lfloor p \rfloor$–connected manifolds

We prove the following approximation result.

Theorem 4.6. *If $p < m = \dim \mathcal{M}$ and if the manifold $\mathcal{N}$ is $\lfloor p \rfloor$–connected, then every map $u \in W^{1,p}(\mathcal{N}, \mathcal{M})$ can be approximated by maps in $C^\infty(\mathcal{M}, \mathcal{N})$.*

The proof is based on the following constrution.

Proposition 4.7. *If $\mathcal{N}$ is ℓ–connected and if $\rho, \eta > 0$ are small enough, then for every $\varepsilon > 0$ small enough, there exists a map $\Phi_\varepsilon \in C^\infty(\mathbb{R}^\nu, \mathcal{N} + \mathbb{B}^\nu_{\delta_{\mathcal{N}/2}})$ such that*

(i) for every $y \in (\mathcal{N} + \mathbb{B}^\nu_\rho) \setminus (\bigcup \mathcal{L}^{\nu-\ell-1}_\rho + \mathbb{B}^\nu_\varepsilon)$, $\Phi_\varepsilon(y) = y$,

(ii) $\|\mathrm{D}\Phi_\varepsilon\|_{L^\infty} \le C/\varepsilon$, where the constant can be taken independently of ε,

(iii) the map Φ_ε is constant outside a compact set which can be taken uniformly with respect to ε.

Proof. Let $\Pi : \mathcal{N} + \mathbb{B}^\nu_{\delta_\mathcal{N}} \to \mathcal{N}$ be the retraction of lemma 2.2. We consider $\mathcal{U}^\ell_\eta = \{Q \in \mathcal{K}^\ell_\eta \mid Q \subset \mathcal{N} + \mathbb{B}^\nu_{\delta_{\mathcal{N}/2}}(0)\}$. If η is small enough, then $(\mathcal{N} + \mathbb{B}^\nu_\rho) \cap \bigcup \mathcal{K}^\ell_\eta \subset \bigcup \mathcal{U}^\ell_\eta$. We set

$$\mathcal{V} \triangleq \big\{Q \in \mathcal{K}^\ell_\eta \mid Q \cap \bigcup \mathcal{U}^\nu_\eta = \varnothing\big\},$$

we choose $b \in \mathcal{N}$ and we define $\check{\Pi}(y)$ for each $y \in \bigcup \mathcal{V}^\nu_\eta$ by $\check{\Pi}(y) = b$. Since the manifold $\mathcal{N}$ is ℓ–connected and Π is a deformation retraction, the set $\mathcal{N} + \mathbb{B}^\nu_{\delta_\mathcal{N}/2}$ is ℓ–connected. Hence the map $\check{\Pi}|_{\bigcup \mathcal{U}^\ell_\eta/2}$ is homotopic to a constant map on $\bigcup \mathcal{U}^\ell_\eta \cup \bigcup \mathcal{V}^\nu_\eta$. Hence there exists a Lipschitz-continuous map $\check{\Pi} : \mathbb{R}^\nu \to \mathcal{N} + \mathbb{B}^\nu_{\delta_\mathcal{N}}$ such that $\check{\Pi} = \Pi$ on $\bigcup \mathcal{U}^\ell_\eta$ and such that $\check{\Pi} = b$ on $\bigcup \mathcal{V}^\nu_\eta$.

We define for each $(y,t) \in \mathbb{R}^\nu \times \{1\} \cup \mathbb{R}^\nu \setminus (\bigcup \mathcal{L}^{\nu-\ell-1}) \times [0,1]$,

$$\Upsilon(y,t) = (1-t)\,(\Theta^\ell_\eta \circ \cdots \circ \Theta^{\nu-1}_\eta)(y) + ty$$

and we define for every $\varepsilon > 0$, the function $d_\varepsilon : \mathbb{R}^\nu \to \mathcal{N}$ by setting for each $y \in \mathbb{R}^\nu$

$$d_\varepsilon(y) = \begin{cases} -1 & \text{if } \hat{d}_\varepsilon(y) \le -1, \\ \hat{d}_\varepsilon(y) & \text{if } -1 < \hat{d}_\varepsilon(y) < -1, \\ 1 & \text{if } \hat{d}_\varepsilon(y) \ge 1. \end{cases}$$

where

$$\hat{d}_\varepsilon(y) = \min\left\{1, \frac{6\,\mathrm{dist}(y, K^{\nu-\ell-1}_\eta)}{\varepsilon} - 2, 1 - \frac{\mathrm{dist}(y,\mathcal{N}) - \rho}{\lambda}\right\}$$

We then set

$$\Phi_\varepsilon(y) = \begin{cases} \Upsilon\big(y, d_\varepsilon(y)\big) & \text{if } d_\varepsilon(y) \ge 0, \\ \check{\Pi}\big(\Upsilon\big(y, -d_\varepsilon(y)\big)\big) & \text{if } d_\varepsilon(y) < 0. \end{cases}$$

By construction of $\check{\Pi}$, if $d_\varepsilon(y) < 0$, then $\Pi_\varepsilon(y) = \check{\Pi}(\Upsilon(y, -d_\varepsilon(y))) \in \mathcal{N} + \mathbb{B}^\nu_\rho$. If $d_\varepsilon(y) \ge 0$, then we have $\mathrm{dist}(y,\mathcal{N}) \le \rho + \lambda$, and thus if $\rho + \lambda + \eta\sqrt{\nu} \le \delta_\mathcal{N}/2$, hence $\Upsilon\big(y, d_\varepsilon(y)\big) \in \mathcal{N} + \mathbb{B}_{\rho+\lambda+\eta\sqrt{\nu}}$ and thus $\Phi_\varepsilon(y) = \Upsilon(y, d_\varepsilon(y)) \in \mathcal{N} + \mathbb{B}^\nu_{\rho+\lambda+\eta\sqrt{\nu}}$.

If moreover $d_\varepsilon(y) = 0$, then $\Phi_\varepsilon(y) = \check{\Pi}(\Upsilon(y,0)) = \Pi(\Upsilon(y,0))$, since $\Upsilon(y,0) \in \bigcup \mathcal{K}^\ell_\eta$ and by construction of $\check{\Pi}$, so that $\check{\Upsilon}_\varepsilon$ is continuous.

We observe that if $y \in \mathcal{N} + \mathbb{B}^\nu_\rho \setminus (\bigcup \mathcal{K}^{\nu-\ell-1}_\eta + \mathbb{B}^\nu_{\varepsilon/2})$, then $\mathrm{dist}(y, \bigcup \mathcal{K}^{\nu-\ell-1}_\eta) \ge \varepsilon/2$ and $\mathrm{dist}(y,\mathcal{N}) \le \rho$, and thus $d_\varepsilon(y) = 1$, so that $\Phi_\varepsilon(y) = \Upsilon(y,1) = \Pi_\rho(y) = y$.

If $d_\varepsilon(y) = -1$, then we clearly have

$$\|D\Phi_\varepsilon(x)\|_{L^\infty} \le C_4.$$

If $d_\varepsilon(y) \in (-1,1)$, we have

$$\mathrm{dist}\big(y, \bigcup \mathcal{L}^{\nu-\ell-1}_\rho\big) \ge \frac{\varepsilon}{6},$$

and thus by the chain rule

$$\|D\Phi_\varepsilon(y)\|_{L^\infty} \le \frac{C_5}{\varepsilon}.$$

In order to have smooth map, we convolve with a *radial* mollifying kernel at a scale $\varepsilon/2$ and by using the fact that $\rho + \lambda + \eta\sqrt{\nu} < \delta_{\mathcal{N}}/2$. By radiality of the kernel, it preserves the fact that the map is the identity and by uniform convergence, it does not move the points too far away from the target set. □

Problem 4.3 (★★). Prove that Φ_ε can be chosen to satisfy for every $k \in \mathbb{N}$,

$$|\mathrm{D}^k \Phi_\varepsilon| \le \frac{C_k}{\varepsilon^k},$$

for some constant C_k independent of ε.

Proof of theorem 4.6. We choose $\rho = \delta_{\mathcal{N}}/4$. We first show that there exists a family h_ε in $\mathbb{B}^\nu_\rho$ such that

$$\lim_{\varepsilon \to 0} \|(\Phi_\varepsilon(u - h_\varepsilon) + h_\varepsilon) - u\|_{W^{1,p}(\mathcal{M})} = 0.$$

Indeed, we have for every vector $h \in \mathbb{B}^\nu_\rho$,

$$\int_{\mathcal{M}} |\mathrm{D}(\Phi_\varepsilon \circ (u - h) + h) - \mathrm{D}u|^p = \int_{u^{-1}(\bigcup \mathcal{L}^{\nu-\ell-1}_\eta + \mathbb{B}^\nu_\varepsilon(h))} |\mathrm{D}(\Phi_\varepsilon \circ (u - h)) - \mathrm{D}u|^p.$$

Thus by integrating with respect to h we get, if $\mathcal{N} \subset \mathbb{B}^\nu_R$

$$\begin{aligned}\int_{\mathbb{B}^\nu_\rho} \int_{\mathcal{M}} &|\mathrm{D}((\Phi_\varepsilon(u - h) + h) - \mathrm{D}u|^p \\ &\le C_6 \int_{\mathcal{M}} |\mathrm{D}u|^p \int_{\mathbb{B}^\nu_{R+\rho} \cap (\bigcup \mathcal{L}^{\nu - \lfloor p \rfloor - 1}_\eta + \mathbb{B}^\nu_\varepsilon)} |\mathrm{D}\Phi_\varepsilon(z) - \mathrm{id}|^p \,\mathrm{d}z \,\mathrm{d}x \\ &\le C_7 \,\varepsilon^{\lfloor p \rfloor + 1 - p} \int_{\mathcal{M}} |\mathrm{D}u|^p.\end{aligned}$$

We take now $h_\varepsilon \in \mathbb{B}^\nu_\rho$ such that

$$\int_{\mathcal{M}} |\mathrm{D}((\Phi_\varepsilon \circ (u - h_\varepsilon) + h_\varepsilon)) - \mathrm{D}u|^p \le C_8 \,\varepsilon^{\lfloor p \rfloor + 1 - p} \int_{\mathcal{M}} |\mathrm{D}u|^p.$$

By continuous differentiability of Π and since $\Pi \circ u = u$, we have

$$\lim_{\varepsilon \to 0} \|\Pi \circ (\Phi_\varepsilon(u - h_\varepsilon) + h_\varepsilon) - u\|_{W^{1,p}(\mathcal{M})} = 0.$$

We consider $(v_j)_{j \in \mathbb{N}}$ to a be a sequence in $C^\infty(\mathcal{M}, \mathbb{R}^\nu)$ converging to u in $W^{1,p}(\mathcal{M}, \mathbb{R}^\nu)$. By continuity, we have

$$\limsup_{j \to \infty} \|\Pi \circ (\Phi_\varepsilon(v_j - h_\varepsilon) + h_\varepsilon) - u\|_{W^{1,p}(\mathcal{M})} \le \|\Pi \circ (\Phi_\varepsilon(u - h_\varepsilon) + h_\varepsilon) - u\|_{W^{1,p}(\mathcal{M})}$$

and thus

$$\lim_{\varepsilon \to 0} \limsup_{j \to \infty} \|\Pi \circ (\Phi_\varepsilon(v_j - h_\varepsilon) + h_\varepsilon) - u\|_{W^{1,p}(\mathcal{M})} = 0.$$ □

4.4 Comments

The notion of ℓ–connectedness is classical in homotopy theory (see [Hu59, §II.9; Hat02, §4.1]).

Theorem 4.3 and its proof are due to Robert Hardt and Lin Fanghua [HL87, theorem 6.2]. Proposition 4.4 is essentially due to Hardt and Lin [HL87, lemma 6.1] (see also [BPVS14, lemma 2.2; Hop16, lemma 4.5]). A difference from the classical treatment is that the singular retraction Ψ is a retraction on a neighbourhood of the target manifold, so that it is not needed to triangulate the manifold and its neighbourhood and conjugation with a small translation still leaves a neighbourhood of the manifold invariant.

Theorem 4.6 are due to Piotr Hajłasz [Haj94, theorem 1]. The idea and the analytical part of the proof are due to Fabrice Béthuel and Zheng Xiaomin for $\mathcal{N} = \mathbb{S}^n$ [BZ88, theorem 1]. The present proof of theorem 4.6 differs by the fact that Φ_ε is supposed to be the identity in a neighbourhood of the manifold $\mathcal{N}$ except a neibourhood of a dual skeleton. Again, our approach avoids triangulating the manifold and its neighbourhood and conjugation of Φ_ε with a small translation preserves its properties. Proposition 4.7 is a modification of corresponding result of Piotr Hajłasz [Haj94] (see also [BPVS13, proposition 2.1] in the higher-order case). The construction of $\hat{\Pi}_\rho$ in proposition 4.7 corresponds to [Haj94, lemma 1], where a cone construction is performed.

5 Analytical obstruction and lack of linear estimates

5.1 Quantitative obstruction for the extension of traces

The next theorem shows that there is a quantitative obstruction to the extension problem.

Theorem 5.1. *If $2 \le p < m$ and if for every map $u \in W^{1-1/p,p}(\partial\mathcal{M},\mathcal{N})$ there exists a map $U \in W^{1-1/p,p}(\mathcal{M},\mathcal{N})$ such that $\operatorname{tr}_{\partial\mathcal{M}} U = u$, then for every $j \in \{1,\dots,\lfloor p-1\rfloor\}$, the homotopy group $\pi_j(\mathcal{N})$ is finite.*

The combination of theorem 3.6 and theorem 5.1 gives as a necessary condition that the groups $\pi_1(\mathcal{N}),\dots,\pi_{\lfloor p-2\rfloor}(\mathcal{N})$ are finite and that the group $\pi_{\lfloor p-1\rfloor}(\mathcal{N})$ is trivial. On the other hand by theorem 4.3, when all the homotopy groups are trivial $\pi_1(\mathcal{N}) \simeq \dots \simeq \pi_{\lfloor p-1\rfloor}(\mathcal{N}) \simeq \{0\}$. It is also known that when the group $\pi_1(\mathcal{N})$ is finite and $\pi_2(\mathcal{N}) \simeq \dots \simeq \pi_{\lfloor p-1\rfloor}(\mathcal{N}) \simeq \{0\}$, then any map has an extension [MVS].

If $w \in W^{1,p}(\mathbb{B}^\ell,\mathcal{N})$ and $w = b \in \mathcal{N}$, we define the *topological Sobolev energy*

$$\begin{aligned}\mathcal{E}^{1,p}_{\mathrm{top}}(w) \triangleq \inf\biggl\{\int_{\mathbb{B}^\ell} |Dv|^p \;\big|\; &v \in W^{1,p}(\mathbb{B}^\ell,\mathcal{N}),\ \operatorname{tr} v_{\partial\mathbb{B}^\ell} = b\\ &\text{and } v \text{ is homotopic to } w \text{ relatively to } \partial\mathbb{B}^\ell \text{ in } W^{1,\ell}(\mathbb{B}^\ell,\mathcal{N})\biggr\}\end{aligned} \tag{5.1}$$

If $p > \ell$, by the fractional Sobolev–Morrey embedding, we have $v = v'$ and $w = w'$ almost everywhere on $\mathbb{B}^\ell$ for some $v', w' \in C^{0,1-\frac{\ell}{p}}(\mathbb{B}^\ell,\mathcal{N})$ and the homotopy in (5.1) can be understood in the classical sense (see section 2.2).

Lemma 5.2. *If $\ell \le p-1$, then there exists a constant $C > 0$ such that if $\mathcal{N}$ is a compact manifold, if $U \in W^{1,p}(\mathbb{B}^{\ell+1} \cap \mathbb{R}^{\ell+1}_+,\mathcal{N})$ and if $\operatorname{tr}_{\mathbb{B}^\ell\times\{0\}} U = b$ on $(\mathbb{B}^\ell_1 \setminus \mathbb{B}^\ell_{1/2}) \times \{0\}$ for some constant $b \in \mathcal{N}$, then*

$$\mathcal{E}^{1,p}_{\mathrm{top}}(\operatorname{tr}_{\mathbb{B}^\ell\times\{0\}} U) \le C \int_{\mathbb{B}^{\ell+1}\cap\mathbb{R}^{\ell+1}_+} |DU|^p.$$

Proof of lemma 5.2. By Fubini's theorem, for almost every $r \in (\frac{1}{2},1)$, we have $U_r \triangleq U|_{\partial\mathbb{B}^{\ell+1}_r\cap\mathbb{R}^\ell} = \operatorname{tr}_{\partial\mathbb{B}^{\ell+1}_r\cap\mathbb{R}^\ell} U$, $U_r \in W^{1,p}(\partial\mathbb{B}^{\ell+1}_r \cap \mathbb{R}^\ell,\mathcal{N}) \simeq W^{1,p}(\mathbb{B}^\ell,\mathcal{N})$ and by proposition 2.11, U_r and $\operatorname{tr}_{\mathbb{B}^\ell\times\{0\}} U$ are homotopic relatively to $\partial\mathbb{B}^\ell$. We then have for every $r \in (\frac{1}{2},1)$, by lemma 5.2

$$\mathcal{E}^{1,p}_{\mathrm{top}}(\operatorname{tr}_{\mathbb{B}^\ell\times\{0\}} U) \le C_1 \int_{\partial\mathbb{B}^{\ell+1}_r\cap\mathbb{R}^\ell} |DU|^p,$$

where $\mu \in [0, +\infty)$ denote the infimum in the conclusion. By integration with respect to r over the interval $(\frac{1}{2}, 1)$, we reach the conclusion. □

Lemma 5.3. *Let $\mathcal{N}$ be a Riemannian manifold. If $\ell < p$ and if the set $\pi_\ell(\mathcal{N})$ is infinite, then for every $b \in \mathcal{N}$,*

$$\sup\left\{\mathcal{E}^{1,p}_{\mathrm{top}}(w) \mid w \in W^{1,p}(\mathbb{B}^\ell, \mathcal{N}) \text{ and } \operatorname{tr}_{\partial\mathbb{B}^\ell} w = b\right\} = +\infty.$$

Proof. Since $p > \ell$, for every $M \in \mathbb{R}$, the set

$$S_M = \left\{v \in C(\mathbb{B}^\ell, \mathcal{N}) \cap W^{1,p}(\mathbb{B}^m, \mathcal{N}) \mid v|_{\partial\mathbb{B}^\ell} = b \text{ and } \mathcal{E}^{1,p}(v) \le M\right\}$$

is precompact in $C(\mathbb{B}^\ell, \mathcal{N})$ by the Morrey–Sobolev embedding and by the Ascoli–Arzela compactness criterion. Therefore, there exists a finite set of maps in $C(\mathbb{B}^\ell, \mathcal{N})$ to which each map in the set S_M is homotopic relatively to $\partial\mathbb{B}^\ell$. Since by assumption the group $\pi_\ell(\mathcal{N})$ is infinite, there are infinitely many such homotopy classes and thus the conclusion holds. □

Proposition 5.4. *Let $\mathcal{N}$ be a compact manifold, let $\ell \in \mathbb{N}$ and $p \in [1, +\infty)$. If there exists $\ell \in \{1, \dots, \min\{\lfloor p - 1\rfloor, m - 1\}\}$ such that $\pi_\ell(\mathcal{N})$ is infinite, then for every $m \ge \ell + 1$, there exists a constant $C > 0$ such that for every $M > 0$, there exists a mapping $u \in W^{1,p}(\mathbb{B}^{m-1}, \mathcal{N})$ such that*

(i) $u = b$ *on* $\mathbb{B}^{m-1} \setminus \mathbb{B}^{m-1}_{1/2}$,

(ii) $\displaystyle\int_{\mathbb{B}^{m-1}} |\mathrm{D}u|^p \ge M$,

(iii) for every map $U \in W^{1,p}(\mathbb{B}^m \cap \mathbb{R}^m_+, \mathcal{N})$ such that $\operatorname{tr}_{\mathbb{B}^{m-1}\times\{0\}} U = u$, one has

$$\int_{\mathbb{B}^m \cap \mathbb{R}^m_+} |\mathrm{D}U|^p \ge C \int_{\mathbb{B}^{m-1}} |\mathrm{D}u|^p.$$

Proof. When $m = \ell + 1$, by lemma 5.3 and by eq. (5.1), there exists a map such that

$$M \le \mathcal{E}^{1,p}_{\mathrm{top}}(u) \le \int_{\mathbb{B}^{m-1}} |\mathrm{D}u|^p \le 2\,\mathcal{E}^{1,p}_{\mathrm{top}}(u).$$

By lemma 5.2, if $U \in W^{1,p}(\mathbb{B}^m \cap \mathbb{R}^m_+, \mathcal{N})$ and if $\operatorname{tr}_{\mathbb{B}^{m-1}\times\{0\}} U = u$, one has

$$\int_{\mathbb{B}^m \cap \mathbb{R}^m_+} |\mathrm{D}U|^p \ge C_2\,\mathcal{E}^{1,p}_{\mathrm{top}}(u) \ge \frac{C_2}{2} \int_{\mathbb{B}^{m-1}} |\mathrm{D}u|^p$$

If $m > \ell + 1$, we let $u_* : \mathbb{B}^{\ell+1} \to \mathcal{N}$ be the map given by the first part of the proof. Let $\Psi : G \to \mathbb{B}^{\ell+1} \times \mathbb{S}^{m-\ell-1}$ be a diffeomorphism for some $G \subset \mathbb{B}^m$ such that $\bar{G} \subset \mathbb{B}^m$ and define

$$u(x) = \begin{cases} (u_* \circ P_1 \circ \Psi)(x) & \text{if } x \in G, \\ b & \text{otherwise,} \end{cases}$$

where $P_1 : \mathbb{B}^{\ell+1} \times \mathbb{S}^{m-\ell-1} \to \mathbb{B}^{\ell+1}$ is the canonical projection on the first component. The conclusion follows from the application of Fubini's theorem. □

Proposition 5.5. *If $1 \le q < p < m$ and if every map $u \in W^{1,q}(\partial\mathcal{M},\mathcal{N})$ is the trace of some map $U \in W^{1-1/p,p}(\mathcal{M},\mathcal{N})$, then for every $j \in \{1,\dots,\lfloor p-1\rfloor\}$, the group $\pi_j(\mathcal{N})$ is finite.*

Proof. Assume by contradiction that there exists $\ell \in \{1,\dots,\min\{\lfloor p-1\rfloor, m-1\}\}$ such that $\pi_\ell(\mathcal{N})$ is infinite We fix $b \in \mathcal{N}$ and we construct by theorem 5.6 a sequence of maps $(u_j)_{j\in\mathbb{N}}$ in $W^{1,p}(\mathbb{B}^{m-1},\mathcal{N})$ such that

(a) $u_j = b$ on $\mathbb{B}^{m-1}_1 \setminus \mathbb{B}^{m-1}_{1/2}$,

(b) $\displaystyle\int_{\mathbb{B}^{m-1}} |\mathrm{D}u_j|^p \ge 2^{j(m-p)}$

(c) there exists a constant $C_3 > 0$ for every $U \in W^{1,p}(\mathbb{B}^m \cap \mathbb{R}^m_+,\mathcal{N})$ such that $\operatorname{tr}_{\mathbb{B}^{m-1}\times\{0\}} U = u_j$, we have
$$\int_{\mathbb{B}^m\cap\mathbb{R}^m_+} |\mathrm{D}U|^p \ge C_3 \int_{\mathbb{B}^{m-1}} |\mathrm{D}u_j|^p.$$

We choose a sequence of radii $(\rho_j)_{j\in\mathbb{N}}$ in such a way that
$$\rho_j^{m-p} \int_{\mathbb{B}^m} |\mathrm{D}u_j|^p = 1. \tag{5.2}$$

By (b) in our construction, we have for every $j \in \mathbb{N}$, $\rho_j \le 2^{-j}$, and thus $\rho_j \in (0,1)$ and the sequence $(\rho_j)_{j\in\mathbb{N}}$ converges to 0. If we set $R = 3\sqrt{m}/2$, there exists a sequence of points $(a_j)_{j\in\mathbb{N}}$ converging to 0 such that the balls $B_{\rho_j}(a_j)$ are pairwise disjoint and $\bigcup_{j\in\mathbb{N}} B_{\rho_j}(a_j) \subset B_R$. We define the map $u : B_R \to \mathcal{N}$ for each $x \in B_R$ by
$$u(x) = \begin{cases} u_j\big(\frac{x-a_j}{\rho_j}\big) & \text{if } x \in B_{\rho_j}(a_j) \text{ for some } j \in \mathbb{N},\\ b & \text{otherwise.}\end{cases}$$

We have by construction, by the Hölder inequality and by (5.2)
$$\begin{aligned}\sum_{j\in\mathbb{N}} \int_{B_{\rho_j}(a_j)} |\mathrm{D}u|^q &\le \sum_{j\in\mathbb{N}} \rho_j^{m-q} \int_{\mathbb{B}^m} |\mathrm{D}u_j|^q\\ &\le C_4 \sum_{j\in\mathbb{N}} \rho_j^{m-q} \Big(\int_{\mathbb{B}^m} |\mathrm{D}u_j|^p\Big)^{\frac{q}{p}}\\ &= C_4 \sum_{j\in\mathbb{N}} \rho_j^{m(1-\frac{q}{p})} \Big(\rho_j^{m-p}\int_{\mathbb{B}^m} |\mathrm{D}u_j|^p\Big)^{\frac{q}{p}}\\ &\le C_4 \sum_{j\in\mathbb{N}} \frac{1}{2^{jm(1-q/p)}} < +\infty,\end{aligned}$$

and thus $u \in W^{1,q}(B_R^{m-1},\mathcal{N})$.

Next we observe that if $U \in W^{1,p}(\mathbb{B}^m_R \cap \mathbb{R}^m_+, \mathcal{N})$, then we have for every $j \in \mathbb{N}$, by (c) and by (5.2)

$$\int_{\mathbb{B}^m_{\rho_j}(a_j)\cap\mathbb{R}^m_+} |DU|^p \ge C_3 \rho_j^{m-p} \int_{\mathbb{B}^{m-1}} |Du_j|^p = C_3,$$

and thus since the balls in the sequence $(B_{\rho_j}(a_j))_{j\in\mathbb{N}}$ are disjoint,

$$\int_{\mathbb{B}^m_R\cap\mathbb{R}^m_+} |DU|^p \ge \sum_{j\in\mathbb{N}} \int_{\mathbb{B}^m_{\rho_j}(a_j)\cap\mathbb{R}^m_+} |DU|^p = \sum_{j\in\mathbb{N}} C_3 = +\infty,$$

which is a contradiction. □

Proof of the main proposition. If $p > 2$, let $q = p - 1$. If $u \in W^{1,1-p}(\mathcal{M}, \mathcal{N})$, then by the fractional Gagliardo–Nirenberg inequality (proposition 1.9)

$$\mathcal{E}^{1-\frac{1}{p},p}(u) \le C_5\, \mathcal{E}^{1,p-1}(u) < +\infty.$$

and thus $u \in W^{1-\frac{1}{p},p}(\mathcal{M}, \mathcal{N})$ and we get the conclusion by proposition 5.5 with $q = p - 1$.

If $p = 2$, we observe that if $u \in W^{1,\frac{3}{2}}(\mathcal{M}, \mathcal{N})$, then y the Gagliardo–Nirenberg inequality (proposition 1.9) and by the boundedness of $\mathcal{M}$, we get

$$\mathcal{E}^{\frac{1}{2},2}(u) \le \mathcal{E}^{\frac{3}{4},2}(u) \le \mathcal{E}^{1,\frac{3}{2}}(u) < +\infty. \qquad \square$$

The conclusion follows then by proposition 5.5 with $q = \frac{3}{2}$.

Problem 5.1. (★★★) Prove that if the group $\pi_{\lfloor \ell \rfloor}(\mathcal{N})$ is infinite for some $\ell \in \{1, \dots, \lfloor p-1 \rfloor\}$ and if $p < m$, then there exists a sequence of smooth maps $(u_j)_{j\in\mathbb{N}}$ in $C^\infty(\mathbb{B}^m, \mathbb{S}^m)$ converging to a map $u \in W^{1-\frac{1}{p},p}(\partial\mathcal{M}, \mathcal{N})$ which is not the trace of any map $U \in W^{1,p}(\mathcal{M}, \mathcal{N})$.

Theorem 5.6. *If $p \in [1, +\infty)$, if $sq < p$ and if there exists $\ell \in \{1, \dots, m-1\}$ such that $\pi_\ell(\mathcal{N})$ is infinite, then there exists a sequence of maps $(u_j)_{j\in\mathbb{N}}$ in $W^{1-\frac{1}{p},p}(\mathcal{M}, \mathcal{N})$ such that*

$$\lim_{j\to\infty} \mathcal{E}^{1,p}_{\mathrm{ext}}(u_j) = +\infty \qquad \text{and} \qquad \liminf_{j\to\infty} \frac{\mathcal{E}^{1,p}_{\mathrm{ext}}(u_j)^{\frac{sq}{p}}}{\mathcal{E}^{s,q}(u)}, \tag{5.3}$$

where

$$\mathcal{E}^{1,p}_{\mathrm{ext}} = \inf\left\{ \mathcal{E}^{1,p}(U) \mid U \in W^{1,p}(\mathcal{M}, \mathcal{N}) \text{ and } \operatorname{tr}_{\partial\mathcal{M}} U = u \right\}.$$

Theorem 5.6 shows that the extension of proposition 2.9 cannot be bounded linearly when some lower-dimensional homotopy group is infinite in contrast with the linear case where for each $u \in W^{1-\frac{1}{p},p}(\mathcal{M}, \mathbb{R}^\nu)$ there exists an extension $U \in W^{1,p}(\mathcal{M}, \mathbb{R}^\nu)$ that satisfies the estimate

$$\int_{\mathcal{M}} |DU|^p \le \mathcal{E}^{1-\frac{1}{p},p}(u).$$

Proof of theorem 5.6. This follows from theorem 5.6.Theorem 5.6 means that there is a sequence of maps $(u_j)_{j\in\mathbb{N}}$ in $W^{1,p}(\mathbb{B}^{m-1}\times\{0\},\mathcal{N})$ such that for every sequence $(U_j)_{j\in\mathbb{N}}$ in $W^{1,p}(\mathbb{B}^m\cap\mathbb{R}^m_+,\mathcal{N})$ such that $\operatorname{tr}_{\mathbb{B}^{m-1}\times\{0\}} U_j = u_j$,

$$\lim_{j\to\infty}\int_{\mathbb{B}^{m-1}}|Du_j|^p = +\infty \qquad\text{and}\qquad \inf_{j\in\mathbb{N}}\frac{\displaystyle\int_{\mathbb{B}^m\cap\mathbb{R}^m_+}|DU_j|^p}{\displaystyle\int_{\mathbb{B}^{m-1}}|Du_j|^p} > 0.$$

By the Hölder inequality this implies that for every $r\in[1,p)$,

$$\liminf_{j\to\infty}\frac{\left(\displaystyle\int_{\mathbb{B}^m\cap\mathbb{R}^m_+}|DU_j|^p\right)^{\frac{q}{p}}}{\displaystyle\int_{\mathbb{B}^{m-1}}|Du_j|^r} > 0.$$

If $1<sq<p$ we set $q=sp$, by the fractional Gagliardo–Nirenberg interpolation inequality (proposition 1.9) we obtain

$$\lim_{j\to\infty}\frac{\left(\displaystyle\int_{\mathbb{B}^m\cap\mathbb{R}^m_+}|DU_j|^p\right)^{\frac{sq}{p}}}{\mathcal{E}^{s,q}(u_j)} = +\infty,$$

the case $sq<1$ is similar. □

5.2 Quantitative obstruction for the lifting problem

We also have the following necessary condition for the existence of a lifting.

Theorem 5.7. *If $\mathcal{N}$ is compact, $s\in(0,1)$, $p\in[1,+\infty)$, $1\le sp<m$ and if $\pi:\widetilde{\mathcal{N}}\to\mathcal{N}$ is a Riemannian covering and if for every $u\in W^{s,p}(\mathcal{M},\widetilde{\mathcal{N}})$ there exists a map $\widetilde{u}\in W^{s,p}(\mathcal{M},\mathcal{N})$ such that $\pi\circ\widetilde{u}=u$, then $\widetilde{\mathcal{N}}$ is compact.*

The main step will be to construct a map with prescribed regularity properties.

Proposition 5.8. *If $1<sp<m$. If $\pi:\widetilde{\mathcal{N}}\to\mathcal{N}$ is a Riemannian covering map and if the manifold $\widetilde{\mathcal{N}}$ is not compact, then for every $\widetilde{b}\in\mathcal{N}$, there exists a map $\widetilde{u}\in C(\mathbb{B}^m\setminus\{0\})$ such that*

(i) for every $q\in[1,+\infty)$ such that $\frac{1}{q}>\frac{1}{p}+\frac{1-s}{m}$, one has $\widetilde{u}\in W^{1,q}(\mathbb{B}^m,\mathcal{N})$,

(ii) $\widetilde{u}\notin W^{s,p}(\mathbb{B}^m,\mathcal{N})$,

(iii) $\widetilde{u}=\widetilde{b}$ in $\mathbb{B}^m\setminus\mathbb{B}^m_{1/2}$.

Proposition 5.8 in turn will follow from the iteration of the following elementary construction.

Lemma 5.9. *Let $m \in \mathbb{N}$, $s \in (0,1)$ and $p \in [1,+\infty)$. There exists a constant $C > 0$. If $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ is a Riemannian covering map, then for every $\widetilde{c}, \widetilde{b} \in \widetilde{\mathcal{N}}$ and every $M \in \mathbb{R}$, there exists a map $\widetilde{u} \in C(\mathbb{B}^m, \widetilde{\mathcal{N}})$ such that*

(a) $\widetilde{u} = b$ in $\mathbb{B}^m_1 \setminus \mathbb{B}^m_{1/2}$,

(b) for every $x \in \mathbb{B}^m$,

$$|D\widetilde{u}(x)| \le 4\, d_{\widetilde{\mathcal{N}}}(\widetilde{b}, \widetilde{c}),$$

(c) for every $s \in (0,1)$ and $p \in [1,+\infty)$,

$$\mathcal{E}^{s,p}(\widetilde{u}) \ge C\, d_{\widetilde{\mathcal{N}}}(\widetilde{b}, \widetilde{c})^p.$$

Proof. Let $\widetilde{f} : [0,1] \to \widetilde{\mathcal{N}}$ be a constant velocity geodesic from $\widetilde{c}$ to $\widetilde{b}$. That is, $\widetilde{f}(0) = \widetilde{c}$, $\widetilde{f}(1) = \widetilde{b}$ and $|f'| = d_{\widetilde{\mathcal{N}}}(\widetilde{b}, \widetilde{c})$ on $[0,1]$. We define the map $\widetilde{u} \in C(\mathbb{B}^m, \widetilde{\mathcal{N}})$ for each $x \in \mathbb{B}^m$ by

$$\widetilde{u}(x) = \begin{cases} \widetilde{c} & \text{if } |x| \le \frac{1}{4}, \\ \widetilde{f}(2 - 4|x|) & \text{if } \frac{1}{4} \le |x| \le \frac{1}{2}, \\ \widetilde{b} & \text{if } \frac{1}{2} \le |x| \le 1. \end{cases}$$

We have

$$\mathcal{E}^{s,p}(\widetilde{u}) \ge d_{\widetilde{\mathcal{N}}}(\widetilde{b}, \widetilde{c})^p \int_{\mathbb{B}^m_1 \setminus \mathbb{B}^m_{1/2}} \int_{\mathbb{B}^m_{1/4}} \frac{1}{|y - x|^{m+sp}} \,\mathrm{d}x\, \mathrm{d}y,$$

where the last integral is positive. □

Proof of proposition 5.8. Since $\widetilde{\mathcal{N}}$ is connected and not compact, for every $\tilde{b} \in \widetilde{\mathcal{N}}$ and every $j \in \mathbb{N}$, there exists a point $\widetilde{c}_j \in \widetilde{N}$ such that $d_{\mathcal{N}}(\widetilde{b}, \widetilde{c}_j) = 2^{j(\frac{m}{p} - s)}$. We let $\widetilde{u}_j$ be given by lemma 5.9. We define for each $j \in \mathbb{N}$, $\rho_j \triangleq 2^{-j}$. If we set $R = 3\sqrt{m}/2$, there exists a sequence of points $(a_j)_{j \in \mathbb{N}}$ converging to 0 such that the balls $B_{\rho_j}(a_j)$ are pairwise disjoint and $\bigcup_{j \in \mathbb{N}} B_{\rho_j}(a_j) \subset B_R$. We define the map $u : B_R \to \mathcal{N}$ for every $x \in B_R$ by

$$\widetilde{u}(x) = \begin{cases} \widetilde{u}_j\big(\frac{x - a_j}{\rho_j}\big) & \text{if } x \in B_{\rho_j}(a_j) \text{ for some } j \in \mathbb{N}, \\ \widetilde{b} & \text{otherwise.} \end{cases}$$

We have by lemma 5.9 and by the definition of $\widetilde{c}_j$ and of ρ_j

$$\mathcal{E}^{s,p}(\widetilde{u}) \ge \sum_{j \in \mathbb{N}} \rho_j^{m-sp} \mathcal{E}^{s,p}(\widetilde{u}_j) \ge C_6 \sum_{j \in \mathbb{N}} \rho_j^{m-sp} d_{\mathcal{N}}(\widetilde{b}, \widetilde{c}_j)^p \ge C_6 \sum_{j \in \mathbb{N}} 1 = +\infty.$$

and on the other hand

$$\begin{aligned} \int_{\mathbb{B}^m_R} |Du|^q \le \sum_{n \in \mathbb{N}} \int_{\mathbb{B}^m_{\rho_j}(a)} |D\widetilde{u}|^q &\le \sum_{j \in \mathbb{N}} \rho_j^{m-q} \int_{\mathbb{B}^m} |D\widetilde{u}_j|^p \\ &\le C_7 \sum_{j \in \mathbb{N}} \rho_j^{m-q} d_{\mathcal{N}}(\widetilde{b}, \widetilde{c}_j)^q \\ &\le \sum_{j \in \mathbb{N}} \frac{C_8}{2^{jq(m(\frac{1}{q} - \frac{1}{p}) - (1-s))}} < +\infty, \end{aligned}$$

since by assumption $q(m(\frac{1}{q}-\frac{1}{p})-(1-s))=qm(\frac{1}{q}-\frac{1}{p}-\frac{1-s}{m})>0$. □

Proof of theorem 5.7. Let $b\in\mathcal{N}$. The set $\pi^{-1}(\{b\})\subset\widetilde{\mathcal{N}}$ is countable and can thus be written as $\pi^{-1}(\{b\})=\{\widetilde{b}_j\mid j\in\mathbb{N}\}$. We set $q=sp$ if $sp>1$ and $q=\frac{1+s}{2}p$ if $sp=1$.

For every $j\in\mathbb{N}$, let $\widetilde{u}_j$ be the map given by proposition 5.8 with $\widetilde{b}=\widetilde{b}_j$. We consider a sequence of balls $(B_{\rho_j}(a_j))_{j\in\mathbb{N}}$ such that the balls $(B_{\rho_j}(a_j))_{j\in\mathbb{N}}$ are disjoint, and for every $j\in\mathbb{N}$, we have

$$\rho_j^{m-sp}\int_{\mathbb{B}^m}|D\widetilde{u}_j|^q\le\frac{1}{2^j}.$$

and $(a_j)_{j\in\mathbb{N}}$ converges to 0. We define the map $u:\mathbb{B}^m\to\mathcal{N}$ by

$$u(x)=\begin{cases}(\pi\circ\widetilde{u}_j)\big(\frac{x-a_j}{\rho_j}\big) & \text{if } x\in B_{\rho_j}(a_j) \text{ for some } j\in\mathbb{N},\\ b & \text{otherwise.}\end{cases}$$

We immediately have $u\in W^{1,q}(\mathbb{B}^m,\mathcal{N})$. It follows from the Gagliardo–Nirenberg fractional interpolation inequality that $u\in W^{\frac{q}{p},p}(\mathbb{B}^m,\mathcal{N})$. If $sp>1$, then since $q=sp$, we have $u\in W^{s,p}(\mathbb{B}^m,\mathcal{N})$; if $sp=1$, then $q=\frac{1+p}{2}$ and $u\in W^{\frac{1+s}{2},p}(\mathbb{B}^m,\mathcal{N})\subset W^{s,p}(\mathbb{B}^m,\mathcal{N})$.

In order to conclude, assume that there exists $\widetilde{u}\in W^{s,p}(\mathbb{B}^m,\mathcal{N})$ such that $\pi\circ\widetilde{u}=$. Since $u\in C(\mathbb{B}^m\setminus\{0,a_0,a_1,\dots\},\mathcal{N})$, we have by proposition 5.8 $\widetilde{u}\in C(\mathbb{B}^m\setminus\{0,a_0,a_1,\dots\},\mathcal{N})$. By construction of u, the map $u=b$ on $\mathbb{B}^m\setminus\{0\}\setminus\bigcup_{j\in\mathbb{N}}B_{\rho_j/2}(a_j)$. Since the latter set is connected, there exists thus $j\in\mathbb{N}$ such that $\widetilde{u}=\widetilde{b}_j$ on that set. Hence, we have $\widetilde{u}(x)=\widetilde{u}_j(\frac{x-a}{\rho_j})$ for every $x\in B_{\rho_j}(a_j)$, in contradiction with the fact that $\widetilde{u}_j\notin W^{s,p}(\mathbb{B}^m,\mathcal{M})$ in view of proposition 5.8. □

Problem 5.2 (★★). Show that a counterexample to theorem 5.7 can be constructed in such a way that $u\in C^\infty(B_R\setminus\{0\},\mathcal{N})$.

Problem 5.3 (★★). Show that a counterexample to theorem 5.7 can be constructed in such a way that $\widetilde{u}$ is a limit in $W^{s,p}(\mathcal{M},\mathcal{N})$ of smooth maps.

Lemma 5.9

Theorem 5.10. *If $\mathcal{N}$ is compact, $s\in(0,1)$, $p\in[1,+\infty)$, if $\pi:\widetilde{\mathcal{N}}\to\mathcal{N}$ is a covering map and if $\widetilde{\mathcal{N}}$ is not compact, then there exists a sequence $\widetilde{u}_j\in W^{s,p}(\mathcal{M},\mathcal{N})$ such that*

$$\lim_{j\to\infty}\mathcal{E}^{s,p}(\widetilde{u}_j)=+\infty \qquad \text{and} \qquad \liminf_{j\to\infty}\frac{\mathcal{E}^{s,p}(\widetilde{u}_j)^s}{\mathcal{E}^{s,p}(\pi\circ\widetilde{u}_j)}>0.$$

In particular, if the covering $\pi:\widetilde{\mathcal{N}}\to\mathcal{N}$, then the lifting is unique up to a deck transformation, which preserves the energy of the lifting. Hence there is no linear estimate on the lifting. Theorem 5.10 extends the phenomenon of the nonexistence of lifting for subcritical dimensions $m>sp$ to the absence of linear estimates to (sub)critical dimensions $m\le sp$.

Proof of theorem 5.10. Let $\widetilde{b} \in \widetilde{\mathcal{N}}$. Since the manifold $\widetilde{\mathcal{N}}$ is not compact, there exists a sequence $(\widetilde{c}_j)_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty} d_{\widetilde{\mathcal{N}}}(\widetilde{c}_j, \widetilde{b}) = +\infty$. Let $\widetilde{u}_j$ be given by lemma 5.9. We estimate then

$$\mathcal{E}^{s,p}(\pi \circ \widetilde{u}) \le C_9\, d_{\widetilde{\mathcal{N}}}(\widetilde{c}_j, \widetilde{b})^{sp}$$

and

$$d_{\widetilde{\mathcal{N}}}(\widetilde{c}_j, \widetilde{b})^{sp} \le \mathcal{E}^{s,p}(\pi \circ \widetilde{u}).$$

The conclusion then follows. □

5.3 Comments

Theorem 5.1 is due to Bethuel [Bet14], with a statement of a necessary condition that $\pi_1(\mathcal{N})$ is finite and $\pi_2(\mathcal{N}) \simeq \dots \simeq \pi_{\lfloor p-1 \rfloor}(\mathcal{N})$ are trivial. A particular case of theorem 5.1 corresponding to $\mathcal{N} = \mathbb{S}^1$ was obtained by Fabrice Béthuel and Françoise Demengel [BD95, theorem 6]. The passage from theorem 5.6 to proposition 5.5 and theorem 5.1 is in fact a particular instance of nonlinear uniform boundedness principles for Sobolev mappings [MVS19].

Theorem 5.7 was proved for $\mathcal{N} = \mathbb{S}^1$ by Jean Bourgain, Haïm Brezis and Petru Mironescu [BBM00, theorem 2 (b)], with an example of the form $(\cos|x|^{-\alpha}, \sin|x|^{-\alpha})$ treated through the fractional Gagliardo–Nirenberg interpolation inequality. This counterexample was transfered to noncompact universal coverings by Fabrice Béthuel and David Chiron [BC07, proposition 2]; the proof is based on the existence of a ray (unbounded minimizing geodesic) and goes through any noncompact Riemannian covering. The proof of theorem 5.7 presented here highlights the connection of the analytical obstruction with the failure of linear bounds, which can also be connected through a general uniform boundedness principle.

The same approach of bundling together maps satisfying worsening bounds has been used by Fabrice Béthuel and David Chiron to prove the existence of maps in $W^{1,p}(\mathcal{M}, \mathbb{S}^2)$ that can be lifting by the Hopf fibration $\pi : \mathbb{S}^3 \to \mathbb{S}^2$ [BC07, theorem 4 d)].

Theorem 5.10 is due for $\pi : \mathbb{R} \to \mathbb{S}^1$ when $m = 1$ to Benoît Merlet [Mer06, theorem 1.1] and when $m \ge 2$ to Petru Mironescu and Ioana Molnar [MM15, proposition 5.7].

6 Liftings

6.1 Construction of a lifting

We first construct a lifting for maps in $W^{1,p}(\mathcal{M},\mathcal{N})$ when the domain manifold $\mathcal{M}$ is simply-connected.

Theorem 6.1. *Assume that the manifold $\mathcal{M}$ is compact and simply-connected and $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ is a covering map. For every $u \in W^{1,p}(\mathcal{M},\mathcal{N})$, there exists a map $\widetilde{u} \in W^{1,p}(\mathcal{M},\widetilde{\mathcal{N}})$ such that $\pi \circ \widetilde{u} = u$.*

Since the Riemannian covering map $\pi : \mathcal{M} \to \mathcal{N}$ is by definition a local isometry, if $\widetilde{u} \in W^{1,p}(\mathcal{M},\widetilde{\mathcal{N}})$, then for almost every $x \in \mathcal{M}$,

$$|\mathrm{D}(\pi \circ \widetilde{u})(x)| = |\mathrm{D}\widetilde{u}(x)|,$$

and thus for every $q \in [1,+\infty)$, $\widetilde{u} \in W^{1,p}(\mathcal{M},\widetilde{\mathcal{N}})$ if and only if $\pi \circ \widetilde{u} \in W^{1,p}(\mathcal{M},\widetilde{\mathcal{N}})$.

Lemma 6.2. *If $\ell \geq 2$, if $u \in W^{1,2}(\mathbb{Q}^\ell,\mathcal{N})$, if $\widetilde{g} \in W^{1,1}(\mathbb{Q}^\ell,\widetilde{\mathcal{N}})$ and if $\operatorname{tr}_{\partial \mathbb{Q}^\ell} u = \pi \circ \widetilde{g}$ almost everywhere on $\partial\mathbb{Q}^\ell$, then there exists a map $\widetilde{u} \in W^{1,2}(\mathbb{Q}^\ell,\mathcal{N})$ such that $\pi \circ \widetilde{u} = u$ on $\mathbb{Q}^\ell$ and $\operatorname{tr}_{\partial \mathbb{Q}^\ell} \widetilde{u} = \widetilde{g}$ on $\partial\mathbb{Q}^\ell$.*

The proof of lemma 6.2 rests on two lemmas describing the behaviour of a Sobolev functions on almost every one-dimensional line and on almost every two-dimensional plane.

Lemma 6.3. *For every $u \in W^{1,p}(\mathbb{Q}^\ell,\mathbb{R}^\nu)$, then there exists a negligible sets $E \subset \mathbb{Q}^\ell$ and $F \subset \mathbb{Q}^{\ell-1}$ such that*

(i) if $y \in \mathbb{Q}^{\ell-1} \setminus F$, then $u|_{\mathbb{Q}^1\times\{y\}} \in W^{1,p}(\mathbb{Q}^1 \times \{y\},\mathbb{R}^\nu)$ and $\mathrm{D}(u|_{\mathbb{Q}^1\times\{y\}}) = (\mathrm{D}u)|_{\mathbb{Q}^1\times\{y\}}$,

(ii) if $y \in \mathbb{Q}^{\ell-1} \setminus F$, then $\operatorname{tr}_{\partial\mathbb{Q}^1\times\{y\}} u|_{\mathbb{Q}^1\times\{y\}} = (\operatorname{tr}_{\partial\mathbb{Q}^{\ell-1}} u)|_{\partial\mathbb{Q}^1\times\{y\}}$,

(iii) if $(x,y) \in \mathbb{Q}^\ell \setminus (E \cup \mathbb{Q}^1 \times F)$, then $\operatorname{tr}_{\{(x,y)\}} u|_{\mathbb{Q}^1\times\{y\}} = u(x,y)$.

Proof. Let $(u_j)_{j\in\mathbb{N}}$ be a sequence of functions in $C^1(\mathbb{Q}^\ell,\mathbb{R}^\nu)$ such that for every $j \in \mathbb{N}$,

$$\int_{\mathbb{Q}^\ell} |\mathrm{D}u_j - \mathrm{D}u|^p + |u_j - u|^p \leq \frac{1}{2^j}. \tag{6.1}$$

By elementary trace theory, we have

$$\int_{\partial\mathbb{Q}^1\times\mathbb{Q}^{\ell-1}} |u_j - u|^p \le \frac{C_1}{2^j}. \tag{6.2}$$

We define the sets

$$E \triangleq \Big\{(x,y) \in \mathbb{Q}^1 \times \mathbb{Q}^{\ell-1} = \mathbb{Q}^\ell \mid \sum_{j\in\mathbb{N}} |u_j(x,y) - u(x,y)|^p = +\infty\Big\}$$

and

$$F \triangleq \left\{y \in \mathbb{Q}^{\ell-1} \mid \sum_{j\in\mathbb{N}} \int_{\mathbb{Q}^1\times\{y\}} |\mathrm{D}u_j - \mathrm{D}u|^p + |u_j - u|^p = +\infty \text{ or } \partial\mathbb{Q}^1 \times \{y\} \notin E\right\}$$

In view of (6.1) and (6.2), both sets E and F are negligible.

Moreover, for every $y \in \mathbb{Q}^{\ell-1} \smallsetminus F$, the sequence $(u_j|_{[0,1]\times\{y\}})_{j\in\mathbb{N}}$ converges to the function $u|_{[0,1]\times\{y\}}$ and for every $(x,y) \in \mathbb{Q}^\ell \smallsetminus E$, the sequence of vectors $(u_j(x,y))_{j\in\mathbb{N}}$ converges to the vector $u(x,y)$. Moreover, $\partial\mathbb{Q}^1 \times (\mathbb{Q}^{\ell-1} \smallsetminus F) \subseteq \mathbb{Q}^\ell \smallsetminus E$. The conclusion follows from continuity of weak derivatives and of the trace in Sobolev spaces. □

We also use the two-dimensional version of this lemma.

Lemma 6.4. *For every $u \in W^{1,p}(\mathbb{Q}^\ell, \mathbb{R}^\nu)$, then there exists a negligible sets $E \subset \mathbb{Q}^{\ell-1}$ and $F \subset \mathbb{Q}^{\ell-2}$ such that*

(i) if $z \in \mathbb{Q}^{\ell-2} \smallsetminus G$, then $u|_{\mathbb{Q}^2\times\{z\}} \in W^{1,p}(\mathbb{Q}^2 \times \{z\}, \mathbb{R}^\nu)$ and $\mathrm{D}(u|_{\mathbb{Q}^2\times\{z\}}) = (\mathrm{D}u)|_{\mathbb{Q}^2\times\{z\}}$,

(ii) if $z \in \mathbb{Q}^{\ell-2} \smallsetminus G$, then $\operatorname{tr}_{\partial\mathbb{Q}^2\times\{z\}} u|_{\mathbb{Q}^2\times\{z\}} = (\operatorname{tr}_{\partial\mathbb{Q}^\ell} u)|_{\partial\mathbb{Q}^2\times\{z\}}$,

(iii) if $(y,z) \in \mathbb{Q}^{\ell-1} \smallsetminus (E \cup \mathbb{Q}^1 \times F)$, then $\operatorname{tr}_{\mathbb{Q}^1\times\{(y,z)\}} u|_{\mathbb{Q}^2\times\{z\}} = u|_{\mathbb{Q}^1\times\{(y,z)\}}$.

The proof of lemma 6.4 is similar to the proof of lemma 6.3.

Problem 6.1 (★). Prove lemma 6.4.

Proof of lemma 6.2. In view lemma 6.3 and of the lifting of continuous maps on an interval, for almost every $(\ell - 1)$–dimensional face $Q \subset \partial\mathbb{Q}^\ell$, we define the map $\widetilde{u}_Q : \mathbb{Q}^\ell \to \widetilde{N}$ almost everywhere on $\mathbb{Q}^\ell$ by requiring it to be continuous on almost every segment perpendicular to Q and coinciding with $\widetilde{g}$ on $\mathbb{Q}^\ell$. Moreover, $\widetilde{u}_Q$ is measurable and weakly differentiable in the direction normal to Q.

For every $x \in \mathbb{Q}^\ell$ on which $\widetilde{u}_{Q_1}(x)$ and $\widetilde{u}_{Q_2}(x)$ are defined, let Σ be a rectangle consisting of the two segments that have been used to define this values (which are colinear if Q_1 and Q_2 are opposite sides) and the other sides are taken in $\partial\mathbb{Q}^\ell$. By applying lemma 6.3 and lemma 6.4 for almost every $x \in \mathbb{Q}^\ell$, we have $u \in W^{1,2}(\Sigma, \mathcal{N})$ and $\operatorname{tr}_{\partial\Sigma} u|_\Sigma = u|_{\partial\Sigma} \in W^{1,1}(\partial\Sigma)$ and we have $\widetilde{g}|_{\Sigma\cap\partial\mathbb{Q}^\ell} \in W^{1,1}(\Sigma \cap \partial\mathbb{Q}^\ell)$. By proposition 2.11, $u|_{\partial\Sigma}$ is almost everywhere equal to the restriction of a continuous function (and is equal at the vertices), and hence by classical properties of lifting of continuous mappings, one should have $\widetilde{u}_{Q_1}(x)$ and $\widetilde{u}_{Q_2}(x)$.

We define the map $\widetilde{u}$ to be the common value of these liftings on a full measure set. By construction $\widetilde{u}$ is weakly differentiable in all the directions of axis and has the required trace on each face. □

Problem 6.2 (★★★★). Prove that u_Q is measurable.

Proof of theorem 6.1. Since $\mathcal{M}$ is a compact manifold, it can be embedded isometrically in some Euclidean space $\mathbb{R}^\mu$. Let $\Pi_{\mathcal{M}} : \mathcal{M} + \mathbb{B}^\mu_{\delta_{\mathcal{M}}} \to \mathcal{M}$ denote the nearest point retraction. We define the map $U = u \circ \Pi$. We check that $U \in W^{1,p}(\mathcal{M} + \mathbb{B}^\mu_{\delta_{\mathcal{M}}}, \mathcal{N})$.

Let $\mathcal{U}^\mu_\eta = \{Q \in \mathcal{L}^\mu_\eta \mid Q \subset \mathcal{M} + \mathbb{B}^\mu_{\delta_{\mathcal{M}}/2}$. When $\eta > 0$ is small enough, $\bigcup \mathcal{U}^\mu_\eta Q$ is a deformation retract of $\mathcal{M} + \mathbb{B}^\mu_{\delta_{\mathcal{M}}/2}$. We denote by $\mathcal{U}^\ell_\eta$ the set of ℓ-dimensional faces of the cubes in $\mathcal{U}^\mu_\eta$.

By lemma 6.3, for almost every $h \in \mathbb{B}^\mu_{\delta_{\mathcal{M}}/2}$, $U|_{\bigcup \mathcal{U}^1_\eta + h} \in W^{1,p}(\bigcup \mathcal{U}^1_\eta + h)$, that is the function is in the Sobolev space on every segment and the traces and the vertices coincide. Moreover, we can ensure by lemma 6.4, that $U|_{\bigcup \mathcal{U}^2_\eta + h} \in W^{1,p}(\bigcup \mathcal{U}^1_\eta + h)$ and that $\operatorname{tr}_{\bigcup \mathcal{U}^1_\eta + h} U|_{\bigcup \mathcal{U}^2_\eta + h} = \operatorname{tr}_{\bigcup \mathcal{U}^1_\eta + h} U$. By proposition 2.11, the map $U|_{\bigcup \mathcal{U}^1_\eta + h}$ is almost everywhere equal to the restriction of a continuous map $F \in C(\bigcup \mathcal{U}^2_\eta + h, \mathcal{N})$. Since $\mathcal{M}$ is simply connected, the set $\bigcup \mathcal{U}^\mu_\eta + h$ is simply connected and the set $\bigcup \mathcal{U}^2_\eta + h$ is also simply connected. By classical results on the lifting, there exists $\widetilde{F} \in C(\bigcup \mathcal{U}^2_\eta + h, \widetilde{\mathcal{N}})$ such that $\pi \circ \widetilde{F} = F$. In particular, we have $\widetilde{F}|_{\bigcup \mathcal{U}^1_\eta + h} \in W^{1,p}(\bigcup \mathcal{U}^1_\eta + h, \widetilde{\mathcal{N}})$. We set $\widetilde{U} = \widetilde{F}$ on $\bigcup \mathcal{U}^1_\eta + h$. We apply now lemma 6.2, to define successively the map $\widetilde{U}$ on the sets $\bigcup \mathcal{U}^2_\eta + h, \dots, \bigcup \mathcal{U}^\mu_\eta + h$.

Finally, we observe that since $\widetilde{U} \in W^{1,p}(\bigcup \mathcal{U}^\mu_\eta + h)$, there exists $\widetilde{u} \in W^{1,p}(\mathcal{M}, \widetilde{\mathcal{N}}$ such that $\widetilde{U} = \widetilde{u} \circ \Pi_{\mathcal{M}}$ on $\mathcal{M} + \mathbb{B}^\mu_{\delta_{\mathcal{M}}/4}$. □

Problem 6.3 (★). Prove theorem 6.1 when $\mathcal{M} = \mathbb{R}^\nu$.

Problem 6.4 (★★★). Prove theorem 6.1 when $\mathcal{M} \subset \mathbb{R}^\nu$ is an open set, with a smooth boundary.

6.2 Fractional construction of a lifting

When $0 < sp < 1$, Sobolev maps are not regular enough to guarantee for example some uniqueness property for the lifting; this leaves much room for the construction of a lifting, which is possible without any restriction.

Theorem 6.5. *Let $s \in (0,1)$ and $p \in [1, +\infty)$. If $sp < 1$ and if $\pi : \widetilde{\mathcal{M}} \to \mathcal{N}$ is a Riemannian covering map, then for every $u \in W^{s,p}(\mathcal{M}, \mathcal{N})$, there exists $\widetilde{u} \in W^{s,p}(\mathcal{M}, \widetilde{\mathcal{N}})$ such that $\pi \circ \widetilde{u} = u$.*

Proof of theorem 6.5 when $\mathcal{M} = \mathbb{Q}^m \triangleq [0,1]^m$. We define for each $j \in \mathbb{N}$, the set of cubes

$$\mathcal{Q}_j = \left\{\left[\tfrac{k_1}{2^m}, \tfrac{k_1+1}{2^m}\right) \times \dots \times \left[\tfrac{k_m}{2^j}, \tfrac{k_m+1}{2^j}\right] \mid k_1, \dots, k_m \in \{0, \dots, 2^j - 1\}\right\}$$

and we define the function $v_j : [0,1)^m \to \mathbb{R}^\nu$ by defining if $Q \in \mathcal{Q}_j$ and $x \in Q_j$,

$$v_j(x) = \fint_Q u.$$

By classical properties of measurable functions, the sequence $(v_j)_{j\in\mathbb{N}}$ converges almost everywhere to u.

Let $\Pi : \mathcal{N} + \mathbb{B}^\nu_{\delta_\mathcal{N}} \to \mathcal{N}$ be the nearest-point retraction given by lemma 2.2 and let $b \in \mathcal{N}$. We define

$$u_j(x) = \begin{cases} \Pi(u_j(x)) & \text{if } u_j(x) \in \mathcal{N} + \mathbb{B}^\nu_{\delta_\mathcal{N}}, \\ b & \text{otherwise.} \end{cases}$$

By definition of u_j, there exists $a \in \mathcal{N}$ such that $u_0 = a$ in $[0,1)^m$. We choose $\widetilde{a} \in \mathcal{N}$ such that $\pi(\widetilde{a}) = a$. We assume now that $\widetilde{u}_j$ has been defined. Since the manifold $\mathcal{N}$ is connected, and thus path-connected, for every $x \in [0,1)^m$, there exists a minimizing geodesic $\gamma \in C^1([0,1], \mathcal{N})$ such that $\gamma(0) = u_j(x)$ and $\gamma(1) = u_{j+1}(x)$. By the classical lifting theory, there exists $\widetilde{\gamma} \in C^1([0,1], \widetilde{N})$ such that $\pi \circ \widetilde{\gamma} = \gamma$ and $\widetilde{\gamma}(0) = \widetilde{u}_j(x)$. We define $\widetilde{u}_{j+1}(x) \triangleq \widetilde{\gamma}(1)$ and we observe that $\pi \circ \widetilde{u}_{j+1} = u_{j+1}$ and that

$$d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1}(x), \widetilde{u}_j(x)) = d_\mathcal{N}(u_{j+1}(x), u_j(x)) \tag{6.3}$$

We claim that for every $x \in [0,1)^m$,

$$d_\mathcal{N}(u_{j+1}(x), u_j(x)) \le C_2\big(d_\mathcal{N}(v_{j+1}(x), u(x)) + d_\mathcal{N}(v_j(x), u(x))\big) \tag{6.4}$$

If $\operatorname{dist}(v_j(x), \mathcal{N}) \le \delta_\mathcal{N}$ and $\operatorname{dist}(v_{j+1}(x), \mathcal{N}) \le \delta_\mathcal{N}$, then by the Lipschitz-continuity of Π and by the triangle inequality, we have

$$\begin{aligned} d_\mathcal{N}(u_{j+1}(x), u_j(x)) &\le C_3 d_\mathcal{N}(u_{j+1}(x), u_j(x)) \\ &\le C_3\big(d_\mathcal{N}(v_{j+1}(x), u(x)) + d_\mathcal{N}(v_j(x), u(x))\big). \end{aligned} \tag{6.5}$$

Otherwise, we have

$$d_\mathcal{N}(u_{j+1}(x), u_j(x)) \le \frac{\operatorname{diam} \mathcal{N}}{\delta_\mathcal{N}}\big(d_\mathcal{N}(v_{j+1}(x), u(x)) + d_\mathcal{N}(v_j(x), u(x))\big). \tag{6.6}$$

The inequality (6.4) follows then from (6.5) and (6.6).

From (6.4) we deduce that

$$\begin{aligned} &\int_{[0,1)^m} d_\mathcal{N}(u_{j+1}(x), u_j(x))^p \,\mathrm{d}x \\ &\qquad \le C_4\left(\int_{[0,1)^m} d_\mathcal{N}(v_{j+1}(x), u(x))^p \,\mathrm{d}x + \int_{[0,1)^m} d_\mathcal{N}(v_j(x), u(x))^p \,\mathrm{d}x\right). \end{aligned} \tag{6.7}$$

We compute for every $j \in \mathbb{N}$,

$$\int_{[0,1)^m} d_{\mathcal{N}}(v_j(x), u(x))^p \,\mathrm{d}x = \sum_{Q \in \mathcal{Q}_j} 2^{jm} \int_Q \int_Q d_{\mathcal{N}}(u(y), u(x))^p \,\mathrm{d}y \,\mathrm{d}x.$$

We observe now that if $x, y \in Q$ for some $Q \in \mathcal{Q}_j$, then by Pythagoras' theorem, $|y - x| \le \sqrt{m} 2^{-j}$, and thus

$$\sum_{\substack{j \in \mathbb{N} \\ \{x,y\} \subset Q \\ Q \in \mathcal{Q}_j}} \sum_{x,y \in Q} 2^{j(m+sp)} \le \frac{C_5}{|y - x|^{m+sp}},$$

and

$$\sum_{j \in \mathbb{N}} 2^{jsp} \int_{[0,1)^m} d_{\mathcal{N}}(v_j(x), u(x))^p \,\mathrm{d}x \le C_6 \iint\limits_{[0,1)^m \times [0,1)^m} \frac{d_{\mathcal{N}}(u(y), u(x))^p}{|y - x|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x. \tag{6.8}$$

and finally by (6.7)

$$\sum_{j \in \mathbb{N}} 2^{jsp} \int_{[0,1)^m} d_{\mathcal{N}}(u_{j+1}(x), u_j(x))^p \,\mathrm{d}x \le C_7 \iint\limits_{[0,1)^m \times [0,1)^m} \frac{d_{\mathcal{N}}(u(y), u(x))^p}{|y - x|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x. \tag{6.9}$$

By (6.3) and (6.9), we have

$$\sum_{j \in \mathbb{N}} 2^{jsp} \int_{[0,1)^m} d_{\mathcal{N}}(\widetilde{u}_{j+1}(x), \widetilde{u}_j(x))^p \,\mathrm{d}x \le C_7 \iint\limits_{[0,1)^m \times [0,1)^m} \frac{d_{\mathcal{N}}(u(y), u(x))^p}{|y - x|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x. \tag{6.10}$$

In particular, for almost every $x \in [0,1)^m$,

$$\sum_{j \in \mathbb{N}} 2^{jsp} d_{\mathcal{N}}(\widetilde{u}_{j+1}(x), \widetilde{u}_j(x))^p < +\infty,$$

and thus if $k < \ell$, one has by Hölder's inequality

$$\begin{aligned} d_{\mathcal{N}}(\widetilde{u}_\ell(x), \widetilde{u}_k(x)) &\le \sum_{j=k}^{\ell-1} d_{\mathcal{N}}(\widetilde{u}_{j+1}(x), \widetilde{u}_j(x)) \\ &\le \left(\sum_{j=k}^{\ell-1} 2^{jsp} d_{\mathcal{N}}(\widetilde{u}_{j+1}(x), \widetilde{u}_j(x))^p \right)^{\frac{1}{p}} \left(\sum_{j=k}^{\ell-1} \frac{1}{2^{\frac{jsp}{p-1}}} \right)^{1-\frac{1}{p}} \\ &\le \frac{C_8}{2^{ks}} \left(\sum_{j \in \mathbb{N}} 2^{jsp} d_{\mathcal{N}}(\widetilde{u}_{j+1}(x), \widetilde{u}_j(x))^p \right)^{\frac{1}{p}}. \end{aligned}$$

The sequence $(\widetilde{u}_j(x))_{j \in \mathbb{N}}$ thus a Cauchy sequence in $\widetilde{\mathcal{N}}$ that converges to some $\widetilde{u}(x) \in \mathcal{N}$ and for almost every $x \in [0,1)^m$, $\pi \circ \widetilde{u}(x) = \lim_{j \to \infty} \pi \circ \widetilde{u}_j(x) = \lim_{j \to \infty} u_j(x) = u(x)$.

It remains to prove that $u \in W^{s,p}([0,1)^m, \mathcal{N})$. We define for each $j \in \mathbb{N}$ the set

$$\Delta_j \triangleq \bigcup_{Q \in \mathcal{Q}_j} Q \times Q \subset [0,1)^m \times [0,1)^m.$$

For every $x, y \in [0,1)^m \times [0,1)^m$, if $(x,y) \in \Delta_k$ and $\ell \geq k$, we have by the triangle inequality

$$\begin{aligned} d_{\widetilde{\mathcal{N}}}(\widetilde{u}_\ell(y), \widetilde{u}_\ell(x)) &\leq \sum_{j=k}^{\ell} d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1}(y), \widetilde{u}_j(y)) + d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1}(x), \widetilde{u}_j(x)) + d_{\widetilde{\mathcal{N}}}(\widetilde{u}_k(y), \widetilde{u}_k(x)) \\ &= \sum_{j=k}^{\ell} d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1}(y), \widetilde{u}_j(y)) + d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1}(x), \widetilde{u}_j(x)), \end{aligned}$$

since by construction $\widetilde{u}_k(y) = \widetilde{u}_k(x)$ if $(x,y) \in \Delta_\ell$. By definition of $\widetilde{u}$ we have for every $x, y \in \Delta_k$,

$$d_{\widetilde{\mathcal{N}}}(\widetilde{u}(y), \widetilde{u}(x)) \leq \sum_{j=k}^{\infty} d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1}(y), \widetilde{u}_j(y)) + d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1}(x), \widetilde{u}_j(x)).$$

If $p = 1$, this implies that for every $k \in \mathbb{N}$,

$$\iint\limits_{\Delta_k \setminus \Delta_{k+1}} \frac{d_{\widetilde{\mathcal{N}}}(\widetilde{u}(y), \widetilde{u}(x))}{|y-x|^{m+s}} \,\mathrm{d}y \,\mathrm{d}x \leq \sum_{j=k}^{\infty} \iint\limits_{\Delta_k \setminus \Delta_{k+1}} \frac{d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1}(x), \widetilde{u}_j(x))}{|y-x|^{m+s}} \,\mathrm{d}y \,\mathrm{d}x.$$

and thus by summing over $k \in \mathbb{N}$, we obtain

$$\begin{aligned} \iint\limits_{[0,1)^m \times [0,1)^m} \frac{d_{\widetilde{\mathcal{N}}}(\widetilde{u}(y), \widetilde{u}(x))}{|y-x|^{m+s}} \,\mathrm{d}y \,\mathrm{d}x &\leq \sum_{j=0}^{\infty} \sum_{k=0}^{j} \iint\limits_{\Delta_k \setminus \Delta_{k+1}} \frac{d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1}(x), \widetilde{u}_j(x))}{|y-x|^{m+s}} \,\mathrm{d}y \,\mathrm{d}x \\ &= \sum_{j=0}^{\infty} \iint\limits_{[0,1)^m \times [0,1)^m \setminus \Delta_{j+1}} \frac{d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1}(x), \widetilde{u}_j(x))}{|y-x|^{m+s}} \,\mathrm{d}y \,\mathrm{d}x. \end{aligned} \tag{6.11}$$

If $p > 1$, have for every $x \in [0,1)^m$,

$$\begin{aligned} &\Bigl(\sum_{j=k}^{\infty} d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1}(x), \widetilde{u}_j(x))\Bigr)^p \\ &\quad\leq 2^{p-1} \Biggl(\Bigl(\sum_{\substack{j \geq k \\ 2^j|y-x| \geq 1}} d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1}(x), \widetilde{u}_j(x))\Bigr)^p + \Bigl(\sum_{\substack{j \geq k \\ 2^j|y-x| < 1}} d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1}(x), \widetilde{u}_j(x))\Bigr)^p\Biggr). \end{aligned} \tag{6.12}$$

By the discrete Hölder inequality, we bound the first term in the right-hand side of

(6.12) for every $\delta > 0$ by

$$\begin{aligned}\Bigg(\sum_{\substack{j\ge k\\ 2^j|y-x|\ge 1}} d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1}(x),\widetilde{u}_j(x))\Bigg)^p &\le \sum_{\substack{j\ge k\\ 2^j|y-x|\ge 1}} 2^{j\delta} d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1}(x),\widetilde{u}_j(x))^p \Bigg(\sum_{\substack{j\ge k\\ 2^j|y-x|\ge 1}} 2^{-j\frac{\delta}{p-1}}\Bigg)^{p-1}\\ &\le C_9 \sum_{j\ge k} 2^{j\delta}\,|y-x|^{\delta} d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1}(x),\widetilde{u}_j(x))^p.\end{aligned}$$

Similarly, we have for the second term in (6.12) for every $\delta > 0$ by Hölder's inequality

$$\begin{aligned}\Bigg(\sum_{\substack{j\ge k\\ 2^j|y-x|<1}} d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1}(x),\widetilde{u}_j(x))\Bigg)^p &\le \sum_{\substack{j\ge k\\ 2^j|y-x|<1}} 2^{-\delta j} d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1}(x),\widetilde{u}_j(x))^p \Bigg(\sum_{\substack{j\ge k\\ 2^j|y-x|<1}} 2^{j\frac{\delta}{p-1}}\Bigg)^{p-1}\\ &\le C_{10} \sum_{j\ge k} \frac{d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1}(x),\widetilde{u}_j(x))^p}{2^{j\delta}|y-x|^{\delta}}.\end{aligned}$$

Therefore, by integration and by symmetry we have for every $k \in \mathbb{N}$,

$$\begin{aligned}\iint\limits_{\Delta_k\setminus\Delta_{k+1}} \frac{d_{\widetilde{\mathcal{N}}}(\widetilde{u}(y),\widetilde{u}(x))^p}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x &\le C_{11}\sum_{j=k}^{\infty} 2^{j\delta} \iint\limits_{\Delta_k\setminus\Delta_{k+1}} \frac{d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1}(x),\widetilde{u}_j(x))^p}{|y-x|^{m+sp-\delta}}\,\mathrm{d}y\,\mathrm{d}x\\ &\quad + C_{12}\sum_{j=k}^{\infty} 2^{-j\delta} \iint\limits_{\Delta_k\setminus\Delta_{k+1}} \frac{d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1}(x),\widetilde{u}_j(x))^p}{|y-x|^{m+sp+\delta}}\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

By summing over $k \in \mathbb{N}$, we get

$$\begin{aligned}
\iint\limits_{[0,1)^m\times[0,1)^m} \frac{d_{\widetilde{\mathcal{N}}}(\widetilde{u}(y),\widetilde{u}(x))^p}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x
&\le C_{11}\sum_{j=0}^{\infty}\sum_{k=0}^{j}2^{j\delta}\iint\limits_{\Delta_k\setminus\Delta_{k+1}} \frac{d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1}(x),\widetilde{u}_j(x))^p}{|y-x|^{m+sp-\delta}}\,\mathrm{d}y\,\mathrm{d}x\\
&\quad + C_{12}\sum_{j=0}^{\infty}\sum_{k=0}^{j}2^{-j\delta}\iint\limits_{\Delta_k\setminus\Delta_{k+1}} \frac{d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1}(x),\widetilde{u}_j(x))^p}{|y-x|^{m+sp+\delta}}\,\mathrm{d}y\,\mathrm{d}x\\
&= C_{11}\sum_{j=0}^{\infty}2^{j\delta}\iint\limits_{[0,1)^m\times[0,1)^m\setminus\Delta_{j+1}} \frac{d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1}(x),\widetilde{u}_j(x))^p}{|y-x|^{m+sp-\delta}}\,\mathrm{d}y\,\mathrm{d}x\\
&\quad + C_{12}\sum_{j=0}^{\infty}2^{-j\delta}\iint\limits_{[0,1)^m\times[0,1)^m\setminus\Delta_{j+1}} \frac{d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1}(x),\widetilde{u}_j(x))^p}{|y-x|^{m+sp+\delta}}\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}\tag{6.13}$$

We observe that for if $Q \in \mathcal{Q}_{j+1}$, since by construction, the maps $\widetilde{u}_j$ and $\widetilde{u}_{j+1}$ are constant on Q, we have for every $x \in Q$,

$$d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1}(x),\widetilde{u}_j(x))^p = \frac{1}{2^{jm}}\int_Q d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1},\widetilde{u}_j)^p.$$

hence for every $\beta \in (0,1)$

$$\begin{aligned}
\iint\limits_{Q\times[0,1)^m\setminus Q\times Q} \frac{d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1}(x),\widetilde{u}_j(x))^p}{|y-x|^{m+\beta}}\,\mathrm{d}y\,\mathrm{d}x
&= \frac{1}{2^{jm}}\int_Q d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1},\widetilde{u}_j)^p \iint\limits_{Q\times(\mathbb{R}^m\setminus Q)} \frac{1}{|y-x|^{m+\beta}}\,\mathrm{d}y\,\mathrm{d}x\\
&= C_{13}2^{j\beta}\int_Q d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1},\widetilde{u}_j)^p,
\end{aligned}\tag{6.14}$$

since $\beta \in (0,1)$, the constant depending on β. By summing over $Q \in \mathcal{Q}_{j+1}$ the estimate (6.14), we deduce that

$$\iint\limits_{[0,1)^m\times[0,1)^m\setminus\Delta_{j+1}} \frac{d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1}(x),\widetilde{u}_j(x))^p}{|y-x|^{m+\beta}}\,\mathrm{d}y\,\mathrm{d}x \le C_{14}2^{j\beta}\int_{[0,1)^m} d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1},\widetilde{u}_j)^p. \tag{6.15}$$

Thus in view of (6.11) if $p = 1$, or (6.13) if $p > 1$ and if we choose $\delta > 0$ such that $0 < sp - \delta$ and $sp + \delta < 1$, we deduce that

$$\iint\limits_{[0,1)^m\times[0,1)^m} \frac{d_{\widetilde{\mathcal{N}}}(\widetilde{u}(y),\widetilde{u}(x))^p}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x \le C_{15}\sum_{j\in\mathbb{N}}2^{jsp}\int_{[0,1)^m} d_{\widetilde{\mathcal{N}}}(\widetilde{u}_{j+1},\widetilde{u}_j)^p. \tag{6.16}$$

By (6.10), we conclude that

$$\iint\limits_{[0,1)^m\times[0,1)^m} \frac{d_{\widetilde{\mathcal{N}}}(\widetilde{u}(y),\widetilde{u}(x))^p}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x \le C_{16} \iint\limits_{[0,1)^m\times[0,1)^m} \frac{d_{\mathcal{N}}(u(y),u(x))^p}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x. \quad \square \tag{6.17}$$

Problem 6.5 (★★)**.** Show that for every measurable $u : \mathcal{M} \to \mathcal{N}$ there exists a bounded and measurable map $\widetilde{u} : \mathcal{M} \to \widetilde{\mathcal{N}}$ such that $\pi \circ \widetilde{u} = u$ on $\mathcal{M}$.

6.3 Remarks and comments

Theorem 6.1 for $\mathcal{M} = \mathbb{B}^m$ and $\mathcal{N} = \mathbb{S}^1$ due to Fabrice Béthuel and Zheng Xiaomin [BZ88, lemma 1 (i)] and to Fabrice Béthuel and David Chiron when $\mathcal{M}$ is simply connected and $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ is the universal covering of $\mathcal{N}$ [BC07, theorem 1]. John Ball and Argir Zarnescu, and Domenico Mucci have proved theorem 6.1 when $\pi : \mathbb{R}P^2 \to \mathbb{S}^2$ is the (universal) double covering of the projective plane $\mathbb{R}P^n$ by the sphere $\mathbb{S}^n$ [BZ11, theorem 2; Muc12, theorem 1.1]).

Our proof of theorem 6.1 follows the strategy of Fabrice Béthuel and Zheng Xiaomin [BZ88, lemma 1 (i)] and its adaptation to the universal covering by Fabrice Béthuel and David Chiron [BC07, theorem 1] and provides the detail on the adaptation to the case where $\mathcal{M}$ is not homeomorphic to a ball. An alternative would be to follow Mucci's approach [Muc12, theorem 3.5] and to prove that maps that are continuous outside singularities of codimension 3 are strongly dense in $W^{1,p}(\mathcal{M},\mathcal{N})$ (see [Bet91, theorem 2; HL03a, theorem 6.1]), that such maps have a lifting with the same Sobolev energy and thus the sequence of liftings of the approximations converges (Ball and Zarnescu [BZ11, theorem 2] follow the same strategy but use the more delicate property of weak density of smooth maps [PR03] whose full power does seem to be required). When $\pi : \mathbb{R} \to \mathbb{S}^1$, Carbou has constructed the lifting $\widetilde{u}$ by first computing its derivative in terms of the derivative of u and showing that this equation admits a solution [Car92, proposition 1] (see also [Mir07a, proof of theorem 3.1 (ii)]).

Theorem 6.5 is due for $\mathcal{N} = \mathbb{S}^1$ to Jean Bourgain, Haïm Brezis and Petru Mironescu [BBM00] and was extended to the case where $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ is a universal covering by Fabrice Béthuel and David Chiron [BC07]. The construction of the lifting as a limit of liftings on approximations on dyadic cubes that are as close as possible comes from Bourgain, Brezis and Mironescu [BBM00], who deduce convergence and boundedness estimates (6.9) and (6.16) that can be obtained from Bourdaud's [Bou95]; they prove also these estimates with a particular attention to the behaviour of the constants in the equivalences. Bethuel and Chiron [BC07] follow the same strategy and propose a simplified argument for the estimate (6.17) (it was not clear to us how the insertion in [BC07] of (A.13) and (A.14) gives (A.12) is performed when $p > 1$). Our proof still follows the same strategy and relies on the same estimates, that we have written out at an elementary level and in a nonlinear formulation.

The constant appearing in the construction of theorem 6.5 is not sharp when in the limit $sp \to 1$. When $p = 2$, $s < \frac{1}{2}$ and $\pi : \mathbb{R} \to \mathbb{S}^1$, Jean Bourgain, Haïm Brezis and Petru

Mironescu have given a construction with sharp constant [BBM00, theorem 4]. These bounds on lifting yield in turn sharp estimates on Ginzburg–Landau functionals in terms of norms of the boundary data in the trace space $W^{1/2,2}(\partial\mathcal{M}, \mathbb{S}^1)$ [BBM00, theorem 5]; this bound on the Ginzburg–Landau was obtained by Tristan Rivière with a proof in the spirit of the proof of the extension for 0–connected manifolds theorem 4.3 [Riv00, proposition 2.1]. Petru Mironescu and Ioana Molnar have constructed liftings satisfying the sharp bound when $p \in [1, +\infty)$ [MM15, theorem 1.3].

The only property used in theorem 6.5 is the fact that for every path $\gamma \in C^1([0,1], \mathcal{N})$, there exists a path $\widetilde{\gamma} \in C^1([0,1], \widetilde{\mathcal{N}}$ such that $\pi \circ \widetilde{\gamma} = \gamma$ on $[0,1]$ and $|\widetilde{\gamma}'|_{\widetilde{\mathcal{N}}} \le |\widetilde{\gamma}'|_{\mathcal{N}}$. This property is still satisfied when and thus theorem 6.5 still holds under the condition that $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ is a *Riemannian submersion* (or *isometric submersion*), that is the differential of π is at every point an isometry on the orthogonal to its kernel [Kli82, definition 1.11.9].

Theorem 6.6. *Let $s \in (0,1)$ and $p \in [1, +\infty)$. If $sp < 1$ and if $\pi : \widetilde{\mathcal{M}} \to \mathcal{N}$ is a Riemannian submersion, then for every $u \in W^{s,p}(\mathcal{M}, \mathcal{N})$, there exists $\widetilde{u} \in W^{s,p}(\mathcal{M}, \widetilde{\mathcal{N}})$ such that $\pi \circ \widetilde{u} = u$.*

7 Approximation of Sobolev mappings

7.1 The approximation theorem in slightly supercritical dimension

We will prove the following approximation theorem in slightly supercritical dimension.

Theorem 7.1. *Let $p \in [1,+\infty)$. If $m = \dim(\mathcal{M})$, if $m-1 \le p < m$ and if $\pi_{m-1}(\mathcal{N}) \simeq \{0\}$, then $C^\infty(\mathcal{M},\mathcal{N})$ is dense in $W^{1,p}(\mathcal{M},\mathcal{N})$.*

In view of proposition 3.1, the condition $\pi_{m-1}(\mathcal{N}) \simeq \{0\}$ is necessary for the density of smooth maps in $W^{1,p}(\mathcal{M},\mathcal{N})$.

The first construction in the proof of theorem 7.1 consists in modifying a function inside a ball on which it has small energy in such a way that its image is small.

Lemma 7.2. *Let $\delta \in (0,\delta_{\mathcal{N}})$. There exists $\kappa > 0$, such that for every $u \in W^{1,p}(\mathbb{B}^m_\rho,\mathcal{N})$ that satisfies the estimate*

$$\int_{\mathbb{B}^m_\rho} |\mathrm{D}u|^p \le \kappa \rho^{m-p}\delta^p,$$

then there exists $\sigma \in (\frac{\rho}{2},\rho)$, a map $v \in W^{1,p}(\mathbb{B}^m_\rho,\mathcal{N})$ and a set $A \subset \mathbb{B}^m_\rho$ such that

(i) $v = u$ *on* $\mathbb{B}^m_\rho \setminus \mathbb{B}^m_\sigma$,

(ii) $\operatorname{osc}(v,\mathbb{B}^m_\sigma) \le \delta$,

(iii) $\displaystyle\int_{\mathbb{B}^m_\rho} |\mathrm{D}u - \mathrm{D}v|^p + \frac{|u-v|^p}{\rho^p} \le C\int_A |\mathrm{D}u|^p$,

(iv) $\displaystyle\mathcal{L}^m(A) \le C\frac{\rho^p}{\delta^p}\int_{\mathbb{B}^m_\rho} |\mathrm{D}u|^p$.

A first tool to prove lemma 7.2 is the following construction of restrictions.

Lemma 7.3. *If $u \in W^{1,p}(\mathbb{B}^m_\rho,\mathbb{R}^\nu)$, then*

(i) for almost every $r \in (0,\rho)$, one has $u|_{\mathbb{S}^{m-1}_r} \in W^{1,p}(\mathbb{S}^{m-1}_r,\mathbb{R}^\nu)$ and $\mathrm{D}(u|_{\mathbb{S}^{m-1}_r}) = (\mathrm{D}u)|_{\mathbb{S}^{m-1}_r}$ almost everywhere on $\mathbb{S}^{m-1}_r$,

(ii) for almost every $r \in (0,\rho)$, one has $\operatorname{tr}_{\mathbb{S}^{m-1}_r} u = u|_{\mathbb{S}^{m-1}_r}$,

(iii) $\displaystyle\int_0^\rho \left(\int_{\mathbb{S}^{m-1}_r} |\mathrm{D}u|_{\mathbb{S}^{m-1}_r}|^p\right)\mathrm{d}r \le \int_{\mathbb{B}^m} |\mathrm{D}u|^p.$

The proof of lemma 7.3 is similar to the proof of lemma 6.3.

Lemma 7.4. *There exists a constant C such that if $\delta > 0$ is small enough and $y \in \mathcal{N}$, there exists a smooth map $\Pi_y^\delta : \mathcal{N} \to \mathbb{B}^\mu_\delta(y) \cap \mathcal{N}$ such that*

(i) if $z \in \mathbb{B}^\mu_\delta(y) \cap \mathcal{N}$, then $\Pi_y^\delta(z) = z$,

(ii) $\|D\Pi_y^\delta\|_{L^\infty} \le C$.

Proof. We take a function $\theta : [0, +\infty) \to (0, +\infty)$, such that $\theta = 1$ on $[0, 1/2]$ and $\theta = 0$ and $[1, +\infty)$. We set then for every $z \in \mathcal{N}$,

$$\Pi_y^\delta(z) = \Pi\biggl(\theta\Bigl(\frac{|z-y|}{\delta}\Bigr)z + \biggl(1 - \theta\Bigl(\frac{|z-y|}{\delta}\Bigr)\biggr)y\biggr).$$

We have then, since Π is smooth,

$$|D\Pi_y^\delta(z)| \le C_1\biggl(1 + \biggl|\theta'\Bigl(\frac{|z-y|}{\delta}\Bigr)\biggr| \frac{|z-y|}{\delta}\biggr) \le C_2,$$

since $\theta' = 0$ on $[1, +\infty)$. □

Proof of lemma 7.2. By lemma 7.3, there is some $\sigma \in (\rho/2, \rho)$ such that $\operatorname{tr}_{\mathbb{S}^{m-1}_r} u = u|_{\mathbb{S}^{m-1}_r} \in W^{1,p}(\mathbb{S}^{m-1}_r, \mathbb{R}^\nu)$, $\mathrm{D}(u|_{\mathbb{S}^{m-1}_r}) = (\mathrm{D}u)|_{\mathbb{S}^{m-1}_r}$ and

$$\int_{\mathbb{S}^{m-1}_\sigma} |\mathrm{D} \operatorname{tr}_{\mathbb{S}^m_r} u|^p \le \frac{2}{\rho} \int_{\mathbb{B}^m_\rho} |\mathrm{D}u|^p. \tag{7.1}$$

Since $p > m - 1$, by Morrey's embedding on $\mathbb{S}^{m-1}_\sigma$, we have for almost every $y, z \in \mathbb{S}^{m-1}_\sigma$, in view of (7.1),

$$\begin{aligned}|u(y) - u(x)| &\le C_3\biggl(\int_{\mathbb{S}^{m-1}_\sigma} |\mathrm{D}u|^p\biggr)^{\frac{1}{p}} |y - x|^{1-\frac{m-1}{p}} \le C_4\biggl(\frac{1}{\rho}\int_{\mathbb{B}^{m-1}_\rho} |\mathrm{D}u|^p\biggr)^{\frac{1}{p}} \sigma^{1-\frac{m-1}{p}}\\ &\le C_5\bigl(\kappa\rho^{m-p-1}\delta^p\bigr)^{\frac{1}{p}}\rho^{1-\frac{m-1}{p}} \le C_5\kappa^{\frac{1}{p}}\delta.\end{aligned}$$

In particular if $\kappa > 0$ is small enough, there exists a point $y_* \in \mathcal{N}$, such that for almost every $x \in \mathbb{S}^{m-1}_\sigma$,

$$u(x) \in \mathbb{B}^\nu_{\delta/4}(y_*). \tag{7.2}$$

We now define the map $v : \mathbb{B}^m_\rho \to \mathcal{N}$ for each $x \in \mathbb{B}^m_\rho$ by

$$v(x) = \begin{cases} u(x) & \text{if } |x| > \sigma, \\ \Pi_{y_*}(u(x)) & \text{if } |x| \le \sigma, \end{cases}$$

where $\Pi_y^\delta : \mathcal{N} \to \mathcal{N}$ is given by lemma 7.4. In view of (7.2), we have almost everywhere on $\mathbb{S}^{m-1}_\sigma$, $\operatorname{tr}_{\mathbb{S}^{m-1}_\sigma} u = \operatorname{tr}_{\mathbb{S}^{m-1}_\sigma} \Pi_{y_*} \circ u = \Pi_{y_*} \circ \operatorname{tr}_{\mathbb{S}^{m-1}_\sigma} u$ and thus $v \in W^{1,p}(\mathbb{B}^m_\rho, \mathcal{N})$.

Finally, we observe that, since $u = v$ on $\mathbb{B}^m_\sigma$, we have

$$\int_{\mathbb{B}^m_\rho} |\mathrm{D}u - \mathrm{D}v|^p = \int_A |\mathrm{D}u - \mathrm{D}v|^p \le 2^{p-1} \int_A |\mathrm{D}u|^p + |\mathrm{D}v|^p \le 2^{p-1}\bigl(1 + \|\mathrm{D}\Pi^\delta_y\|^p_{L^\infty}\bigr) \int_A |\mathrm{D}u|^p.$$

where the set A is defined as

$$A \triangleq \bigl\{x \in \mathbb{B}^m_\mu \mid u(x) \notin \mathbb{B}^m_{\delta/2}(y_*)\bigr\}.$$

We define the function $f : \mathbb{B}^m_\rho \to [0, +\infty)$ for each $x \in \mathbb{B}^m_\sigma$ by

$$f(x) = \bigl(|u(x) - y_*| - \delta/4\bigr)_+.$$

By the chain rule for Sobolev functions, $f \circ u \in W^{1,p}_0(\mathbb{B}^m_\sigma)$ and $f(x) \ge \frac{\delta}{4}$ on the set A. Hence, by the Chebyshev and Poincaré inequalities,

$$\mathcal{L}^m(A) \le C_6 \int_{\mathbb{B}^m_\sigma} |f|^p \le \frac{C_7 \sigma^p}{\delta^p} \int_{\mathbb{B}^m_\sigma} |\mathrm{D}f|^p \le \frac{C_8 \sigma^p}{\delta^p} \int_{\mathbb{B}^m_\sigma} |\mathrm{D}u|^p \le \frac{C_9 \rho^p}{\delta^p} \int_{\mathbb{B}^m_\rho} |\mathrm{D}u|^p.$$

Finally, by the Poincaré inequality, we have, since $u = v$ on $\mathbb{B}^m_\rho \setminus \mathbb{B}^m_\sigma$,

$$\int_{\mathbb{B}^m_\rho} |u - v|^p = \int_{\mathbb{B}^m_\sigma} |u - v|^p \le C_{10}\, \sigma^p \int_{\mathbb{B}^m_\sigma} |\mathrm{D}(u - v)|^p \le C_{10}\, \rho^p \int_{\mathbb{B}^m_\rho} |\mathrm{D}(u - v)|^p. \qquad \square$$

The second construction in theorem 7.1, is a controlled modification of the map u inside a ball, with a quality of the approximation controlled by the energy of the original map on the ball.

Lemma 7.5. *If $\pi_{m-1}(\mathcal{N}) \simeq \{0\}$ and if $u \in W^{1,p}(\mathbb{B}^m_\rho, \mathcal{N})$, then there exists $\sigma \in (\frac{\rho}{2}, \rho)$ and $v \in W^{1,p}(\mathbb{B}^m_\rho, \mathcal{N})$ such that*

(i) $u = v$ *on* $\mathbb{B}^m_\rho \setminus \mathbb{B}^m_\sigma$,

(ii) v *is continuous on* $\bar{\mathbb{B}}^m_\sigma$,

(iii) $\displaystyle\int_{\mathbb{B}^m_\rho} |\mathrm{D}u - \mathrm{D}v|^p + \frac{|u - v|^p}{\rho^p} \le C \int_{\mathbb{B}^m_\rho} |\mathrm{D}u|^p.$

The core construction in the proof of lemma 7.5 is the possibility to extend a map on a sphere of subcritical dimension with a controlled on the energy of the extension. The homotopy assumption plays a role in this precise lemma.

Lemma 7.6. *If $m - 1 < p < m$, if $\pi_{m-1}(\mathcal{N}) \simeq \{0\}$ and if $w \in W^{1,p}(\mathbb{S}^{m-1}_\sigma, \mathcal{N})$, then there exists a map $v \in (W^{1,p} \cap C)(\mathbb{B}^m_\sigma, \mathcal{N})$ such that $\operatorname{tr}_{\mathbb{S}^{m-1}_\sigma} v = w$ and*

$$\int_{\mathbb{B}^m_\sigma} |\mathrm{D}v|^p \le C\sigma \int_{\mathbb{S}^{m-1}_\sigma} |\mathrm{D}w|^p.$$

Proof. By scaling, we can assume that $\sigma = 1$. We first observe that if $\int_{\mathbb{B}^m} |\mathrm{D}w|^p = 0$, then there exists a point $b \in \mathcal{N}$ such that $w = b$ almost everywhere on $\mathbb{S}^{m-1}$. The conclusion follows then by taking $v \triangleq w$ on $\mathbb{B}^m$.

We assume now that $\int_{\mathbb{B}^m} |\mathrm{D}w|^p > 0$. Since $w \in W^{1,p}(\mathbb{S}^{m-1})$ and $p > m-1$, $w = \breve{w}$ almost everywhere, with $\breve{w} \in (W^{1,p} \cap C)(\mathbb{S}^{m-1}, \mathcal{N})$. Without loss of generality, we assume tha $\breve{w} = w$. By classical approximation of continuous maps by smooth maps, there exists a map $f \in C^1(\mathbb{S}^{m-1}, \mathcal{N})$ such that $|f - \breve{w}| \le \delta_{\mathcal{N}}$. Since by assumption $\pi_{m-1}(\mathcal{N}) \simeq \{0\}$, there exists a map $F \in C^1(\mathbb{B}^m, \mathcal{N})$ such that $F|_{\mathbb{S}^{m-1}} = f$ on $\mathbb{S}^{m-1}$. We define then for every $\lambda \in (0,1]$, the map $v_\lambda : \mathbb{B}^m \to \mathcal{N}$ for each $x \in \mathbb{B}^m$ by

$$v_\lambda(x) = \begin{cases} w\bigl(\frac{x}{|x|}\bigr) & \text{if } |x| \ge \lambda, \\ \Pi_{\mathcal{N}}\Bigl(\bigl(\frac{2|x|}{\lambda} - 1\bigr) w\bigl(\frac{x}{|x|}\bigr) + 2\bigl(1 - \frac{|x|}{\lambda}\bigr) f\bigl(\frac{x}{|x|}\bigr)\Bigr) & \text{if } \frac{\lambda}{2} \le |x| \le \lambda, \\ F\bigl(\frac{x}{2\lambda}\bigr) & \text{if } |x| \le \frac{\lambda}{2}. \end{cases}$$

We have $v_\lambda \in (W^{1,p} \cap C)(\mathbb{B}^m, \mathcal{N})$. Moreover, since $p < m$, we have

$$\begin{aligned} \int_{\mathbb{B}^m} |\mathrm{D}v_\lambda|^p &\le \int_\lambda^1 \Bigl(\int_{\mathbb{S}^{m-1}} |\mathrm{D}w|^p\Bigr) r^{m-1-p} \,\mathrm{d}r + \lambda^{m-p} \int_{\mathbb{B}^m} |\mathrm{D}v_1|^p \\ &\le \frac{1}{m-p} \int_{\mathbb{S}^{m-1}} |\mathrm{D}w|^p + \lambda^{m-p} \int_{\mathbb{B}^m} |\mathrm{D}v_1|^p. \end{aligned}$$

By taking $\lambda \in (0,1)$ in such a way that

$$\lambda^{m-p} \int_{\mathbb{B}^m} |\mathrm{D}v_1|^p \le \frac{1}{m-p} \int_{\mathbb{S}^{m-1}} |\mathrm{D}w|^p$$

we conclude that

$$\int_{\mathbb{B}^m} |\mathrm{D}v_\lambda|^p \le \frac{2}{m-p} \int_{\mathbb{S}^{m-1}} |\mathrm{D}w|^p. \qquad \square$$

Proof of lemma 7.5. By lemma 7.3, there exists $\sigma \in (\rho/2, \rho)$ such that if $\operatorname{tr}_{\mathbb{S}^{m-1}_r} u = u|_{\mathbb{S}^{m-1}_\sigma} \in W^{1,p}(\mathbb{S}^{m-1}_\sigma, \mathbb{R}^\nu)$, then $\mathrm{D}(u|_{\mathbb{S}^{m-1}_\sigma}) = (\mathrm{D}u)|_{\mathbb{S}^{m-1}_\sigma}$ and

$$\int_{\mathbb{S}^{m-1}_\sigma} |\mathrm{D}\operatorname{tr}_{\mathbb{S}^m_\sigma} u|^p \le \frac{2}{\rho} \int_{\mathbb{B}^m_\rho} |\mathrm{D}u|^p.$$

We apply lemma 7.6 to the map $w \triangleq \operatorname{tr}_{\mathbb{S}^{m-1}_r} u$, to obtain a map $v : \mathbb{B}^m_\sigma \to \mathbb{R}^m$. Since $\operatorname{tr}_{\mathbb{S}^{m-1}_\sigma} v = \operatorname{tr}_{\mathbb{S}^{m-1}_\sigma} u$, we can extend v by u to $\mathbb{B}^m_\rho$ to obtain a map $v \in W^{1,p}(\mathbb{B}^m_\sigma, \mathcal{N})$.

By construction, we have

$$\begin{aligned} \int_{\mathbb{B}^m_\rho} |\mathrm{D}u - \mathrm{D}v|^p &= \int_{\mathbb{B}^m_\sigma} |\mathrm{D}u - \mathrm{D}v|^p \le 2^{p-1} \int_{\mathbb{B}^m_\sigma} |\mathrm{D}u|^p + |\mathrm{D}v|^p \\ &\le C_{11}\Bigl(\int_{\mathbb{B}^m_\rho} |\mathrm{D}u|^p + \frac{1}{\sigma} \int_{\mathbb{S}^{m-1}_\rho} |\mathrm{D}\operatorname{tr}_{\mathbb{S}^{m-1}_\sigma} u|^p\Bigr) \le C_{12} \int_{\mathbb{B}^m_\rho} |\mathrm{D}u|^p. \end{aligned} \tag{7.3}$$

Finally, we note that since $u = v$ on $\mathbb{B}^m_\rho \setminus \mathbb{B}^m_\sigma$, by the Poincaré inequality with vanishing trace on the ball $\mathbb{B}^m_\sigma$, we have

$$\int_{\mathbb{B}^m_\rho} |u - v|^p = \int_{\mathbb{B}^m_\sigma} |u - v|^p \le C_{13}\sigma^p \int_{\mathbb{B}^m_\sigma} |\mathrm{D}(u - v)|^p \le C_{13}\rho^p \int_{\mathbb{B}^m_\rho} |\mathrm{D}(u - v)|^p$$

and we conclude by (7.3). □

We can summarize the constructions lemma 7.2 and lemma 7.5 in the following statement.

Lemma 7.7. *If $\pi_{m-1}(\mathcal{N}) \simeq \{0\}$. Then there exists a constant $C > 0$ such that if if $u \in W^{1,p}(\mathbb{B}^m_\rho, \mathcal{N})$, then there exists $\sigma \in (\frac{\rho}{2}, \rho)$, $v \in W^{1,p}(\mathbb{B}^m_\rho, \mathcal{N})$ and a set $A \subset \mathbb{B}^m_\rho$ such that*

(i) $u = v$ *on* $\mathbb{B}^m_\rho \setminus \mathbb{B}^m_\sigma$,

(ii) $\limsup_{\varepsilon \to 0} \sup\Big\{|v(y) - v(x)| \mid x, y \in \mathbb{B}^m_\sigma \text{ and } |y - x| \le \varepsilon\Big\} \le \delta$,

(iii) $\displaystyle\int_{\mathbb{B}^m_\rho} |\mathrm{D}u - \mathrm{D}v|^p + \frac{|u - v|^p}{\rho^p} \le C \int_{\mathbb{B}^m_\rho} |\mathrm{D}u|^p$,

(iv) $\displaystyle\mathcal{L}^m(A) \le C\frac{\rho^p}{\delta^p} \int_{\mathbb{B}^m_\rho} |\mathrm{D}u|^p$.

Proof. If

$$\int_{\mathbb{B}^m_\rho} |\mathrm{D}u|^p \le \kappa \rho^{m-p}\delta^p,$$

we apply lemma 7.2. Otherwise we have

$$\int_{\mathbb{B}^m_\rho} |\mathrm{D}u|^p > \kappa \rho^{m-p}\delta^p, \tag{7.4}$$

we apply lemma 7.5, we set $A = \mathbb{B}^m_\rho$ and we and we note that in view of (7.4)

$$\mathcal{L}^m(A) = C_{14}\rho^m \le C_{15}\frac{\rho^p}{\delta^p} \int_{\mathbb{B}^m_\rho} |\mathrm{D}u|^p,$$

and thus the conclusion holds. □

Lemma 7.8. *Let $\mathcal{M}$ be a compact manifold. There exists an integer k, such that if $\rho > 0$ is small enough, then there exists finite sets a^i_j with $1 \le i \le k_j$, such that*

$$\mathcal{M} = \bigcup_{i=1}^{k} \bigcup_{j=1}^{k_j} B^{\mathcal{M}}_{\rho/2}(a^i_j)$$

and for every $i \in \{1, \dots, k\}$ and $j, \ell \in \{1, \dots, k_j\}$, if $i \ne j$ then $\bar{B}^{\mathcal{M}}_{\rho/2}(a^i_j) \cap \bar{B}^{\mathcal{M}}_{\rho/2}(a^i_\ell) = \varnothing$.

Proof. Since $\mathcal{M}$ is compact it suffices to perform the construction locally.

In order to do this, we observe that in $\mathbb{R}^m$, $\bar{\mathbb{B}}^m(\rho/2)$ covers a cube of edge-length $2\rho/\sqrt{m}$. If we arrange the cubes on an array $\frac{2\rho q}{\sqrt{m}}\mathbb{Z}^m$ with $q \geq \sqrt{m}$, then we find $k = q^m$ families of nonintersecting balls covering $\mathbb{R}^n$. $\square$

Problem 7.1 (★★). Perform the construction of lemma 7.8 for $\mathcal{M} = \mathbb{S}^m$.

Proof of theorem 7.1. For every $\rho > 0$, let $(a^i_j)_{1\leq i\leq k, 1\leq j\leq k_i}$ be the points given by lemma 7.8. Let $\delta \triangleq \delta_{\mathcal{N}}/k$.

We set $u^\rho_0 = u$ and given u^ρ_{i-1} we construct u^ρ_i by applying lemma 7.7 on $B_\rho(a^j_i)$, which yields also a set $A^j_i \subset B_\rho(a^j_i)$. We define $A^\rho_i = \bigcup_{j=1}^{k_i} A^i_j$.

We claim that we have the following properties

(a) if $H^\rho_i = \bigcup_{\ell=1}^{i} \bigcup_{1=j}^{k_\ell} B_{\rho/2}(a^j_i)$, then

$$\lim_{\varepsilon\to 0} \sup\{|u^\rho_i(y) - u^\rho_i(x)| \mid x, y \in H^\rho_i \text{ and } d(y,x) \leq \varepsilon\} \leq i\delta_{\mathcal{N}},$$

(b) there exists a set $F^\rho_i \subset \mathcal{M}$ such that

$$\int_{\mathcal{M}} |\mathrm{D}u^\rho_i - \mathrm{D}u|^p + \frac{|u^\rho_i - u|^p}{\rho^p} \leq C_{16} \int_{F^\rho_i} |\mathrm{D}u|^p,$$

and

$$\mathcal{H}^m(F^\rho_i) \leq C_{17}\rho^p \int_{\mathcal{M}} |\mathrm{D}u|^p.$$

The property a follows from lemma 7.7 and by induction. Next, we have

$$\begin{aligned}\int_{\mathcal{M}} &|\mathrm{D}u^\rho_i - \mathrm{D}u|^p + \frac{|u^\rho_i - u|^p}{\rho^p}\\ &\leq 2^{p-1}\left(\int_{\mathcal{M}} |\mathrm{D}u^\rho_i - \mathrm{D}u^\rho_{i-1}|^p + \frac{|u^\rho_i - u_{i-1}|^p}{\rho^p} + \int_{\mathcal{M}} |\mathrm{D}u^\rho_{i-1} - \mathrm{D}u|^p + \frac{|u^\rho_{i-1} - u|^p}{\rho^p}\right)\end{aligned}$$

By lemma 7.7, and by our induction assumption,

$$\begin{aligned}
\int_{\mathcal{M}} |\mathrm{D}u_i^\rho - \mathrm{D}u_{i-1}^\rho|^p + \frac{|u_i^\rho - u|^p}{\rho^p} &\le \sum_{j=1}^{k_i} \int_{\mathbb{B}^m_\rho(a_i^j)} |\mathrm{D}u_i^\rho - \mathrm{D}u_{i-1}^\rho|^p + \frac{|u_i^\rho - u|^p}{\rho^p}\\
&\le C_{18} \sum_{j=1}^{k_i} \int_{A_i^j} |\mathrm{D}u_{i-1}^\rho|^p\\
&\le C_{19} \int_{A_i^\rho} |\mathrm{D}u_{i-1}^\rho|^p\\
&\le C_{20} \Bigl(\int_{A_i^\rho} |\mathrm{D}u|^p + \int_{\mathcal{M}} |\mathrm{D}u_{i-1}^\rho - \mathrm{D}u|^p \Bigr)\\
&\le C_{21} \int_{A_i^\rho \cup F_{i-1}^\rho} |\mathrm{D}u|^p.
\end{aligned}$$

The set A_i^ρ satisfies, in view of our induction assumption,

$$\begin{aligned}
\mathcal{H}^m(A_i^\rho) &\le C_{22} \rho^p \int_{\mathcal{M}} |\mathrm{D}u_{i-1}^\rho|^p\\
&\le C_{23} \rho^p \Bigl(\int_{\mathcal{M}} |\mathrm{D}u_{i-1}^\rho - \mathrm{D}u|^p + \int_{\mathcal{M}} |\mathrm{D}u|^p \Bigr)\\
&\le C_{24} \rho^p \int_{\mathcal{M}} |\mathrm{D}u|^p.
\end{aligned}$$

If we set $F_i^\rho = A_i^\rho \cup F_{i-1}^\rho$, we have

$$\mathcal{H}^m(F_i^\rho) \le \mathcal{H}^m(A_i^\rho) + \mathcal{H}^m(F_{i-1}^\rho) \le C_{25} \rho^p \int_{\mathcal{M}} |\mathrm{D}u|^p.$$

We set now $u^\rho = u_k^\rho$ and $F^\rho = F_k^\rho$ and we observe that by Lebesgue's dominated convergence theorem,

$$\limsup_{\rho \to 0} \int_{\mathcal{M}} |\mathrm{D}u_i^\rho - \mathrm{D}u_{i-1}^\rho|^p + \frac{|u_i^\rho - u|^p}{\rho^p} \le C_{26} \limsup_{\rho \to 0} \int_{F^\rho} |\mathrm{D}u|^p = 0,$$

since

$$\mathcal{H}^m(F_i^\rho) \le C_{27} \rho^p \int_{\mathcal{M}} |\mathrm{D}u|^p.$$

It remains to remark that in view of (a), we have

$$\limsup_{\varepsilon \to 0} \sup \{ |u(y) - u(x)| \mid x, y \in \mathcal{M} \text{ and } d(y,x) \le \varepsilon \} \le \delta_{\mathcal{N}}. \qquad \square$$

Problem 7.2 (★). Approximate u_k^ρ by smooth mappings.

7.2 Other approximation results

Theorem 7.9. *Let $p \in [1, +\infty)$. If $m = \dim(\mathcal{M})$, if $1 \le p < m$ and if $\pi_{\lfloor p \rfloor}(\mathcal{N}) \simeq \{0\}$, then $C^\infty(\mathbb{B}^m, \mathcal{N})$ is dense in $W^{1,p}(\mathbb{B}^m, \mathcal{N})$.*

Theorem 7.10. *Let $p \in [1, +\infty)$. If $m = \dim(\mathcal{M})$ and if $1 \le p < m$, then maps that are smooth outside a set of dimension $m - \lfloor p + 1 \rfloor$ are dense in $W^{1,p}(\mathbb{B}^m, \mathcal{N})$.*

Proof of theorem 7.10 when $m - 1 < p < m$. The proof follows the proof of theorem 7.1, with a homogeneous extension replacing the extension in lemma 7.6.

□

7.3 Remarks and comments

Theorem 7.1, theorem 7.9 and theorem 7.10 is due to Fabrice Bethuel [Bet91].

We present here a proof of theorem 7.1 due to Augusto Ponce with the author [PVS09]; this proof has been adapted to higher-order Sobolev spaces $W^{k,p}(\mathcal{M}, \mathcal{N})$ when $m - 1 \le kp < m$ [GN11; BPVS08, §3]. This approach can be adapted to the case $m = p + 1$, but does not seem to work any more when $m > p + 1$.

The original proofs of theorem 7.9 of Fabrice Bethuel [Bet91, §1] and of Lin Fanghua and Hang Fengbo [HL03a] is based on a two-scale decomposition into cubes of the domain. It is possible to work with a single decomposition into cubes provided the cubes are not aligned with each other by relying on an *opening* construction [BPVS15].

Lemma 7.4 is due to Fabrice Béthuel [Bet91, lemma A.5].

8 Perspectives

8.1 The trace problem

The trace problem is not currently completely solved:

Open problem 8.1. For $4 \le p < m$, determine whether every map in $W^{1-\frac{1}{p},p}(\mathbb{B}^{m-1},\mathcal{N})$ is the trace of a map in $W^{1,p}(\mathbb{B}^{m-1}\times(0,1),\mathcal{N})$, when $\pi_1(\mathcal{N})$ is finite, $\pi_{\lfloor p-1\rfloor}(\mathcal{N}) \simeq \{0\}$, and the groups $\pi_2(\mathcal{N}),\dots,\pi_{\lfloor p-2\rfloor(\mathcal{N})}$ are finite but not all trivial.

8.2 Relaxed questions

8.2.1 Approximation by smooth maps outside a singular set

If $\pi_{\lfloor p\rfloor}(\mathcal{N}) \not\simeq \{0\}$, maps in $W^{1,p}(\mathcal{M},\mathcal{N})$ cannot be in general approximated strongly by smooth maps in $C^\infty(\mathcal{M},\mathcal{N})$ (see proposition 3.1). This leads to the question about what are the smoothest maps that can be used to approximate maps in $W^{1,p}(\mathcal{M},\mathcal{N})$. The general answer is that they can be approximated by smooth maps outside a singular set of codimension $\lfloor p+1\rfloor$.

More precisely, one defines

$$
\begin{aligned}
R^\infty_\ell(\mathcal{M},\mathcal{N}) \triangleq \bigl\{u \in C^\infty(\mathcal{M}\setminus K,\mathcal{N}) \mid K \subset \mathcal{M} &\text{ is compact and}\\
\text{is contained in a union of } \ell\text{–dimensional submanifolds of } &\mathcal{M}, \text{ and}\\
\text{for every } k \in \mathbb{N}_*, \text{ there exists } C_k \text{ such that for each } &x \in \mathcal{M}\setminus K\\
|D^k u(x)| \le C_k/\operatorname{dist}(x,K)^k&\bigr\}.
\end{aligned}
$$

One has then the following density result:

Theorem 8.1. *If $\mathcal{M}$ and $\mathcal{N}$ are compact Riemannian manifolds, then $R^\infty_{m-\lfloor p+1\rfloor}(\mathcal{M},\mathcal{N})$ is dense in $W^{1,p}(\mathcal{M},\mathcal{N})$.*

Theorem 8.1 is due to Fabrice Béthuel and Zheng Xiaomin when $p < m \le p+1$ [BZ88, theorem 4]) and to Fabrice Béthuel in the general case [Bet91, theorem 2] (see also [HL03a, theorem 5.1]).

8.2.2 Singular extensions

If $\pi_{\lfloor p-1\rfloor}(\mathcal{N}) \not\simeq \{0\}$, then maps in $W^{1-1/p,p}(\partial\mathcal{M},\mathcal{N})$ are not necessarily the traces of maps in $W^{1,p}(\mathcal{M},\mathcal{N})$ (theorem 3.6).

This obstruction can be bypassed by relaxed by considering for $U \in W^{1,p}(\mathcal{M},\mathcal{N})$, an energy of the form

$$\mathcal{J}_\varepsilon(U) = \int_{\mathcal{M}} |DU|^p + \frac{1}{\varepsilon^p}\int_{\mathcal{M}} \operatorname{dist}(u,\mathcal{M})^p.$$

When $p = 2$ and $\mathcal{N} = \mathbb{S}^1$, we have for every $y \in \mathbb{R}^2$ with $\operatorname{dist}(y,\mathcal{M})^2 = (1 - |y|)^2$, and thus if $|y| \le 1$, $\frac{1}{2}(1 - |y|^2)^2 \le \operatorname{dist}(y,\mathcal{M}) \le (1 - |y|^2)^2$, and $\mathcal{J}_\varepsilon(U)$ is equivalent to the *Ginzburg–Landau* energy.

There exists a constant C such that for $\varepsilon > 0$ small enough, for every $u \in W^{1/2,2}(\mathcal{M},\mathbb{S}^1)$ one has [BBM00, theorem 5] (see also [Riv00, proposition 2.1])

$$\inf\{\mathcal{G}^{1,p}_\varepsilon(U) \mid U \in W^{1,2}(\mathcal{M},\mathbb{S}^1) \text{ and } \operatorname{tr}_{\partial\mathcal{M}} U = u\} \le C\,\mathcal{E}^{1/2,2}(u)\ln\frac{1}{\varepsilon}.$$

The proof of the estimates shows that in general if $\pi_1(\mathcal{N}) \simeq \dots \simeq \pi_{p-2}(\mathcal{N}) \simeq \{0\}$ and if $p \in \mathbb{N}$, then

$$\inf\{\mathcal{G}^{1,p}_\varepsilon(U) \mid U \in W^{1,p}(\mathcal{M},\mathcal{N}) \text{ and } \operatorname{tr}_{\partial\mathcal{M}} U = u\} \le C\,\mathcal{E}^{1-1/p,p}(u)\ln\frac{1}{\varepsilon}.$$

In the critical case $m = p$, where estimates for extensions fail, it has been proved that any map in $W^{1-\frac{1}{m},m}(\mathbb{S}^{m-1},\mathcal{N})$ has an extension whose gradient in the weak Marcinkiewicz space is controlled [PVS17] (see also [PR14]).

Open problem 8.2. If $m > p \in \mathbb{N}$ and $\pi_1(\mathcal{N}) \simeq \dots \simeq \pi_{p-2}(\mathcal{N}) \simeq \{0\}$, is it possible for every $u \in W^{1-\frac{1}{p},p}(\mathcal{M},\mathcal{N})$ to construct $U \in W^{1,1}(\mathcal{M},\mathcal{N})$ such that $\operatorname{tr}_{\partial\mathcal{M}} U = u$ and for every $\lambda \in (0,+\infty)$,

$$\lambda^{\frac{1}{p}}\mathcal{H}^m(\{x \in \mathcal{M} \mid |DU(x)| \ge \lambda\}) \le C\mathcal{E}^{1-\frac{1}{p},p}(u)?$$

8.2.3 Relaxed lifting

If $s \in (0,1)$ and $sp \le 2 < m$, the analytical obstruction to the lifting problem (theorem 5.7) tells somehow that the space in which we are searching a lifting might not be the right space.

When $\mathcal{N} = \mathbb{S}^1$ and $\pi : \mathbb{R} \to \mathbb{S}^1$ is the universal covering, then any map can be lifted in $(W^{s,p} + W^{1,sp})(\mathcal{M},\mathbb{R})$.

Theorem 8.2. *If $0 < s < 1$ and $2 \le sp < m$, then for every map $u \in W^{s,p}(\mathbb{B}^m,\mathbb{S}^1)$, there exists a map $\widetilde{u} \in (W^{s,p} + W^{1,sp})(\mathbb{B}^m,\mathbb{R})$ such that $(\cos\widetilde{u},\sin\widetilde{u}) = u$.*

Theorem 8.2 is due to Petru Mironescu [Mir08]. When $s = \frac{1}{2}$, $p = 2$, $m \ge 2$ and u is the strong limit of smooth maps, theorem 8.2 is due to Jean Bourgain and Haïm Brezis [BB03, theorem 4] and when $m = 1$ and $s = \frac{1}{p}$ to Nguyen Hoai-Minh [Ngu08, theorem 2].

As a sort of converse to theorem 8.2, if $\widetilde{u} \in (W^{s,p} + W^{1,sp})(\mathbb{B}^m,\mathbb{R})$ and if $p > 1$ then by the chain rule and by the fractional Gagliardo–Nirenberg interpolation inequality $(\cos\widetilde{u},\sin\widetilde{u}) \in W^{s,p}(\mathcal{M},\mathcal{N})$.

Open problem 8.3. For every $s \in (0,1)$ and $p \in [1,+\infty)$ define a space $\widetilde{W}^{s,p}(\mathcal{M},\widetilde{\mathcal{N}})$ such that $\pi(\widetilde{W}^{s,p}(\mathcal{M},\widetilde{\mathcal{N}}) \subset W^{s,p}(\mathcal{M},\widetilde{\mathcal{N}})$ and equality holds if $sp \geq 2$ and if the domain $\mathcal{M}$ is simply connected.

When $\widetilde{\mathcal{N}}$, one can take $\widetilde{W}^{s,p}(\mathcal{M},\widetilde{\mathcal{N}}) = W^{s,p}(\mathcal{M},\widetilde{\mathcal{N}})$ [MVS].

8.3 Global obstructions

8.3.1 Global obstruction to the approximation problem

Besides the local obstruction of proposition 3.1, there is a local obstruction.

Theorem 8.3. *If $\mathcal{M}^{\lfloor p \rfloor}$ is the $\lfloor p \rfloor$–dimensional skeleton of a triangulation of $\mathcal{M}$. Then $C^\infty(\mathcal{M},\mathcal{N})$ is dense in $W^{1,p}(\mathcal{M},\mathcal{N})$ if and only if the restriction operator $f \in C(\mathcal{M},\mathcal{N}) \mapsto f|_{\mathcal{M}^{\lfloor p \rfloor}} \in C(\mathcal{M}^{\lfloor p \rfloor},\mathcal{N})$ is surjective.*

Theorem 8.3 is due to Hang Fengbo and Lin Fanghua [HL03a, theorem 1.3].

The assumption is stronger than $\pi_{\lfloor p \rfloor}(\mathcal{N}) \simeq \{0\}$, this assumption is not satisfied when when $\mathcal{M} = \mathbb{R}P^m$ and $\mathcal{N} = \mathbb{R}^n$, and $1 \leq p < n+1 \leq m$, or $\mathcal{M} = \mathbb{C}P^m$ and $\mathcal{N} = \mathbb{C}^n$, $m > n$ and $2 \leq p < 2n+1 \leq 2m$ [HL03a, Corrolary 1.5], whereas .

Particular cases where the assumption of theorem 8.3 is satisfied are when $\pi_1(\mathcal{M}) \simeq \dots \simeq \pi_j(\mathcal{M}) \simeq \pi_{j+1}(\mathcal{N}) \simeq \dots \simeq \pi_{\lfloor p \rfloor}(\mathcal{N})$ for some $j \in \{1,\dots,\lfloor p-1 \rfloor\}$ (leading to theorem 4.6 when $j = 0$), and when $\pi_{\lfloor p \rfloor}(\mathcal{N}) \simeq \dots \simeq \pi_{m-1}(\mathcal{N}) \simeq \{0\}$ (leading to theorem 7.1 when $m \leq p+1$).

8.3.2 Global obstruction for the lifting problem

We have stated our lifting results under the condition that the domain $\pi(\mathcal{M})$ should be simply connected. In fact it is possible to relax this assumption.

Theorem 8.4. *If $p \geq 2$ and $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ be a covering map. Then $W^{1,p}(\mathcal{M},\mathcal{N}) = \pi(W^{1,p}(\mathcal{M},\widetilde{\mathcal{N}}))$ if and only if $C(\mathcal{M}^2,\mathcal{N}) = \pi(C(\mathcal{M}^2,\mathcal{N}))$.*

Theorem 8.4 does not seem to have been written in the literature.

The condition can be weaker than $\pi_1(\mathcal{M})$ simply connected. Indeed, it will be satisfied if for example any homomorphism of $\pi_1(\mathcal{M})$ into $\pi_1(\mathcal{N})$ is trivial; this would be the case if for example $\pi_1(\mathcal{M}) = \mathbb{Z}_i$ and $\pi_1(\mathcal{N}) = \mathbb{Z}_j$ and i and j have no nontrivial common divisor.

8.3.3 Global obstruction for the extension problem

For the extension problem, the local obstruction has a corresponding global obstruction.

Theorem 8.5. *If $p \geq 2$ and if the trace operator $U \in W^{1,p}(\mathcal{M},\mathcal{N}) \mapsto \operatorname{tr}_{\partial\mathcal{M}} U$ is surjective, then the restriction operator $F \in C(\mathcal{M}^{\lfloor p \rfloor},\mathcal{N}) \mapsto F|_{\partial\mathcal{M}^{\lfloor p-1 \rfloor}} \in C(\partial\mathcal{M}^{\lfloor p-1 \rfloor},\mathcal{N})$ is surjective.*

Theorem 8.5 is essentially due Isobe Takeshi [Iso03, theorems 1.2 and 1.3].

8.3.4 Global obstruction for the homotopy problem

Theorem 8.6. *The set $W^{1,p}(\mathcal{M},\mathcal{N})$ is path-connected if and only if the image of the restriction operator $F \in C(\mathcal{M}^{\lfloor p \rfloor},\mathcal{N}) \mapsto F|_{\mathcal{M}^{\lfloor p-1 \rfloor}} \in C(\mathcal{M}^{\lfloor p-1 \rfloor},\mathcal{N})$ is path-connected in $C(\mathcal{M}^{\lfloor p-1 \rfloor},\mathcal{N})$.*

Theorem 8.6 is due to Hang Fengbo and Lin Fanghua [HL03a, theorem 5.1]. When $1 \le p < 2$, theorem 8.6 gives immediately the path-connectedness of $W^{1,p}(\mathcal{M},\mathcal{N})$ without any restriction on the domain, which is due to Haïm Brezis and Li Yanyan [BL01, theorem 0.2].

In particular, if there exists $j \in \{1,\dots,\lfloor p-1 \rfloor\}$ such that $\pi_1(\mathcal{M}) \simeq \dots \simeq \pi_j(\mathcal{M}) \simeq \pi_{j+1}(\mathcal{N}) \simeq \dots \simeq \pi_{\lfloor p-1 \rfloor}(\mathcal{N})$, then $W^{1,p}(\mathcal{M},\mathcal{N})$ is path-connected [HL03a, theorem 5.1] (the result was due to Haïm Brezis and Yanyan Li when $j = 0$, [BL01, theorem 0.3], when $\mathcal{M} = \mathbb{S}^m$ [BL01, proposition 0.1]).

Theorem 8.7. *Every map in $W^{1,p}(\mathcal{M},\mathcal{N})$ is connected to a smooth map if and only if the restriction operators $F \in C(\mathcal{M},\mathcal{N}) \mapsto F|_{\mathcal{M}^{\lfloor p-1 \rfloor}} \in C(\mathcal{M}^{\lfloor p-1 \rfloor},\mathcal{N})$ and $f \in C(\mathcal{M}^{\lfloor p \rfloor},\mathcal{N}) \mapsto f|_{\mathcal{M}^{\lfloor p-1 \rfloor}} \in C(\mathcal{M}^{\lfloor p-1 \rfloor},\mathcal{N})$ have the same image.*

Theorem 8.7 is due to Hang Fengbo and Lin Fanghua [HL03a, corollary 5.4].

8.4 Properties of individual mappings

8.4.1 Cohomological tools

If $f \in W^{1,\ell}(\mathcal{M},\mathcal{N})$, $\omega \in C^\infty(\mathcal{N},\bigwedge^\ell \mathcal{N})$ and $\zeta \in C^\infty(\mathcal{M},\bigwedge^{m-\ell-1} \mathcal{M})$, we define the distribution $\operatorname{Hur}_\ell(f)[\omega]$,

$$\langle \operatorname{Hur}_\ell(f)[\omega],\zeta\rangle = \int_{\mathcal{M}} f^*\omega \wedge d\zeta.$$

By construction, $\operatorname{Hur}_\ell[\omega]$ depends linearly on ω. If moreover $\omega = d\eta$, for some $\eta \in C^\infty(\mathcal{N},\bigwedge^{\ell-1}\mathcal{N})$, then

$$\langle \operatorname{Hur}_\ell(f)[d\eta],d\zeta\rangle = \int_{\mathcal{M}} u^* d\eta \wedge d\zeta = \int_{\mathcal{M}} d(u^*\eta) \wedge d\zeta = (-\eta)^\ell \int_{\mathcal{M}} f^*\eta \wedge d^2\zeta = 0.$$

In particular, $\langle \operatorname{Hur}_\ell(f)[\omega],\zeta\rangle$ is well-defined on the ℓ–th order de Rham cohomology group $H^\ell_{\mathrm{dR}}(\mathcal{N})$, and thus by duality $\langle \operatorname{Hur}_\ell(f),\zeta\rangle \in H_\ell(\mathcal{N},\mathbb{R}) \simeq H_\ell(\mathcal{N},\mathbb{Q}) \simeq H_\ell(\mathcal{N},\mathbb{Z}) \otimes \mathbb{Q}$.

Moreover, we observe that if $f \in W^{1,\ell+1}(\mathcal{M},\mathcal{N})$,

$$\langle \operatorname{Hur}_\ell(f)[d\eta],d\zeta\rangle = \int_{\mathcal{M}} u^* d\eta \wedge d\zeta = \int_{\mathcal{M}} d(u^*\eta) \wedge d\zeta = (-\eta)^{\ell+1} \int_{\mathcal{M}} u^*\eta \wedge d^2\zeta = 0.$$

This implies the following proposition:

Theorem 8.8. *If $u \in W^{1,p}(\mathcal{M},\mathcal{N})$ is the strong limit of smooth mappings, then* $\operatorname{Hur}_{\lfloor p \rfloor}(f) = 0$.

Theorem 8.8 has a converse under the assumption that the $\lfloor p \rfloor$–th homotopy group describes the homology.

Theorem 8.9. *If the Hurewicz homomorphism $\pi_{\lfloor p \rfloor}(\mathcal{N}) \to H_{\lfloor p \rfloor}(\mathcal{N}, \mathbb{Q})$ is an isomorphism and if every the restriction operators $F \in C(\mathcal{M}, \mathcal{N}) \to F|_{\mathcal{M}^{\lfloor p-1 \rfloor}} \in C(\mathcal{M}^{\lfloor p-1 \rfloor}, \mathcal{N})$ and $F \in C(\mathcal{M}^{\lfloor p \rfloor}, \mathcal{N}) \to F|_{\mathcal{M}^{\lfloor p-1 \rfloor}} \in C(\mathcal{M}^{\lfloor p-1 \rfloor}, \mathcal{N})$ have the same image, then if $u \in W^{1,p}(\mathcal{M}, \mathcal{N})$ and* $\operatorname{Hur}_{\lfloor p \rfloor}(u) = 0$, *then u has a strong approximation by smooth maps.*

Theorem 8.9 is due to Fabrice Béthuel, Jean-Michel Coron, Françoise Demengel and Frédéric Hélein [BCDH91, theorem 1].

When $\mathcal{N} = \mathbb{S}^n$, then any $\omega \in C^\infty(\mathbb{S}^n, \bigwedge^n \mathbb{S}^n)$ such that $d\omega = 0$ is a constant multiple of the volume form on $\mathbb{S}^n$, and Hur_n is the *distributional Jacobian* introduced by John Ball in elasticity [Bal76, (6.10)], for continuous Sobolev mappings with finitely many singularities by Haïm Brezis, Jean-Michel Coron and Eliot H. Lieb [BCL86, appendix B] and by Fabrice Béthuel, Haïm Brezis and Jean-Michel Coron for Sobolev mappings [BBC90]. In particular, theorem 8.9 was obtained for $\mathcal{M} = \mathbb{B}^3$, $\mathcal{N} = \mathbb{S}^2$ and $p = 2$ by Fabrice Béthuel [Bet90] and for $\mathcal{M} = \mathbb{B}^m$, $\mathcal{N} = \mathbb{S}^1$ and $p = 1$ by Françoise Demengel [Dem90].

The drawback of the homological approach is that it seems that sharp answers in the theory of Sobolev maps are formulated in homotopy theory and that homological tools cannot capture the full picture of homotopies.

8.4.2 Parametrized families of skeletons

Sobolev mappings in $W^{1,p}(\mathcal{M}, \mathcal{N})$ that can be approximated smoothly can be characterized by the behaviour on a parametrized family of $\lfloor p \rfloor$–dimensional components of a triangulation for a set of parameters with positive measure [HL03a, remark 6.1] (see also [Iso05]).

The existence of a path between two maps of $W^{1,p}(\mathcal{M}, \mathcal{N})$ can similarly be determined by examining whether they are homotopic on a parametrized family of $\lfloor p-1 \rfloor$–dimensional triangulations for a set of positive measure of the parameter [HL03a, theorem 1.1].

8.5 Weak-bounded approximation by smooth maps

The question of the approximation treated up to now can be weakened as the question of the *weak-bounded approximation*. That is, given $u \in W^{1,p}(\mathcal{M}, \mathcal{N})$, does there exists a sequence $(u_j)_{j \in \mathbb{N}}$ in $C^\infty(\mathcal{M}, \mathcal{N})$ such that $(u_j)_{j \in \mathbb{N}}$ converges almost everywhere to u and $(u_j)_{j \in \mathbb{N}}$ is bounded in $W^{1,p}(\mathcal{M}, \mathcal{N})$.

When $p > 1$, a weak-bounded approximating sequence is an approximating sequence in the weak topology; when $p = 1$ this is not anymore the case (the existence of approximating sequences in the weak topology turns then to be equivalent to the existence of norm approximating sequences [Han02]).

The first result concerns the case where $p \notin \mathbb{N}$.

Theorem 8.10. *Let $p \in [1, \infty)$. If $p \notin \mathbb{N}$. Every map in $W^{1,p}(\mathcal{M}, \mathcal{N})$ has a weak-bounded approximation if and only if the restriction operator $F \in C(\mathcal{M}, \mathcal{N}) \mapsto F|_{\mathcal{M}^{\lfloor p \rfloor}}$ is surjective.*

In view of theorem 8.10, the strong approximation problem and the weak-bounded approximation problem are equivalent. It suffices thus to show the necessity in theorem 8.10. This was done locally by Fabrice Béthuel [Bet91, theorem 3]; the global obstruction can be proved by following Hang Fengbo and Lin Fanghua [HL03a].

In general, Hang Fengbo and Lin Fanghua have proved the following condition.

Theorem 8.11. *Let $p \in [1, \infty)$. If every map in $W^{1,p}(\mathcal{M}, \mathcal{N})$ has a weak-bounded approximation, then the restriction operators $F \in C(\mathcal{M}, \mathcal{N}) \mapsto F|_{\mathcal{M}^{\lfloor p-1 \rfloor}}$ and $F \in C(\mathcal{M}^{\lfloor p \rfloor}, \mathcal{N}) \mapsto F|_{\mathcal{M}^{\lfloor p-1 \rfloor}}$ have the same image.*

If p is an integer, then the condition of theorem 8.11 is sufficient for the closure of $C^\infty(\mathcal{M}, \mathcal{N})$ to be $W^{1,p}(\mathcal{M}, \mathcal{N})$ [HL03a, §7; Bet91, theorem 5]. Since the weak topology is not metrisable, this does not imply the existence of weak-boundedly approximating sequences.

When $\mathcal{M} = \mathbb{B}^3$ and $\mathcal{N} = \mathbb{S}^2$. For every $u \in W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$, Fabrice Béthuel, Haïm Brezis and Jean-Michel Coron have prove that there exists a sequence of smooth maps $(u_j)_{j \in \mathbb{N}}$ such that $u_j \to u$, and

$$\limsup_{j \to \infty} \int_{\mathbb{B}^3} |Du_j|^2 \le \int_{\mathbb{B}^3} |Du_j|^2 + 2 \inf \Big\{ \int_{\mathbb{B}^3} u^* \omega \wedge d\zeta \mid \zeta \in C_c^\infty(\mathbb{B}^3, \textstyle\bigwedge^0 \mathbb{R}^3) \Big\},$$

where $\omega \in C^\infty(\mathbb{S}^2, \bigwedge^2 \mathbb{S}^2)$ is a normalized volume form on $\mathbb{S}^2$, that is $d\omega = 0$ and $\int_{\mathbb{S}^2} \omega = 1$ [BBC90, theorem $\tilde{2}$].

For $p = 1$ and $p = 2$ and if the necessary condition of Hang and Lin holds, Mohammad Reza Pakzad and Tristan Rivière have proved that any map has a weak-bounded approximation [Pak03, theorem 1; PR03, theorems I and IV]. The situation is similar but more delicate for $W^{1,2}(M, \mathbb{S}^2)$ when $\dim M \ge 4$ [Pak02, GMS98a, GMS98b, TF05] (see also [GM06, GMM08, GM07a, GM07b, GM06]). The same approach has also been used when $p = 1$ and $N = \mathbb{S}^1$ [DH92, Ign05].

It is also known that if $p \in \mathbb{N}$ and $\pi_1(\mathcal{N}) \simeq \dots \simeq \pi_{p-1}(\mathcal{N}) \simeq \{0\}$, then every map in $W^{1,p}(\mathcal{M}, \mathcal{N})$ has a weak-bounded approximation [Haj94].

The weak-bounded approximation problem satisfies a uniform boundedness principle [HL03b, theorem 9.6] (see also [MVS19]).

For $W^{1,3}(\mathbb{B}^4, \mathbb{S}^2)$, Fabrice Béthuel has given in a recent preprint [Bet14] a quantitative obstruction to the weak-bounded approximation.

Open problem 8.4. When $p \in \mathbb{N}$ and $p \ge 2$ determine whether every $W^{1,p}(\mathcal{M}, \mathcal{N})$ has weak-bounded approximation by smooth maps.

8.6 Quantitative estimates

When $\mathcal{M} = \mathcal{N} = \mathbb{S}^n$, the connected components of $C(\mathbb{S}^m, \mathbb{S}^m)$ are classified by the Brouwer topological degree. The degree of smooth map $f \in C^1(\mathbb{S}^1, \mathbb{S}^1)$ can be computed through the Kronecker formula

$$\deg f = \int_{\mathbb{S}^m} f^* \omega,$$

where $\omega \in C^\infty(\mathbb{S}^m, \bigwedge^m \mathbb{S}^m)$ is the standard volume form.

In view of the classical inequality between the geometric and quadratic means, we have

$$|f_\sharp \omega_{\mathbb{S}^m}| = |\det Df|\, \omega_{\mathbb{S}^n} \le \frac{|Df|^m}{m^{m/2}}\, \omega_{\mathbb{S}^m}$$

everywhere on the sphere $\mathbb{S}^m$. An integral estimate on the degree (see [BBM05, Remark 0.7]) then follows

$$|\deg(f)| \le \frac{1}{m^{m/2}} \fint_{\mathbb{S}^m} |Df|^m = \frac{1}{m^{m/2}|\mathbb{S}^m|} \mathcal{E}^{1,m}(f). \tag{8.1}$$

This formula extends to fractional Sobolev maps: for every $p \in (m, +\infty)$, there exists a constant $C_{m,p}$ such that for every map $f \in W^{m/p,p}(\mathbb{S}^m, \mathbb{S}^m)$, one has [BBM05, theorem 0.6] (see also [BdMBGP91, theorem A.3; Mir07b, theorem 2.3])

$$|\deg(f)| \le C_{m,p}\, \mathcal{E}^{m/p,p}(f). \tag{8.2}$$

The inequality (8.2) follows from *gap potential estimates* obtained by Jean Bourgain, Haïm Brezis, Petru Mironescu and Nguyên Hoài-Minh [BBN05, theorem 1.1; BBM05, open problem 2; Ngu07] (see also [Ngu14]): for every $\varepsilon \in (0, \sqrt{2(1 + \frac{1}{m+1})})$, there exists a constant $C_{\varepsilon,m}$ such that for every map $f \in C(\mathbb{S}^m, \mathbb{S}^m)$, one has

$$|\deg(f)| \le C_{\varepsilon,m} \iint\limits_{\substack{(x,y) \in \mathbb{S}^m \times \mathbb{S}^m \\ |f(y) - f(x)| > \varepsilon}} \frac{1}{|y - x|^{2m}}\, \mathrm{d}x\, \mathrm{d}y. \tag{8.3}$$

If $m \ge 2$, the constant $C_{\varepsilon,m}$ can be taken to satisfy $C_{\varepsilon,m} \le C_m \varepsilon^m$ [Ngu17].

Open problem 8.5. What is the optimal scaling of the constant in (8.3) for $m = 1$?

In the general case, one can wonder whether a bound on the Sobolev energies implies that the number of connected components to which maps can belong remains bounded. In general the answer is negative and is related to the fact that thanks the action of $\pi_1(\mathcal{N})$ on $\pi_m(\mathcal{N})$ can produce maps in infinitely many connected components and such that the Sobolev energy remains bounded, an explicit counterexample is provided by gluing $\mathbb{S}^1 \times \mathbb{S}^{2m}$ to $\mathbb{S}^m \times \mathbb{S}^{m+1}$ through a trivial sphere $\mathbb{S}^{2m}$ [VS]. It has been proved that sets satisfying a bound on a critical Sobolev energy are generated by finitely many maps and the action of $\pi_1(\mathcal{N})$ on $\pi_m(\mathcal{N})$ when $m = 1$, $s = \frac{1}{2}$ and $p = 2$ by Ernst Kuwert [Kuw98], when $m \ge 1$, $s = 1$ by Frank Duzaar and Ernst Kuwert [DK98, theorem 4], when $m \ge 1$ and $s = 1 - \frac{1}{m+1}$ by Thomas Müller [Mül00, theorem 5.1], when $m = 2$ and $s = 1$ by Richard Schoen and Jon Wolfson [SW01, lemma 5.2]. The general result for bounded sets in critical fractional Sobolev spaces follows from a decomposition result under a bound of a quantity of the form of the right-hand side of (8.3) [VS].

In the case when $\mathcal{M} = \mathbb{S}^{2n-1}$ and $\mathcal{N} = \mathbb{S}^n$, one can define the Hopf invariant, which classifies homotopy classes when $n = 2$. When n is odd it is always trivial, whereas when n is even, it is nontrivial and only finitely many connected components of $C(\mathbb{S}^{2n-1}, \mathbb{S}^n)$ share the same Hopf invariant. Tristan Rivière has proved that [Riv98, lemma III.1] (see also [Gro99a; Gro99b, Lemma 7.12])

$$|\deg_H(f)| \le C \left(\int_{\mathbb{S}^{2n-1}} |Df|^{2n-1} \right)^{1+\frac{1}{2n-1}}$$

(the proof is written when $n = 2$ but goes to higher dimension) and that the power is optimal. In comparison with the corresponding estimate of the topological degree (8.1), the estimate saws the appearance of a power $1 + \frac{1}{2n-1}$ applied to the integral coming from the Whitehead formula for the Hopf invariant [Whi47].

In the fractional case, Armin Schikorra and the author Jean Van Schaftingen have proved that

$$|\deg_H(f)| \le C\, \mathcal{E}^{s,\frac{2n-1}{s}}(u)^{1+\frac{1}{2n-1}}, \tag{8.4}$$

when $s \ge 1 - \frac{1}{2n}$ [SVS].

Open problem 8.6. Prove (8.4) when $0 < s < 1 - \frac{1}{2n}$.

8.7 Higher-order Sobolev mappings

The higher order Sobolev spaces $W^{k,p}(\mathcal{M}, \mathcal{N})$ are motivated by problems such as the biharmonic maps.

In contrast with first-order Sobolev spaces, the space and the convergence depend in general on the embedding and are not equivalent with an intrinsic deinition [CVS].

By the Gagliardo–Nirenberg interpolation inequality, one has the embedding of the spaces $W^{k,p}(\mathcal{M}, \mathcal{N}) \subset W^{1,kp}(\mathcal{M}, \mathcal{N})$, and one expects both spaces to have similar properties.

For the lifting problem, once one has a lifting $\widetilde{u}$ in $W^{1,kp}(\mathcal{M}, \mathcal{N})$, one can write

$$D\widetilde{u} = (D\pi\widetilde{u})^{-1}[Du]$$

and then study the higher differentiability properties of the lifting by the chain rule.

For the approximation problem, any map in $W^{k,p}(\mathbb{B}^m, \mathcal{N})$ has a strong approximation by maps in $C^\infty(\mathbb{B}^m, \mathcal{N})$ if and only if $\pi_{\lfloor kp \rfloor}(\mathcal{N}) \simeq \{0\}$ [BPVS15].

The extension problem is delicate because one would then like to prescribe a value of the function and of the normal derivative.

Open problem 8.7. Let $m > 2p$. Given $u \in W^{2,p}(\mathbb{B}^m_1, \mathcal{N})$, does there exist $v \in W^{2,p}(\mathbb{B}^m_2, \mathcal{N})$ such that $v|_{\mathbb{B}^m_1} = u$ on $\mathbb{B}^m_1$?

Equivalently, open problem 8.7 asks whether the space of traces of u and its derivative that is independent on the side of the boundary on which one takes the trace.

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