GROUNDSTATES FOR THE NONLINEAR SCHRÖDINGER EQUATION WITH POTENTIAL VANISHING AT INFINITY

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Abstract. Groundstates of the stationary nonlinear Schrödinger equation

$$-\Delta u + Vu = Ku^{p-1},$$

are studied when the nonnegative function V and K are neither bounded away from zero, nor bounded from above. A special care is paid to the case of a potential V that goes to 0 at infinity. Conditions on compact embeddings that allow to prove in particular the existence of groundstates are established. The fact that the solution is in $L^2(\mathbb{R}^N)$ is studied and decay estimates are derived using Moser iteration scheme. The results depend on whether V decays slower than $|x|^{-2}$ at infinity.

Contents

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1. Introduction

In this paper, we consider the following problem for the time-independent nonlinear Schrödinger equation:

$$\begin{cases} -\Delta u + Vu = Ku^{p-1} & \text{in } \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} u(x) = 0. \end{cases} \tag{$\mathcal{P}_{V,K}$}$$

Here $u : \mathbb{R}^N \to \mathbb{R}$ is an unknown function, while $V : \mathbb{R}^N \to \mathbb{R}^+$ and $K : \mathbb{R}^N \to \mathbb{R}^+$ are given potentials. Solutions to $(\mathcal{P}_{V,K})$ can be used to represent a standing wave to the time-dependent nonlinear Schödinger equation; they also appear as stationary solutions in models of cross-diffusion [12]. The study of such problems was initiated by Floer and Weinstein [9] by perturbation methods.

Afterwards, Rabinowitz showed how the variational methods could be applied to this problem. Indeed, the solutions of $(\mathcal{P}_{V,K})$ are — at least formally — critical points of the action functional

$$I(u) = \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{2} + V\frac{|u|^2}{2} - K\frac{|u|^p}{p}.$$

The quadratic part of the functional naturally defines the Hilbert space

$$H^1_V(\mathbb{R}^N) = \bigl\{u \in W^{1,1}_{\text{loc}}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 < \infty\bigr\};$$

the functional $I : H^1_V(\mathbb{R}^N) \to \mathbb{R} \cup \{-\infty\}$ is then well-defined. The groundstate is the nontrivial weak solution to $(\mathcal{P}_{V,K})$ in $H^1_V(\mathbb{R}^N)$ which has the least energy $I(u)$ among all solutions in H^1_V. The classical scheme to prove the existence of groundstates consists in minimizing I on the Nehari manifold

$$\mathcal{N} = \Bigl\{u \in H^1_V(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 = \int_{\mathbb{R}^N} K|u|^p\Bigr\}$$

The particularization of one result of Rabinowitz to our setting is

Theorem 1 (Rabinowitz [16]). *Let $V \in C(\mathbb{R}^N; \mathbb{R}^+_0)$ and $K \in C(\mathbb{R}^N; \mathbb{R})$. If $2 < p < 2N/(N-2)$,*

(i) $\sup_{\mathbb{R}^N} K < \infty$,
(ii) $\inf_{\mathbb{R}^N} V > 0$,
(iii) $\lim_{|x|\to\infty} V(x) = +\infty$,

then problem $(\mathcal{P}_{V,K})$ has a groundstate $u \in H^1_V(\mathbb{R}^N)$.

Rabinowitz could also handle cases in which V is bounded from above on $\mathbb{R}^N$. Further applications of variational methods have yield existence of solutions that are not groundstates, for problems that might also not have a groundstate, see e.g. [7, 8].

All the works mentioned are built on the assumption that V has a postive lower bound and that K is bounded. In a recent work, Ambrosetti, Felli and Malchiodi have investigated groundstates when V tends to zero at infinity. One of the problems arising is that the natural space $H^1_V(\mathbb{R}^N)$ is not anymore embedded in $L^2(\mathbb{R}^N)$. They obtained

Theorem 2 (Ambrosetti, Felli and Malchiodi [2]). *Assume $N \geq 3$, $V \in C(\mathbb{R}^N; \mathbb{R}_0^+)$ and $K \in C(\mathbb{R}^N; \mathbb{R})$. If $2 < p < 2N/(N-2)$, $0 < \alpha < 1$,*

$$\beta > (1-\alpha)\Big(N - p\Big(\frac{N}{2} - 1\Big)\Big), \tag{1}$$

(i) $\sup_{x\in\mathbb{R}^N} (1+|x|)^\beta K < +\infty$,
(ii) $\inf_{x\in\mathbb{R}^N} (1+|x|)^{2-2\alpha} V(x) > 0$,

then problem $(\mathcal{P}_{V,K})$ has a groundstate $u \in H^1_V(\mathbb{R}^N)$. Moreover, $u \in L^2(\mathbb{R}^2)$ and

$$u(x) \leq C e^{-\lambda|x|^\alpha}$$

for some $C > 0$ and $\lambda > 0$.

One should note that the solution is constructed as an element of $H^1_V(\mathbb{R}^N)$, and need therefore not be a priori in $L^2(\mathbb{R}^N)$. However, some regularity theory allows to show afterwards that u is indeed square integrable. The fact that $u \in L^2(\mathbb{R}^N)$ has an interpretation in the model of the nonlinear Schrödinger equation: since $|u|^2$ corresponds to the probability density of a particle, this means that the particle is localized, and that the solution corresponds to a boundstate. The study of boundstates which are not necessary groundstates with potentials vanishing at infinity has also been recently studied [3, 5].

The aim of the present work consists in giving more insights on Theorem 2. A first question is the existence question: What conditions should V and K satisfy so that problem $(\mathcal{P}_{V,K})$ has a groundstate? A second question is whether the groundstate solution is in $L^2(\mathbb{R}^N)$. We provide here an unified approach which allows to handle potentials V that vanish at infinity or potentials K that explode at infinity. Unbounded potentials have been considered by several authors, see e.g. [18].

A classical tool to prove the existence of groundstates of $(\mathcal{P}_{V,\mu})$ is

Theorem 3. *If one has the continuous embedding*

$$H^1_V(\mathbb{R}^N) \subset L^p(\mathbb{R}^N, K\mathcal{L}^N),$$

then the functional $I : H^1_V(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$I(u) = \int_{\mathbb{R}^N} \frac{|\nabla u^2|}{2} + V\frac{|u|^2}{2} - \int_{\mathbb{R}^N} |u|^p \, d\mu$$

is well-defined and continuously differentiable on $H^1_V(\mathbb{R}^N)$.
If moreover this embedding is compact, then there exists a groundstate solution to problem $(\mathcal{P}_{V,\mu})$.

The applicability of Theorem 3 depends just on the answer to a question about continuous and compact embeddings. The assumptions of Theorem 2 are one way to ensure these embeddings, but there are other ways. A first tool is the function

$$\mathcal{W}(x) = \frac{K(x)}{V(x)^{\frac{N}{2} - \frac{p}{2}(\frac{N}{2}-1)}}.$$

Using Hölder's inequality and Sobolev inequality, one can prove the following result.

Theorem 4. *Let $K : \mathbb{R}^N \to \mathbb{R}^+$ and $V : \mathbb{R}^N \to \mathbb{R}^+$ be measurable functions.*
i) If $\mathcal{W} \in L^\infty(\mathbb{R}^N)$ and $2 \le p \le \frac{2N}{N-2}$, then one has the continuous embedding

$$H^1_V(\mathbb{R}^N) \subset L^p(\mathbb{R}^N, K\mathcal{L}^N).$$

ii) If moreover $K \in L^\infty_{loc}(\mathbb{R}^N)$, $p < \frac{2N}{N-2}$ and for every $\varepsilon > 0$,

$$\mathcal{L}^N(\{x \in \mathbb{R}^N \mid \mathcal{W}(x) > \varepsilon\} < \infty,$$

then this embedding is compact.

Theorem 4 is related to Theorem 18.6 in [14] by which $H^1_V(\mathbb{R}^N) \subset L^p_K(\mathbb{R}^N)$ when there exists $R > 0$ and $r : \mathbb{R}^N \setminus B(0,R) \to \mathbb{R}^+$ such that

$$\frac{1}{\sqrt{V(x)}} \le r(x) \le \frac{|x|}{3} \qquad \text{for every } x \in \mathbb{R}^N \setminus B(0,R),$$

$$0 < c^{-1} \le \frac{r(y)}{r(x)} \le c \qquad \text{for every } x \in \mathbb{R}^N \setminus B(0,R) \text{ and } y \in B(x, r(x)),$$

$$\sup_{x \in \mathbb{R}^N \setminus B(0,R)} \sup_{y \in B(x,r(x))} K(y) r(x)^{N - p(\frac{N}{2} - 1)} < \infty.$$

Since

$$\mathcal{W}(x) \le K(x) r(x)^{N-p(\frac{N}{2}-1)} \le \sup_{y \in B(x, r(x)} K(y) r(x)^{N-p(\frac{N}{2}-1)},$$

these assumptions are stronger than those of Theorem 4, and that they may fail for highly oscillating potentials while those of Theorem 4 hold.

In the case where $V(x) = (1+|x|)^{2\alpha-2}$, Theorem 4 allows for potentials K such that

$$\lim_{|x|\to\infty} |x|^\beta K(x) = 0,$$

with

$$\beta = (1-\alpha)\Big(N - p\big(\frac{N}{2} - 1\big)\Big), \tag{2}$$

which is a small improvement in view of Theorem 2. In the case of unbounded potentials, we recover the embeddings of [18].

While the condition of Theorem 4 allows V and K to oscillate strongly, their oscillation should be coordinated. A second tool provides embedding theorems with a condition without interplay between K and V, in terms of Marcinkiewicz spaces. Setting

$$\|f\|_{L^{r,\infty}} = \sup_{E \subset \mathbb{R}^N} \frac{1}{\mathcal{L}^N(E)^{1-\frac{1}{r}}} \int_E |f|,$$

for $p > 1$, recall that the space $L^{r,\infty}(\mathbb{R}^N)$ is the space of measurable functions $f : \mathbb{R}^N \to \mathbb{R}$ such that $\|f\|_{L^{r,\infty}} < +\infty$. Its subspace $L^{\infty,r}_0(\mathbb{R}^N)$ is the closure of $(L^\infty \cap L^1)(\mathbb{R}^N)$ in $L^{r,\infty}(\mathbb{R}^N)$.

In the sequel, we denote by $\dot{H}^1(\mathbb{R}^N)$ the homogeneous Sobolev space, i.e. $H^1_V(\mathbb{R}^N)$ with $V \equiv 0$.

Theorem 5. *Assume $N \ge 3$.*
i) If $2 \le p \le \frac{2N}{N-2}$

$$p\Big(\frac{1}{2} - \frac{1}{N}\Big) + \frac{1}{r} = 1$$

and $K \in L^{r,\infty}(\mathbb{R}^N, \mathbb{R}^+)$, then the embedding

$$\dot{H}^1(\mathbb{R}^N) \subset L^p(\mathbb{R}^N, K\mathcal{L}^N)$$

is continuous.
ii) If moreover $p < \frac{2N}{N-2}$ and $K \in L_0^{r,\infty}(\mathbb{R}^N)$, then this embedding is compact.

The first part of the result has been obtained by Visciglia [20]. Whereas the combination of Theorems 4 and 5 allows K not to be controlled pointwise by V, it still requires when V is bounded that K should not be locally worst than a function in $L^{r,\infty}$. On the other hand, when p is small enough, trace theorems show that $|u|^p$ is locally integrable on subsurfaces. This brings us to embeddings theorem for a general measure. Here we state the result with an explicit shape of a model potential V. Define

$$[\mu]_\alpha = \sup\Bigl\{ \frac{\mu(B(x,\rho))^{\frac{1}{p}}}{\rho^{\frac{N}{2}-1}} \mid x \in \mathbb{R}^N \text{ and } 0 < \rho < \frac{1}{2}(1+|x|)^{1-\alpha} \Bigr\}. \tag{3}$$

Theorem 6. *Let $N \geq 3$, $\alpha \geq 0$, $V(x) = (1+|x|)^{2\alpha-2}$ and μ be a Radon measure. Then,*
(i) $[\mu]_\alpha$ is finite if and only if there exists $c > 0$ such that for every $u \in H^1_V(\mathbb{R}^N)$,

$$\|u\|_{L^p(\mathbb{R}^N,\mu)} \leq c\|u\|_{H^1_V},$$

the quantity $[\mu]_\alpha$ being equivalent to the optimal constant in the inequality ;

(ii) the embedding $H^1_V(\mathbb{R}^N) \subset L^p(\mathbb{R}^N, \mu)$ is compact if and only if

$$\lim_{\delta \to 0} \sup\Bigl\{ \frac{\mu(B(x,\rho))^{\frac{1}{p}}}{\rho^{\frac{N}{2}-1}} \mid x \in \mathbb{R}^N \textit{ and } 0 < \rho < \delta(1+|x|)^{1-\alpha} \Bigr\} = 0, \tag{4}$$

$$\lim_{|x| \to \infty} \sup\Bigl\{ \frac{\mu(B(x,\rho))^{\frac{1}{p}}}{\rho^{\frac{N}{2}-1}} \mid 0 < \rho < \frac{1}{2}(1+|x|)^{1-\alpha} \Bigr\} = 0. \tag{5}$$

When $\alpha = 0$, then $H^1_V(\mathbb{R}^N) = D^{1,2}(\mathbb{R}^N)$; then the continuity part of Theorem 6 was proven by Maz'ja [11, Theorem 1.4.4/1] and the compactness part by Schneider [17, Theorem 2.1]. When $\alpha = 1$, it is due to Maz'ja [11, Theorems 1.4.4/2 and 1.4.6/1].

Whereas we do not have counterparts of Theorems 4 and 5 when $N = 2$, Theorem 6 remains true when $N = 2$ provided $\rho^{\frac{N}{2}-1}$ is replaced by $(\log \rho(1+|x|)^{\alpha-1})^{-1}$ everywhere in the statement (see Theorem 11). When $p < \frac{2N}{N-2}$, Theorem 6 allows the measure to be singular with respect to the Lebesgue measure. Another situation in which Theorem 6 works while the previous theorems fail is the following: $\alpha = 1$ and $K \in L^r_{\text{loc}}(\mathbb{R}^N) \setminus L^\infty(\mathbb{R}^N)$ is periodic.

We now draw our interest to the question whether the solutions to

$$\begin{cases} -\Delta u + Vu = u^{p-1}\mu & \text{in } \mathbb{R}^N, \\ \lim_{|x| \to \infty} u(x) = 0. \end{cases} \tag{$\mathcal{P}_{V,\mu}$}$$

are in $L^2(\mathbb{R}^N)$, as it is the case in Theorem 2. Observe that we have replaced the potential K by a postive Radon measure μ. The solution is then understood in the distributional sense.

Let us first point out a necessary condition. Indeed, if $u \neq 0$, and

$$\limsup_{|x|\to\infty} V(x)|x|^2 < \lambda(\lambda + 2 - N), \tag{6}$$

then, by the maximum principle, we have, for some $c > 0$,

$$u(x) \geq \frac{c}{(1+|x|)^\lambda}.$$

In particular, if (6) holds with $\lambda = \frac{N}{2}$, then $u \notin L^2(\mathbb{R}^N)$. This decay of V is in fact critical for u to be square-integrable.

Theorem 7. *Assume that $H^1_V(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$, and that*

$$\liminf_{|x|\to\infty} |x|^2 V(x) > 1 - (\tfrac{N}{2} - 1)^2 > 0, \tag{7}$$

then $u \in L^2(\mathbb{R}^N)$.

The proof proceeds by multiplication of the equation by a test function of the form $u(1 + |x|)$.

We will go further in this analysis, and try to catch as much information as possible about the decay of a solution.

Theorem 8. *Assume that $H^1(\mathbb{R}^N, V) \subset L^p(\mathbb{R}^N, \mu)$ and $u \in H^1_V(\mathbb{R}^N)$ solves*

$$-\Delta u + V u = u^{p-1}\mu.$$

(i) If there exists $\lambda > 0$ such that

$$\liminf_{|x|\to\infty} V(x)|x|^2 > \lambda(\lambda + 2 - N),$$

then there exists $C < \infty$ such that

$$u(x) \leq \frac{C}{(1+|x|)^\lambda}.$$

(ii) If moreover there exists $\alpha > 0$ and $\lambda > 0$ such that

$$\liminf_{|x|\to\infty} V(x)|x|^{2-2\alpha} > \lambda^2,$$

then there exists $C < \infty$ such that

$$u(x) \leq C e^{-\lambda(1+|x|)^\alpha}.$$

Theorem 2 gives the same decay rate than the last part of the theorem. However, our result allows equality in (1) — provided a solution exists. The limit case where equality holds in (1) brings us some complications in the proof. In the previous situation, the condition (1) implies that $H^1_V(\mathbb{R}^N) \subset L^q(\mathbb{R}^N, \mu)$ for some $q > p$. This allows to start immediately a bootstrap argument. In the present setting, a first step is required to prove that $H^1_V(\mathbb{R}^N) \subset L^q(\mathbb{R}^N, \mu)$ for some $q > p$.

The sequel of the paper is organized as follows. In Section 2, we work out the continuous and compact embeddings ; in particular, we prove Theorems 4, 5 and 6. Section 3 is devoted to decay estimates and contains the proofs of Theorems 7 and 8. Finaly, Section 4 deals with some extensions of our decay estimates to other frameworks that we do not cover with details.

2. Embedding theorems

In this section, we consider conditions that ensure continuity or compactness of the imbedding of $H^1_V(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N, K\mathcal{L}^N)$. We shall use three different methods: one based on the concentration function, the second based on Marcinkiewicz weak L^p–spaces and the last on the measure of balls, which will lead respectively to Theorems 4, 5 and 6 which are independent.

2.1. **Concentration function method.**

A first technique to obtain embeddings of $H^1_V(\mathbb{R}^N)$ consists in interpolating between $L^2(\mathbb{R}^N, V\mathcal{L}^N)$ and a space in which $H^1_V(\mathbb{R}^N)$ is contained : $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$.

Proof of Theorem 4. For every measurable set $A \subset \mathbb{R}^N$, since $2 \le p \le 2^*$, using Hölder's inequality, we infer that for any $u \in H^1_V(\mathbb{R}^N)$,

$$\int_A K|u|^p \le \|\mathcal{W}\|_{L^\infty(A)} \left(\int_A V|u|^2\right)^{\frac{N}{2}-\frac{p}{2}(\frac{N}{2}-1)} \left(\int_A |u|^{\frac{2N}{N-2}}\right)^{(\frac{p}{2}-1)(\frac{N}{2}-1)}. \quad (8)$$

Taking $A = \mathbb{R}^N$, we deduce the first statement of the Theorem from the Sobolev inequality.

To prove the second statement, it is sufficient to show that for any $\varepsilon > 0$, there exists a set $A \subset \mathbb{R}^N$ of finite-measure such that for every $u \in H^1_V(\mathbb{R}^N)$ with $\|u\|_{H^1_V} \le 1$,

$$\int_{A^c} K(x)|u|^p < \varepsilon.$$

Choosing $A_\delta = \{x \in \mathbb{R}^N \mid \mathcal{W}(x) \ge \delta\}$ in (8), we get

$$\int_{\mathbb{R}^N\setminus A_\delta} K(x)|u|^p \le \delta \left(\int_{\mathbb{R}^N} V|u|^2\right)^{\frac{N}{2}-\frac{p}{2}(\frac{N}{2}-1)} \left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}}\right)^{(\frac{p}{2}-1)(\frac{N}{2}-1)},$$

so that our claim follows from the Sobolev inequality. □

As mentioned in the introduction, Theorem 4 implies that $H^1_V(\mathbb{R}^N) \subset L^p(\mathbb{R}^N, K\mathcal{L}^N)$ when $V(x) = |x|^{2-2\alpha}$ and $K(x) = |x|^{-\beta}$, with β given by (2).

It should be pointed out that not only the proof of Theorem 4 fails in dimension 2: one can find counter-examples. A weaker statement will be proved in Section 2.3.3.

2.2. **Marcinkiewicz spaces method.**

Another point of view to obtain embedding, consists in using only the information about the Sobolev embedding of $H^1_V(\mathbb{R}^N)$.

Proof of Theorem 5. By [15], see also [21, Chapter 2], the Sobolev space $\dot{H}^1(\mathbb{R}^N)$ is continuousloy embedded in the Lorentz space $L^{\frac{2N}{N-2},2}(\mathbb{R}^N)$, i.e. the estimate

$$\|u\|_{L^{\frac{2N}{N-2},2}} \le C\|\nabla u\|_{L^2}$$

holds. One has then, by Hölder's inequality for Lorentz spaces and by the embedding $L^{\frac{2N}{N-2},p}(\mathbb{R}^N) \subset L^{\frac{2N}{N-2},2}(\mathbb{R}^N)$, and for every measurable set $A \subset$

$\mathbb{R}^N$

$$\begin{aligned}\int_A K|u|^p &\le \|K\|_{L^{r,\infty}(A)} \|u\|^p_{L^{\frac{2N}{N-2},p}}\\ &\le \|K\|_{L^{r,\infty}(A)} \|u\|^p_{L^{\frac{2N}{N-2},2}}\\ &\le C\|K\|_{L^{r,\infty}(A)} \|\nabla u\|^p_{L^2(\mathbb{R}^N)}.\end{aligned}$$

Under assumption ii), the compactness of the embedding can be proved easily. □

Let us compare Theorems 4 and 5 in the case where $V(x) \ge (1+|x|)^{2\alpha-2}$ and $K(x) \le (1+|x|)^\beta$. The first gives a continuous embedding when

$$\beta \ge (1-\alpha)\Big(N - p(1-\frac{N}{2})\Big)$$

while the latter requires

$$\beta \ge N - p(1-\frac{N}{2}).$$

If $\alpha \ge 0$, the condition of Theorem 4 is weaker than the condition of Theorem 5; when $\alpha \le 0$, one has the converse situation. The criticality of the rate $\alpha = 0$ can be explained by the Hardy inequality: $H^1_V(\mathbb{R}^N)$ is a strict subspace of $\dot H^1(\mathbb{R}^N)$ if, and only if, $\alpha > 0$.

As a byproduct of Theorems 4 and 5, one has

Corollary 2.1. *Assume that*

$$p\Big(\frac{1}{2} - \frac{1}{N}\Big) + \frac{1}{s} + \frac{2t}{N} = 1,$$

with $2 \le p \le \frac{2N}{N-2}$ *and* $t > 0$.
i) If $KV^{-t} \in L^{s,\infty}(\mathbb{R}^N)$, *then the embedding* $H^1_V(\mathbb{R}^N) \subset L^p(\mathbb{R}^N, K\mathcal{L}^N)$ *holds.*
ii) If $p < \frac{2N}{N-2}$ *and* $KV^{-t} \in L^{s,\infty}_0(\mathbb{R}^N)$, *this embedding is compact.*

Proof. Taking $\theta = \frac{1}{t}\big(\frac{N}{2} - \frac{p}{2}(\frac{N}{2}-1)\big)$ and using Hölder's inequality, we infer

$$\int_{\mathbb{R}^N} K|u|^p \le \Big(\int_{\mathbb{R}^N} V^{\frac{N}{2}-\frac{p}{2}(\frac{N}{2}-1)}|u|^p\Big)^{\frac{1}{\theta}} \Big(\int_{\mathbb{R}^N} \big(KV^{-t}\big)^{\frac{\theta}{\theta-1}}|u|^p\Big)^{1-\frac{1}{\theta}}.$$

One checks that the first factor is bounded by Theorem 4 while the second is bounded by Theorem 5. We then conclude that

$$\int_{\mathbb{R}^N} K|u|^p \le C\|KV^{-t}\|_{L^{s,\infty}} \|u\|^p_{H^1_V}.$$

Under the assumptions in ii), one obtains the compactness in a straightforward way. □

2.3. **Trace-type inequalities.** We now examine the special case where $V(x) = (1+|x|)^\alpha$. In this case, one can find necessary and sufficient conditions on a Radon measure μ so that one has the continuous embedding $H^1_V(\mathbb{R}^N) \subset L^p(\mathbb{R}^N, \mu)$, or so that it is compact. This approach is based on the corresponding work of Maz′ja on $\dot H^1(\mathbb{R}^N)$. We first explain how the case $N > 2$ is treated before sketching out how to adapt the arguments to the dimension $N = 2$.

2.3.1. *The subcritical case.* A first tool in the proof of Theorem 6 is a characterizations of the measures for which $H^1_V(\mathbb{R}^N) \subset L^p(\mathbb{R}^N, \mu)$ when $N > 2$. Define

$$[\mu] = \sup\Bigl\{ \frac{\mu(B(x,\rho))^{\frac{1}{p}}}{\rho^{\frac{N}{2}-1}} \mid x \in \mathbb{R}^N \text{ and } \rho > 0 \Bigr\}.$$

Theorem 9 (Adams [1], Maz′ja [11, Theorems 1.4.4/1 and 1.4.6/1])**.** *Let $N > 2$, μ be a Radon measure and $p > 2$. Then,*
(i) $[\mu]$ is finite if and only if there exists $C > 0$ such that for every $u \in \dot{H}^1(\mathbb{R}^N)$,

$$\|u\|_{L^p(\mathbb{R}^N,\mu)} \le C\, \|\nabla u\|_{L^2},$$

the quantity $[\mu]$ being equivalent to the optimal constant in the inequality ;

(ii) the embedding $\dot{H}^1(\mathbb{R}^n) \subset L^p(\mathbb{R}^N, \mu)$ is compact if and only if

$$\lim_{\delta \to 0} \sup\Bigl\{ \frac{\mu(B(x,\rho))^{\frac{1}{p}}}{\rho^{\frac{N}{2}-1}} \mid x \in \mathbb{R}^N \text{ and } 0 < \rho < \delta \Bigr\} = 0,$$

$$\lim_{|x| \to \infty} \sup\Bigl\{ \frac{\mu(B(x,\rho))^{\frac{1}{p}}}{\rho^{\frac{N}{2}-1}} \mid \rho > 0 \Bigr\} = 0.$$

Remark 1. Since for every Radon measure $\mu \ne 0$,

$$\liminf_{\rho \to 0} \sup_{x \in \mathbb{R}^N} \frac{\mu(B(x,\rho))}{\rho^N} > 0,$$

Theorem 9 essentially applies only if $p < \frac{2N}{N-2}$.

In order to prove Theorem 6, we first prove that Theorem 9 applies to the restriction of the measure μ to the ball $B(x, \frac{1}{2}(1 + |x|)^\alpha)$. Recall that $[\mu]_\alpha$ has been defined in (3).

Lemma 2.2. *Under the assumptions of Theorem 6, one has*
(i) for every x, $y \in \mathbb{R}^N$ and $\rho > 0$,

$$\frac{\mu(B(y,\rho) \cap B(x,r))^{\frac{1}{p}}}{\rho^{\frac{N}{2}-1}} \le C[\mu]_\alpha,$$

where $r = \frac{1}{2}(1 + |x|)^{1-\alpha}$;

(ii) for every $R > 0$ and $\delta > 0$,

$$\begin{aligned} \sup\Bigl\{ \frac{\mu(B(x,\rho) \cap B(0,R))}{\rho^{\frac{N}{2}-1}} \mid x \in \mathbb{R}^N \text{ and } \rho < \delta \Bigr\} \\ \le \sup\Bigl\{ \frac{\mu(B(x,\rho) \cap B(0,R))}{\rho^{\frac{N}{2}-1}} \mid x \in \mathbb{R}^N \\ \text{and } \rho < \delta \frac{(1+|x|)^{1-\alpha}}{\min(1, (1+\delta+R)^{1-\alpha})} \Bigr\} \end{aligned} \tag{9}$$

and

$$\begin{aligned}\sup\Big\{\frac{\mu(B(x,\rho)\setminus B(0,R))^{\frac{1}{p}}}{\rho^{\frac{N}{2}-1}}\mid x\in\mathbb{R}^N \textit{ and } \rho<\frac{1}{2}(1+|x|)^{1-\alpha}\Big\}\\ \le\sup\Big\{\frac{\mu(B(x,\rho))^{\frac{1}{p}}}{\rho^{\frac{N}{2}-1}}\mid |x|>\frac{2R-1}{3} \textit{ and } 0<\rho<\frac{1}{2}(1+|x|)^{1-\alpha}\Big\}.\end{aligned}\tag{10}$$

Proof. When $\rho<\frac{1}{2}(1+|y|)^{1+\alpha}$, one has trivially

$$\frac{\mu(B(y,\rho)\cap B(x,r))^{\frac{1}{p}}}{\rho^{\frac{N}{2}-1}}\le\frac{\mu(B(y,\rho))^{\frac{1}{p}}}{\rho^{\frac{N}{2}-1}}\le[\mu]_\alpha.$$

Assume now that $\rho\ge\frac{1}{2}(1+|y|)^{1-\alpha}$. If $\frac{1}{3}(1+|x|)\le(1+|y|)\le 3(1+|x|)$, one has $\rho\ge 3^{-|1-\alpha|}r$, and thus

$$\begin{aligned}\frac{\mu(B(y,\rho)\cap B(x,r))^{\frac{1}{p}}}{\rho^{\frac{N}{2}-1}}&\le\frac{\mu(B(x,r))^{\frac{1}{p}}}{\rho^{\frac{N}{2}-1}}\\ &\le 3^{|1-\alpha|(\frac{N}{2}-1)}\frac{\mu(B(x,r)^{\frac{1}{p}}}{r^{\frac{N}{2}-1}}\\ &\le 3^{|1-\alpha|(\frac{N}{2}-1)}[\mu]_\alpha.\end{aligned}\tag{11}$$

If $3(1+|y|)<1+|x|$, assume without loss of generality that $B(x,r)\cap B(y,\rho)\neq\emptyset$. One has then, since $r\le\frac{1}{2}(1+|x|)$,

$$\frac{|x|-1}{2}\le|x|-r<|y|+\rho\le\frac{|x|-2}{3}+\rho$$

so that

$$\rho\ge\frac{|x|+1}{6}>\frac{r}{3}.$$

Reasoning as in (11), one obtains

$$\frac{\mu(B(y,\rho)\cap B(x,r))^{\frac{1}{p}}}{\rho^{\frac{N}{2}-1}}\le 3^{(\frac{N}{2}-1)}[\mu]_\alpha.$$

Finally, when $3(1+|x|)<1+|y|$ and $B(x,r)\cap B(y,\rho)\neq\emptyset$, one has

$$3|x|+2-\rho\le|y|-\rho<|x|+r\le\frac{3|x|+1}{2}$$

so that

$$\rho\ge\frac{3}{2}(|x|+1)>3r,$$

and, as before,

$$\frac{\mu(B(y,\rho)\cap B(x,r))^{\frac{1}{p}}}{\rho^{\frac{N}{2}-1}}\le\frac{1}{3^{\frac{N}{2}-1}}[\mu]_\alpha.$$

For the second statement, assume that $\rho\le\delta$ and $B(x,\rho)\cap B(0,R)\neq\emptyset$. One has then $|x|\le\rho+R\le\delta+R$, so that

$$\rho\le\delta\frac{(1+|x|)^{1-\alpha}}{\min(1,(1+\delta+R)^{1-\alpha})}.$$

For the last statement, if $B(x,\rho) \not\subset B(0,R)$, then $R \le |x| + \rho \le (3|x|+1)/2$ and $|x| \ge (3R-1)/2$; the conclusion follows. □

The third tool to prove Theorem 6 is

Theorem 10 (Besicovitch's covering theorem, see e.g. [10, Theorem 2.7])**.** *If $A \subset \mathbb{R}^N$ is bounded and $\mathcal{B}$ is a family of closed balls such that each point of A is the center of some ball of $\mathcal{B}$, then there exists a finite or countable collection of balls $B_i \in \mathcal{B}$ that covers A and such that every point of $\mathbb{R}^N$ belong to at most $P(N)$ balls.*

We can now prove the main result of this section

Proof of Theorem 6. By Lemma 2.2 and Theorem 9, for every $x \in \mathbb{R}^N$ and $v \in \dot{H}^1(\mathbb{R}^N)$,

$$\|v\|^2_{L^p(B(x,r/2),\mu)} \le \|v\|^2_{L^p(B(x,r),\mu)} \le C[\mu]_\alpha \int_{\mathbb{R}^N} |\nabla v|^2,$$

where $r = \frac{1}{2}(1+|x|)^{1-\alpha}$. Recall that every $u \in H^1(B(0,1/2))$ has an extension $v \in H^1(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} |\nabla v|^2 \le C \int_{B(0,1/2)} |\nabla u|^2 + |u|^2.$$

By translation and scaling, every $u \in H^1(B(x,r/2))$ has an extension $v \in H^1(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} |\nabla v|^2 \le C \int_{B(x,r/2)} |\nabla u|^2 + r^{-2}|u|^2.$$

By the choice of r, for every $y \in B(x,r)$,

$$\begin{aligned}\frac{3}{2}(1+|x|) \le 1 + |x| - \frac{(1+|x|)^{1-\alpha}}{2} &\le 1 + |y| \\ &\le 1 + |x| + \frac{(1+|x|)^{1-\alpha}}{2} \le \frac{3}{2}(1+|x|),\end{aligned}$$

so that

$$\int_{\mathbb{R}^N} |\nabla v|^2 \le C' \int_{B(x,r/2)} |\nabla u|^2 + V|u|^2.$$

One has thus, for every $u \in H^1_V(\mathbb{R}^N)$,

$$\Bigl(\int_{B(x,r/2)} |u|^p\Bigr)^{\frac{2}{p}} \le C[\mu]_\alpha \int_{B(x,r/2)} |\nabla u|^2 + V|u|^2.$$

For every $R > 0$, applying now Theorem 10 to $A = \bar{B}(0,R)$ and $\mathcal{B} = B(x, \frac{1}{2}(1+|x|)^{1-\alpha})$, one obtains a collection of balls $(\bar{B}(x_i, r_i/2))_{i\in I}$ such that $A \subset \bigcup_{i \in I} \bar{B}(x_i, r_i/2)$, with $r_i = \frac{1}{2}(1+|x_i|)^{1-\alpha}$ and $\sum_{i\in I} \chi_{\bar{B}(x_i,r_i/2)} \le P(N)$, so that

$$\begin{aligned}\Big(\int_{B(0,R)} |u|^p \,d\mu\Big)^{\frac{2}{p}} &\le \Big(\sum_{i\in I} \int_{B(x_i,r_i)} |u|^p \,d\mu\Big)^{\frac{2}{p}}\\ &\le \sum_{i\in I} \Big(\int_{B(x_i,r_i)} |u|^p \,d\mu\Big)^{\frac{2}{p}}\\ &\le C[\mu]_\alpha \sum_{i\in I} \int_{B(x_i,r_i)} |\nabla u|^2 + V|u|^2\\ &\le CP(N)[\mu]_\alpha \int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2.\end{aligned}$$

One obtains the continuous embedding by letting $R \to \infty$.

For the converse statement, let φ be a compactly supported smooth function such that $\varphi = 1$ on $B(0, \frac{1}{2})$ and $\operatorname{supp} \varphi \subset B(0, 3/4)$ and set $\varphi_{x,\rho}(y) = \varphi((x-y)/\rho)$. If $\rho < \frac{1}{2}(1+|x|)^{1-\alpha}$, then $\frac{1}{2}(1+|y|) \le (1+|x|) \le 2(1+|y|)$ for $y \in B(x,\rho)$, so that

$$\int_{\mathbb{R}^N} V|\varphi_{x,\rho}|^2 \le \frac{C\rho^N}{(1+|x|)^{2-2\alpha}} \le C'\rho^{N-2}. \tag{12}$$

One has thus

$$\mu(B(x,\rho))^{\frac{1}{p}} \le \|\varphi_{x,\rho}\|_{L^p(\mathbb{R}^N,\mu)} \le c\Big(\int_{\mathbb{R}^N} |\nabla \varphi_{x,\rho}|^2 + V|\varphi_{x,\rho}|\Big)^{\frac{1}{2}} \le cC\rho^{\frac{N}{2}-1}.$$

For the compactness part, first note that we deduce from (9) of Lemma 2.2 and Theorem 9 that $\dot H^1(\mathbb{R}^N)$ is compactly embedded in $L^p(B(0,R),\mu)$ for every $R > 0$. Therefore the map $u \mapsto \chi_{B(0,R)} u$ is a compact operator from $H^1_V(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N,\mu)$. By the first part of this theorem and (10) of Lemma 2.2,

$$\begin{aligned}&\frac{\|u - \chi_{B(0,R)} u\|_{L^p(\mathbb{R}^N,\mu)}}{\|u\|_{H^1_V}}\\ &\quad\le \sup\Big\{\frac{\mu(B(x,\rho) \setminus B(0,R))^{\frac{1}{p}}}{\rho^{\frac{N}{2}-1}} \mid x \in \mathbb{R}^N \text{ and } \rho < \frac{1}{2}(1+|x|)^{1-\alpha}\Big\} \to 0\end{aligned}$$

as $R \to \infty$. Therefore the embedding $H^1_V(\mathbb{R}^N) \subset L^p(\mathbb{R}^N,\mu)$ is compact as a limit in the operator norm of compact operators.

For the necessity part, let $\delta_k \to 0$ and $(x_k)_k \subset \mathbb{R}^N$. Set $\rho_k = \delta_k(1+|x|)^{1-\alpha}$. The sequence $u_k = \rho_k^{-(N-2)/2} \varphi_{x_k,\rho_k}$ is bounded in $H^1_V(\mathbb{R}^N)$ (see (12)) and converges weakly to 0. Since $H^1_V(\mathbb{R}^N)$ is embedded compactly in $L^p(\mathbb{R}^N,\mu)$, one obtains

$$\frac{\mu(B(x_k,\rho_k))^{\frac{1}{p}}}{\rho_k^{\frac{N}{2}-1}} \le C\|u_k\|_{L^p(\mathbb{R}^N,\mu)} \to 0.$$

as $k \to \infty$. This proves (4). Assuming that $|x_k| \to \infty$ and taking $\delta_k = \frac{1}{2}$ instead of $\delta_k \to 0$, one obtains similarly (5). $\square$

Remark 2. In view of [11], it is clear that similar results apply to the Sobolev spaces $W^{1,q}(\mathbb{R}^N)$, with $q < N$. For example, one has

$$\Big(\int_{\mathbb{R}^N} |u|^p \, d\mu\Big)^{\frac{1}{p}} \leq [\mu]_{q,\alpha}\Big(\int_{\mathbb{R}^N} \sum_{i=0}^{k} \frac{|D^i u|^p}{(1+|x|)^{(1-\alpha)(k-i)p}}\Big)^{\frac{1}{q}},$$

where

$$[\mu]_{\alpha,q} = \sup\Big\{\frac{\mu(B(x,\rho))^{\frac{1}{p}}}{\rho^{\frac{N}{p}-k}} \mid x \in \mathbb{R}^N \text{ and } 0 < \rho < \frac{1}{2}(1+|x|)^{1-\alpha}\Big\}.$$

Remark 3. One can also consider spaces with a weight on the gradient. For example, set

$$H = \{u \in \mathrm{W}^{1,1}_{\mathrm{loc}} \mid \int_{\mathbb{R}^N} (1+|x|)^{2\tau} |\nabla u|^2 + (1+|x|)^{2\alpha+2\tau-2}|u|^2\}.$$

One has then $H \in L^p(\mathbb{R}^N, \mu)$ if and only if

$$\sup\Big\{\frac{\mu(B(x,\rho))^{\frac{1}{p}}}{\rho^{\frac{N}{2}-1}(1+|x|)^{\tau}} \mid x \in \mathbb{R}^N \text{ and } 0 < \rho < \frac{1}{2}(1+|x|)^{1-\alpha}\Big\} < \infty.$$

2.3.2. *The critical case.* In two dimensions, one has a similar result. Define

$$[\mu]_{\alpha,2} = \sup\{|\log \rho| \, \mu(B(x, \rho(1+|x|)^{1-\alpha}))^{\frac{1}{p}} \mid x \in \mathbb{R}^N \text{ and } 0 < \rho < \frac{1}{2}\}.$$

Theorem 11. *Assume $\alpha \geq 0$, $V(x) = (1+|x|)^{2\alpha-2}$ and let μ be a Radon measure. Then,*
(i) $[\mu]_{\alpha,2}$ is finite if and only if there exixts $C > 0$ such that for every $u \in \dot{H}^1(\mathbb{R}^2)$,

$$\|u\|_{L^p(\mathbb{R}^2,\mu)} \leq C\|u\|_{H^1_V},$$

the quantity $[\mu]_{\alpha,2}$ being equivalent to the optimal constant in the inequality ;
(ii) the embedding $H^1_V(\mathbb{R}^2) \subset L^p(\mathbb{R}^2, \mu)$ is compact if and only if

$$\lim_{\delta \to 0} \sup\{|\log \rho| \, \mu(B(x, \rho(1+|x|)^{1-\alpha}))^{\frac{1}{p}} \mid x \in \mathbb{R}^N \textit{ and } 0 < \rho < \delta\} = 0,$$
$$\lim_{|x|\to\infty} \sup\{|\log \rho| \, \mu(B(x, \rho(1+|x|)^{1-\alpha}))^{\frac{1}{p}} \mid 0 < \rho < \frac{1}{2}\} = 0.$$

Instead of Theorem 9, the main tool to prove Theorem 11 is

Theorem 12 (see [11, Corollary 8.6/1])**.** *Let μ be a Radon measure, $p > 2$ and*

$$[\mu]_2 = \sup\{|\log \rho| \, \mu(B(x,\rho))^{\frac{1}{p}} \mid x \in \mathbb{R}^N \textit{ and } 0 < \rho < 1\}.$$

Then,
(i) $[\mu]_2$ is finite if and only if there exixts $C > 0$ such that for every $u \in H^1(\mathbb{R}^2)$,

$$\|u\|_{L^p(\mathbb{R}^2,\mu)} \leq C(\|\nabla u\|_{L^2} + \|u\|_{L^2}),$$

the quantity $[\mu]_2$ being equivalent to the optimal constant in the inequality ;
(ii) the embedding $H^1(\mathbb{R}^2) \subset L^p(\mathbb{R}^2, \mu)$ is compact if and only if

$$\lim_{\delta \to 0} \sup\{|\log \rho|\, \mu(B(x,\rho))^{\frac{1}{p}} \mid x \in \mathbb{R}^N \text{ and } 0 < \rho < \delta\} = 0,$$

$$\lim_{|x| \to \infty} \sup\{|\log \rho|\, \mu(B(x,\rho))^{\frac{1}{p}} \mid 0 < \rho < 1\} = 0.$$

Proof of Theorem 6. By a variant of Lemma 2.2 and Theorem 12 together with a scaling argument, one obtains that for every $v \in H^1(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$,

$$\|v\|^2_{L^p(B(x,R/2),\mu)} \le C[\mu] \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{v^2}{R^2},$$

where $R = \frac{1}{2}(1+|x|)^{1-\alpha}$. The proof continues then as the proof of Theorem 6. □

Remark 4. Remark 2 still applies for $W^{k,q}(\mathbb{R}^N)$, with $kq = N$ and

$$[\mu]_{q,\alpha} = \sup\Big\{|\log \rho|^{q-1} \mu(B(x,\rho))^{\frac{1}{p}} \mid x \in \mathbb{R}^N \text{ and } 0 < \rho < \frac{1}{2}(1+|x|)^{1-\alpha}\Big\}.$$

2.3.3. *Power-like potentials.* When $N \ge 2$ and $V(x) = (1+|x|)^{2\alpha-2}$, Theorems 6 and 11, show that when $K(x) = (1+|x|)^{-\beta}$, where β is given by (2),

$$H^1_V(\mathbb{R}^N) \subset L^p(\mathbb{R}^N, K\mathcal{L}^N).$$

While Theorem 4 fails when $N = 2$, the preceding conclusion holds in this particular case. We prove it as a lemma that we keep for future reference in section 3. As this remains true when $N = 1$, we provide a direct proof that works for all dimensions:

Lemma 2.3. *Let $N \ge 1$, $\alpha > 0$, $2 \le p \le \frac{2N}{N-2}$ if $N \ge 3$ and $2 \le p < \infty$ otherwise, and β be given by (2). If $p < \infty$, $V(x) = (1+|x|)^{2\alpha-2}$ and $K(x) = (1+|x|)^{-\beta}$, then $H^1_V(\mathbb{R}^N) \subset L^p(\mathbb{R}^N, K\mathcal{L}^N)$.*

Proof. First note that by Gagliardo–Nirenberg's inequality [13] and by scale invariance, for every $R > 0$

$$\int_{B(0,2R)\setminus B(0,R)} \frac{|u(x)|^p}{|x|^\beta}\,dx \le C\Big(\int_{B(0,2R)\setminus B(0,R)} \frac{|u(x)|^2}{|x|^{2-2\alpha}}\,dx\Big)^{\frac{N}{2}-\frac{p}{2}(\frac{N}{2}-1)}$$
$$\Big(\int_{B(0,2R)\setminus B(0,R)} |\nabla u(x)|^2 + \frac{|u(x)|^2}{|x|^2}\,dx\Big)^{(\frac{p}{2}-1)\frac{N}{2}}.$$

Summing this for $R = 2^k$, $k \ge 0$, we obtain since $\alpha \ge 0$,

$$\int_{\mathbb{R}^N\setminus B(0,1)} \frac{|u(x)|^p}{|x|^\beta}\,dx \le C\Big(\int_{\mathbb{R}^N\setminus B(0,1)} \frac{|u(x)|^2}{|x|^{2-2\alpha}}\,dx\Big)^{\frac{N}{2}-\frac{p}{2}(\frac{N}{2}-1)}$$
$$\Big(\int_{\mathbb{R}^N\setminus B(0,1)} |\nabla u(x)|^2 + \frac{|u(x)|^2}{|x|^2}\,dx\Big)^{(\frac{p}{2}-1)\frac{N}{2}}$$
$$\le C\Big(\int_{\mathbb{R}^N\setminus B(0,1)} |\nabla u(x)|^2 + \frac{|u(x)|^2}{|x|^{2-2\alpha}}\,dx\Big)^{\frac{p}{2}}.$$

The conclusion follows then from Sobolev's embedding Theorem. □

One could similarly obtain some conditions for the compactness of the embedding. As a corollary, one has in $\mathbb{R}^2$,

$$\int_{\mathbb{R}^2} \frac{|u(x)|^p}{|x|^2}\,dx \le C\Big(\int_{\mathbb{R}^2} |\nabla u(x)|^2 + \frac{|u(x)|^2}{|x|^2}\,dx\Big)^{\frac{p}{2}}.$$

In contrast with the higher-dimensional case, the previous lemma cannot be improved when $N = 2$ and $\alpha > 0$ by replacing $(1 + |x|)$ by $|x|$. If one sets $V(x) = (1 + |x|^{2\alpha-2})$ and $K(x) = 1 + |x|^{-\beta}$, then the conclusion of the Lemma holds provided $\alpha \le 0$.

3. Decay estimates

We now turn out to the decay property of solutions to $(\mathcal{P}_{V,\mu})$. The first improvement is to obtain that u multiplied by some function is still in the energy space $H^1_V(\mathbb{R}^N)$. The latter method also allows that the same holds for a small power of u. By Moser's iteration technique, we show then that a solution u satisfies some decay estimates at infinity.

3.1. **Linear estimates.** We begin by considering the L^2 theory of decay of finite-energy. These are special cases of the sequel, but give an useful insight on the proof of the exact decay estimates.

Assumption 1. Let μ be a Radon measure, $f \in L^{p/(p-2)}(\mathbb{R}^N, \mu)$ and $u \in H^1_V(\mathbb{R}^N)$ be such that

(i) the embedding $H^1_V \subset L^p(\mathbb{R}^N, \mu)$ is continuous,
(ii) u satisfies

$$-\Delta u + V u = f u \mu. \tag{13}$$

Proposition 3.1. *Under Assumption 1, if*

$$\nu := \liminf_{|x|\to\infty} |x|^2 V(x) > \lambda^2 - (\tfrac{N}{2} - 1)^2 > 0, \tag{14}$$

then $(1 + |x|)^\lambda\, u \in H^1_V(\mathbb{R}^N)$.

Let us first show how Theorem 7 follows:

Proof of Theorem 7. Under the assumptions of Theorem 7, the assumptions of Proposition 3.1 hold with $f = |u|^{p-2} \in L^{\frac{p}{p-2}}(\mathbb{R}^N, \mu)$ and $\lambda = 1$. We have thus $(1 + |x|)u \in H^1_V$ and it easily follows that $u \in L^2(\mathbb{R}^N)$. □

The proof roughly goes as follow. Take $|x|^{2\lambda} u$ as a test function in (13), integrate on $\mathbb{R}^N \setminus B(0, R)$ and apply Hölder's inequality to obtain

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B(0,R)} &|\nabla(|x|^\lambda u)|^2 + V(x)||x|^\lambda u|^2 \\ &\le \Big(\int_{\mathbb{R}^N\setminus B(0,R)} f^{p/(p-2)}\,d\mu\Big)^{1-2/p}\Big(\int_{\mathbb{R}^N\setminus B(0,R)} ||x|^\lambda u|\Big)^{1/p} \\ &\qquad + \lambda^2 \int_{\mathbb{R}^N} \frac{|u|x|^\lambda|^2}{|x|^2} + \int_{\partial B(0,R)} u \frac{\partial}{\partial\nu}(u|x|^{2\lambda}).\end{aligned}$$

When R is large enough, by the assumption on f, μ and λ, the two first terms in the right-hand side can be absorbed, so that the conclusion follows.

As usual, we need to be careful in the estimates of quantities that might not be finite.

Proof of Proposition 3.1. For every $\Omega \subset \mathbb{R}^N$ and for every $\varphi \in W_0^{1,\infty}(\Omega)$ such that $\nabla\varphi$ has compact support in Ω, recall that $\varphi^2 u$ and $\varphi u \in H^1_V(\mathbb{R}^N)$,

$$|\nabla(\varphi u)|^2 = \nabla u \cdot \nabla(\varphi^2 u) + |\nabla\varphi|^2|u|^2. \tag{15}$$

so that, by Hölder's inequality and the embedding $H^1_V(\mathbb{R}^N) \subset L^p(\mathbb{R}^N, \mu)$, we get

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla(\varphi u)|^2 + V|\varphi u|^2 &= \int_{\mathbb{R}^N} f\varphi^2|u|^2\,d\mu + |\nabla\varphi|^2|u|^2\\
&\le \Bigl(\int_\Omega |f|^{\frac{p}{p-2}}\,d\mu\Bigr)^{1-\frac{2}{p}}\Bigl(\int_{\mathbb{R}^N}|\varphi u|^p\,d\mu\Bigr)^{\frac{2}{p}} + \int_{\mathbb{R}^N}|\nabla\varphi|^2|u|^2\\
&\le C\Bigl(\int_\Omega |f|^{\frac{p}{p-2}}\,d\mu\Bigr)^{1-\frac{2}{p}}\Bigl(\int_{\mathbb{R}^N}|\nabla(\varphi u)|^2 + V|\varphi u|^2\Bigr) + \int_{\mathbb{R}^N}|\nabla\varphi|^2|u|^2.
\end{aligned}$$

Let $\delta = C\Bigl(\int_\Omega |f|^{\frac{p}{p-2}}\,d\mu\Bigr)^{1-\frac{2}{p}}$. Since $f \in L^{\frac{p}{p-2}}(\mathbb{R}^N, \mu)$, we can choose $\Omega = \mathbb{R}^N \setminus B(0, R)$ in such a way that $0 < \delta < 1$. The preceding estimates then yield a control on the norm of φu

$$(1-\delta)\int_{\mathbb{R}^N}|\nabla(\varphi u)|^2 + V|\varphi u|^2 \le \int_{\mathbb{R}^N}|\nabla\varphi|^2|u|^2. \tag{16}$$

Taking (14) into account and increasing R if necessary, we can assume that for every $x \in \Omega$,

$$V(x) \ge \frac{\nu-\delta}{|x|^2} \tag{17}$$

and

$$(\nu-\delta)(1-\delta) \ge \frac{\lambda^2}{1-\delta} - (1-\delta)\Bigl(\frac{N}{2}-1\Bigr)^2, \tag{18}$$

where we recall that $\nu = \liminf_{|x|\to\infty}|x|^2V(x)$.

Choose now $\psi \in C_c^\infty(\Omega)$ such that $\psi \equiv 1$ on $\mathbb{R}^N \setminus B(0, 2R)$ and, for $k > 0$, set $\varphi_k(x) = \psi(x)\min(k, |x|^\lambda)$. We infer from (16) and (17) that

$$\begin{aligned}
(1-\delta)\int_{\mathbb{R}^N}|\nabla(\varphi_k u)|^2 + \Bigl(\delta V + (1-\delta)\frac{\nu-\delta}{|x|^2}\Bigr)|\varphi_k u|^2&\\
&\le \int_{\mathbb{R}^N}|\nabla\varphi_k|^2|u|^2\\
&\le \int_{\mathbb{R}^N}\frac{\lambda^2}{|x|^2}|\varphi_k u|^2 + C\int_{B(0,2R)\setminus B(0,R)}|u|^2,
\end{aligned}$$

where the constant C depends only on ψ, R and λ. Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^N}|\nabla(\varphi_k u)|^2 + \Bigl(\delta V + \Bigl((1-\delta)(\nu-\delta) - \frac{\lambda^2}{1-\delta}\Bigr)\frac{1}{|x|^2}\Bigr)|\varphi_k u|^2&\\
&\le \frac{C}{1-\delta}\int_{B(0,2R)\setminus B(0,R)}|u|^2.
\end{aligned}$$

Now, using (18), we infer that

$$\delta\left(\int_{\mathbb{R}^N}|\nabla(\varphi_k u)|^2+V|\varphi_k u|^2\right)$$
$$+(1-\delta)\left(\int_{\mathbb{R}^N}|\nabla(\varphi_k u)|^2-\Big(\frac{N}{2}-1\Big)^2\frac{|\varphi_k u|^2}{|x|^2}\right)$$
$$\leq C'\int_{B(0,2R)\setminus B(0,R)}|u|^2$$

and Hardy's inequality then yields

$$\delta\left(\int_{\mathbb{R}^N}|\nabla(\varphi_k u)|^2+V|\varphi_k u|^2\right)\leq C'\int_{B(0,2R)\setminus B(0,R)}|u|^2.$$

By letting $k\to\infty$, we deduce from Fatou's lemma that

$$\int_{\mathbb{R}^N}|\nabla(\varphi u)|^2+V|\varphi u|^2\leq C'\int_{B(0,2R)\setminus B(0,R)}|u|^2,$$

with $\varphi(x)=\psi(x)|x|^\lambda$. Since local estimates are straightforward, we easily conclude that $|x|^\lambda\, u\in H^1_V(\mathbb{R}^N\setminus B(0,1))$.

To complete the proof, we need to show that $\nabla((1+|x|)^\lambda u)\in L^2(\mathbb{R}^N)$. For this purpose, it is enough to observe that

$$(1+|x|)^\lambda u=\frac{(1+|x|)^\lambda}{|x|^\lambda}|x|^\lambda u$$

and to use the fact that $\nabla(|x|^\lambda u)\in L^2(\mathbb{R}^N)$. □

A similar method works in the case where V decays slowly at the infinity:

Proposition 3.2. *Under Assumption 1, if*

$$\nu_\alpha:=\liminf_{|x|\to\infty}|x|^{2-2\alpha}V(x)>\lambda^2, \tag{19}$$

then $e^{\lambda(1+|x|)^\alpha}u\in H^1_V(\mathbb{R}^N)$.

Proof. Arguing as in the proof of Proposition 3.1, we choose the radius R in such a way that $\delta<1$,

$$\nu_\alpha>\frac{\lambda^2}{(1-\delta)^2}+\delta. \tag{20}$$

and

$$V(x)>\frac{\nu_\alpha-\delta}{|x|^{2-2\alpha}}, \tag{21}$$

for every $x\in U$. Let $\psi\in C^\infty_c(U)$ be such that $\psi\equiv 1$ on $\mathbb{R}^N\setminus B(0,2R)$ and, for $k>0$, set $\varphi_k(x)=\psi(x)\min(k,e^{\lambda|x|^\alpha})$. By (16), (20) and (21), we deduce that

$$\int_{\mathbb{R}^N}|\nabla(\varphi_k u)|^2+V|\varphi_k u|^2\leq C\int_{B(0,2R)\setminus B(0,R)}|u|^2.$$

Letting $k\to\infty$ and applying Fatou's lemma, we conclude that

$$\int_{\mathbb{R}^N}|\nabla(\varphi u)|^2+V|\varphi u|^2\leq C\int_{B(0,2R)\setminus B(0,R)}|u|^2,$$

with $\varphi(x)=\psi(x)e^{\lambda|x|^\alpha}$. One concludes therefrom and from local estimates that $e^{\lambda'(1+|x|)^\alpha}u\in H^1_V(\mathbb{R}^N)$ for every $\lambda'<\lambda$. □

Remark 5. The statement uses the weight $e^{\lambda(1+|x|)^\alpha}$ instead of the simpler one $e^{\lambda|x|^\alpha}$ because the latter is not Lipschitz when $0 < \alpha < 1$.

3.2. **Nonlinear estimates.** The method of proof of Propositions 3.1 and 3.2 allows in fact to obtain information about $((1+|x|)^\lambda\, u)^\gamma$ or $(e^{\lambda(1+|x|)^\alpha}\, u)^\gamma$ for $\gamma > 1$.

Lemma 3.3. *Under Assumption 1, assuming moreover that* $\gamma > 1$, $u \in L^{2\gamma}_{\mathrm{loc}}(\mathbb{R}^N)$ *and one of the following hypothesis holds*

(i)
$$\lambda < (\tfrac{N}{2} - 1)\tfrac{2\gamma-1}{\gamma^2-\gamma},$$
and
$$\nu = \liminf_{|x|\to\infty} |x|^2 V(x) > \bigl(\lambda + \tfrac{\gamma-1}{\gamma}(\tfrac{N}{2}-1)\bigr)^2 - \bigl(\tfrac{N}{2}-1\bigr)^2 > 0,$$

(ii)
$$\nu > (1 + \tfrac{(\gamma-1)^2}{2\gamma-1})\lambda^2,$$

we have $((1+|x|)^\lambda\, u)^\gamma \in H^1_V(\mathbb{R}^N)$.

The statement of Theorem 3.3 is a perturbation of Proposition 3.1 in the sense that for every λ that satisfies (14), there exists $\bar\gamma(\nu,\lambda) > 1$ such that Theorem 3.3 is applicable for $1 \le \gamma < \bar\gamma(\nu,\lambda)$. On the other hand, Theorem 3.3 will only be useful when γ is small. Indeed, starting with $u \in H^1_{loc}$, Sobolev's embedding Theorem only says $u \in L^{2\gamma}_{\mathrm{loc}}(\mathbb{R}^N)$ for $\gamma \le N/(N-2)$. Iterating the Lemma, one obtains successively that $u \in L^{2\gamma_k}_{\mathrm{loc}}(\mathbb{R}^N)$ for $\gamma_k = N^k/(N-2)^k$ for every k. For every fixed $\lambda > 0$, the iteration process will cease giving global integrability information about $((1+|x|)^\lambda u)^\gamma$ when γ is too large.

The proof of Lemma 3.3 follows the strategy used to prove that solutions $u \in H^1(B(0,1))$ of the critical problem
$$-\Delta u = u^{\frac{N+2}{N-2}}$$
are in $L^q(B(0,\frac{1}{2}))$ for $q < 2N^2/(N-2)^2$ [4, 6, 19]. The proof proceeds as follows. We first establish by integration by parts the inequality (25). A suitable choice of test functions yields that $((1+|x|)^\lambda\, u)^\gamma \in H^1_V(\mathbb{R}^N \setminus B(0,2R))$ for some large $R > 0$. Finally we prove that one also has that for every $y \in \mathbb{R}^N$, $((1+|x|)^\lambda\, u)^\gamma \in H^1_V(B(y,\rho))$ for some $\rho > 0$. Since by Besicovitch's covering theorem, $\mathbb{R}^N$ can be written as the union of a finite collection of such balls together with $\mathbb{R}^N \setminus B(0,2R)$, the claim will follow.

Proof of Lemma 3.3. First note that if $v \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^N)$ is locally bounded and if φ is locally Lipschitz, one has
$$\begin{aligned}|\nabla((\varphi v)^\gamma)|^2 = \tfrac{\gamma^2}{2\gamma-1}\nabla v\cdot\nabla(\varphi^{2\gamma}v^{2\gamma-1}) + \tfrac{2\gamma^2-2\gamma}{2\gamma-1} v^\gamma\varphi^{\gamma-1}\nabla\varphi\cdot\nabla(\varphi v)^\gamma \\ + \tfrac{\gamma^2}{2\gamma-1}|\nabla\varphi|^2 v^{2\gamma}\varphi^{2\gamma-2} \qquad (22)\end{aligned}$$
and thus, for every $\eta > 0$,
$$\begin{aligned}(1 - \eta\tfrac{\gamma^2-\gamma}{2\gamma-1})|\nabla((\varphi v)^\gamma|)^2 \\ \le \tfrac{\gamma^2}{2\gamma-1}\nabla v\cdot\nabla(\varphi^{2\gamma}v^{2\gamma-1}) + (\tfrac{\gamma^2}{2\gamma-1} + \tfrac{1}{\eta}\tfrac{\gamma^2-\gamma}{2\gamma-1})|\nabla\varphi|^2 v^{2\gamma}\varphi^{2\gamma-2}. \qquad (23)\end{aligned}$$

On the other hand, by (15), and since $\gamma > 1$,

$$\begin{aligned}(1 - \eta\tfrac{\gamma^2-\gamma}{2\gamma-1})|\nabla(\varphi v)|^2 &\leq \tfrac{\gamma^2}{2\gamma-1}|\nabla(\varphi v)|^2\\ &= \tfrac{\gamma^2}{2\gamma-1}\nabla v\cdot\nabla(\varphi^2 v) + \tfrac{\gamma^2}{2\gamma-1}|\nabla\varphi|^2 v^2\\ &\leq \tfrac{\gamma^2}{2\gamma-1}\nabla v\cdot\nabla(\varphi^2 v) + (\tfrac{\gamma^2}{2\gamma-1} + \tfrac{1}{\eta}\tfrac{\gamma^2-\gamma}{2\gamma-1})|\nabla\varphi|^2 v^2.\end{aligned}\tag{24}$$

We will use this last estimates successively to obtain a first estimate at infinity and a second one on small balls.

First step - a basic inequality. Define the truncation sequences $(v_k)_k$ and $(w_k)_k$ by

$$v_k = \min((u\varphi_k)^\gamma, ku\varphi_k) \text{ and } w_k = \min((u\varphi_k)^{2\gamma-1}, k^2 u\varphi_k),$$

where the choice of φ_k will be specified later. By applying successively (23) and (24) to v_k, we get the estimate

$$(1 - \eta\tfrac{\gamma^2-\gamma}{2\gamma-1})|\nabla v_k|^2 \leq \tfrac{\gamma^2}{2\gamma-1}\nabla u\cdot\nabla(\varphi_k w_k) + (\tfrac{\gamma^2}{2\gamma-1} + \tfrac{1}{\eta}\tfrac{\gamma^2-\gamma}{2\gamma-1})\frac{|\nabla\varphi_k|^2}{\varphi_k^2}v_k^2.$$

If the support of φ_k lies in some open set $\Omega \subset \mathbb{R}^N$, choosing $\varphi_k w_k$ as test function, applying Hölder's inequality and the embedding $H^1_V(\mathbb{R}^N) \subset L^p(\mathbb{R}^N, \mu)$, we infer that

$$\begin{aligned}&\int_{\mathbb{R}^N}(\tfrac{2\gamma-1}{\gamma^2} - \eta\tfrac{\gamma-1}{\gamma})|\nabla v_k|^2 + V|v_k|^2\\ &\qquad\leq \int_{\mathbb{R}^N} f|v_k|^2\,d\mu + (1 + \tfrac{1}{\eta}\tfrac{\gamma-1}{\gamma})\int_{\mathbb{R}^N}\frac{|\nabla\varphi_k|^2}{|\varphi_k|^2}|v_k|^2\\ &\qquad\leq \Big(\int_\Omega |f|^{\frac{p}{p-2}}\,d\mu\Big)^{1-\frac{2}{p}}\Big(\int_{\mathbb{R}^N}|v_k|^p\,d\mu\Big)^{\frac{2}{p}} + (1 + \tfrac{1}{\eta}\tfrac{\gamma-1}{\gamma})\int_{\mathbb{R}^N}\frac{|\nabla\varphi_k|^2}{|\varphi_k|^2}|v_k|^2\\ &\qquad\leq C\Big(\int_\Omega |f|^{\frac{p}{p-2}}\,d\mu\Big)^{1-\frac{2}{p}}\Big(\int_{\mathbb{R}^N}|\nabla v_k|^2 + V|v_k|^2\Big) + (1+\tfrac{1}{\eta}\tfrac{\gamma-1}{\gamma})\int_{\mathbb{R}^N}\frac{|\nabla\varphi_k|^2}{|\varphi_k|^2}|v_k|^2.\end{aligned}\tag{25}$$

Let us set again $\delta = C\Big(\int_\Omega |f|^{\frac{p}{p-2}}\,d\mu\Big)^{1-\frac{2}{p}}$. The preceding estimate then leads to

$$(\tfrac{2\gamma-1}{\gamma^2} - \eta\tfrac{\gamma-1}{\gamma} - \delta)\int_{\mathbb{R}^N}|\nabla v_k|^2 + (1-\delta)\int_{\mathbb{R}^N}V|v_k|^2 \leq (1+\tfrac{1}{\eta}\tfrac{\gamma-1}{\gamma})\int_{\mathbb{R}^N}\frac{|\nabla\varphi|^2}{|\varphi|^2}|v_k|^2.\tag{26}$$

Second step - An estimate at infinity. Assume first that (i) holds. We then choose $\eta = \lambda/(\frac{N}{2} - 1)$. Since $f \in L^{\frac{p}{p-2}}(\mathbb{R}^N, \mu)$, we can take $\Omega = \mathbb{R}^N \setminus B(0, R)$ in such a way that

$$\delta(2-\delta) \leq \tfrac{2\gamma-1}{\gamma^2} - \tfrac{2\lambda}{N-2}\tfrac{\gamma-1}{\gamma}.$$

On the other hand, increasing R if necessary, we can assume that

$$(\nu - \delta) \geq \frac{(\lambda + \frac{\gamma}{\gamma-1}(\frac{N}{2}-1))^2}{(1-\delta)^2} - (\tfrac{N}{2} - 1)^2\tag{27}$$

and

$$V(x) \geq \frac{\nu - \delta}{|x|^2},$$

for every $x \in \Omega$. Let $\psi \in C_c^\infty(\Omega)$ be such that $\psi \equiv 1$ on $\mathbb{R}^N \setminus B(0, 2R)$. For $k > 0$, set $\varphi_k(x) = \psi(x) \min(k, |x|^\lambda)$. By (16), for k and R large enough, we have

$$\begin{aligned}\int_{\mathbb{R}^N} (\tfrac{2\gamma-1}{\gamma^2} - \eta\tfrac{\gamma-1}{\gamma} - \delta)|\nabla v_k|^2 + \Bigl((1-\delta)\delta V + (1-\delta)^2 \frac{\nu-\delta}{|x|^2}\Bigr)|v_k|^2 \\ \le (1 + \tfrac{1}{\eta}\tfrac{\gamma-1}{\gamma}) \int_{\mathbb{R}^N} \frac{|\nabla \varphi_k|^2}{|\varphi_k|^2} |v_k|^2 \\ \le (1 + \tfrac{1}{\eta}\tfrac{\gamma-1}{\gamma})\Bigl(\int_{\mathbb{R}^N} \frac{\lambda^2}{|x|^2}|v_k|^2 + C\int_{B(0,2R)\setminus B(0,R)} |u|^{2\gamma}\Bigr),\end{aligned}$$

where the constant C does not depend on k. Taking (27) into account, we deduce that

$$\begin{aligned}(\tfrac{2\gamma-1}{\gamma^2} - \eta\tfrac{\gamma-1}{\gamma} - \delta(2-\delta))\Bigl(\int_{\mathbb{R}^N} |\nabla v_k|^2 - (\tfrac{N}{2}-1)^2\frac{|v_k|^2}{|x|^2}\Bigr) \\ + (1-\delta)\delta\Bigl(\int_{\mathbb{R}^N} |\nabla v_k|^2 + V|v_k|^2\Bigr) \\ \le C\int_{B(0,2R)\setminus B(0,R)} |u|^{2\gamma}\end{aligned}$$

Applying Hardy's inequality yields

$$\int_{\mathbb{R}^N} |\nabla v_k|^2 + V|v_k|^2 \le C' \int_{B(0,2R)\setminus B(0,R)} |u|^{2\gamma},$$

and letting $k \to \infty$, we conclude that

$$\int_{\mathbb{R}^N} |\nabla(\varphi u)^\gamma|^2 + V|(\varphi u)^\gamma|^2 \le C' \int_{B(0,2R)\setminus B(0,R)} |u|^2,$$

with $\varphi(x) = \psi(x)|x|^\lambda$. Arguing as in the proof of Proposition 3.1, we deduce that $((1+|x|)^\lambda u)^\gamma \in H^1_V(\mathbb{R}^N \setminus B(0,2R))$.

If (ii) holds, we proceed similarly, choosing the radius R sufficiently large and $\eta > 0$ such that

$$\eta\tfrac{\gamma-1}{\gamma} + 2\delta - \delta^2 \le \tfrac{2\gamma-1}{\gamma^2}, \qquad \lambda^2(1 + \tfrac{1}{\eta}\tfrac{\gamma-1}{\gamma}) \le (\nu-\delta)(1-\delta)^2$$

instead of (27).

Third step - the local estimates. Keeping the same notations, we now fix $x_0 \in \mathbb{R}^N$, choose $\eta = 1/(\gamma-1)$, $\Omega = B(x_0, \rho)$, $\varphi \in C_c^\infty(\Omega)$ such that $\varphi = 1$ on $B(x_0, \rho/2)$ and we set $\psi_k = \varphi$ for every k. Taking ρ in such a way that

$$\delta \le \tfrac{\gamma-1}{2\gamma^2},$$

we deduce from (16) that

$$\tfrac{\gamma-1}{2\gamma^2}\int_{B(x_0,\rho)} |\nabla v_k|^2 + V|v_k|^2 \le C\int_{B(x_0,\rho)} |v_k|^2 \le C' \int_{B(x_0,\rho)} |u|^{2\gamma}.$$

Letting $k \to \infty$, we conclude that $\nabla(u^\gamma) \in L^2(B(x_0, \rho/2))$, and therefore $((1+|x|)^\lambda u)^\gamma \in H^1_V(B(x_0, \rho/2)$.

Conclusion. Taking all the previous estimates into account, the conclusion now follows from a standard application of Besicovitch's covering theorem. □

In view of Theorem 8, one would have expected to have conditions (i) or (ii) replaced by the weaker assumption

$$\nu > (\lambda - \tfrac{\gamma-1}{\gamma}(\tfrac{N}{2} - 1))^2 - (\tfrac{N}{2} - 1)^2.$$

Observe that the sign in front of $\frac{\gamma-1}{\gamma}$ has changed. This can be explained partially roughly as follows. If λ is optimal, one expects u to behave as $|x|^{-\lambda-(\frac{N}{2}-1)/\gamma}$ and

$$2u^\gamma |x|^{\lambda(\gamma-1)} \nabla |x|^\lambda \cdot \nabla (|x|^\lambda u)^\gamma \sim -\frac{\lambda(N-2)}{|x|^N}.$$

When passing from (22) to (23), the latter quantity can be bounded by

$$\eta |\nabla (u|x|^\lambda)^\gamma|^2 + \frac{1}{\eta} u^{2\gamma} |x|^{2\lambda(\gamma-1)} |\nabla |x|^\lambda|^2$$

so that choosing $\eta = \lambda/(\frac{N}{2} - 1)$ as in the proof, yields $\lambda(N-2)/|x|^N$, i.e. the opposite quantity. (One would like thus to take $\eta = -\lambda/(\frac{N}{2} - 1)$.)

The method of proof also works for $\frac{1}{2} < \gamma < 1$. In this case, the second term on the right-hand side of (22) has a negative coefficient, so that one (23) holds for $\eta < 0$. The conditions on γ, λ and ν are the same excepted that the second inequality in (i) becomes

$$\nu > \bigl(\lambda - \tfrac{\gamma-1}{\gamma}(\tfrac{N}{2} - 1)\bigr)^2 - \bigl(\tfrac{N}{2} - 1\bigr)^2.$$

In view of the previous remark, the case $\gamma < 1$ is slightly better.

Finally, in the same fashion, one obtains the counterpart of Proposition 3.2:

Lemma 3.4. *Under Assumption 1, if $u \in L^{2\gamma}_{\mathrm{loc}}(\mathbb{R}^N)$ with $\gamma > 1$, and if*

$$\nu_\alpha = \liminf_{|x|\to\infty} |x|^{2-2\alpha} V(x) > \bigl(1 + \tfrac{(\gamma-1)^2}{2\gamma-1}\bigr)\lambda^2,$$

then $\bigl(e^{\lambda(1+|x|)^\alpha} u\bigr)^\gamma \in H^1_V(\mathbb{R}^N)$.

As for Lemma 3.3, the condition on ν_α and λ are stonger than the condition $\nu_\alpha > \lambda^2$ that is stated in Theorem 8.

Whereas Lemma 3.3 plays a crucial role in the sequel, Lemma 3.4 is not really needed, since Lemma 3.6 only requires information on the integrability of $|u|^{p-2}$ with a power-type weight.

3.3. **Moser iteration scheme.** We now show that whenever u and f are in slighlty better spaces than $H^1_V(\mathbb{R}^N)$ and $L^{p/(p-2)}(\mathbb{R}^N, \mu)$, this information can be upgraded to a uniform decay of u at infinity.

Lemma 3.5. *Assume that* (14) *holds, $H^1(\mathbb{R}^N, V) \subset L^p(\mathbb{R}^N, \mu)$ and*

$$f(1+|x|)^{(N-2)(\eta-1)} \in L^q(\mathbb{R}^N, \mu),$$

where

$$\eta = \frac{p}{2}\Bigl(1 - \frac{1}{q}\Bigr) > 1.$$

Then, if $(1+|x|)^{\lambda} u \in H^1_V(\mathbb{R}^N)$ *and* u *solves* (13)*, there exists* $C < \infty$ *such that*

$$u(x) \le \frac{C}{(1+|x|)^{\lambda+(N-2)/2}}.$$

Proof. Assume that $((1+|x|)^{\sigma} u)^{\gamma} \in H^1_V(\mathbb{R}^N)$ for some $\gamma \ge 1$ and $\sigma > 0$. Setting $\gamma' = \eta\gamma$,

$$\begin{aligned} \sigma' &= \sigma + (\tfrac{N}{2} - 1)\frac{\eta - 1}{\gamma'}, \\ w(x) &= u^{2\gamma'-1}(1+|x|)^{2\gamma'\sigma'} \end{aligned}$$

and

$$v(x) = ((1+|x|)^{\sigma'} u)^{\gamma'},$$

one has, see (24),

$$|\nabla v|^2 = \tfrac{\gamma'^2}{2\gamma'-1} \nabla u \cdot \nabla w + 2\sigma' \tfrac{\gamma'(\gamma'-1)}{2\gamma'-1} \frac{v}{1+|x|} \frac{x \cdot \nabla v}{|x|} + \tfrac{\gamma'^2}{2\gamma'-1} \sigma'^2 \frac{|v|^2}{(1+|x|)^2},$$

so that

$$|\nabla v|^2 \le \tfrac{2\gamma'^2}{2\gamma'-1} \nabla u \cdot \nabla w + \gamma'^2 \sigma'^2 (1 + \tfrac{1}{(2\gamma'-1)^2}) \frac{|v|^2}{(1+|x|)^2}.$$

By a suitable limiting argument, one has therefore

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla v|^2 \le \tfrac{2\gamma'^2}{2\gamma'-1} \int_{\mathbb{R}^N} f v^2 \, d\mu - \tfrac{2\gamma'^2}{2\gamma'-1} \int_{\mathbb{R}^N} V v^2 \\ + \gamma'^2 \sigma'^2 (1 + \tfrac{1}{(2\gamma'-1)^2}) \int_{\mathbb{R}^N} \frac{|v|^2}{(1+|x|)^2}. \end{aligned}$$

One has by Hölder's inequality and the embedding $H^1_V \subset L^p(\mathbb{R}^N, \mu)$

$$\begin{aligned} \int_{\mathbb{R}^N} f v^2 \, d\mu &= \int_{\mathbb{R}^N} f (1+|x|)^{(N-2)(\eta-1)} |u(x)(1+|x|)^{\sigma}|^{2\gamma'} \, d\mu \\ &\le C \Big(\int_{\mathbb{R}^N} |f (1+|x|)^{(N-2)(\eta-1)}|^q \, d\mu \Big)^{\frac{1}{q}} \Big(\int_{\mathbb{R}^N} |(1+|x|)^{\sigma} u|^{\gamma p} \, d\mu \Big)^{1-\frac{1}{q}} \\ &\le C \Big(\int_{\mathbb{R}^N} |f (1+|x|)^{(N-2)(\eta-1)}|^q \, d\mu \Big)^{\frac{1}{q}} \| ((1+|x|)^{\sigma} u)^{\gamma} \|^{2\eta}_{H^1_V}. \end{aligned}$$

Observing that $\eta < p \le 2N/(N-2)$ and combining this with (14), we infer that Lemma 2.3 is applicable and yields

$$\int_{\mathbb{R}^N} \frac{|v|^2}{(1+|x|)^2} = \int_{\mathbb{R}^N} \frac{(|(1+|x|)^{\sigma} u|^{\gamma})^{2\eta}}{(1+|x|)^{2-(\frac{N}{2}-1)(2\eta-2)}} \le C \| ((1+|x|)^{\sigma} u)^{\gamma} \|^{2\eta}_{H^1_V}.$$

One concludes thus that

$$\| ((1+|x|)^{\sigma'} u)^{\gamma'} \|_{H^1_V} \le C(1 + \gamma' + \sigma'\gamma'^2) \| ((1+|x|)^{\sigma} u)^{\gamma} \|^{\eta}_{H^1_V}.$$

Setting now $\gamma_k = \eta^k$ and

$$\sigma_k = \lambda + (1 - \frac{1}{\eta^k}) \frac{N-2}{2},$$

we get

$$\|((1+|x|)^{\sigma_{k+1}} u)^{\gamma_{k+1}}\|_{H^1_V}^{1/\gamma_{k+1}} \le [C(1+\eta^{2(k+1)})]^{1/\eta^{k+1}} \|((1+|x|)^{\sigma_k} u)^{\gamma}\|_{H^1_V}^{1/\gamma_k}.$$

Therefore, the quantity

$$\|((1+|x|)^{\sigma_k} u)^{\gamma_k}\|_{H^1_V}^{1/\gamma_k}$$

is bounded uniformly in k. In particular, by Lemma 2.3 again, we infer that

$$\Bigl(\int_{\mathbb{R}^N} \frac{((1+|x|)^{\lambda+(N-2)/2} u)^{2\eta^k}}{(1+|x|)^N}\Bigr)^{1/(2\eta^k)}$$

is bounded uniformly in k, so that

$$(1+|x|)^{\lambda+(N-2)/2} u \in L^\infty(\mathbb{R}^N). \qquad \square$$

The same can be done when the potential decays slowly at infinity.

Lemma 3.6. *Assume* (19) *holds,* $H^1(\mathbb{R}^N, V) \subset L^p(\mathbb{R}^N, \mu)$ *and*

$$f(1+|x|)^{(1-\alpha)(N-2)(\eta-1)} \in L^q(\mathbb{R}^N, \mu),$$

where

$$\eta = \frac{p}{2}\Bigl(1-\frac{1}{q}\Bigr) > 1.$$

If $e^{\lambda(1+|x|)^\alpha} u \in H^1_V(\mathbb{R}^N)$ *and* u *solves* (13), *then there exists* $C<\infty$ *such that*

$$u(x) \le \frac{Ce^{-\lambda(1+|x|)^\alpha}}{(1+|x|)^{(1-\alpha)(N-2)/2}}.$$

Proof. We argue as in the proof of the previous lemma, taking $\gamma' = \eta\gamma$,

$$\sigma' = \sigma + (1-\alpha)(\tfrac{N}{2}-1)\frac{\eta-1}{\gamma'}$$

$$w(x) = (1+|x|)^{2\gamma'\sigma'} e^{2\gamma'\lambda(1+|x|)^\alpha} u^{2\gamma'-1}(x),$$

and

$$v(x) = ((1+|x|)^{\sigma'} e^{\lambda(1+|x|)^\alpha} u(x))^{\gamma'}.$$

One obtains similarly

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla v|^2 \le & \tfrac{2\gamma'^2}{2\gamma'-1} \int_{\mathbb{R}^N} f v^2\, d\mu - \tfrac{2\gamma'^2}{2\gamma'-1} \int_{\mathbb{R}^N} V v^2 \\ & + \gamma'^2 (|\sigma'| + \lambda\alpha)^2 (1 + \tfrac{1}{(2\gamma'-1)^2}) \int_{\mathbb{R}^N} \frac{|v|^2}{(1+|x|)^{2-2\alpha}}.\end{aligned}$$

From the embedding $H^1_V \subset L^p(\mathbb{R}^N, \mu)$ and Lemma 2.3, we deduce

$$\|((1+|x|)^{\sigma'} u)^{\gamma'}\|_{H^1_V} \le C(1+\gamma' + (|\sigma|+\lambda\alpha)\gamma'^2) \|((1+|x|)^{\sigma} u)^{\gamma}\|_{H^1_V}^{2\eta}.$$

Setting now $\gamma_k = \eta^k$ and

$$\sigma_k = \lambda + (1-\alpha)(1-\frac{1}{\eta^k})\frac{N-2}{2},$$

and iterating as before, one has that

$$\|(e^{\lambda(1+|x|)^\alpha} u)^{\gamma_k}\|_{H^1_V}^{1/\gamma_k}$$

is bounded uniformly in k. In particular, by Lemma 2.3

$$\Bigl(\int_{\mathbb{R}^N} \frac{((1+|x|)^{(1-\alpha)(N-2)/2} e^{\lambda(1+|x|)^\alpha} u)^{2\eta^k}}{(1+|x|)^{N(1-\alpha)}}\Bigr)^{1/(2\eta^k)}$$

is bounded uniformly in k, so that

$$(1+|x|)^{(1-\alpha)(N-2)/2} e^{\lambda(1+|x|)^\alpha} u \in L^\infty(\mathbb{R}^N). \qquad \square$$

3.4. **Proof of Theorem 8.** We can now bring together the results of the previous sections in order to deduce the decay at infinity.

Proof of Theorem 8. Consider first the statement (i). Since we know that $|u|^{p-2} \in L^{p/(p-2)}(\mathbb{R}^N, \mu)$ and, by assumption, we have

$$\liminf_{|x|\to\infty} V(x)|x|^2 > (\lambda - (\tfrac{N}{2} - 1))^2 - (\tfrac{N}{2} - 1)^2,$$

we deduce from Proposition 3.1 that $u(1+|x|)^{\lambda-(\frac{N}{2}-1)} \in H^1_V(\mathbb{R}^N)$.

Next, when $\gamma > 1$ is sufficiently small, Lemma 3.3 shows that

$$(u(1+|x|)^{\frac{\gamma-1}{\gamma}(\frac{N}{2}-1)})^\gamma \in H^1_V(\mathbb{R}^N) \subset L^p(\mathbb{R}^N, \mu).$$

Setting $q = \frac{\gamma p}{p-2}$ and

$$\eta = \tfrac{p}{2}(1 - \tfrac{1}{q}) = 1 + \tfrac{\gamma-1}{\gamma}(\tfrac{p}{2} - 1),$$

one reaches the conclusion by using Lemma 3.5.

The proof of (ii) is similar. We start from Proposition 3.2 which states $e^{\lambda(1+|x|)^\alpha} u \in H^1_V(\mathbb{R}^N)$. On the other hand, in view of Lemma 3.3, there exists $\gamma > 1$ such that

$$(u(1+|x|)^{\frac{\gamma-1}{\gamma}(\alpha-1)(\frac{N}{2}-1)})^\gamma \in H^1_V(\mathbb{R}^N)$$

Taking q and η as above, by Lemma 3.6,

$$(1+|x|)^{(1-\alpha)(N-2)/2} e^{\lambda(1+|x|)^\alpha} u \in L^\infty(\mathbb{R}^N).$$

This gives the conclusion if $\alpha \le 1$. Otherwise, one just need to notice that the range of admissible λ is open. $\square$

4. Further comments

The method that we have followed is known to be very flexible. Let us highlight some similar situations that can be treated as above.

4.1. **Fast decay for exploding potential.** By the Kelvin transform the estimates around infinity are equivalent to local estimates with a singular potential. Indeed, if $u \in H^1_V(\mathbb{R}^N)$ satisfies $(\mathcal{P}_{V,\mu})$, then

$$\bar u(x) = \frac{1}{|x|^{N-2}} u\Bigl(\frac{x}{|x|^2}\Bigr).$$

satisfies

$$-\Delta \bar u + \bar V u = u^{p-1} \bar\mu,$$

where

$$\bar V(x) = \frac{1}{|x|^4} V\Bigl(\frac{x}{|x|^2}\Bigr)$$

and the measure $\bar{\mu}$ is defined by

$$\int_{\mathbb{R}^N} \varphi \, d\bar{\mu} = \int_{\mathbb{R}^N} \varphi\Big(\frac{x}{|x|^2}\Big) \frac{1}{|x|^{(N-2)p}} \, d\mu.$$

As a consequence of Theorem 8, one has that if

$$\liminf_{x \to 0} |x|^2 V(x) > \lambda(\lambda + N - 2)$$

for $\lambda > 0$, then in a a neighbourhood of 0, $u(x) \le C|x|^\lambda$. Similarly, if

$$\liminf_{x \to 0} |x|^{2+2\alpha} V(x) > \lambda^2,$$

then $u(x) \le e^{-\lambda/|x|^\alpha}$ in a a neighbourhood of 0.

4.2. **Divergence-form operators.** The Laplacian can be replaced by an elliptic operator in divergence form. Assume that u solves,

$$-\operatorname{div} \cdot A \nabla u + V u = |u|^{p-2} u \mu,$$

where $A : \mathbb{R}^N \to \mathbb{R}^{N \times N}$ is measurable and $A(x)$ is symmetric for every $x \in \mathbb{R}^N$ and there exist $0 < \underline{a} \le \overline{a} < \infty$ such that

$$\underline{a} |\xi|^2 \le \xi \cdot A \xi \le \overline{a} |\xi|^2. \tag{28}$$

If

$$\liminf_{|x| \to \infty} |x|^2 V(x) > \overline{a} \lambda^2 - \underline{a} (\tfrac{N}{2} - 1)^2 > 0$$

then $(1 + |x|)^\lambda \in H^1_V(\mathbb{R}^N)$. Similarily, if

$$\liminf_{|x| \to \infty} |x|^{2-2\alpha} V(x) > \overline{a} \lambda^2,$$

then $e^{\lambda(1+|x|)^\alpha} u \in H^1_V(\mathbb{R}^N)$. The proof of Lemmas 3.5 and 3.6 apply directy, so that $u(x) \le C(1+|x|)^{-\lambda+1-\frac{N}{2}}$ and $u(x) \le C e^{-\lambda(1+|x|)^\alpha} (1+|x|)^{(\alpha-1)(\frac{N}{2}-1)}$.

4.3. **Nonuniformly elliptic operators.** If the matrix A is not anymore uniformly elliptic, but satisfies

$$\frac{\underline{a}}{(1 + |x|)^{2\tau}} |\xi|^2 \le \xi \cdot A \xi \le \frac{\overline{a}}{(1 + |x|)^{2\tau}} |\xi|^2,$$

instead of (28). One has then the following extension: if

$$\liminf_{|x| \to \infty} |x|^2 V(x) > \overline{a} \lambda^2 - \underline{a} (\tfrac{N}{2} - \tau - 1)^2 > 0,$$

then $(1 + |x|)^\lambda u \in H$, where H is defined in Remark 3, and if

$$\liminf_{|x| \to \infty} |x|^{2-2\alpha} V(x) > \overline{a} \lambda^2,$$

then $e^{\lambda(1+|x|)^\alpha} u \in H$. Suitable adaptations of Lemmas 3.5 allow also to show that

$$u(x) \le C(1 + |x|)^{-\lambda - (\frac{N}{2} - 1 - \tau)}$$

and

$$u(x) \le C e^{-\lambda(1+|x|)^\alpha} (1 + |x|)^{(\alpha-1)(\frac{N}{2} - 1 - \tau)}$$

respectively.

References

[1] D. Adams, *A trace inequality for generalized potentials,* Studia Math. **48** (1973), 99–105.

[2] A. Ambrosetti, V. Felli, and A. Malchiodi, *Ground states of nonlinear Schrödinger equations with potentials vanishing at infinity*, J. Eur. Math. Soc. (JEMS) **7** (2005), no. 1, 117–144.

[3] A. Ambrosetti, A. Malchiodi, D. Ruiz, *Bound states of Nonlinear Schrödinger Equations with Potentials Vanishing at Infinity,* J. Anal. Math. **98** (2006), 317–348.

[4] T. Aubin and Y. Y. Li, *On the best Sobolev inequality*, J. Math. Pures Appl. (9) **78** (1999), no. 4, 353–387.

[5] D. Bonheure and J. Van Schaftinegn, *Bound state solutions for a class of nonlinear Schrödinger equations*, Rev. Mat. Iberoamericana, 24 (2008), 297-351.

[6] H. Brezis and T. Kato, *Remarks on the Schrödinger operator with singular complex potentials*, J. Math. Pures Appl. (9) **58** (1979), no. 2, 137–151.

[7] M. del Pino, P. Felmer, Local mountain passes for semilinear elliptic problems in unbounded domains. Calc. Var. Partial Differential Equations 4 (1996), 121-137.

[8] N. del Pino, P. Felmer, Semi-classical states of nonlinear Schrödinger equations: a variational reduction method. Math. Ann. 324 (2002), no. 1, 1-32.

[9] A. Floer, and A. Weinstein, *Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential*, J. Funct. Anal. 69 (1986), no. 3, 397–408.

[10] P. Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge University Press, Cambridge, 1995, Fractals and rectifiability.

[11] V. G. Maz′ja, *Sobolev spaces*, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1985, Translated from the Russian by T. O. Shaposhnikova.

[12] W-M Ni, *Diffusion, cross-diffusion, and their spike-layer steady states*, Notices Amer. Math. Soc. **45** (1998), no. 1, 9–18.

[13] L. Nirenberg, *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa (3) **13** 1959, 115–162.

[14] B. Opic and A. Kufner, *Hardy-type inequalities*, Pitman Research Notes in Mathematics Series, 219, Longman Scientific & Technical, Harlow, 1990.

[15] J. Peetre, *Espaces d'interpolation et théorème de Soboleff*, Annales de l'institut Fourier, **16** (1966) no. 1, 279–317

[16] P. H. Rabinowitz, *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys. **43** (1992), no. 2, 270–291.

[17] M. Schneider, *Compact embeddings and indefinite semilinear elliptic problems*, Nonlinear Anal. **51** (2002), no. 2, Ser. A: Theory Methods, 283–303.

[18] B. Sirakov, *Existence and multiplicity of solutions of semi-linear elliptic equations in* $\mathbb{R}^N$, Calc. Var and PDE 11 (2000), 119–142.

[19] N. S. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa (3) **22** (1968), 265–274.

[20] N. Visciglia, *A note about the generalized Hardy-Sobolev inequality with potential in* $L^{p,d}(\mathbf{R}^n)$, Calc. Var. Partial Differential Equations 24 (2005), no. 2, 167–184.

[21] W. Ziemer, *Weakly differentiable functions. Sobolev spaces and functions of bounded variation*, Graduate Texts in Mathematics, 120, Springer-Verlag, New York, 1989.

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