BOUNDARY ESTIMATES FOR ELLIPTIC SYSTEMS WITH L^1–DATA

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Abstract.

1. Introduction

Recently, new estimates for L^1–vector fields have been discovered by Bourgain and Brezis [1, 2], which yield in particular improved estimates for the solutions of elliptic systems in $\mathbf{R}^N$ or in a cube $Q \subset \mathbf{R}^N$ with periodic boundary conditions. Simplified proof of some of the results have been given by Van Schaftingen [8]. Here are two typical results:

Theorem 1.1. *Let $f \in L^1(\mathbf{R}^N; \mathbf{R}^N)$, $N \geq 3$. If $\operatorname{div} f = 0$, then the system*

$$-\Delta u = f \qquad \textit{in } \mathbf{R}^N,$$

admits a unique solution $u \in L^{N/(N-2)}(\mathbf{R}^N; \mathbf{R}^N)$ with $\nabla u \in L^{N/(N-1)}$.

A similar concluion holds for the same problem in a cube with periodic boundary conditions.

Theorem 1.2. *Let $f \in L^1(\mathbf{R}^2; \mathbf{R}^2)$. If $\operatorname{div} f = 0$, then the system*

$$-\Delta u = f \qquad \textit{in } \mathbf{R}^N,$$

admits a unique solution $u \in (L^\infty \cap C)(\mathbf{R}^2; \mathbf{R}^2)$ with $\nabla u \in L^2$.

Our main goal in this paper is to address similar questions in domains $\Omega \subset \mathbf{R}^N$ with Dirichlet or Neumann boundary conditions. Interior estimates can be easily derived from the results in [1, 2, 8]. However the question of estimates up to the boundary requires some further work.

In section 2, we study the system

$$-\Delta u = f \qquad \text{in } \Omega \subset \mathbf{R}^2,$$

together with the Dirichlet boundary condition $u = 0$ on $\partial\Omega$ or the Neumann boundary condition $\partial u/\partial n = 0$ on $\partial\Omega$. For the Dirichlet problem, we show that if $f \in L^1(\Omega; \mathbf{R}^2)$ and $\operatorname{div} f = 0$, then $u \in C(\overline{\Omega}; \mathbf{R}^2) \cap W^{1,2}(\Omega; \mathbf{R}^2)$. For the Neumann problem, we get the same conclusion under the additionnal assumption that $(f \cdot n) = 0$ on $\partial\Omega$; such a condition plays an essential role, see Remark 2.3. The proofs are elementary; they involve sharp estimates for the Green's functions. These are well-known to the experts and are presented in Appendix A for the convenience of the reader.

In section 3, we start with the system

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with $\Omega \subset \mathbf{R}^N$, $N \geq 2$ and $f \in \mathrm{L}^1(\mathbf{R}^N;\mathbf{R}^N)$. If $\operatorname{div} f = 0$, the heart of the matter is the inequality

$$\Big|\int_\Omega f \cdot \varphi\Big| \leq \|f\|_{\mathrm{L}^1} \|D\varphi\|_{\mathrm{L}^N}, \tag{1.2}$$

for every $\varphi \in (\mathrm{W}^{1,N}_0 \cap \mathrm{L}^\infty)(\Omega;\mathbf{R}^N)$, which we derive from similar estimates in [1, 2, 8]. Therefore it admits an elementary proof in the spirit of [8]. Next we combine (1.2) with standard L^p regularity theory to conclude that $u \in \mathrm{W}^{1,N/(N-1)}(\Omega;\mathbf{R}^N)$ when $f \in \mathrm{L}^1(\Omega;\mathbf{R}^N)$ and $\operatorname{div} f = 0$.

A much more delicate result asserts that if $f \in \mathrm{L}^1(\Omega;\mathbf{R}^2)$ and $\operatorname{div} f \in (\mathrm{W}^{2,N} + \mathrm{W}^{1,N}_0)^*$, one still has $u \in \mathrm{W}^{1,N/(N-1)}(\Omega;\mathbf{R}^N)$. The main ingredient is due to Bourgain and Brezis and asserts that every vector field in $\mathrm{W}^{1,N}_0$ belongs to L^∞ modulo gradients (see the precise statements in Theorem 3.2 and Lemma 3.3).

The remainder of section 3 is devoted to the pure Neumann boundary conditions and to various mixtures of Dirichlet and Neumann boundary conditions. We also consider the problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In section 4, we present estimates up to the boundary for the problem

$$\begin{cases} \operatorname{div} Z = 0 & \text{in } \Omega, \\ \operatorname{curl} Z = Y & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^3$, together with the boundary conditions $Z \cdot n = 0$ or $Z \times n = 0$. Next we present some results for first-order systems of k–forms, $2 \leq k \leq N-2$, such as

$$\begin{cases} d\omega = \alpha & \text{in } \Omega, \\ \delta\omega = \beta & \text{in } \Omega. \end{cases}$$

2. Elliptic systems in $\mathbf{R}^2$

Theorem 2.1. *Let $\Omega \subset \mathbf{R}^2$ be a smooth simply-connected domain and let $f \in \mathrm{L}^1(\Omega;\mathbf{R}^2)$. If $\operatorname{div} f = 0$ in the sense of distributions, i.e.,*

$$\int_\Omega f \cdot \nabla\zeta = 0, \quad \forall \zeta \in C^1_0(\overline{\Omega}),$$

then the problem

$$\begin{cases} -\Delta u = f & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega, \end{cases}$$

has a unique solution $u \in \mathrm{W}^{1,2}(\Omega;\mathbf{R}^2) \cap C(\overline{\Omega};\mathbf{R}^2)$ satisfying

$$\|u\|_{\mathrm{W}^{1,2}} + \|u\|_{\mathrm{L}^\infty} \leq C\|f\|_{\mathrm{L}^1}. \tag{2.1}$$

Proof. By classical regularity estimates, there is a solution $u \in \mathrm{W}^{1,q}$, for $q < 2$. Since $\operatorname{div} f = 0$, there exists $F \in \mathrm{W}^{1,1}(\Omega)$ such that $f = (-\partial_2 F, \partial_1 F)$

and $\int_\Omega F = 0$. By Sobolev's inequality, $\|F\|_{\mathrm{L}^2} \le C\|f\|_{\mathrm{L}^1}$. For every $\varphi \in C_c^\infty(\Omega; \mathbf{R}^2)$, one has

$$\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f \cdot \varphi = \int_\Omega F(\partial_2 \varphi_1 - \partial_1 \varphi_2) \le C\|f\|_{\mathrm{L}^1} \|\nabla \varphi\|_{\mathrm{L}^2}.$$

Therefore $u \in \mathrm{W}^{1,2}(\Omega; \mathbf{R}^2)$ with the required estimate.

We now prove that $u \in \mathrm{L}^\infty$. Let G denote the Green's function associated to Dirichlet boundary condition for Ω. Integrating by parts, one has, for every $x \in \Omega$,

$$\begin{aligned} u(x) &= \int_\Omega G(x,y) f(y)\,dy \\ &= -\int_\Omega (x-y)(\nabla_y G(x,y) \cdot f(y))\,dy - \int_\Omega G(x,y)(x-y) \operatorname{div} f(y)\,dy \\ &= -\int_\Omega (x-y)(\nabla_y G(x,y) \cdot f(y))\,dy, \end{aligned} \tag{2.2}$$

since $\operatorname{div} f = 0$. By Proposition A.1, $|\nabla G(x,y)||x-y|$ is uniformly bounded for $x, y \in \Omega$. Hence u satisfies the required estimate. To prove that u is continuous, assume that $x_n \to x$. Then $u(x_n) \to u(x)$ by (2.2) and Lebesgue's dominated convergence Theorem. □

Remark 2.1. In the more general case where $f \in \mathrm{L}^1(\Omega; \mathbf{R}^2)$ and $\operatorname{div} f \in \mathrm{L}^1(\Omega)$, one obtains the continuity and the boundedness of u from the boundedness of $|\nabla_y G(x,y)||x-y|$ on $\Omega \times \Omega$. Using the the estimate

$$\int_\Omega |\nabla u|^2 = \int_\Omega u \cdot f \le \|u\|_{\mathrm{L}^\infty} \|f\|_{\mathrm{L}^1} \le C(\|f\|_{\mathrm{L}^1} + \|\operatorname{div} f\|_{\mathrm{L}^1})\|f\|_{\mathrm{L}^1},$$

one concludes $u \in \mathrm{W}^{1,2}(\Omega; \mathbf{R}^2)$. Theorem 3.1 in the next section gives $\mathrm{W}^{1,2}$ (but not L^∞!) estimates under the weaker condition $\operatorname{div} f \in (\mathrm{W}^{2,2} \cap \mathrm{W}_0^{1,2})^*$.

Remark 2.2. The conclusion of Theorem 2.1 remains valid if f is a measure (more precisely, f belongs to the dual space of $C_0(\overline{\Omega})$). The proof of the bounds in $\mathrm{W}^{1,2}$ and in L^∞ are unchanged. To prove that u is continuous, assume that $x_n \to x \in \overline{\Omega}$. Then

$$(x_n - y)(\nabla_y G(x_n, y)) \to (x-y)(\nabla_y G(x,y))$$

for every $y \in \overline{\Omega} \setminus \{0\}$. On the other hand, the measure f belongs to the dual space of $\mathrm{W}_0^{1,2}(\Omega)$ and thus it does not charge points. One can conclude as before by Lebesgue's dominated convergence Theorem.

Theorem 2.2. *Let $\Omega \subset \mathbf{R}^2$ be a smooth simply-connected domain and let $f \in \mathrm{L}^1(\Omega; \mathbf{R}^2)$. If $\operatorname{div} f = 0$ in Ω and $f \cdot n = 0$ on $\partial\Omega$ in the sense that*

$$\int_\Omega f \cdot \nabla \zeta = 0, \quad \forall \zeta \in C^1(\overline{\Omega}),$$

then $\int_\Omega f = 0$ and the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \\ \int_\Omega u = 0, \end{cases}$$

has a unique solution $u \in \mathrm{W}^{1,2}(\Omega; \mathbf{R}^2) \cap C(\overline{\Omega}; \mathbf{R}^2)$ *and satisfying* (2.1).

Proof. The proof of Theorem 2.2 is similar to the proof of Theorem 2.1. Since $f \cdot n = 0$ on $\partial\Omega$, one can construct F such that $F = 0$ on $\partial\Omega$ (in place of $\int_\Omega F = 0$ in the previous proof). Since $F \in \mathrm{W}^{1,2}_0(\Omega)$, one has

$$\int_\Omega f = \int_\Omega (\nabla F)^\perp = 0.$$

One concludes as before that $u \in \mathrm{W}^{1,2}$ and, using Proposition A.3, that u is bounded and continuous up to the boundary. □

Remark 2.3. If one replaces the condition $f \cdot n = 0$ on $\partial\Omega$ in Theorem 2.2 by $\int_\Omega f = 0$, the conclusion is not necessarily true. Assume without loss of generality that $0 \in \partial\Omega$ and that the normal to the boundary at 0 is parallel to the first coordinate axis. Choose a function $\rho \in C^\infty_c(\mathbf{R}^2)$ such that $\rho \geq 0$ and $\int_\mathbf{R} \rho(0, x_2)\, dx_2 = 1$. Define $\rho_\varepsilon(x) = \rho(x/\varepsilon)/\varepsilon$ and

$$f_\varepsilon = (\nabla \rho_\varepsilon)^\perp - \frac{1}{|\Omega|} \int_\Omega (\nabla \rho_\varepsilon)^\perp.$$

For every $\varphi \in C(\overline{\Omega}; \mathbf{R}^2)$,

$$\int_\Omega f_\varepsilon \varphi \to e_1 \Bigl(\varphi(0) - \frac{1}{|\Omega|} \int_\Omega \varphi\Bigr).$$

Let now $u_\varepsilon \in \mathrm{W}^{1,2}(\Omega; \mathbf{R}^2)$ be the solution of

$$\begin{cases} -\Delta u_\varepsilon = f_\varepsilon & \text{in } \mathbf{R}^2, \\ \frac{\partial u_\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega, \\ \int_\Omega u_\varepsilon = 0, \end{cases}$$

and assume by contradiction that $\|\nabla u_\varepsilon\|_{\mathrm{L}^2}$ remains bounded as $\varepsilon \to 0$. One has then, for every $\varphi \in C^\infty(\overline{\Omega}; \mathbf{R}^2)$,

$$\Bigl|\varphi(0) - \frac{1}{|\Omega|} \int_\Omega \varphi\Bigr| = \lim_{\varepsilon \to 0} \Bigl|\int_\Omega f_\varepsilon \varphi\Bigr| = \lim_{\varepsilon \to 0} \Bigl|\int_\Omega \nabla u_\varepsilon \cdot \nabla \varphi\Bigr| \leq C \|\nabla \varphi\|_{\mathrm{L}^2}.$$

This is not possible, since $\mathrm{W}^{1,2}(\mathbf{R}^2)$ is not imbedded in $\mathrm{L}^\infty(\mathbf{R}^2)$. Note that

$$\int_\Omega |\nabla u_\varepsilon|^2 \leq \|f_\varepsilon\|_{\mathrm{L}^1} \|u_\varepsilon\|_{\mathrm{L}^\infty},$$

and thus $\|u_\varepsilon\|_{\mathrm{L}^\infty}$ is not bounded as $\varepsilon \to 0$.

3. Second-order systems in higher dimensions

In this section, assume $N \geq 2$.

3.1. **The Laplace equation with zero Dirichlet boundary condition.**

Theorem 3.1. *Let* $\Omega \subset \mathbf{R}^N$ *be a smooth bounded domain and let* $f \in \mathrm{L}^1(\Omega; \mathbf{R}^N)$. *If*

$$(3.1) \quad [f] = \sup\Bigl\{\int_\Omega f \cdot \nabla \zeta \ :\ \zeta \in C^2(\overline{\Omega}), \\ \zeta = 0 \text{ on } \partial\Omega \text{ and } \|D^2 \zeta\|_{\mathrm{L}^N} \leq 1\Bigr\} < \infty,$$

then the system

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique weak solution $u \in W^{1,N/(N-1)}(\Omega;\mathbf{R}^N)$ *satisfying*

$$\|u\|_{W^{1,N/(N-1)}} \le C(\|f\|_{L^1} + [f]).$$

Remark 3.1. If $\operatorname{div} f = 0$ in the sense of distributions, then $[f] = 0$. More generally,

$$[f] = \|\operatorname{div} f\|_{(W^{2,N}\cap W_0^{1,N})^*}.$$

In order to prove Theorem 3.1, first recall

Theorem 3.2 (Bourgain and Brezis [2])**.** *For every* $\varphi \in W_0^{1,N}(Q;\mathbf{R}^N)$*, there exist* $\psi \in (W_0^{1,N} \cap L^\infty)(Q;\mathbf{R}^N)$ *and* $\eta \in W_0^{2,N}(Q)$ *such that*

$$\varphi = \psi + \nabla\eta,$$

where Q *denotes the unit cube in* $\mathbf{R}^N$*. Moreover,*

$$\|\psi\|_{W^{1,N}} + \|\psi\|_{L^\infty} + \|\eta\|_{W^{2,N}} \le C\|\varphi\|_{W^{1,N}}.$$

Theorem 3.2 can be extended to any smooth domain:

Lemma 3.3. *Let* $\Omega \subset \mathbf{R}^N$ *be a smooth domain. For every* $\varphi \in W_0^{1,N}(\Omega;\mathbf{R}^N)$*, there exist* $\psi \in (W_0^{1,N} \cap L^\infty)(\Omega;\mathbf{R}^N)$ *and* $\eta \in W_0^{2,N}(\Omega)$ *such that*

$$\varphi = \psi + \nabla\eta.$$

Moreover,

$$\|\psi\|_{W^{1,N}} + \|\psi\|_{L^\infty} + \|\eta\|_{W^{2,N}} \le C\|\varphi\|_{W^{1,N}}.$$

Proof. In the case where there is a bi-Lipschitzian homeomorphism H from Ω to Q and G is its inverse, consider $\tilde\varphi$ defined by

$$\tilde\varphi_i(x) = \sum_{j=1}^N \frac{\partial H_j}{\partial x_i}(\varphi_j \circ H).$$

By Theorem 3.2, there is $\tilde\psi \in (W_0^{1,N} \cap L^\infty)(Q;\mathbf{R}^N)$ and $\tilde\eta \in W_0^{2,N}(Q)$. One checks immediately that $\eta = \tilde\eta \circ G$ and

$$\tilde\psi_i = \sum_{j=1}^N \frac{\partial G_j}{\partial x_i}(\psi_j \circ G)$$

satisfy the conclusion with estimates independent of φ. The case of a general smooth domain follows then by partition of unity. □

With Lemma 3.3, we can now prove that the data f is in the dual of $W_0^{1,N}(\Omega)$, more precisely:

Lemma 3.4. *Let* $\Omega \subset \mathbf{R}^N$ *be a smooth bounded domain and let* $f \in L^1(\Omega;\mathbf{R}^N)$ *be such that* (3.1) *holds. For every* $\varphi \in (W_0^{1,N} \cap L^\infty)(\Omega;\mathbf{R}^N)$*,*

$$\Big|\int_\Omega f\cdot\varphi\Big| \le C(\|f\|_{L^1} + [f])\|D\varphi\|_{L^N}.$$

Proof. Write $\varphi \in (\mathrm{W}_0^{1,N} \cap L^\infty)(\Omega; \mathbf{R}^N)$ as $\varphi = \psi + \nabla \eta$ according to Lemma 3.3. One has

$$\int_\Omega f \cdot \varphi = \int_\Omega f \cdot (\psi + \nabla \eta)$$
$$\leq \|f\|_{\mathrm{L}^1} \|\psi\|_{\mathrm{L}^\infty} + [f] \|D^2 \eta\|_{\mathrm{L}^N} \leq C(\|f\|_{\mathrm{L}^1} + [f]) \|D\varphi\|_{\mathrm{L}^N}.$$

□

Remark 3.2. Starting from the estimate

$$\Big| \int_{\mathbf{R}^N} f \cdot \varphi \Big| \leq C(\|f\|_{\mathrm{L}^1} \|D\varphi\|_{\mathrm{L}^N} + \|\operatorname{div} f\|_{\mathrm{L}^1} \|\varphi\|_{\mathrm{L}^1}), \tag{3.2}$$

which has an elementary proof [8], it is possible to obtain

$$\Big| \int_\Omega f \cdot \varphi \Big| \leq C(\|f\|_{\mathrm{L}^1} \|\varphi\|_{\mathrm{W}^{1,N}} + \|\operatorname{div} f\|_{\mathrm{L}^1} \|\varphi\|_{\mathrm{L}^1}), \tag{3.3}$$

for every $f \in \mathrm{L}^1(\Omega; \mathbf{R}^N)$ and $\varphi \in (\mathrm{W}_0^{1,N} \cap \mathrm{L}^\infty)(\Omega; \mathbf{R}^N)$. This is proved by extension of f to a small neighbourhood of Ω and multiplication of this extension by a suitable cutoff function. Note that, trivially,

$$[f] \leq C \|\operatorname{div} f\|_{\mathrm{L}^1},$$

and thus (3.3) is an immediate consequence of Lemma 3.4. However, we call the attention of the reader to the fact that (3.2) has an elementary proof, while the proof of Lemma 3.3 is quite elaborate.

The other tool to obtain our regularity result is an elliptic regularity result for data in $\mathrm{W}^{-1,p}(\Omega)$ that is well-known to the experts, but difficult to find in the litterature. Much more general estimates are obtained e.g. in [4].

Lemma 3.5. *Let $\Omega \subset \mathbf{R}^N$ be a smooth domain. Let $F \in \mathrm{L}^p(\Omega)$, $1 < p < \infty$. There is a unique $u \in \mathrm{W}_0^{1,p}(\Omega)$ that solves*

$$\begin{cases} -\Delta u = -\operatorname{div} F & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover,

$$\|u\|_{\mathrm{W}^{1,p}} \leq C \|F\|_{\mathrm{L}^p}.$$

The proof of Theorem 3.1 is a direct consequence of Lemmas 3.4 and 3.5.

Remark 3.3. Most of the results of this paper can be easily localized. Here is a typical localization: Let $u \in \mathrm{W}^{1,1}(\Omega \cap B(x_0, R); \mathbf{R}^N)$ and let $f \in \mathrm{L}^1(\Omega \cap B(x_0, R); \mathbf{R}^N)$. If

$$\begin{cases} -\Delta u = f & \text{in } \Omega \cap B(x_0, R), \\ u = 0 & \text{on } (\partial\Omega) \cap B(x_0, R), \end{cases}$$

and

$$\sup\Big\{ \int_\Omega f \cdot \nabla \zeta \ : \ \zeta \in C^2\big(\overline{\Omega} \cap B(x_0, R)\big),$$
$$\zeta = 0 \text{ on } \partial\big(\Omega \cap B(x_0, R)\big) \text{ and } \|D^2 \zeta\|_{\mathrm{L}^N} \leq 1 \Big\} < \infty$$

then $u \in \mathrm{W}^{1,N/(N-1)}\big(\Omega \cap B(x_0, R/2); \mathbf{R}^N\big)$.

3.2. **The Laplace equation with Neumann boundary condition.**

Theorem 3.6. *Let $\Omega \subset \mathbf{R}^N$ be a smooth and bounded domain. Let $f \in \mathrm{L}^1(\Omega; \mathbf{R}^N)$ and $g \in \mathrm{L}^1(\partial\Omega; \mathbf{R}^N)$. If*

$$(3.4)\quad [f,g] = \sup\Big\{ \int_\Omega f \cdot \nabla\zeta + \int_{\partial\Omega} g \cdot \nabla\zeta \ :\ \zeta \in C^2(\overline{\Omega}) \\ \text{and } \|D^2\zeta\|_{\mathrm{L}^N} \le 1 \Big\} < \infty,$$

then the system

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = g & \text{on } \partial\Omega, \\ \int_\Omega u = 0, \end{cases}$$

has a unique weak solution $u \in \mathrm{W}^{1,N/(N-1)}(\Omega; \mathbf{R}^N)$ and

$$\|u\|_{\mathrm{W}^{1,N/(N-1)}} \le C(\|f\|_{\mathrm{L}^1} + [f,g]).$$

Remark 3.4. If $\operatorname{div} f = 0$ in Ω, $f \cdot n = 0$, $g \cdot n = 0$ and $\operatorname{div} g = 0$ on $\partial\Omega$, then $[f,g] = 0$. As explained in Remark 2.3, the conditions $g = 0$ on $\partial\Omega$, $\operatorname{div} f = 0$ in Ω and $\int_\Omega f = 0$ do not imply the conclusion of Theorem 3.6.

A first thing to note is that the necessary condition for the existence of solution $\int_\Omega f + \int_{\partial\Omega} g = 0$ is satisfied.

Lemma 3.7. *If $\Omega \subset \mathbf{R}^N$ be a smooth and bounded domain, then $f \in \mathrm{L}^1(\Omega; \mathbf{R}^N)$ and $g \in \mathrm{L}^1(\partial\Omega; \mathbf{R}^N)$ satisfy the assumptions of Theorem 3.6 if and only if $\int_\Omega f + \int_{\partial\Omega} g = 0$ and*

$$(3.5)\quad [f,g]' = \sup\Big\{ \int_\Omega f \cdot \nabla\zeta + \int_{\partial\Omega} g \cdot \nabla\zeta \ :\ \zeta \in C^2(\overline{\Omega}) \\ \text{and } \|\zeta\|_{\mathrm{W}^{2,N}} \le 1 \Big\} < \infty.$$

Proof. First assume f satisfies the assumptions of Theorem 3.6. It is then clear that $[f,g] < \infty$. Moreover, for every $1 \le i \le N$, taking $\zeta_i(x) = x_i$, one has $D^2\zeta_i = 0$, so that since $K < \infty$, $\int_\Omega f_i + \int_{\partial\Omega} g_i = 0$.

On the other hand, assume $\int_\Omega f + \int_{\partial\Omega} g = 0$ and (3.5) holds. For every $\zeta \in C^2(\Omega)$, there is ζ' such that $D^2\zeta' = D^2\zeta$, $\int_\Omega \zeta' = 0$ and $\int_\Omega \nabla\zeta' = 0$, so that $\nabla\zeta - \nabla\zeta'$ is constant and by Poincaré's inequality, $\|\zeta'\|_{\mathrm{W}^{2,N}} \le C\|D^2\zeta\|_{\mathrm{L}^N}$. Therefore,

$$\int_\Omega f \cdot \nabla\zeta + \int_{\partial\Omega} g \cdot \nabla\zeta = \int_\Omega f \cdot \nabla\zeta' + \int_{\partial\Omega} g \cdot \nabla\zeta' \\ \le [f,g]'\|\zeta'\|_{\mathrm{W}^{2,N}} \le C[f,g]'\|D^2\zeta\|_{\mathrm{W}^{2,N}}.$$

□

In order to prove Theorem 3.1, also recall

Theorem 3.8 (Bourgain and Brezis [2]). *For every $\varphi \in \mathrm{W}^{1,N}(Q; \mathbf{R}^N)$, there exist $\psi \in (\mathrm{W}^{1,N} \cap \mathrm{L}^\infty)(Q; \mathbf{R}^N)$ and $\eta \in \mathrm{W}^{2,N}(Q)$ such that*

$$\varphi = \psi + \nabla\eta.$$

Moreover,

$$\|\psi\|_{\mathrm{W}^{1,N}} + \|\psi\|_{\mathrm{L}^\infty} + \|\eta\|_{\mathrm{W}^{2,N}} \le C\|\varphi\|_{\mathrm{W}^{1,N}}.$$

Theorem 3.8 can be extended to any smooth domain:

Lemma 3.9. *Let* $\Omega \subset \mathbf{R}^N$ *be a smooth domain. For every* $\varphi \in \mathrm{W}^{1,N}(\Omega;\mathbf{R}^N)$, *there exist* $\psi \in (\mathrm{W}^{1,N} \cap \mathrm{L}^\infty)(\Omega;\mathbf{R}^N)$ *and* $\eta \in \mathrm{W}^{2,N}(\Omega)$ *such that*

$$\varphi = \psi + \nabla\eta.$$

Moreover,

$$\|\psi\|_{\mathrm{W}^{1,N}} + \|\psi\|_{\mathrm{L}^\infty} + \|\eta\|_{\mathrm{W}^{2,N}} \le C\|\varphi\|_{\mathrm{W}^{1,N}}.$$

Proof. Since Ω is bounded, up to translation and scaling, $\Omega \subset Q$. The conclusion is obtained by extending φ to Q, applying Theorem 3.8 and restricting ψ and η to Ω. □

With Lemma 3.9, we can now prove

Lemma 3.10. *Let* $\Omega \subset \mathbf{R}^N$ *be a smooth and bounded domain. Let* $f \in \mathrm{L}^1(\Omega;\mathbf{R}^N)$ *and* $g \in \mathrm{L}^1(\partial\Omega;\mathbf{R}^N)$. *If* (3.4) *holds, then for every* $\varphi \in (\mathrm{W}^{1,N} \cap L^\infty)(\Omega;\mathbf{R}^N)$,

$$\Big|\int_\Omega f\cdot\varphi + \int_{\partial\Omega} g\cdot\varphi\Big| \le C(\|f\|_{\mathrm{L}^1} + \|g\|_{\mathrm{L}^1} + [f,g])\|D\varphi\|_{\mathrm{L}^N}.$$

Proof. Write $\varphi \in (\mathrm{W}^{1,N}\cap L^\infty)(\Omega;\mathbf{R}^N)$ as $\varphi = \psi + \nabla\eta$ according to Lemma 3.9. One has

$$\begin{aligned}\int_\Omega f\cdot\varphi + \int_{\partial\Omega} g\cdot\varphi &= \int_\Omega f\cdot(\psi+\nabla\eta) + \int_{\partial\Omega} g\cdot(\psi+\nabla\eta)\\ &\le (\|f\|_{\mathrm{L}^1} + \|g\|_{\mathrm{L}^1})\|\psi\|_{\mathrm{L}^\infty} + [f,g]\|D^2\eta\|_{\mathrm{L}^N}\\ &\le C(\|f\|_{\mathrm{L}^1} + \|g\|_{\mathrm{L}^1} + [f,g])\|D\varphi\|_{\mathrm{L}^N}.\end{aligned}$$

□

Remark 3.5. As in Remark 3.2, using the estimate (3.2), one can obtain the weaker estimate

$$\begin{aligned}\Big|\int_\Omega f\cdot\varphi + \int_{\partial\Omega} g\cdot\varphi\Big| \le C\big(&(\|f\|_{\mathrm{L}^1} + \|g\|_{\mathrm{L}^1(\partial\Omega)})\|D\varphi\|_{\mathrm{L}^N}\\ &+ (\|\operatorname{div} f\|_{\mathrm{L}^1} + \|f\cdot n\|_{\mathrm{L}^1(\partial\Omega)} + \|\operatorname{div} g\|_{\mathrm{L}^1(\partial\Omega)})\|\varphi\|_{\mathrm{L}^N}\big),\end{aligned}$$

for every $f \in \mathrm{L}^1(\Omega;\mathbf{R}^N)$, $g \in \mathrm{L}^1(\Omega;\mathbf{R}^N)$ and $\varphi \in (\mathrm{W}^{1,N} \cap \mathrm{L}^\infty)(\Omega;\mathbf{R}^N)$. The proof consists in noting that (3.2) remains valid for measures and in extending f to $\mathbf{R}^N$ by 0 and taking a $\mathrm{W}^{1,N}$ extension of u to $\mathbf{R}^N$ and applying the estimate (3.2) on $\mathbf{R}^N$.

The last tool to obtain our regularity result is the counterpart of Lemma 3.5 for Neumann boundary condition.

Lemma 3.11. *Let* $\Omega \subset \mathbf{R}^N$ *be a bounded smooth domain and let* $1 < p < \infty$. *If* $F \in \mathrm{L}^p(\Omega;\mathbf{R}^N)$, *there is a unique* $u \in \mathrm{W}^{1,p}(\Omega)$ *such that* $\int_\Omega u = 0$ *that solves*

$$\int_\Omega \nabla u\cdot\nabla\varphi = \int_\Omega F\cdot\nabla\varphi, \qquad \forall\varphi \in C^1(\overline{\Omega}).$$

Moreover,

$$\|u\|_{\mathrm{W}^{1,p}} \le C\|F\|_{\mathrm{L}^p}.$$

3.3. **The Laplace equation with nonzero Dirichlet boundary condition.**

Theorem 3.12. *Let $\Omega \subset \mathbf{R}^N$ be a smooth and bounded open set. Let $g \in \mathrm{L}^1(\partial\Omega; \mathbf{R}^N)$. If*

$$[g] = \sup\Big\{ \int_{\partial\Omega} g \cdot \nabla\zeta \ : \ \zeta \in C^2(\overline{\Omega}) \text{ and } \|\zeta\|_{\mathrm{W}^{2,N}} \le 1 \Big\} < \infty,$$

then the system

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

has a unique weak solution $u \in \mathrm{L}^{N/(N-1)}(\Omega; \mathbf{R}^N)$ and

$$\|u\|_{\mathrm{L}^{N/(N-1)}} \le C(\|g\|_{\mathrm{L}^1} + [g]).$$

In particular, when g is tangent to $\partial\Omega$ and $\operatorname{div} g = 0$ on $\partial\Omega$ in the sense of distributions, i.e.,

$$\int_{\partial\Omega} g \cdot \nabla\varphi = 0 \qquad \forall \varphi \in C^1(\partial\Omega),$$

then $u \in \mathrm{L}^{N/(N-1)}(\Omega; \mathbf{R}^N)$ and

$$\|u\|_{\mathrm{L}^{N/(N-1)}} \le C\|g\|_{\mathrm{L}^1}.$$

Proof. Assume without loss of generality that $\int_{\partial\Omega} g = 0$. By classical regularity estimates, there exists a solution $u \in \mathrm{L}^q$ for $q < N/(N-1)$. Let now $\varphi \in C^\infty(\overline{\Omega}; \mathbf{R}^N)$ and let Φ be the solution of

$$\begin{cases} -\Delta \Phi = \varphi & \text{in } \Omega, \\ \Phi = 0 & \text{on } \partial\Omega. \end{cases}$$

By the classical regularity estimates, $\Phi \in W^{2,N}(\mathbf{R}^N; \mathbf{R}^N)$ and

$$\|\Phi\|_{W^{2,N}(\mathbf{R}^N;\mathbf{R}^N)} \le \|\varphi\|_{\mathrm{L}^N(\mathbf{R}^N;\mathbf{R}^N)}.$$

Combining the definitions of Φ and u, one has

$$\int_\Omega u \cdot \varphi = -\int_{\partial\Omega} g \cdot \frac{\partial \Phi}{\partial n}. \tag{3.6}$$

Since Ω is smooth, the smooth vector field n defined on $\partial\Omega$ can be extended smoothly to $\overline{\Omega}$; we denote it $\tilde{n}$. The vector field g satisfies the assumptions of Lemma 3.10 with $f = 0$, one has therefore

$$\begin{aligned} \int_{\partial\Omega} g \cdot \frac{\partial\Phi}{\partial n} &\le C(\|g\|_{\mathrm{L}^1} + [g])\|\tilde{n} \cdot D\Phi\|_{\mathrm{W}^{1,N}} \\ &\le C'(\|g\|_{\mathrm{L}^1} + [g])\|\Phi\|_{W^{2,N}} \le C''(\|g\|_{\mathrm{L}^1} + [g])\|\varphi\|_{\mathrm{L}^N}. \end{aligned}$$

Combining this with (3.6), one deduces, since φ is arbitrary, that $u \in \mathrm{W}^{1,N/(N-1)}$ with the desired estimate. □

3.4. **The Laplace equation with other boundary conditions.** Since we are dealing with systems, we can prescribe Dirichlet on some components and Neumann on the others. The assumptions on f on the boundary have to be made accordingly. The condition on the tangential component plays a distinguished role.

Theorem 3.13. *Let $\Omega \subset \mathbf{R}^N$ be a bounded smooth domain and let $f \in \mathrm{L}^1(\Omega; \mathbf{R}^N)$. If (3.1) holds, then the system*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u - (u \cdot n)n = 0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial n} \cdot n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.7}$$

has a unique weak solution $u \in \mathrm{W}^{1,N/(N-1)}(\Omega; \mathbf{R}^N)$ and

$$\|u\|_{\mathrm{W}^{1,N/(N-1)}} \le C(\|f\|_{\mathrm{L}^1} + [f]).$$

Theorem 3.14. *Let $\Omega \subset \mathbf{R}^N$ be a bounded Lipschitz domain. Let $f \in \mathrm{L}^1(\Omega; \mathbf{R}^N)$. If (3.4) holds with $g = 0$, then the system*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u \cdot n = 0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial n} - (\frac{\partial u}{\partial n} \cdot n)n = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique weak solution $u \in \mathrm{W}^{1,N/(N-1)}(\Omega; \mathbf{R}^N)$ and

$$\|u\|_{\mathrm{W}^{1,N/(N-1)}} \le C(\|f\|_{\mathrm{L}^1} + [f, 0]).$$

Problem (3.7) is weakly formulated as

$$\int_\Omega Du \cdot D\varphi = \int_\Omega f \cdot \varphi,$$

for every $\varphi \in C^\infty(\overline{\Omega}; \mathbf{R}^N)$ such that $\varphi - (\varphi \cdot n)n = 0$ on $\partial\Omega$. In order to pursue the strategy of the proof of Theorem 3.1, we need to refine Lemma 3.4 into

Lemma 3.15. *Let $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain and let $f \in \mathrm{L}^1(\Omega; \mathbf{R}^N)$ be such that (3.1) holds. For every $\varphi \in (\mathrm{W}^{1,N} \cap L^\infty)(\Omega; \mathbf{R}^N)$ such that $\varphi - (\varphi \cdot n)n = 0$ on $\partial\Omega$,*

$$\Big| \int_\Omega f \cdot \varphi \Big| \le C(\|f\|_{\mathrm{L}^1} + [f]) \|D\varphi\|_{\mathrm{L}^N}.$$

Lemma 3.15 is proved as Lemma 3.4, using, instead of Lemma 3.3, the following

Lemma 3.16. *Let $\Omega \subset \mathbf{R}^N$ be a smooth domain. For every $\varphi \in \mathrm{W}^{1,N}(\Omega; \mathbf{R}^N)$ such that $\varphi - (\varphi \cdot n)n = 0$ on $\partial\Omega$, there exist $\psi \in (\mathrm{W}^{1,N}_0 \cap \mathrm{L}^\infty)(\Omega; \mathbf{R}^N)$ and $\eta \in \mathrm{W}^{2,N}(\Omega)$, such that $\eta = 0$ on $\partial\Omega$ and*

$$\varphi = \psi + \nabla \eta.$$

Moreover,

$$\|\psi\|_{\mathrm{W}^{1,N}} + \|\psi\|_{\mathrm{L}^\infty} + \|\eta\|_{\mathrm{W}^{2,N}} \le C \|\varphi\|_{\mathrm{W}^{1,N}}.$$

Proof. There is $\tilde{\varphi} \in \mathrm{W}_0^{1,N}(\Omega; \mathbf{R}^N)$ and $\theta \in \mathrm{W}^{2,N}(\Omega)$ such that

$$\varphi = \tilde{\varphi} + \nabla \theta$$

and $\theta = 0$ on $\partial\Omega$ (see e.g. [2, Lemma 2]). By applying Lemma 3.3 to $\tilde{\varphi}$, one obtains the conclusion. □

4. First-order elliptic systems

4.1. div − curl systems in 3–d. Here we consider the system

$$\begin{cases} \operatorname{div} Z = 0, \\ \operatorname{curl} Z = Y, \end{cases}$$

together with boundary conditions either on the normal or on the tangential part of Z.

Theorem 4.1. *Let $\Omega \subset \mathbf{R}^3$ be a contractible smooth domain, and let $Y \in \mathrm{L}^1(\Omega; \mathbf{R}^3)$. If $\operatorname{div} Y = 0$, then there exists a unique $Z \in \mathrm{L}^{3/2}(\Omega; \mathbf{R}^3)$ such that*

$$\begin{cases} \operatorname{div} Z = 0 & \text{in } \Omega, \\ \operatorname{curl} Z = Y & \text{in } \Omega, \\ Z \cdot n = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. The classical construction yields a solution $Z \in \mathrm{L}^q(\Omega; \mathbf{R}^3)$, for $q < 3/2$. Let $X \in C^\infty(\overline{\Omega}; \mathbf{R}^N)$. By the classical Hodge decomposition, there are $\varphi \in C^\infty(\overline{\Omega})$ and $A \in C^\infty(\overline{\Omega}; \mathbf{R}^N)$ such that

$$X = \nabla \varphi + \operatorname{curl} A,$$

$A \times n = 0$ on $\partial\Omega$, and $\|\varphi\|_{\mathrm{W}^{1,3}} + \|A\|_{\mathrm{W}^{1,3}} \le C\|X\|_{\mathrm{L}^3}$. Integrating by parts, and applying Lemma 3.15, one obtains,

$$\begin{aligned} \int_\Omega Z \cdot X = \int_\Omega Z \cdot (\nabla \varphi + \operatorname{curl} A) = \int_\Omega Y \cdot A \\ \le C\|Y\|_{\mathrm{L}^1}\|A\|_{\mathrm{W}^{1,3}} \le C'\|Y\|_{\mathrm{L}^1}\|X\|_{\mathrm{L}^3}. \end{aligned}$$

Since X is arbitrary, $Z \in \mathrm{L}^{3/2}$ with the required estimate. □

Remark 4.1. Since these estimates can be localized (see Remark 3.3), the result extends to smooth domains which are not contractible provided the suitable conditions of orthogonality with harmonic fields are met by the data.

One can also prescribe the tangential part of Z on $\partial\Omega$:

Theorem 4.2. *Let $\Omega \subset \mathbf{R}^3$ be a contractible smooth domain, let $Y \in \mathrm{L}^1(\Omega; \mathbf{R}^3)$ and $V \in \mathrm{L}^1(\partial\Omega; \mathbf{R}^3)$ be such that $V \cdot n = 0$ on $\partial\Omega$. If $\operatorname{div} Y = 0$ in Ω and $Y \cdot n = \operatorname{div} V$ on $\partial\Omega$ in the sense that*

$$\forall \varphi \in C^\infty(\overline{\Omega}), \qquad \int_\Omega Y \cdot \nabla \varphi = -\int_{\partial\Omega} V \cdot \nabla \varphi, \tag{4.1}$$

then there exists $Z \in \mathrm{L}^{3/2}(\Omega; \mathbf{R}^3)$ such that

$$\begin{cases} \operatorname{div} Z = 0 & \text{in } \Omega, \\ \operatorname{curl} Z = Y & \text{in } \Omega, \\ Z \wedge n = V & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Remark 4.2. Clearly (4.1) is a necessary condition for the solvability of (4.2).

4.2. **Systems of differential forms.** If $\omega : \Omega \to \Lambda^k \mathbf{R}^N$ is a differential form, $\delta\omega$ denotes its exterior codifferential and $\mathbf{t}\omega$ its tangential component on the boundary. An extension to higher dimensions and to k–forms is

Theorem 4.3. *Let $N \geq 4$, $\Omega \subset \mathbf{R}^N$ be a contractible smooth domain, $2 \leq k \leq N-2$, $\alpha \in \mathrm{L}^1(\Omega; \Lambda^{k+1})$, $\beta \in \mathrm{L}^1(\Omega; \Lambda^{k-1}\mathbf{R}^N)$ and $\gamma \in \mathrm{L}^1(\partial\Omega; \Lambda^k \partial\Omega)$. If $d\alpha = 0$, $\delta\beta = 0$ and $\mathbf{t}\alpha = d\gamma$ in the sense that*

$$\int_\Omega \alpha \wedge d\varphi = \int_{\partial\Omega} \gamma \wedge d\varphi \qquad \forall \varphi \in C^1(\overline{\Omega}; \Lambda^{N-k-2}\mathbf{R}^N), \tag{4.3}$$
$$\int_\Omega \beta \wedge \delta\varphi = 0 \qquad \forall \varphi \in C^1(\overline{\Omega}; \Lambda^{N-k+2}\mathbf{R}^N) \text{ s.t. } \mathbf{n}\varphi = 0,$$

then there exists a unique $\omega \in \mathrm{L}^{N/(N-1)}(\Omega; \Lambda^k \mathbf{R}^N)$ such that

$$\begin{cases} d\omega = \alpha & \text{in } \Omega, \\ \delta\omega = \beta & \text{in } \Omega, \\ \mathbf{t}\omega = \gamma & \text{on } \partial\Omega. \end{cases}$$

Remark 4.3. The counterpart in the whole space can be found in [2] and [5].

The analogue of Lemma 3.3 for k-forms is

Lemma 4.4. *Let $\Omega \subset \mathbf{R}^N$ be a smooth domain and let $1 \leq k \leq N$. For every $\varphi \in \mathrm{W}^{1,N}(\Omega; \Lambda^k \mathbf{R}^N)$, there exist $\psi \in (\mathrm{W}^{1,N} \cap \mathrm{L}^\infty)(\Omega; \Lambda^k \mathbf{R}^N)$ and $\eta \in \mathrm{W}^{2,N}(\Omega; \Lambda^{k+1})$ such that*

$$\varphi = \psi + d\eta,$$

satisfying

$$\|\psi\|_{\mathrm{W}^{1,N}} + \|\psi\|_{\mathrm{L}^\infty} + \|\eta\|_{\mathrm{W}^{2,N}} \leq C\|\varphi\|_{\mathrm{W}^{1,N}}.$$

If moreover $\mathbf{t}\varphi = 0$ on $\partial\Omega$, then one can choose ψ and η such that $\psi = 0$ on $\partial\Omega$ and $\eta = 0$ on $\partial\Omega$.
If, in addition, $\varphi = 0$ on $\partial\Omega$, then one can take $\eta \in \mathrm{W}^{2,N}_0(\Omega; \Lambda^{k-1}\mathbf{R}^N)$.

Proof. The k-form φ can be written as

$$\varphi = \sum_{i_1 < \dots < i_k} \varphi_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

By Lemma 3.9, there exist $\psi_{i_1 \dots i_k} \in (\mathrm{W}^{1,N} \cap \mathrm{L}^\infty)(\Omega; \Lambda^1 \mathbf{R}^N)$ and $\eta \in \mathrm{W}^{2,N}(\Omega; \Lambda^0 \mathbf{R}^N)$ such that

$$\varphi_{i_1 \dots i_k} dx_{i_1} = \psi_{i_1 \dots i_k} + d\eta_{i_1 \dots i_k}.$$

The conclusion now holds with

$$\psi = \sum_{i_1 < \dots < i_k} \psi_{i_1 \dots i_k} dx_{i_2} \wedge \dots \wedge dx_{i_k},$$
$$\eta = \sum_{i_1 < \dots < i_k} \eta_{i_1 \dots i_k} dx_{i_2} \wedge \dots \wedge dx_{i_k}.$$

In the case where $\varphi = 0$ on $\partial\Omega$, if one uses Lemma 3.3 in place of Lemma 3.9, one constructs $\psi \in (\mathrm{W}^{1,N}_0 \cap \mathrm{L}^\infty)(\Omega; \Lambda^k \mathbf{R}^N)$ and $\eta \in \mathrm{W}^{2,N}_0(\Omega; \Lambda^{k-1}\mathbf{R}^N)$. In the general case, as in the proof of Lemma 3.16, when $\varphi \in \mathrm{W}^{1,N}(\Omega; \Lambda^k \mathbf{R}^N)$

and $\mathbf{t}\varphi = 0$, there are $\tilde\varphi \in \mathrm{W}^{1,N}_0(\Omega; \Lambda^k\mathbf{R}^N)$ and $\tilde\eta$ such that $\varphi = \tilde\varphi + d\tilde\eta$ and $\tilde\eta = 0$ on $\partial\Omega$. We are thus reduced to the previous case with $\tilde\varphi$ replacing φ. □

As Lemmas 3.10 and 3.15 followed from Lemmas 3.9 and 3.16, one deduces from Lemma 4.4 the

Lemma 4.5. *Let $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain and let $1 < k < n$. Let $\alpha \in \mathrm{L}^1(\Omega; \Lambda^k\mathbf{R}^N)$ let $\gamma \in \mathrm{L}^1(\Omega; \Lambda^{k-1}\mathbf{R}^N)$. If $d\alpha = 0$ in Ω, then for every $\varphi \in (\mathrm{W}^{1,N} \cap \mathrm{L}^\infty)(\Omega; \Lambda^{N-k}\mathbf{R}^N)$ such that $\mathbf{t}\varphi = 0$,*

$$\Bigl|\int_\Omega \alpha \wedge \varphi\Bigr| \le C\|\alpha\|_{\mathrm{L}^1}\|D\varphi\|_{\mathrm{L}^N}.$$

If $d\alpha = 0$ in Ω and $d\gamma = \mathbf{t}\alpha$ on $\partial\Omega$ in the sense of (4.3)*, then for every $\varphi \in (\mathrm{W}^{1,N} \cap \mathrm{L}^\infty)(\Omega; \Lambda^{N-k}\mathbf{R}^N)$*

$$\Bigl|\int_\Omega \alpha \wedge \varphi + \int_{\partial\Omega} \gamma \wedge \varphi\Bigr| \le C(\|\alpha\|_{\mathrm{L}^1} + \|\gamma\|_{\mathrm{L}^1})\|D\varphi\|_{\mathrm{L}^N}.$$

Proof of Theorem 4.3. Let $\zeta \in C^\infty(\Omega; \Lambda^{N-k}\mathbf{R}^N)$. By the classical Hodge decomposition for k-forms on domains [7], there are $\varphi \in C^\infty(\overline{\Omega}; \Lambda^{N-k-1}\mathbf{R}^N)$ and $\psi \in C^\infty(\overline{\Omega}; \Lambda^{N-k+1}\mathbf{R}^N)$ such that $\zeta = d\varphi + \delta\psi$, $\mathbf{n}\psi = 0$ and

$$\|\varphi\|_{\mathrm{W}^{1,N}} + \|\psi\|_{\mathrm{W}^{1,N}} \le C\|\zeta\|_{\mathrm{L}^N}.$$

One has thus

$$\int_\Omega \omega \wedge \zeta = \int_\Omega \omega \wedge (d\varphi + \delta\psi).$$

One estimates then, by Lemma 4.5

$$\begin{aligned}\int_\Omega \omega \wedge d\varphi &= \int_{\partial\Omega} \gamma \wedge \varphi + \int_\Omega \alpha \wedge \varphi\\ &\le C(\|\alpha\|_{\mathrm{L}^1} + \|\gamma\|_{\mathrm{L}^1})\|D\varphi\|_{\mathrm{L}^N} \le C'(\|\alpha\|_{\mathrm{L}^1} + \|\gamma\|_{\mathrm{L}^1})\|\zeta\|_{\mathrm{L}^N}\end{aligned}$$

Similarly, since $d(*\beta) = 0$ and $\mathbf{t} * \psi = 0$, by Lemma 4.5 again

$$\int_\Omega \omega \wedge \delta\psi = \int_\Omega *\beta \wedge *\psi \le C\|\beta\|_{\mathrm{L}^1}\|D\varphi\|_{\mathrm{L}^N} \le C\|\beta\|_{\mathrm{L}^1}\|\zeta\|_{\mathrm{L}^N}.$$ □

Appendix A. Estimates for Green's functions in 2-d

Throughout this appendix Ω is a smooth bounded domain in $\mathbf{R}^2$.

Definition A.1. The *Green's function G of Ω with Dirichlet boundary condition* is defined by $G(x,y) = 0$ if $x \in \partial\Omega$ and $G(\cdot,y) - \frac{1}{2\pi}\log\frac{1}{\cdot - y}$ is harmonic in Ω for every $y \in \Omega$.

Definition A.2. The *Green's function G of Ω with Neumann boundary condition* (also called *Neumann's function*) is defined by

$$\nabla_x G(x,y) \cdot n = 0$$

for $(x,y) \in \partial\Omega \times \Omega$, where n denotes the normal to the boundary at x, and

$$-\Delta\Bigl(G(\cdot,y) - \frac{1}{2\pi}\log\frac{1}{|\cdot - y|}\Bigr) = -\frac{1}{|\Omega|},$$

in Ω, where $|\Omega|$ denotes the area of Ω.

The Green's functions can be computed explicitly for the unit disk $D \subset \mathbf{R}^2 \equiv \mathbf{C}$: One has, for $x, y \in D$,

$$G(x,y) = \frac{1}{2\pi} \log \frac{|1-\bar{y}x|}{|x-y|},$$

where $\bar{y}$ is the complex conjugate of y and $\bar{y}x$ is the product of $\bar{y}$ and x in $\mathbf{C}$. For the Neumann boundary condition one has

$$G(x,y) = \frac{1}{2\pi}\Big(\log \frac{1}{|x-y||1-\bar{y}x|} + \frac{|x|^2+|y|^2+1}{2}\Big).$$

One has the following estimate on the derivative of Green's function with Dirichlet boundary condition.

Proposition A.1 (Bramble and Payne). *Let Ω be a smooth bounded domain and G be Green's function with Dirichlet's boundary condition. Then there is a constant C such that for every $x, y \in \Omega$,*

$$|\nabla_y G(x,y)| \le \frac{C}{|x-y|}. \tag{A.1}$$

This is proved by the maximum principle for the Green's function with Dirichlet boundary conditions, see [3].

Another method to obtain it is to express Green's function by conformal mapping.

Proposition A.2. *Let Ω be a simply-connected domain and G be the associated Green's function with Dirichlet boundary conditions. If $\psi : \Omega \to D$ is conformal, then*

$$G(x,y) = \frac{1}{2\pi} \log \frac{|1-\overline{\psi(y)}\psi(x)|}{|\psi(x)-\psi(y)|}$$

Thus, in view of Theorem A.4, (A.1) *holds.*

Proposition A.3. *Let Ω be a bounded simply-connected domain and G be the associated Green's function with Neumann boundary conditions. If $\psi \in C^{k,\alpha}(\overline{\Omega}; D)$ is conformal, then there exists $w \in C^{k+1,\alpha}(\overline{\Omega})$ such that*

$$G(x,y) = \frac{1}{2\pi} \log \frac{1}{|\psi(x)-\psi(y)||1-\overline{\psi(y)}\psi(x)|} + w(x) + w(y). \tag{A.2}$$

In particular,

$$\sup_{x,y\in\partial\Omega} \Big|\nabla_x G(x,y)\Big| \le \frac{C}{|x-y|}.$$

Proof. Let G_D denote Green's function on the disk with Neumann boundary condition and define

$$\tilde{G}(x,y) = G_D(\psi(x),\psi(y)).$$

For every $\varphi \in C^1(\overline{\Omega})$ and $y \in \Omega$, since ψ is conformal,

$$\begin{aligned}\int_\Omega \nabla_x \tilde{G}(x,y)\cdot\nabla u(x)\,dx &= \int_D \nabla_\xi G_D(\xi,\psi(y))\cdot\nabla u(\psi^{-1}(\xi))\,d\xi \\ &= u(y) - \frac{1}{2\pi}\int_D u(\psi^{-1}(\xi))\,d\xi = u(y) - \frac{1}{2\pi}\int_\Omega |\psi'(\xi)|^2 u(\xi)\,d\xi,\end{aligned}$$

i.e., for every $y \in \Omega$, $\tilde{G}$ satisfies the equation:

$$\begin{cases} -\Delta \tilde{G}(x,y) = \delta_y - |\psi'(x)|^2/(2\pi) & \text{in } \Omega, \\ \partial \tilde{G}/\partial n(x,y) = 0 & \text{on } \partial\Omega. \end{cases}$$

By the classical Schauder estimates, there is $\tilde{w} \in C^{k+1}(\overline{\Omega})$ to be the solution of the problem

$$\begin{cases} -\Delta \tilde{w} = |\psi'(x)|^2/(2\pi) - 1/|\Omega| & \text{in } \Omega, \\ \partial \tilde{w}/\partial n = 0 & \text{on } \partial\Omega, \\ \int_\Omega w\, dx = 0. \end{cases}$$

Moreover, for every $y \in \Omega$,

$$\begin{aligned} \frac{1}{|\Omega|}\int_\Omega \tilde{G}(x,y)\,dx &= \int_\Omega \tilde{G}(x,y)|\psi'(x)|^2\,dx + \int_\Omega \tilde{G}(x,y)\Big(\frac{1}{|\Omega|} - \frac{|\psi'(x)|^2}{2\pi}\Big)\,dx \\ &= \frac{1}{2\pi}\int_\Omega G_D(\xi,\psi(y))\,d\xi + \int_\Omega \tilde{G}(x,y)\Delta\tilde{w}(x)\,dx \\ &= -\tilde{w}(y) + \int_\Omega \tilde{w}(x)\frac{|\psi'(x)|^2}{2\pi}\,dx = \tilde{w}(y) - \int_\Omega \tilde{w}(x)\Delta\tilde{w}(x)\,dx \\ &= \tilde{w}(y) + \int_\Omega |\nabla\tilde{w}(x)|^2\,dx. \end{aligned}$$

Therefore one has

$$G(x,y) = \tilde{G}(x,y) + \tilde{w}(x) + \tilde{w}(y) + \int_\Omega |\nabla\tilde{w}|^2\,dx.$$

which yields the desired conclusion (A.2). □

Theorem A.4 (Kellogg (1912), Warschawski (1932))**.** *Let $\Omega \subset \mathbf{R}^2$ be a bounded simply-connected domain and let $\psi : \Omega \to D$ be a conformal mapping. If $\partial\Omega$ is of class $C^{k,\alpha}$, then $\psi \in C^{k,\alpha}(\overline{\Omega})$ and $\psi^{-1} \in C^{k,\alpha}(\overline{\Omega})$.*

A modern version of this can be found in [6], where Theorem 3.6 gives the Hölder continuity of the derivatives of ψ^{-1}, and Theorem 3.5 states that $(\psi^{-1})' \neq 0$, and therefore, by the implicit function Theorem $\psi \in C^{k,\alpha}(\overline{\Omega})$.

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