Symmetrizations, Symmetry of Critical Points and L^1 Estimates

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Summary

This thesis is a collection of articles which are accepted or submitted for publication or are already published. It is divided in two independent parts which begin with original introductions.

The first part of this thesis is devoted to symmetrizations. Symmetrizations are tranformations of functions that preserve many properties of functions and enhance their symmetry. In the calculus of variation they are a simple and powerful tool to prove that minimizers of functionals are symmetric functions. In this work, the approximation of symmetrizations by simpler symmetrizations is investigated: The existence of a universal approximating sequence is proved, sufficient conditions for deterministic and random sequences to be approximating are given. These approximation methods are then used to prove some symmetry properties of critical points obtained by minimax methods: For example if there is a solution obtained by the mountain pass theorem, then there is a symmetric solution with the same energy. This part ends with a study of the properties of anisotropic symmetrizations i.e. symmetrizations performed with respect to noneuclidean norms.

The second part is devoted to L^1 estimates. In general, the second derivative of the solution of the Poisson equation with L^1 data fails to be in L^1. Recently it was proved that if the data is a L^1 divergence-free vector-field, then even if in general it is false that the second derivative of the solution is in L^1, all the consequences thereof by Sobolev embeddings hold. Elementary proofs of such results, as well as a generalization with a second order operator replacing the divergence, are given.

Contents

Part 1

Symmetrizations and symmetry of critical points

Introduction

1. Symmetry of solutions of partial differential equations

Consider a semilinear elliptic problem of the form

$$\Delta u = f(x,u) \quad \text{in } \Omega,$$

together with some boundary condition on $\partial\Omega$. When Ω, f and the boundary conditions have some symmetries, the question whether u inherits those symmetries comes naturally. For example, if Ω is a ball or an annulus, $f(x,u) = f(|x|,u)$ and $u = 0$ on $\partial\Omega$, is a solution u a radial function? In general, u does not need to be radial: For $f(x,u) = \lambda u$, there are nonsymmetric solutions u for suitable values of λ. Additional conditions on Ω, f and u are thus required to ensure the symmetry of u. This was done for example by Gidas, Ni and Nirenberg who proved with the maximum principle and the moving-plane method, that if $f \in \mathrm{C}([0,1] \times \mathbf{R})$, $\partial f/\partial u \in \mathrm{C}([0,1] \times \mathbf{R})$, f is increasing with respect to its first argument, $u \in \mathrm{C}^2(\bar{\Omega})$, $u > 0$ on Ω and

$$\begin{cases} \Delta u = f(|x|,u) & \text{in } \mathrm{B}(0,1), \\ u = 0 & \text{on } \partial\mathrm{B}(0,1). \end{cases}$$

then u is spherically symmetric and radially decreasing [10, 11].

When the problem is variationnal, i.e. when the solutions are critical points of a functional φ, the symmetries of minimizing solutions can be investigated. Symmetry breaking can still occur [6, 21]. In some cases, to any function u, a more symmetric function u^* such that $\varphi(u^*) \leqslant \varphi(u)$ is associated. Then, if there is a minimizer of φ, there is a symmetric minimizer of φ. The symmetrization by rearrangement yields such a function u^* for many functionals φ. Moreover it is a nonlinear transformation of functions well suited to nonlinear problems.

2. Symmetrization

While the idea of symmetrizing sets goes back to Steiner as a tool for the proof of the classical isoperimetric theorem and the symmetrization of functions was defined by Hardy, Littlewood and Pólya [12], the use of symmetrization in elliptic variational problem was introduced by Pólya and Szegő [18]. Since then it has been widely studied and exposed [4, 13, 15, 17, 26]. We recall some basic definitions and properties for classical symmetrizations.

The definition of the symmetrization of functions can be based on the symmetrization of sets. We present successively the Schwarz symmetrization, the Steiner symmetrization and the cap symmetrization.

In the simplest setting, the *Schwarz symmetrization* of any Lebesgue measurable set $A \subset \mathbf{R}^N$ is an open ball centered at the origin, which has the same Lebesgue measure as A and which is denoted by A^*. A function u is *admissible* if it is nonnegative, measurable, and if the measure of the set

$$\{x \in \mathbf{R}^N \;:\; u(x) > c\}$$

is finite for every $c > 0$. The *symmetrization of an admissible function* $u : \mathbf{R}^N \to \mathbf{R}$ is the unique function u^* such that for every $c > 0$,

$$\{x \in \mathbf{R}^N \;:\; u^*(x) > c\} = \{x \in \mathbf{R}^N \;:\; u(x) > c\}^* .$$

Alternative definitions of the symmetrization of sets yield alternative symmetrizations of functions. In particular, the *Steiner symmetrization* of a set A with respect to a plane T is the unique set whose intersection with every maximal plane L orthogonal to T is a ball of L centered on the intersection of T and L which has the same measure as the intersection of L and A.

Similarly, the *cap symmetrization* of a set A with respect to a half-plane T is the unique set which has the same intersection with the boundary of T as A, and whose intersection with any sphere L of a maximal orthogonal plane to the boundary of T whose center is in the boundary of T, is an open ball of L centered around the intersection of L and T which has the same measure as the intersection of L and A.

The symmetrization of a set prescribes a measure and a shape for the symmetrized set. The properties of the symmetrizations can be classified as measure-theoretical and geometrical depending on whether their proof relies on the shape of the symmetrized set.

The *mesaure-theoretical* properties can be deduced from two fundamental properties: Rearrangements preserve the inclusions and the measures of sets. These properties are sufficient to ensure that for every admissible function u and for every nonnegative Borel function f,

$$\text{(C)} \qquad \int_{\mathbf{R}^N} f(u^*)\,dx = \int_{\mathbf{R}^N} f(u)\,dx,$$

(*Cavalieri principle*) and that for any $1 \leqslant p \leqslant \infty$ and for any admissible functions $u, v \in \mathrm{L}^p(\mathbf{R}^N)$,

$$\|u^* - v^*\|_p \leqslant \|u - v\|_p.$$

When $p = 2$, the latter is equivalent to the classical *Hardy–Littlewood inequality* [12, 15]:

$$\text{(HL)} \qquad \int_{\mathbf{R}^N} u^* v^* \, dx \geqslant \int_{\mathbf{R}^N} uv \, dx.$$

The proofs of these properties of the rearrangement of functions are based on simple measure-theoretical arguments [8, 13, 15, 25, 26].

The properties of the symmetrization given above did not use the shape of the symmetrized sets, which ensure the *geometrical properties* of the symmetrization of functions. For example, for any Steiner or cap symmetrization $*$, if $u \in \mathrm{W}^{1,p}(\mathbf{R}^N)$ is nonnegative, then $u^* \in \mathrm{W}^{1,p}(\mathbf{R}^N)$ and

$$\text{(PS)} \qquad \int_{\mathbf{R}^N} |\nabla u^*|^p \, dx \leqslant \int_{\mathbf{R}^N} |\nabla u|^p \, dx.$$

This is the *Pólya–Szegő inequality*, which was first proved when $p = 2$ [18]. If $*$ is a Steiner symmetrization with respect to a plane that contains the origin, then the *Riesz–Sobolev rearrangement inequality* also holds:

$$\text{(RS)} \quad \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} u(x)v(y)w(x-y) \, dx \, dy \leqslant \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} u^*(x)v^*(y)w^*(x-y) \, dx \, dy.$$

When $*$ is a cap symmetrization, this is still true when $w(z) = w(|z|)$ and w is decreasing [3, 7], but as shown in Corollary IV.4.3, it does not hold for a general nonnegative compactly supported continuous function w.

In contrast with the measure-theoretical properties, the geometrical properties of the symmetrization of function are not consequences of elementary properties of the symmetrization of sets. There are elementary necessary conditions: A Pólya–Szegő inequality implies that the symmetrization of a ball should be a ball and a Riesz–Sobolev rearrangement inequality implies that the symmetrization of an ellipsoid should be an ellipsoid. Sufficient conditions are more complicated: For the Pólya–Szegő inequality, the capacity of condensators should decrease by symmetrization [24] while for the Riesz–Sobolev inequality, the inequality should hold for characteristic functions of sets [15]. These sufficient conditions are not powerful enough to reduce the proofs to elementary arguments. Another tool is thus needed.

For the Steiner symmetrizations with respect to hyperplanes, the Pólya–Szegő inequality can be proved directly [13]. The inequality is then extended to other Steiner symmetrizations by an approximation argument: For the Steiner symmetrization $*$, a sequence of simpler symmetrizations $S_1, S_2, \dots$ is constructed such that for every function $u \in \mathrm{L}^p(\mathbf{R}^N)$

$$u^{S_1 \cdots S_n} \to u^* \quad \text{in } \mathrm{L}^p(\mathbf{R}^N).$$

Finally, using the fact that the Polyá-Szegő inequality holds for the symmetrizations S_n, the weak convergence $u^{S_1 \cdots S_n} \rightharpoonup u^*$ in $\mathrm{W}^{1,p}(\mathbf{R}^N)$ follows and

$$\int_{\mathbf{R}^N} |\nabla u^*|^p \, dx \leqslant \varliminf_{n \to \infty} \int_{\mathbf{R}^N} |\nabla u^{S_1 \cdots S_n}|^p \, dx \leqslant \int_{\mathbf{R}^N} |\nabla u|^p \, dx, \tag{2.1}$$

by the weak lower semi-continuity of the norm [7]. A similar approach is possible for the Riesz–Sobolev inequality [5, 15].

3. Approximation of symmetrizations

Approximation of symmetrization originated in proofs of the isoperimetric theorem. Once it was known that the Steiner symmetrization decreases the perimeter of compact convex sets, it was sufficient to prove that for any compact convex set of the plane K, there is a sequence of Steiner symmetrizations $(S_n)_{n \geqslant 1}$ such that the sequence $K^{S_1 \cdots S_n}$ converges to a ball in Hausdorff distance. The approximation by Steiner symmetrizations was an essential ingredient for the proof of the Riesz–Sobolev rearrangement inequality in N–dimensional space [5]. Sarvas established isoperimetric inequalities for condensators by the approximation of cap symmetrizations by simpler cap symmetrizations [20].

A still simpler rearrangement is the polarization. The *polarization* u^H of a function $u : \mathbf{R}^N \to \mathbf{R}$ with respect to an affine halfspace H is defined by

$$u^H(x) = \begin{cases} \max\bigl(u(x), u(x_H)\bigr) & \text{if } x \in H, \\ \min\bigl(u(x), u(x_H)\bigr) & \text{if } x \notin H, \end{cases}$$

where x_H denotes the reflection of x with respect to the boundary of H. Is is also called "two-point rearrangement" and it is the simplest non-trivial rearrangement of functions. Its properties are easy to prove and to understand since it operates essentially on two-points sets. Polarizations were introduced by Wolontis to study the behaviour of the capacity of condensators in $\mathbf{R}^2$ under the cap symmetrization [27]. This result was extended by Dubinin to other symmetrizations in higher dimensions [9]. Both proofs considered a set of condensators which could be symmetrized by a finite number of polarizations and then concluded by a density argument. Baernstein used the polarization in a compactness argument in the proof of his "master inequality"

$$\int_{\mathbf{R}^N}\int_{\mathbf{R}^N} u(x)v(y)w(|x-y|)\,dx\,dy \leqslant \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} u^*(x)v^*(y)w(|x-y|)\,dx\,dy, \tag{3.1}$$

where w is a nonincreasing function [3]. He deduced the Pólya–Szegő inequality (2.1) from the inequality (3.1).

The approach of symmetrization by polarization was subsequently developped by Brock and Solynin [7]. They proved that for every nonnegative function $u \in \mathrm{L}^p(\mathbf{R}^N)$ and every Steiner symmetrization $*$, there is a sequence of polarizations $(H_n)_{n\geqslant 1}$ such that

$$u^{H_1\cdots H_n} \to u^*, \qquad \text{in } \mathrm{L}^p(\mathbf{R}^N) \text{ as } n \to \infty.$$

Smets and Willem extended this to cap symmetrizations $*$ [22].

4. Universal approximation

In chapter I, we prove that the sequence of polarizations approximating a given symmetrization can be chosen independently of the function u. We also extend the approximation by polarizations to the increasing rearrangement, which transforms functions into increasing functions. More precisely the increasing rearrangement of a set $A \subset \Omega \times \mathbf{R}$ is the unique set A^* such that for every $x \in \Omega$,

$$A^* \cap (\{x\} \times \mathbf{R}) = \{x\} \times (c(x,A), +\infty)$$

where

$$c(x,A) = \mathcal{H}^1\bigl((\{x\} \times (0,+\infty)) \setminus A\bigr) - \mathcal{H}^1\bigl((\{x\} \times (-\infty,0]) \cap A\bigr)$$

($c(x,A) = -\infty$ if the definition does not make sense). It is extended to functions as in section 2.

We prove that if $*$ is the Steiner, the cap symmetrization or the increasing rearrangement, then there is a sequence of polarizations $(H_n)_{n\geqslant 1}$ such that for every $1 \leqslant p < \infty$ and for every admissible function $u \in \mathrm{L}^p(\mathbf{R}^N)$,

$$u^{H_1\cdots H_n} \to u^*, \qquad \text{in } \mathrm{L}^p(\mathbf{R}^N) \text{ as } n \to \infty.$$

We adapt the arguments of Brock and Solynin, and of Smets and Willem in order to approximate simultaneously the symmetrizations of a countable set of functions. The conclusion comes from a density argument based on the nonexpansiveness of symmetrizations and polarizations.

We also show that the same arguments provide proofs of the approximation of Steiner symmetrizations by Steiner or cap symmetrizations and of cap symmetrizations by cap symmetrizations in a unified framework (Section I.5). These results can be stated in an elegant way by introducing the partial order $\prec$ on the set of affine planes and halfplanes $\mathcal{S}$ defined by $S \prec T$ if $S \subseteq T$ and $\partial S \subseteq \partial T$. Given $S \in \mathcal{S}$ and $\mathcal{T} \subset \mathcal{S}$, there is a sequence $(T_n)_{n\geqslant 1}$ in $\mathcal{T}$ that approximates S, provided for every polarization H with $S \prec H$, there exists $T \in \mathcal{T}$ such that $S \prec T \prec H$ (Theorem III.2.28). This gives a common framework to the Steiner symmetrizations, the cap symmetrizations and the polarizations.

A further question is which sequences are approximating. We give in chapter III a sufficient condition: If S and $\mathcal{T}$ satisfy the assumptions of the preceding paragraph, and if the sequence $(T_n)_{n\geqslant 1}$ in $\mathcal{T}$ contains as a contiguous subsequence any finite subsequence of $\mathcal{T}$ up to an arbitrary small error, then the sequence $(T_n)_{n\geqslant 1}$ approximates the symmetrization S (Theorem III.3.2). While this sufficient condition is not necessary, it is satisfied by many sequences: If the elements of the sequence are chosen randomly and independently, and if the probability that T_n is in a nonempty open set is bounded from 0 uniformly in n, then almost every sequence $(T_n)_{n\geqslant 1}$ approximates S. Mani-Levitska proved previously that for every compact convex set, almost every sequence of Steiner symmetrization converges to a ball in Hausdorff distance d_H, which is defined for $A, B \subset \mathbf{R}^N$ by

$$\mathrm{d}_H(A,B) = \max\bigl(\sup_{x\in A}\inf_{y\in B} \mathrm{d}(x,y), \sup_{y\in B}\inf_{x\in A} \mathrm{d}(y,x)\bigr).$$

He asked whether the same held for nonconvex sets [16]. We give a positive answer in Theorem III.3.13.

5. Symmetry of critical points

As explained above, the symmetrization by rearrangement is a powerful method to prove that a functional achieves its minimum on symmetric functions. It is not directly applicable to critical points which are not minimizers. In chapters II and III we adapt the method to critical points obtained by minimax methods.

In chapter II we apply symmetrization methods to critical points obtained by the *mountain-pass Theorem* of Ambrosetti and Rabinowitz and its generalizations. This theorem can be stated roughly as follows: Suppose φ is a functional defined on a Banach space such that $\varphi(0) = 0$, the functional has a positive lower bound on the boundary of a neighbourdhood of 0, and there is e outside of this neighbourhood such that $\varphi(e) < 0$. Let

$$\Gamma = \{\gamma \in \mathrm{C}([0,1], X) \ : \ \gamma(0) = 0 \text{ and } \gamma(1) = e\}$$

and

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \varphi(\gamma(t)).$$

If φ satisfies an additional Palais–Smale compactness condition, then there is a critical point u such that $\varphi(u) = c > 0$.

When the functional φ decreases by symmetrization, it seems natural that the infimum should be the same if only symmetric functions were considered since any path γ can be replaced by a symmetrized path $\tilde{\gamma}$ defined by $\tilde{\gamma}(t) = \gamma(t)^*$. Unfortunately, in general $\tilde{\gamma}$ is not in Γ because the symmetrization is not a continuous transformation of Sobolev spaces [1]. On the other hand, the polarizations are continuous in Sobolev spaces. Using the fact that Steiner and cap symmetrizations can be approximated by polarizations, we prove that if the function φ decreases by polarizations compatible with a Steiner or cap symmetrization $*$ and if φ satisfies the hypotheses of the mountain-pass Theorem, then there is a critical point u such that $\varphi(u) = c$ and $u^* = u$.

In chapter III, we obtain similarly some symmetry results for critical points obtained using *Krasnoselskii genus*. The Krasnoselskii genus of a subset A of a vector space such that $A = -A$ is the least integer m such that there is a continuous odd mapping of A in S^{m-1}. It is denoted by $\gamma(A)$. If φ is a functional defined on a Banach space and

$$\beta_\ell = \inf_{\substack{A \text{ is closed} \\ A = -A \\ \gamma(A) \geqslant \ell}} \sup_{u \in A} \varphi(u),$$

then it is classical that under an additional Palais–Smale compactness condition, φ has a critical point u_ℓ such that $\varphi(u_\ell) = \beta_\ell$ [19, 23]. As for the mountain-pass Theorem, it is possible to ensure that there are symmetric critical points on the level β_ℓ. Suppose φ is defined on a Sobolev space of functions defined on $\Omega \subset \mathbf{R}^N$ and φ is invariant by all the polarizations whose boundary contains 0. We prove that if $\ell \leqslant N$, then there is a critical point still denoted u_ℓ such that $\varphi(u_\ell) = \beta_\ell$ and $u_\ell^* = u_\ell$, where $*$ is the cap symmetrization with respect to a half straight line in $\mathbf{R}^N$.

The key idea of the proof is to replace any set A appearing in the definition of β_ℓ by a set $\tilde{A}$ of functions which are almost invariant by spherical cap symmetrization: Given $\varepsilon > 0$, there is $\tilde{A}$ such that for every $u \in \tilde{A}$ there is a cap symmetrization $*$ such that

$$\|u - u^*\|_p \leqslant \varepsilon.$$

The dependence of $*$ on u is essential in order to be able to ensure $\tilde{A} = -\tilde{A}$. Moreover,

$$\sup_{\tilde{u} \in \tilde{A}} \varphi(\tilde{u}) \leqslant \sup_{u \in A} \varphi(u).$$

This set $\tilde{A}$ is constructed by polarizations. In order to approximate different cap symmetrizations simulaneously, we need the sufficient condition for a sequence of polarization to approximate a symmetrization that was presented in the preceding section.

6. Anisotropic symmetrization

It was assumed implicitly above that the ambiant space $\mathbf{R}^N$ was endowed with the standard Euclidean norm. One can wonder what happens to the Steiner symmetrization when the Euclidean norm is replaced by a general norm. Alvino, Ferone, Lions and Trombetti initiated this study for symmetrizations with respect to a point [2]. We study the properties of symmetrizations with respect to general norms in chapter IV.

A Steiner symmetrization with respect to an arbitrary norm still preserves the inclusions and the measures of sets. Therefore, as stated in section 2, for every admissible function u and every nonnegative Borel function f,

$$\int_{\mathbf{R}^N} f(u^*)\,dx = \int_{\mathbf{R}^N} f(u)\,dx$$

and for every $1 \leqslant p \leqslant \infty$ and for every admissible $u, v \in \mathrm{L}^p(\mathbf{R}^N)$,

$$\|u^* - v^*\|_p \leqslant \|u - v\|_p.$$

Given an anisotropic symmetrization $*$, there is in general no nontrivial simpler symmetrization $\star$ such that for any set A

$$A^{*\star} = A^*.$$

Therefore, anisotropic symmetrization can not be approximated by simpler symmetrizations, and the methods to prove geometrical inequalities used above for the isotropic Steiner symmetrization fail.

A striking difference between isotropic and anisotropic symmetrizations is that there is no equivalent of the Riesz–Sobolev inequality presented in section 2. In fact, if the inequality

$$\int_{\mathbf{R}^N}\int_{\mathbf{R}^N} u(x)v(y)w(x-y)\,dx\,dy \leqslant \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} u^*(x)v^*(y)w^*(x-y)\,dx\,dy.$$

holds for every nonnegative compactly supported continuous functions u, v and w, then $*$ must be a Steiner symmetrization with respect to an Euclidean norm. Indeed, the Riesz–Sobolev inequality is very delicate and it was already mentioned that it does not hold for cap symmetrizations and polarizations. More surprisingly, there is neither a weakened Riesz–Sobolev inequality like (3.1): Theorem IV.4.4 states that if for some Radon measure μ the inequality

$$\int_{\mathbf{R}^N}\int_{\mathbf{R}^N} u(x-y)v(y)\,d\mu(x)\,dy \leqslant \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} u^*(x-y)v^*(y)\,d\mu(x)\,dy \tag{6.1}$$

holds for every compactly supported continuous function u and v, then either $*$ is a symmetrization with respect to an Euclidean norm or μ is concentrated on the subspace of $\mathbf{R}^N$ parallel to the plane with respect to which the symmetrization is performed. In the latter case the inequality is an easy consequence of the classical Hardy-Littlewood inequality. There is thus no nontrivial generalization of the known convolution inequalities to anisotropic symmetrizations.

The picture is different for the Pólya–Szegő inequality. Let H be a positively homogeneous and lower-semicontinuous function, let

$$H^\circ(x) = \sup_{t\neq 0} \frac{\langle t, x\rangle}{H(t)}.$$

and let $*$ be the symmetrization with respect to $H^\circ(-\cdot)$. If u is admissible and $u \in \mathrm{W}^{1,p}(\mathbf{R}^N)$, then $u^* \in \mathrm{W}^{1,p}(\mathbf{R}^N)$. If moreover $J : \mathbf{R}^+ \to \mathbf{R}^+$ is convex and $J(0) = 0$, then

$$\int_{\mathbf{R}^N} J(H(\nabla u^*))\,dx \leqslant \int_{\mathbf{R}^N} J(H(\nabla u))\,dx.$$

As explained in chapter IV, such inequalities appear in the study of variationnal problems which have an anisotropic energy, e.g. models of cristalline materials. They also yield optimal inequalities in Sobolev spaces when $\mathbf{R}^N$ is endowed with a general norm (section IV.7.2).

Alvino et al. proved this when the symmetrization is performed with respect to a point by the coarea formula and an isoperimetric theorem for an anisotropic surface measure [2]. Our method can deal with general isotropic symmmetrization and is based on a generalization of an inequality of Klimov [14], valid for any isotropic Steiner symmetrization $\star$,

$$\int_{\mathbf{R}^N} \overline{\varphi_\star}(\nabla u^\star)\,dx \leqslant \int_{\mathbf{R}^N} \overline{\varphi}(\nabla u)\,dx.$$

Where the Legendre–Fenchel transform $\overline{\varphi}$ of φ is defined by

$$\overline{\varphi}(y) = \sup_{x \in \mathbf{R}^N} y \cdot x - \varphi(x).$$

and the increasing symmetrization $\varphi_\star$ is given by

$$\varphi_\star(t) = -(-\varphi)^\star(t).$$

We first prove this inequality for Steiner symmetrizations with respect to a hyperplane before being extended by approximation to general isotropic Steiner symmetrizations. The final result is a general Pólya–Szegő inequality, in which the integrand is a function of x, u and Du and is convex in Du. This result is also interesting for isotropic Steiner symmetrizations.

On a more technical level, our proof of the Pólya–Szegő inequality for anisotropic symmetrizations required to develop a good pointwize definition of the symmetrization of functions based on Lebesgue's outer measure. This definition is well-suited for the study of the symmetrization of measurable sets. It coincides with the classical definition for open sets, but it does not for compact sets. As explained in section III.3.3, the classical definition and properties of the symmetrization of a compact set can be easily recovered from our definition.

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CHAPTER I

Universal approximation of symmetrizations by polarizations

1. Introduction

A symmetrization $*$ (or rearrangement) maps any function $u : \Omega \to \mathbf{R}$ to a more symmetrical function $u^* : \Omega^* \to \mathbf{R}$. Under some technical assumptions, it has the following properties:

$$\text{(C)} \qquad \int_\Omega f(u)\, dx = \int_{\Omega^*} f(u^*)\, dx,$$

$$\text{(HL)} \qquad \int_{\Omega^*} |u^* - v^*|^p\, dx \leqslant \int_\Omega |u - v|^p\, dx,$$

$$\text{(PS)} \qquad \int_\Omega |\nabla u|^p\, dx \leqslant \int_{\Omega^*} |\nabla u^*|^p\, dx.$$

Rearrangements are used to prove the symmetry and the existence of solutions of some variational problems.

The symmetrization is defined for sets before being extended to functions. The inequalities (C) and (HL) are straightforward consequences of the preservation of both the inclusions and the measure after rearrangement of sets. Pólya–Szegő's inequality (PS) involves the gradient, and a proof that uses directly the definition of the rearrangement relies on an isoperimetric inequality for sets and on the coarea formula. The inequality (PS) can also be proved using approximation by polarizations, as Brock and Solynin did [5] and as we do in Corollary 6.3. Lieb and Loss [9] and Baernstein [3] deduced it from Riesz-like inequalities that they obtained using approximations.

The first approximation of symmetrizations by simpler symmetrizations appeared in the proof of the classical isoperimetric Theorem. A well-chosen sequence of Steiner symmetrization of a convex body converges with respect to the Hausdorff distance to a ball of the same

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volume. Mani-Levitska proved that random sequences of Steiner symmetrizations converge [10]. Brascamp, Lieb and Luttinger approximated in measure the Schwarz symmetrization of sets by lower order symmetrizations in order to prove a generalized Riesz rearrangement inequality [4,9]. Sarvas approximated the symmetrization of sets by spherical cap and Steiner symmetrizations [12], while Baernstein [3], Brock and Solynin used polarizations [5]. This result was extended to the cap symmetrization by Smets and Willem [13].

For all the methods of approximations of symmetrizations by polarizations quoted above, the sequence of polarizations depends on the function that has to be symmetrized. Our main result (Theorem 4.4) is that there exists a sequence that depends neither on the function nor on the function space, and that the increasing rearrangement can also be approximated by polarizations. This symmetrization coincides in the one-dimensional case, with the rearrangement introduced by Carbou [6] and studied by Alberti [1]. The increasing rearrangement inequalities allow to prove the existence of solutions of variational problems that increase in some direction. Badiale obtained with the moving plane method similar results concerning the monotonicity of solutions of some elliptic systems [2].

By the same method, we prove that cap and Steiner symmetrizations approximate higher order Steiner and cap symmetrization. The approximating symmetrizations can be of any order, but they must be compatible with the symmetrization that they approximate.

The definitions and basic properties of symmetrizations (Section 2) and of polarizations (Section 3) are recalled. Section 4 is devoted to the proof of the main result. Finally the method is briefly extended to approximation by symmetrization (Section 5) and a Pólya–Szegő inequality is proven (Section 6).

2. Symmetrizations

The Lebesgue outer measure on $\mathbf{R}^N$ is denoted $\mathcal{L}^N$, the Hausdorff k-dimensional outer measure on $\mathbf{R}^N$ is denoted by $\mathcal{H}^k$, the scalar product between x and y, $x \cdot y$, and the Euclidean norm $\lvert x\rvert = \sqrt{x \cdot x}$. If $x \in \mathbf{R}^N$, $0 \leqslant r \leqslant +\infty$, then

$$\mathrm{B}(p,r) = \left\{ x \in \mathbf{R}^N \ : \ \lvert x - p\rvert < r \right\} .$$

The characteristic function of a set A is denoted χ_A, and the symmetric difference of the sets A and B is denoted

$$A \,\Delta\, B = (A \setminus B) \cup (B \setminus A) .$$

Definition 2.1. If T is a proper affine subspace of $\mathbf{R}^N$, the *Steiner symmetrization* of a set $A \subset \mathbf{R}^N$ with respect to T is the unique set A^T for which the following holds: for any $p \in T$, if L is the maximal affine subspace orthogonal to T that contains p, then

$$A^T \cap L = \mathrm{B}(p,r) \cap L\,,$$

where r is defined such that

$$\mathcal{H}^{N-\dim T}(\mathrm{B}(p,r) \cap L) = \mathcal{H}^{N-\dim T}(A \cap L)\,.$$

Remark 2.2. The Schwarz symmetrization with respect to $p \in \mathbf{R}^N$ is the Steiner symmetrization with respect to $\{p\}$, it is also sometimes the Steiner symmetrization with respect to a straight line [9, 11].

Definition 2.3. A set $S \subset \mathbf{R}^N$ is a *closed half affine subspace* of $\mathbf{R}^N$ if it is a closed affine halfspace with respect to its affine span. The boundary of S with respect to its affine span is denoted ∂S and its dimension is $\dim S = \dim \partial S + 1$.

Definition 2.4. If S is a closed half affine subspace $\mathbf{R}^N$ and $0 < \dim S < N$, the *cap symmetrization* of a set A with respect to S is the unique set A^S for which the following holds: $A^S \cap \partial S = A \cap \partial S$ and, for any $q \in \partial S$ and any $s > 0$, if L is the maximal affine subspace orthogonal to ∂S that contains q, and p is the only point in the intersection $S \cap (L \cap \partial\mathrm{B}(q,s))$, then

$$A^S \cap L \cap \partial\mathrm{B}(q,s) = \mathrm{B}(p,r) \cap L \cap \partial\mathrm{B}(q,s)\,,$$

where $0 \leqslant r \leqslant +\infty$ is defined by

$$\mathcal{H}^{N-\dim S}\big(\mathrm{B}(p,r) \cap L \cap \partial\mathrm{B}(q,s)\big) = \mathcal{H}^{N-\dim S}\big(A \cap L \cap \partial\mathrm{B}(q,s)\big)\,.$$

Definition 2.5. Let $A \subseteq \mathbf{R}^N$ and $v \in \mathbf{R}^N \cap \partial\mathrm{B}(0,1)$,

$$\begin{aligned} c_v(A) = \mathcal{H}^1\big(\{x \in (A + v\mathbf{R}) \setminus A \;:\; v \cdot x > 0\}\big) \\ - \mathcal{H}^1\big(\{x \in A \;:\; v \cdot x \leqslant 0\}\big)\,, \end{aligned}$$

if the formula makes sense and $c_v(A) = -\infty$ otherwise.

Definition 2.6. The increasing rearrangement of $A \subset \mathbf{R}^N$ with respect to $v \in \mathbf{R}^N \cap \partial\mathrm{B}(0,1)$ is the unique set $A^{(v,\infty)}$ such that for any $x \in \mathbf{R}^N$,

$$(x + v\mathbf{R}) \cap A^{(v,\infty)} = \big\{y \in (x + v\mathbf{R}) \;:\; v \cdot y > c_v\big(A \cap (x + v\mathbf{R})\big)\big\}.$$

In the sequel, $*$ denotes indifferently a Steiner or cap symmetrization, or an increasing rearrangement.

For any sets $A, B \subseteq \mathbf{R}^N$,

$$A \subseteq B \implies A^* \subseteq B^*. \tag{2.1}$$

Proposition 2.7. *If $A \subset \mathbf{R}^N$ is measurable, then A^* is measurable.*

Proof. If $*$ is the increasing rearrangement with respect to $v \in \mathbf{R}^N \cap \partial \mathrm{B}(0,1)$, A^* can be written by definition as

$$A^* = \left\{x \in \mathbf{R}^N \ : \ v \cdot x > c_v(A \cap (x + v\mathbf{R}))\right\}.$$

Fubini's theorem implies that the function $x \mapsto c_v(A \cap (x + v\mathbf{R}))$ is measurable. Hence A^* is measurable. The proof is similar for the Steiner and cap symmetrizations. □

Definition 2.8. A set A is admissible for a Steiner or cap symmetrization $*$ if A is measurable, and $\mathcal{L}^N(A) < +\infty$. If $*$ is the increasing rearrangement with respect to v, A is admissible if and only if

$$\mathcal{L}^N(A \,\Delta\, \left\{x \in \mathbf{R}^N \ : \ v \cdot x > 0\right\}) < \infty.$$

If $A \subset B \subset \mathbf{R}^N$ are admissible sets, then

$$\mathcal{L}^N(B^* \setminus A^*) = \mathcal{L}^N(B \setminus A). \tag{2.2}$$

When the sets A and B may have infinite measure, which is the case for the increasing rearrangement, the second condition is more restrictive than the preservation of the measure of sets ($\mathcal{L}^N(A) = \mathcal{L}^N(A^*)$). If $A, B \subset \mathbf{R}^N$ are admissible sets, then

$$\mathcal{L}^N(B^* \setminus A^*) \leqslant \mathcal{L}^N(B \setminus A). \tag{2.3}$$

Notation 2.9. For any function $u : \Omega \to \mathbf{R}$ and $c \in \mathbf{R}$, we write

$$\{u > c\} = \{x \in \Omega \ : \ u(x) > c\}.$$

Definition 2.10. The symmetrization of a function $u : \mathbf{R}^N \to \mathbf{R}$ is, for $y \in \mathbf{R}^N$,

$$u^*(y) = \sup\left\{c \in \mathbf{R} \ : \ y \in \{u > c\}^*\right\}.$$

Proposition 2.11. *If $*$ is a rearrangement and $u : \mathbf{R}^N \to \mathbf{R}$ is measurable, then u^* is measurable.*

Proof. Since $*$ is monotone on sets, $\{u^* > c\} = \cup_{n \geqslant 1}\{u > c + 1/n\}^*$, and the conclusion follows from Proposition 2.7. □

Definition 2.12. A function $u : \Omega \to \mathbf{R}$ is *admissible* if for any c with $\operatorname{ess\,inf}\, u < c < \operatorname{ess\,sup}\, u$ the set $\{u > c\}$ is admissible.

Definition 2.13. If $*$ is a Steiner or cap symmetrization and $1 \leqslant p < +\infty$, we define $\mathrm{L}^p_*(\mathbf{R}^N) = \mathrm{L}^p_+(\mathbf{R}^N)$ to be the set of nonnegative functions of $\mathrm{L}^p(\mathbf{R}^N)$, $\mathrm{C}_*(\mathbf{R}^N) = \mathrm{C}_{0,+}(\mathbf{R}^N)$ to be the set of nonnegative continuous functions whose limit at the infinity is 0 and $\mathcal{K}_*(\mathbf{R}^N) = \mathcal{K}_+(\mathbf{R}^N)$ to be the set of nonnegative continuous functions with compact support. If $*$

is the increasing rearrangement with respect to $v \in \mathbf{R}^N \cap \partial \mathrm{B}(0,1)$, we write

$$\begin{aligned}\mathrm{L}^p_*(\mathbf{R}^N) =&\{u : \mathbf{R}^N \to [0,1] \ :\ \exists h : \mathbf{R} \to \mathbf{R} \text{ such that } h \text{ is increasing},\\ &\quad (h - \chi_{\mathbf{R}^+}) \in \mathrm{L}^p(\mathbf{R}), \text{ and } (h(v \cdot .) - u) \in \mathrm{L}^p(\mathbf{R}^N)\},\\ \mathrm{C}_*(\mathbf{R}^N) =&\{u : \mathbf{R}^N \to [0,1] \ :\ u \text{ is continuous},\\ &\quad \lim_{v\cdot x \to -\infty} u(x) = 0, \text{ and } \lim_{v\cdot x\to+\infty} u(x) = 1\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{K}_*(\mathbf{R}^N) =&\{u \in \mathrm{C}_*(\mathbf{R}^N) \ :\ \exists h : \mathbf{R} \to \mathbf{R} \text{ such that } h \text{ is increasing},\\ &\quad (h - \chi_{\mathbf{R}^+}) \text{ has compact support, and } (h(v \cdot .) - u) \in \mathcal{K}(\mathbf{R}^N)\}.\end{aligned}$$

The functions of the sets $\mathcal{K}_*(\mathbf{R}^N)$, $\mathrm{C}_*(\mathbf{R}^N)$ and $\mathrm{L}^p_*(\mathbf{R}^N)$ are all admissible. If $u \in \mathcal{K}_*(\mathbf{R}^N)$, then $u^* \in \mathcal{K}_*(\mathbf{R}^N)$ and, for any $c \in \mathbf{R}$,

$$\{u > c\}^* = \{u^* > c\}. \tag{2.4}$$

The preservation of inclusions (2.1) and measure (2.2) imply that the symmetrization of functions is nonexpansive for any L^p norm, $1 \leqslant p \leqslant \infty$. The ideas of Crowe, Zweibel and Rosenbloom [7], and of Alberti [1] can be generalized to encompass the case of the increasing rearrangement.

Proposition 2.14. *For any* $1 \leqslant p \leqslant \infty$, *we have*

$$\|u^* - v^*\|_p \leqslant \|u - v\|_p.$$

Proof. If $1 \leqslant p < \infty$, for any admissible functions u and v, we have

$$\begin{aligned}\int_\Omega |u - v|^p \, dx = \int_{\sigma \leqslant \tau} \bigl(&\mathcal{L}^N(\{v > \tau\} \setminus \{u > \sigma\})\\ &+ \mathcal{L}^N(\{u > \tau\} \setminus \{v > \sigma\})\bigr) p(p-1)|\sigma - \tau|^{p-2} \, d\sigma \, d\tau.\end{aligned}$$

The properties (2.4) and (2.3) yield the conclusion. If $p = \infty$, the conclusion follows from preservation of inclusions. □

3. Polarizations

Definition 3.1. A *polarizer* is a closed affine halfspace of $\mathbf{R}^N$.

Remark 3.2. A set H is a polarizer if and only if there exists $a \in \mathbf{R}^N$, $|a| = 1$ and $b \in \mathbf{R}$ such that $H = \bigl\{x \in \mathbf{R}^N \ :\ a \cdot x \geqslant b\bigr\}$.

Notation 3.3. If $x \in \mathbf{R}^N$ and $H \subseteq \mathbf{R}^N$ is a polarizer, x_H denotes the reflection of x with respect to ∂H. Using the description of Remark 3.2, $x_H = x - 2(a \cdot x - b)a$.

Definition 3.4. The *polarization* of a function $u : \mathbf{R}^N \to \mathbf{R}$ by a polarizer H is the function $u^H : \mathbf{R}^N \to \mathbf{R}$ defined by

$$u^H(x) = \begin{cases} \max\{u(x), u(x_H)\} & \text{if } x \in H, \\ \min\{u(x), u(x_H)\} & \text{if } x \notin H. \end{cases}$$

Remark 3.5. The polarization is also called two-point rearrangement. The polarization by H is the natural extension of the cap symmetrization with respect to $S = H$ when $\dim S = N$ (compare with Definition 2.4).

Notation 3.6. If T is an affine subspace, let

$$\mathcal{H}_T = \left\{H \subset \mathbf{R}^N \ :\ H \text{ is a polarizer and } T \subset H\right\},$$

if S is a closed half affine subspace, let

$$\mathcal{H}_S = \left\{H \subset \mathbf{R}^N \ :\ H \text{ is a polarizer, } S \subset H \text{ and } \partial S \subset \partial H\right\}$$

and, if $v \in \mathbf{R}^N \cap \partial \mathrm{B}(0,1)$, let

$$\mathcal{H}_{(v,\infty)} = \Big\{H \subset \mathbf{R}^N \ :\ H \text{ is a polarizer} \\ \text{and } a = v \text{ in the description of Remark 3.2}\Big\}.$$

For any symmetrization $*$ and for any function $u : \mathbf{R}^N \to \mathbf{R}$,

$$u = u^* \qquad \Longleftrightarrow \qquad \forall H \in \mathcal{H}_*, u = u^H.$$

Polarizations satisfy the properties (2.1) and (2.2). Thus they are nonexpansive. For $u \in \mathrm{L}^p_*(\mathbf{R}^N)$ and $H \in \mathcal{H}_*$, the inequality

$$\|u^H - u^*\|_p = \|u^H - (u^*)^H\|_p \leqslant \|u - u^*\|_p$$

suggests that well chosen polarization can approximate the symmetrization $*$ for a given function. The proof goes in two steps: first the relative compactness of any sequence of iterated polarizations is established (Lemma 3.7), then a convergence condition ensures the convergence to the symmetrized function (Lemma 3.9).

Lemma 3.7. *Let $u \in \mathcal{K}_*(\mathbf{R}^N)$ and $(H_m)_{m \geqslant 0} \subset \mathcal{H}_*$ be a sequence of polarizers and $u_m = u^{H_1 \cdots H_m}$. Then, there is $v \in \mathcal{K}_*(\mathbf{R}^N)$ and an increasing sequence $(m_k)_{k \in \mathbf{N}}$ in $\mathbf{N}$ such that, for any $1 \leqslant p \leqslant \infty$,*

$$\lim_{k \to \infty} \|v - u_{m_k}\|_p = 0.$$

Remark 3.8. This lemma is essentially due to Brock and Solynin [5, Lemmas 6.1 and 6.2], and the main part of the arguments was given by Baernstein [3]. Smets and Willem proved it for the cap symmetrization [13].

Proof. The compactness of the sequence $(u_m)_{m\geqslant 1}$ is proven by Arzelá–Ascoli's Theorem. The sequence is equibounded: for any polarization H, $\|u^H\|_\infty = \|u\|_\infty$ and thus, by induction, $\|u_m\|_\infty = \|u\|_\infty < +\infty$.

Secondly, the sequence is equicontinuous. Let

$$\omega_v(\delta) = \sup\{v(x) - v(y) \ : \ d(x,y) \leqslant \delta\}$$

be the modulus of continuity of a function v. By a tedious analysis of the possible different cases, it can be proved that for any polarization H, $\omega_{u^H} \leqslant \omega_u$, and thus, by induction, $\omega_{u_m} \leqslant \omega_u$. Since $u \in \mathcal{K}_*(\mathbf{R}^N)$ is uniformly continuous, the sequence is equicontinuous.

It remains to prove that the supports are uniformly bounded. For the Steiner or the cap symmetrizations, $u \in \mathcal{K}_*(\mathbf{R}^N)$ implies that, for some p in T or in ∂S, and for some $R > 0$, $\{u > 0\} \subseteq \mathrm{B}(p,R)$. Thus, because polarizations are monotone, $\{u^H > 0\} \subseteq \mathrm{B}(p,R)^H = \mathrm{B}(p,R)$ and, by induction, $\{u_m > 0\} \subseteq \mathrm{B}(p,R)$.

For the increasing rearrangement with respect to v, we have, for some $c \in \mathbf{R}$,

$$\{u > 0\} \subseteq \left\{x \in \mathbf{R}^N \ : \ v \cdot x > c\right\}$$

and

$$\{u^H > 0\} \subseteq \left\{x \in \mathbf{R}^N \ : \ v \cdot x > c\right\}^H = \left\{x \in \mathbf{R}^N \ : \ v \cdot x > c\right\}.$$

Therefore we have

$$\{u_m > 0\} \subseteq \left\{x \in \mathbf{R}^N \ : \ v \cdot x > c\right\}$$

and similarly there exists $d \in \mathbf{R}$ such that

$$\left\{x \in \mathbf{R}^N \ : \ v \cdot x > d\right\} \subseteq \{u_m < 1\}.$$

There exists $R > 0$ such that $u_m(x) \neq h(x)$ implies $\operatorname{dist}(x, x + v\mathbf{R}) \leqslant R$. Therefore, there is a bounded set $B \subset \mathbf{R}^N$ such that $\operatorname{supp}(u_m - h) \subset B$ for $m \in \mathbf{N}$. We conclude, by Arzelá–Ascoli's Theorem that any subsequence has a subsequence converging uniformly to some $v \in \mathcal{K}_*(\mathbf{R}^N)$.

The convergence for $1 \leqslant p < +\infty$ follows from the convergence for $p = +\infty$ and from the fact that all the supports of the functions of the sequence $(u_m - v)$ lie in the same compact set. □

A second lemma states that for any nonsymmetrical function, there exists a polarizer $H \in \mathcal{H}_*$ that makes it closer to its symmetrization.

Lemma 3.9. *Let $u \in \mathcal{K}_*(\mathbf{R}^N)$. If $u \neq u^*$, then there is a polarizer $H \in \mathcal{H}_*$ such that, for any $1 \leqslant p < +\infty$,*

$$\|u^H - u^*\|_p < \|u - u^*\|_p.$$

Remark 3.10. This lemma is due to Brock and Solynin [5] for the Steiner symmetrization and to Smets and Willem [13] for the cap symmetrization.

Proof. Since $u \neq u^*$, there exists $c > 0$ such that the set $\{u > c\} \,\Delta\, \{u^* > c\}$ is not empty. Choose a point $y \in \{u^* > c\} \setminus \{u > c\}$. There is a polarizer $H \in \mathcal{H}_*$ such that $y_H \in \{u > c\} \setminus \{u^* > c\}$. In a sufficiently small neighborhood $N \subset H$ of y, we have then

$$u^H(x) = u(x_H) > c \geqslant u^*(x_H) \quad \text{and} \quad u^*(x) > c \geqslant u(x) = u^H(x_H),$$

whence, for $p \geqslant 1$,

$$\begin{aligned} |u(x) - u^*(x)|^p &+ |u(x_H) - u^*(x_H)|^p \\ &> |u^H(x) - u^*(x)|^p + |u^H(x) - u^*(x_H)|^p. \end{aligned}$$

If $x \in N$, the corresponding nonstrict inequality holds. The integral inequality is obtained by integration of the preceding inequality over N and of the nonstrict inequality on $H \setminus N$. $\square$

4. Approximation by polarizations

We first establish the convergence of a sequence of polarizations for a single function.

Lemma 4.1. *Let $u \in \mathcal{K}_*(\mathbf{R}^N)$, $0 < \kappa < 1$, let $(m_k)_{k\geqslant 1} \subset \mathbf{N}$ be an increasing sequence of indices, and $(H_m)_{m\geqslant 1} \subset \mathcal{H}_*$ a sequence of polarizers such that for all $k \in \mathbf{N}$,*

$$\begin{aligned} (4.1)\quad \|u_{m_k} - u^*\|_1 &- \|{u_{m_k}}^{H_{m_k}} - u^*\|_1 \\ &\geqslant \kappa \sup_{H\in\mathcal{H}_*} \left(\|u_{m_k} - u^*\|_1 - \|{u_{m_k}}^{H} - u^*\|_1\right). \end{aligned}$$

Then the sequence $u_m = u^{H_1\dots H_m}$ converges to u^* for any $1 \leqslant p \leqslant +\infty$.

Remark 4.2. For any function $u \in \mathcal{K}_*(\mathbf{R}^N)$, a sequence of polarizers verifying condition (4.1) can be constructed.

Remark 4.3. We use the same strategy of proof that Smets and Willem [13], except that the inequality (4.1) is weaker than imposing to (H_m) to be optimal as they do.

Proof. By Lemma 3.7, there exists a subsequence $u_{m'_k}$ of $(u_{m_k})_{k\geqslant 1}$ that converges to $v \in \mathcal{K}_*$ for any L^p norm. Since the rearrangement $*$ is nonexpansive,

$$\|u^* - v^*\|_p = \lim_{k\to\infty} \|{u_{m'_k}}^* - v^*\|_p \leqslant \lim_{k\to\infty} \|u_{m'_k} - v\|_p = 0,$$

and $v^* = u^*$. For any polarizer $H \in \mathcal{H}_*$, using the nonexpansiveness of polarizations and equation (4.1), we have then

$$\begin{aligned}\|u_{m'_{k+1}} - u^*\|_1 &\leqslant \|u_{m'_k+1} - u^*\|_1 \\ &\leqslant \|u_{m'_k} - u^*\|_1 + \kappa(\|{u_{m'_k}}^H - u^*\|_1 - \|u_{m'_k} - u^*\|_1) \\ &= (1-\kappa)\|u_{m'_k} - u^*\|_1 + \kappa\|{u_{m'_k}}^H - u^*\|_1 \leqslant \|u_{m'_k} - u^*\|_1.\end{aligned}$$

Passing to the limit, we obtain

$$\|v - u^*\|_1 \leqslant (1-\kappa)\|v - u^*\|_1 + \kappa\|v^H - u^*\|_1 \leqslant \|v - u^*\|_1.$$

Hence and since $u^* = v^*$, we obtain $\|v - v^*\|_1 = \|v^H - v^*\|_1$. By Lemma 3.9 this is absurd if $v \neq u^*$. Therefore the subsequence $(u_{m'_k})_{k\in\mathbf{N}}$ converges to u^* for any L^p norm. The nonexpansiveness of polarizations allows to conclude

$$\lim_{k\to\infty} \|u_k - u^*\|_p \leqslant \lim_{k\to\infty} \|u_{m'_k} - u^*\|_p = 0. \qquad \square$$

Theorem 4.4. *For any symmetrization $*$, there exists a sequence of polarizers $(H)_{m\geqslant 0} \subset \mathcal{H}_*$ such that, for any $1 \leqslant p < \infty$, if $u \in \mathrm{L}^p_*(\mathbf{R}^N)$, the sequence*

$$u_m = u^{H_1\cdots H_m}$$

converges to u^* :

$$\lim_{m\to\infty} \|u_m - u^*\|_p = 0.$$

If $u \in \mathrm{C}_*(\mathbf{R}^N)$, the sequence converges for $p = \infty$.

Proof of Theorem 4.4. If $*$ is a Steiner or spherical cap symmetrization, first note that there is a countable set $N \subset \mathcal{K}_*(\mathbf{R}^N)$ dense in $\mathrm{L}^p_*(\mathbf{R}^N)$ and in $\mathrm{C}_*(\mathbf{R}^N)$ (see [14]). Choose a sequence (H_m) for which (4.1) holds for all $u \in N$. The sequence of iterated polarizations approaches the symmetrization for any $u \in N$.

Let $u \in \mathrm{L}^p_*(\mathbf{R}^N)$ and $\varepsilon > 0$. By density, there is $v \in N$ such that $\|u - v\|_p \leqslant \varepsilon/3$. By contraction, for m sufficiently large and if $v_m = v^{H_1\cdots H_n}$ we obtain

$$\begin{aligned}\|u_m - u^*\|_p \leqslant \|u_m - v_m\|_p + \|v_m - v^*\|_p + \|v^* - u^*\|_p \\ \leqslant 2\|u - v\|_p + \|v_m - v^*\|_p \leqslant \varepsilon.\end{aligned}$$

If $*$ is the increasing rearrangement with respect to v and $h : \mathbf{R} \to [0, 1]$ is a nondecreasing continuous function such that $\operatorname{supp}(h - \chi_{\mathbf{R}^+})$ is compact, then the same reasoning shows the convergence for any $w \in \mathrm{L}^p_*(\mathbf{R}^N) \cap (h(v \cdot .) + \mathrm{L}^p(\mathbf{R}^N))$. Let $u \in \mathrm{L}^p_*(\mathbf{R}^N)$ and

$$C_R = \left\{x \in \mathbf{R}^N \ : \ |(v \cdot x)v - x| \leqslant R\right\}.$$

Consider the function u_R which is equal to u on C_R and equal to h outside of it. Then $u_R \in \mathrm{L}^p_*(\mathbf{R}^N) \cap (h(v\cdot.) + \mathrm{L}^p(\mathbf{R}^N))$ and thus $\int_{C_R} |u_m - u^*|^p \, dx \to 0$. Since

$$\int_{\mathbf{R}^N \setminus C_R} |u_m - u^*|^p \, dx \leqslant 2 \int_{\mathbf{R}^N \setminus C_R} |u - h|^p \, dx,$$

$u_m \to u^*$ follows.

The proof is similar for $u \in \mathrm{C}_*(\mathbf{R}^N)$. □

Remark 4.5. Theorem 4.4 implies that the symmetrization of any set can be approximated in measure and in Hausdorff distance [5, Lemma 7.2]. Conversely, if the symmetrization of any set can be approximated in measure by some fixed sequence of polarizations, then Theorem 4.4 follows by the approximation of functions in $\mathrm{L}^p(\mathbf{R}^N)$ by simple functions.

5. Approximation by symmetrizations

The method of proof of Theorem 4.4 can be extended to approximations of Steiner or cap symmetrizations by lower order Steiner or cap symmetrizations.

Definition 5.1. Let T be an affine subspace. A set of affine subspaces $\mathcal{T}$ *approximates* T if, for any $T' \in \mathcal{T}$, $T \subset T'$, and for any affine subspace $T'' \subset \mathbf{R}^N$ of codimension 1 such that $T \subset T''$, there exists $T' \in \mathcal{T}$ such that $T' \subset T''$.

Theorem 5.2. *Let T be an affine subspace of $\mathbf{R}^N$ and $\mathcal{T}$ be a set of affine subspaces. If $\mathcal{T}$ approximates T, there is a sequence $(T_m)_{m \geqslant 1}$ in $\mathcal{T}$ such that $u^{T_1 \cdots T_m} \to u^T$ for $u \in \mathrm{L}^p_+(\mathbf{R}^N)$ or $u \in \mathrm{C}_0(\mathbf{R}^N)$.*

Definition 5.3. Let T be an affine subspace. A set $\mathcal{S}$ of closed half affine subspaces of $\mathbf{R}^N$ approximates T if, for any $S' \in \mathcal{S}$, $T \subset S'$, and for any affine subspace $T'' \subset \mathbf{R}^N$ of codimension 1 which is parallel to T, there exists $S' \in \mathcal{S}$ such that $\partial S' \subset T''$.

Example 5.4. If $\mathcal{T} = \{T\}$, then $\mathcal{T}$ trivially approximates T. If $T = \{0\}$, the set of polarizers $\mathcal{H}_T$ and the set of closed halflines containing 0 both approximate T.

Definition 5.5. Let S be a closed half affine subspace. A set $\mathcal{S}$ of closed half affine subspaces of $\mathbf{R}^N$ approximates S if, for any $S' \in \mathcal{S}$, $S \subset S'$ and $\partial S \subset S'$, and for any affine subspace $T'' \subset \mathbf{R}^N$ of codimension 1 which is parallel to T, there exists $S' \in \mathcal{S}$ such that $\partial S' \subset T''$.

Theorem 5.6. *Let T be an affine subspace of $\mathbf{R}^N$ (resp. S be a half affine subspace of $\mathbf{R}^N$) and $\mathcal{S}$ be a set of closed half affine subspaces of $\mathbf{R}^N$. If $\mathcal{S}$ approximates T (resp. S), then there exists a sequence $(S_m)_{m\geqslant 1}$ in $\mathcal{S}$ such that $u^{S_1\cdots S_m} \to u^T$ (resp. $u^{S_1\cdots S_m} \to u^S$) for $u \in \mathrm{L}^p_+(\mathbf{R}^N)$ or $u \in \mathrm{C}_0(\mathbf{R}^N)$.*

Proof. The proofs of Theorems 5.2 and 5.6 are similar. The proof is essentially the same as the proof of Theorem 4.4. The modifications in the lemmas are sketched for a closed half affine subspace S in Theorem 5.6. Suppose $u \in \mathcal{K}_+(\mathbf{R}^N)$. It is clear that for any $S' \in \mathcal{S}$, $u^{SS'} = u^S$ and $\|u^{S'} - u^S\|_p \leqslant \|u - u^S\|_p$. Therefore the sequence $\|u^{S_1\cdots S_n} - u^S\|_p$ is nonincreasing. Theorem 4.4 implies that the modulus of continuity decreases along the sequence. This allows to prove an analogue to Lemma 3.7. An analogue of Lemma 3.9 is also needed. Suppose $u \neq u^S$. Then by Lemma 3.9 there exists $H \in \mathcal{H}_S$ such that $\|u^H - u^S\|_p < \|u - u^S\|_p$. Since ∂H is parallel to ∂S and $\mathcal{S}$ approximates S, there exists S' such that $S' \subset H$ and $\partial S' \subset \partial H'$. Hence $u^{HS'} = u^{S'}$ and

$$\|u^{S'} - u^S\|_p \leqslant \|u^H - u^S\|_p < \|u - u^S\|_p.$$

The remaining part of the proof is the same as the proof of Lemma 4.1 and of Theorem 4.4. □

6. Pólya–Szegő's inequality

Definition 6.1. A set Ω is totally invariant with respect to a symmetrization $*$ if for any $H \in \mathcal{H}_*$, Ω is invariant under the reflection with respect to ∂H.

Definition 6.2. If $*$ is a symmetrization, Ω is a totally invariant set and $u : \Omega \to \mathbf{R}$ is a function, then the symmetrization of u is $u^* = \tilde{u}^*|_\Omega$, where $\tilde{u}$ is any extension of u to $\mathbf{R}^N$.

The definition of the symmetrization of $u : \Omega \to \mathbf{R}$ does not depend on the extension $\tilde{u}$ because Ω is totally invariant.

Corollary 6.3. *If Ω is a totally invariant open set, $*$ is a Steiner or cap symmetrization, if $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega)$ is admissible, $1 < p < +\infty$ and $\nabla u \in \mathrm{L}^p(\Omega)$, then*

$$\|\nabla u^*\|_p \leqslant \|\nabla u\|_p. \tag{6.1}$$

Proof. Suppose first $u \in \mathrm{L}^p_*(\Omega)$. Let u_m be the restrictions to Ω of the sequence of iterated polarizations of Theorem 4.4 applied to an extension

$\tilde{u} \in \mathrm{L}^p_*(\mathbf{R}^N)$ of u to $\mathbf{R}^N$. For any compactly supported smooth function $h \in \mathcal{D}(\Omega)^N$,

$$\begin{aligned}-\int_\Omega u^* \operatorname{div} h\,dx = -\lim_{m\to\infty}\int_\Omega u_m \operatorname{div} h\,dx &= \lim_{m\to\infty}\int_\Omega \nabla u_m h\,dx \\ &\leqslant \liminf_{m\to\infty} \|\nabla u_m\|_p \|h\|_{p'} = \|\nabla u\|_p \|h\|_{p'},\end{aligned}$$

since $\|\nabla u^H\|_{\Omega,p} = \|\nabla u\|_{\Omega,p}$ [5] for any $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega)$ such that $\nabla u \in \mathrm{L}^p(\mathbf{R}^N)$, and for any polarizer H. There exist thus $v \in \mathrm{L}^p(\Omega)^N$ that is the weak limit of ∇u_m and the weak gradient of u^*.

In general, if $*$ is an increasing rearrangement, for $m \geqslant 3$, let

$$u_m(x) = \frac{m}{m-2}\min(\max(0, u(x) - 1/m), 1 - 2/m).$$

Since u is admissible, $u_m \in \mathrm{L}^1_*(\Omega)$. From the first part of the proof, $\|\nabla u_m{}^*\|_p \leqslant \|\nabla u_m\|_p$. Since $\frac{m-2}{m}|\nabla u_m| \nearrow |\nabla u|$ and $\frac{m-2}{m}|\nabla u_m{}^*| \nearrow |\nabla u^*|$ almost everywhere, the conclusion comes from the monotone convergence Theorem. The end of the proof is similar for the Steiner and cap symmetrizations. □

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CHAPTER II

Symmetrization and minimax principles

1. Introduction

We are concerned by symmetry properties of symmetric elliptic partial differential equations. Our model problem is

$$
\begin{cases} -\Delta u = f(|x|, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}
$$

where Ω is a ball and u is a real-valued function. When the function f is decreasing in $|x|$ and u is a positive solution continuous up to the boundary, then Gidas, Ni and Nirenberg's celebrated result [6, 7] says that u is radial and $\frac{\partial u}{\partial r} < 0$.

Solutions of (1.1) can be obtained as critical points of the Euler-Lagrange functional φ defined on the Sobolev space $H_0^1(\Omega)$ by

$$
\varphi(u) = \int_\Omega \frac{|\nabla u|^2}{2} - F(|x|, u)\, dx,
$$

where $F(r, t) = \int_0^t f(r, s)\, ds$. In particular one can inquire about the properties of the minimizers of φ. The Schwarz symmetrization maps a nonnegative function $u \in H_0^1(\Omega)$ to a more symmetric one u^*. It can be shown that if $\frac{\partial f}{\partial r} \leqslant 0$, then $\varphi(u^*) \leqslant \varphi(u)$. This proves that if there is a minimizer, then there is a symmetric minimizer. If φ is Gateaux-differentiable, then the minimizer is a critical point. Similarly, using the spherical cap symmetrization, one can ensure that without any sign restriction on u or on $\frac{\partial f}{\partial r}$, the minimum of φ is attained by a function which depends only on the radius and of one angular variable [11].

Continuous symmetrization — a homotopy linking a function to its symmetrization — was used by Brock in order to prove that if $\frac{\partial f}{\partial r} \leqslant 0$, then any nonnegative critical point is locally symmetric, i.e. its domain

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This chapter is an article accepted for publication in Communications in Contemporary Mathematics is the union of annuli on which it is radial and of a set on which $\nabla u = 0$ almost everywhere [3].

In this paper we consider critical points obtained by minimax principles. We modify a general minimax principle of Willem in order to obtain Palais–Smale sequences whose elements are more and more symmetric. This can be applied to prove that some critical levels are achieved by symmetric functions. It also provides an alternative to concentration-compactness.

The paper is organized as follows. Section 2 is devoted to the definition and properties of symmetrizations and polarizations. We briefly recall the classical properties needed for our purpose. In particular, the Schwarz symmetrization and the spherical cap symmetrization can be both approximated by polarizations. We prove that polarizations are continuous in Sobolev spaces. The essential properties are summarized in an axiomatic framework (section 2.4) for the Schwarz symmetrization and the spherical cap symmetrization. Under these assumptions, a homotopy linking a polarization of a function with its symmetrization is constructed. These axioms are easily verified for many variants, e.g. problems in Sobolev-Orlicz spaces and in weighted spaces, and approximation of the Schwarz symmetrization by Steiner symmetrizations.

Section 3 is devoted to our symmetric minimax principle (Theorem 3.5) in the abstract framework of section 2.4. The proof is based on a minimax principle of Willem [15]. The idea of the proof is to replace a path by its symmetrization. The main difficulty is the fact that symmetrizations are not continuous in Sobolev spaces; it is overcome by the approximation of the symmetrization by polarizations.

Finally, section 4 gives examples of applications to semi-linear elliptic partial differential equations. We prove the symmetry of critical points at some critical levels obtained by the mountain pass Theorem of Ambrosetti and Rabinowitz and by Rabinowitz's linking Theorem. We also show how the symmetric minimax principle provides an alternative to concentration-compactness methods in symmetric settings.

All the results in this paper hold for partial symmetrizations ((N, k)–Steiner symmetrization or k–spherical cap symmetrizations). For the sake of clarity, the exposition is made for the Schwarz and $(N-1)$–spherical cap symmetrization, but this restriction can always be removed with no modification in the arguments. Similarly, the results of section 4.1 concerning the spherical cap symmetrization remain valid without any modification for the Neumann boundary conditions.

2. Symmetrization and polarization

2.1. Schwarz symmetrization

For $f : A \to \bar{\mathbf{R}} = \mathbf{R} \cup \{-\infty, +\infty\}$ and $c \in \bar{\mathbf{R}}$, let $\{u > c\} = \{x \in A \ : \ u(x) > c\}$. The set of infinitely differentiable (resp. continuous) functions whose support is compact in $\Omega \subseteq \mathbf{R}^N$ is denoted $\mathcal{D}(\Omega)$ (resp. $\mathcal{K}(\Omega)$).

Definition 2.1. The Schwarz symmetrization of a set $A \subset \mathbf{R}^N$ is the unique open ball centered at the origin A^* such that $\mathcal{L}^N(A^*) = \mathcal{L}^N(A)$, where $\mathcal{L}^N$ denotes the N-dimensional outer Lebesgue measure. If $\mathcal{L}^N(A) = 0$, then $A^* = \phi$ while $A^* = \mathbf{R}^N$ if $\mathcal{L}^N(A) = \infty$.

Definition 2.2. The Schwarz symmetrization of a measurable nonnegative function $u : \Omega \to \bar{\mathbf{R}}$ is the unique function $u^* : \Omega^* \to \bar{\mathbf{R}}$ such that for all $c \in \mathbf{R}$,

$$\{u^* > c\} = \{u > c\}^*.$$

Remark 2.3. The function u^* is also characterized by $u^*(y) = \sup\{c \in \mathbf{R} \ : \ y \in \{u > c\}^*\}$.

Definition 2.4. A measurable function u *vanishes at the infinity* if for all $\varepsilon > 0$, $\mathcal{L}^N(\{|u| > \varepsilon\}) < \infty$.

Definition 2.5. A function is *admissible* for the Schwarz symmetrization if it is nonnegative and it vanishes at the infinity.

Proposition 2.6. *If $u : \Omega \to \bar{\mathbf{R}}$ is admissible, then u^* is admissible and for any Borel measurable function $f : \mathbf{R}^+ \to \mathbf{R}^+$ such that $f(0) = 0$,*

$$\int_{\mathbf{R}^N} f(u^*(x))\, dx = \int_{\mathbf{R}^N} f(u(x))\, dx.$$

In particular, if $u \in \mathrm{L}^p(\Omega)$ is nonnegative, then $u^* \in \mathrm{L}^p(\Omega^*)$ and $\|u\|_p = \|u^*\|_p$.

Remark 2.7. The Steiner symmetrization is an analogue of the Schwarz symmetrization that symmetrizes functions only with respect to certain variables. The (k, N)–Steiner symmetrization of a set $A \in \mathbf{R}^N$ is the unique set A^* such that for all $x'' \in \mathbf{R}^{N-k}$, $\{x' \in \mathbf{R}^k \ : \ (x', x'') \in A^*\} = \{x' \in \mathbf{R}^k \ : \ (x', x'') \in A\}^*$. (The $*$ on the right-hand side denotes the Schwarz symmetrization in $\mathbf{R}^k$ of Definition 2.1.) It is extended to functions as in Definition 2.2; Proposition 2.6 still holds. Steiner-symmetrized functions have cylindrical symmetry: they can be written as $u^*(x', x'') = v(|x'|, x'')$ where $v(\cdot, x'')$ is a decreasing function for each $x'' \in \mathbf{R}^{N-k}$.

2.2. Spherical cap symmetrization

The spherical cap symmetrization is defined following Sarvas [9] (see also [11, 13, 14]).

Definition 2.8. Let $P \in \partial \mathrm{B}(0,1) \cap \mathbf{R}^N$. The spherical cap symmetrization of the set A with respect to P is the unique set A^* such that $A^* \cap \{0\} = A \cap \{0\}$ and, for any $r > 0$,

$$A^* \cap \partial \mathrm{B}(0,r) = B_g(rP, \rho) \cap \partial \mathrm{B}(0,r) \qquad \text{for some } \rho \geqslant 0 \ ,$$
$$\mathcal{H}^{N-1}(A^* \cap \partial \mathrm{B}(0,r)) = \mathcal{H}^{N-1}(A \cap \partial \mathrm{B}(0,r)) \ ,$$

where $\mathcal{H}^{N-1}$ is the outer Hausdorff $(N-1)$-dimensional measure and $B_g(rP, \rho)$ denotes the geodesic ball on the sphere $\partial \mathrm{B}(0,r)$ of center rP and radius ρ. (By definition, $B_g(rP, 0) = \phi$.)

Definition 2.9. The spherical cap symmetrization of a function $u : \Omega \to \bar{\mathbf{R}}$ is the unique function $u^* : \Omega^* \to \bar{\mathbf{R}}$ such that, for all $c \in \mathbf{R}$,

$$\{u^* > c\} = \{u > c\}^* \ .$$

The result of a spherical cap symmetrization is a function that depends on two variables: $u^*(x) = v(|x|, P \cdot x)$, where $v(r, \cdot)$ is a nondecreasing function for any $r \geqslant 0$.

Definition 2.10. A set $\Omega \subset \mathbf{R}^N$ is *invariant* with respect to $*$ if $\Omega^* = \Omega$. It is *totally invariant* if $\Omega^* = \Omega$ and $(\mathbf{R}^N \setminus \Omega)^* = (\mathbf{R}^N \setminus \Omega)$.

Definition 2.11. A function $u : \Omega \to \bar{\mathbf{R}}$ is *admissible* for the spherical cap symmetrization if it is measurable and either Ω is totally invariant or u is nonnegative.

As for the Schwarz symmetrization, we have

Proposition 2.12. *If $u : \Omega \to \bar{\mathbf{R}}$ is admissible, then u^* is measurable and for any Borel measurable function $f : \mathbf{R}^+ \times \bar{\mathbf{R}} \to \mathbf{R}^+$*

$$\int_\Omega f(|x|, u^*(x))\, dx = \int_{\Omega^*} f(|x|, u(x))\, dx.$$

In particular, if $u \in \mathrm{L}^p(\Omega)$, then $u^* \in \mathrm{L}^p(\Omega^*)$.

Remark 2.13. The equivalent of Steiner symmetrization for the spherical cap symmetrization is the k-spherical cap symmetrization with respect to $P \in \mathbf{R}^{k+1}$. The process is the same as in Remark 2.7 and yields symmetrized functions of the form $u^*(x', x'') = v(|x'|, P \cdot x', x'')$.

2.3. Polarizations

Definition 2.14. A set $H \subset \mathbf{R}^N$ is a *polarizer* if it is a closed affine half-space of $\mathbf{R}^N$, i.e. H is the set of all points verifying $a \cdot x \leqslant b$ for some $a \in \mathbf{R}^k, b \in \mathbf{R}$, $|a|_2 = 1$.

Notation 2.15. For any $x \in \mathbf{R}^N$ and any polarizer $H \subseteq \Omega$, x_H denotes the reflection of x with respect to ∂H. With the notation of Definition 2.14, $x_H = x - 2(a \cdot x - b)a$.

Definition 2.16. The *polarization* of a function $u : \mathbf{R}^N \to \mathbf{R}$ by the polarizer H is the function $u^H : \Omega \to \mathbf{R}$, with

$$u^H(x) = \begin{cases} \max\{u(x), u(x_H)\} & \text{if } x \in H, \\ \min\{u(x), u(x_H)\} & \text{if } x \notin H. \end{cases}$$

Definition 2.17 (Extended polarizers). The set of polarizers is compactified by the addition of two polarizers at the infinity, defined by $u^{H_{+\infty}} = u_+$ and $u^{H_{-\infty}} = -u_-$, such that $H_n \to H_{+\infty}$ if $b_n \to \infty$ and $H_n \to H_{-\infty}$ if $b_n \to -\infty$ in the representation of Definition 2.14. The compactified set of polarizers is denoted $\mathcal{H}$ and is homeomorphic to S^N.

Definition 2.18. If $H \in \mathcal{H}_*$ and $\Omega \subset \mathbf{R}^N$, the polarization of $u : \Omega \to \bar{\mathbf{R}}$ with respect to H is defined as $u^H = \tilde{u}^H|_\Omega$, where $\tilde{u}$ is the extension of u to $\mathbf{R}^N$ by 0 outside of Ω.

Proposition 2.19. *Let $H \in \mathcal{H}$. Suppose $\Omega = \Omega^H \subseteq \mathbf{R}^N$, $u, v : \Omega \to \bar{\mathbf{R}}$ are measurable and nonnegative.*

If $g : \Omega \times \bar{\mathbf{R}} \to \bar{\mathbf{R}}^+$ is a Borel measurable function such that $g(x_H, s) = g(x, s)$ for each $(x, s) \in \Omega \times \bar{\mathbf{R}}$, then

$$\int_\Omega g(x, u^H)\, dx = \int_\Omega g(x, u)\, dx.$$

If $G : \Omega \times \bar{\mathbf{R}} \times \bar{\mathbf{R}} \to \bar{\mathbf{R}}^+$ is a Borel measurable function such that $G(x_H, s, t) = G(x, s, t)$ for each $(x, s, t) \in \Omega \times \bar{\mathbf{R}}$ and for any $x \in \Omega$, $a \leqslant b$ and $c \leqslant d$, $G(x, a, c) + G(x, b, d) \geqslant G(x, a, d) + G(x, b, c)$, then

$$\int_\Omega G(x, u^H, v^H)\, dx \geqslant \int_\Omega G(x, u, v)\, dx.$$

In particular, $\|u^H - v^H\|_p \leqslant \|u - v\|_p$.

If $u \in \mathrm{W}_0^{1,p}(\Omega)$, then $u \in \mathrm{W}_0^{1,p}(\Omega)$ and $\|\nabla u^H\|_p = \|\nabla u\|_p$.

If moreover, $(\mathbf{R}^N \setminus \Omega)^H = \mathbf{R}^N \setminus \Omega$, the results remain valid without any sign restriction on u and v and if $u \in \mathrm{W}^{1,p}(\Omega)$, then $u^H \in \mathrm{W}^{1,p}(\Omega)$ and $\|\nabla u^H\|_p = \|\nabla u\|_p$.

For any symmetrization $*$ defined above, the subset of admissible functions in a function space Y is denoted Y_*, and there is a set $\mathcal{H}_* \subset \mathcal{H}$ of polarizers such that for any admissible function $u : \Omega \to \bar{\mathbf{R}}$,

$$u = u^* \quad \iff \quad \forall H \in \mathcal{H}_*, u = u^H.$$

If $*$ is the (N,k)–Steiner symmetrization,

$$\mathcal{H}_* = \Big\{ H \in \mathcal{H} \ : \ \{0\} \times \mathbf{R}^{N-k} \subset H \text{ or } H = H_{+\infty} \Big\}.$$

If $*$ is the k-spherical cap symmetrization,

$$\mathcal{H}_* = \big\{ H \in \mathcal{H} \ : \ \mathbf{R}^+ \times \{0\} \times \mathbf{R}^{N-k-1} \subset H \\ \text{and } \{0\} \times \mathbf{R}^{N-k-1} \subset \partial H \big\}.$$

Because polarizations are contractions in $\mathrm{L}^p(\mathbf{R}^N)$, for any $H \in \mathcal{H}_*$ and $u \in \mathrm{L}^p(\mathbf{R}^N)$, $\|u^H - u^*\|_p \leqslant \|u - u^*\|_p$. In fact they can approximate symmetrizations: in [13], it was shown:

Theorem 2.20. *For any symmetrization $*$, there exists a sequence of polarizers $(H_m)_{m \geqslant 1} \subset \mathcal{H}_*$ such that, for any $1 \leqslant p < \infty$, $\Omega \subset \mathbf{R}^N$ invariant with respect to $*$ and $u \in \mathrm{L}^p_*(\Omega)$, the sequence $u_m = u^{H_1 \cdots H_m}$ converges to u^* :*

$$\lim_{m \to \infty} \|u_m - u^*\|_p = 0.$$

Theorem 2.20 was proved for a fixed function by Brock and Solynin for the Steiner symmetrizations [4] and by Smets and Willem for the spherical cap symmetrization [11].

Lemma 2.21. *If $1 \leqslant p < \infty$, the map*

$$\mathfrak{h} : \bar{\mathcal{H}} \times \mathrm{L}^p(\mathbf{R}^N) \to \mathrm{L}^p(\mathbf{R}^N) : (H, u) \mapsto u^H$$

is continuous at (u, H) if and only if $(u, H) \in \mathcal{H}(\mathrm{L}^p(\mathbf{R}^N))$, where

$$\mathcal{H}(X) = \big\{ (u, H) \in X \times \bar{\mathcal{H}} \ : \ u \geqslant 0 \textit{ if } H = H_{+\infty} \\ \textit{and } u \leqslant 0 \textit{ if } H = H_{-\infty} \big\}.$$

Proof. If (u, H) and (v, L) are in $\mathcal{H}(\mathrm{L}^p(\mathbf{R}^N))$, then

$$\|u^H - v^L\|_p \leqslant \|u^H - v^H\|_p + \|v^H - v^L\|_p.$$

The first term is bounded by $\|u - v\|_p$ because polarizations are nonexpansive. For the second term, since polarizations are nonexpansive we can suppose without loss of generality that v is a compactly supported continuous function. In the latter case $v^H \to v^L$ uniformly on $\mathbf{R}^N$ if $(v, L) \in \mathcal{H}(\mathcal{K}(\mathbf{R}^N))$ and thus in $\mathrm{L}^p(\mathbf{R}^N)$. □

Proposition 2.22. *If $*$ is a symmetrization, Ω is open and invariant with respect to $*$, then the map $\mathfrak{h}$ is continuous from $\mathrm{L}^p_*(\Omega) \times \mathcal{H}_*$ to $\mathrm{L}^p_*(\Omega)$.*

Proof. This is a direct consequence of Lemma 2.21 and of Theorem 2.20. □

The continuity in Sobolev spaces relies on the next standard lemma.

Lemma 2.23. *Let $1 < p < \infty$, $(u_n)_{n\in\mathbf{N}}$ and u be in $\mathrm{W}^{1,p}(\Omega)$ and $|\cdot|$ be a strictly convex norm in $\mathbf{R}^N$. Then $u_n \to u$ in $\mathrm{W}^{1,p}(\Omega)$ if and only if $u_n \to u$ in $\mathrm{L}^p(\Omega)$ and $\||\nabla u_n|\|_p \to \||\nabla u|\|_p$.*

Proof. This is a consequence of the strict convexity of the norm $\||\cdot|\|_p$. □

Proposition 2.24. *If $*$ is a symmetrization, Ω is open and is invariant with respect to $*$ and $1 < p < \infty$, the map $\mathfrak{h}$ is continuous from $\mathrm{W}^{1,p}_{0,+}(\Omega) \times \mathcal{H}_*$ to $\mathrm{W}^{1,p}_0(\Omega)$.*

Moreover, if Ω is totally invariant with respect to $*$ and $*$ is a spherical cap symmetrization, the map $\mathfrak{h}$ is continuous from $\mathrm{W}^{1,p}_0(\Omega) \times \mathcal{H}_*$ to $\mathrm{W}^{1,p}_0(\Omega)$ and from $\mathrm{W}^{1,p}(\Omega) \times \mathcal{H}_*$ to $\mathrm{W}^{1,p}(\Omega)$.

Proof. This is a consequence of Lemma 2.23, together with Proposition 2.22 and the fact that $\||\nabla u^H|\|_p = \||\nabla u|\|_p$. □

2.4. Abstract symmetrizations and polarizations

Assumption 2.25. Let X, V be Banach spaces, $* : S \subset X \to V : u \mapsto u^*$, $\mathcal{H}_*$ be a path-connected topological space and $\mathfrak{h} : S \times \mathcal{H}_* \to S : (u, H) \mapsto u^H$. Assume

(i) X is continuously embedded in V,
(ii) the mapping $\mathfrak{h}$ is continuous,
(iii) for each $u \in S$ and $H \in \mathcal{H}_*$, $u^{*H} = u^{H*} = u^*$ and $u^{HH} = u^H$,
(iv) there is a sequence $(H_m)_{m\geqslant 1} \subset \mathcal{H}_*$ such that for each $u \in S$, $u^{H_1 \dots H_m} \to u^*$ in V as $m \to \infty$,
(v) for each $u, v \in S$ and $H \in \mathcal{H}_*$, $\|u^H - v^H\|_V \leqslant \|u - v\|_V$.

Example 2.26 *(Schwarz symmetrization for nonnegative functions).* Let $\Omega = \mathrm{B}(0,1) \subset \mathbf{R}^N$, $X = \mathrm{W}^{1,p}_0(\Omega)$, $V = (\mathrm{L}^p \cap \mathrm{L}^{p^*})(\Omega))$, with $p^* = Np/(N-p)$, S be the set of nonnegative functions of $\mathrm{W}^{1,p}_0(\Omega)$, $*$ denote the Schwarz symmetrization and $\mathcal{H}_*$ be defined as above. Assumption 2.25 is satisfied by Proposition 2.19, Theorem 2.20 and Proposition 2.24.

Example 2.27 *(Schwarz symmetrization).* Let $\Omega = \mathrm{B}(0,1) \subset \mathbf{R}^N$, $X = \mathrm{W}_0^{1,p}(\Omega)$, $V = (\mathrm{L}^p \cap \mathrm{L}^{p^*})(\Omega))$, with $p^* = Np/(N-p)$, $S = \mathrm{W}_0^{1,p}(\Omega)$, $u^* = |u|^\star$ where $\star$ denotes the Schwarz symmetrization and $\mathcal{H}_*$ is defined as above for the Schwarz symmetrization, but $\mathfrak{h}(u,H) = |u|^H$. Assumption 2.25 is satisfied by Proposition 2.19, Theorem 2.20 and Proposition 2.24.

Example 2.28 *(Spherical cap symmetrization with Dirichlet boundary condition).* Let $\Omega \subset \mathbf{R}^N$ be a ball or an annulus, $X = \mathrm{W}_0^{1,p}(\Omega)$, $V = (\mathrm{L}^p \cap \mathrm{L}^{p^*})(\Omega))$, with $p^* = Np/(N-p)$, $*$ denote the spherical cap symmetrization and $\mathcal{H}_*$ be defined as above. Assumption 2.25 is satisfied by Proposition 2.19, Theorem 2.20 and Proposition 2.24.

Example 2.29 *(Spherical cap symmetrization with Neumann boundary condition).* Let $\Omega \subset \mathbf{R}^N$ be a ball or an annulus, $X = \mathrm{W}^{1,p}(\Omega)$, $V = (\mathrm{L}^p \cap \mathrm{L}^{p^*})(\Omega))$, with $p^* = Np/(N-p)$, $*$ denote the spherical cap symmetrization and $\mathcal{H}_*$ be defined as above. Assumption 2.25 is satisfied by Proposition 2.19, Theorem 2.20 and Proposition 2.24.

Example 2.30 *(Schwarz symmetrization approximated by Steiner symmetrization).* Let $\Omega = \mathrm{B}(0,1) \subset \mathbf{R}^N$, $X = \mathrm{W}_0^{1,p}(\Omega)$, $V = (\mathrm{L}^p \cap \mathrm{L}^{p^*})(\Omega))$, with $p^* = Np/(N-p)$, S be the set of nonnegative function of $\mathrm{W}_0^{1,p}(\Omega)$, $*$ denote the Schwarz symmetrization, $\mathcal{H}_*$ denote the set of hyperplanes passing through the origin and u^H be the Steiner symmetrization with respect to H. Assumption 2.25 is satisfied (see [5] and [13]).

Proposition 2.31. *Under Assumption 2.25, for any $u, v \in S$, $\|u^* - v^*\|_V \leqslant \|u-v\|_V$.*

Proof. By Assumption 2.25, for any $m \geqslant 1$,

$$\begin{aligned}
&\|u^* - v^*\|_V \\
&\quad \leqslant \|u^* - u^{H_1\ldots H_m}\|_V + \|u^{H_1\ldots H_m} - v^{H_1\ldots H_m}\|_V + \|v^{H_1\ldots H_m} - v^*\|_V \\
&\qquad\qquad \leqslant \|u^* - u^{H_1\ldots H_m}\|_V + \|u-v\|_V + \|v^{H_1\ldots H_m} - v^*\|_V.
\end{aligned}$$

The conclusion comes from the property (iv) as $m \to \infty$. □

Proposition 2.32. *Under Assumption 2.25, for any $H_0 \in \mathcal{H}_*$, there exists a continuous mapping $(u,t) \in S \times \mathbf{R}^+ \mapsto u^t$ such that $\lim_{t\to\infty} u^t = u^*$ in V. Furthermore, for each $t \geqslant 0$, there exists $H_t \in \mathcal{H}_*$ such that $u^t = u^{H_0 H_1 \cdots H_{\lfloor t \rfloor} H_t}$, where $\lfloor t \rfloor$ denotes the largest integer less or equal to t.*

Proof. Let H_t be a such that $t \mapsto H_t$ is continuous in $\mathcal{H}_*$. For $t \in [n-1, n]$, $n \in \mathbf{N}$, let

$$u^t = u^{H_0 \cdots H_{n-1} H_t}.$$

This map is well-defined since $u^n = u^{H_0 \cdots H_{n-1} H_n} = u^{H_0 \cdots H_{n-1} H_n H_n}$. It is clear that for any $u \in V$

$$\|u^t - u^*\|_V \leqslant \|u^{\lfloor t \rfloor} - u^*\|_V \to 0 \quad \text{as } t \to \infty.$$

The continuity of $(u, t) \mapsto u^t$ in X comes from the continuity of $(u, H) \mapsto u^H$ and of $t \mapsto H_t$. □

3. Symmetry and variational principles

Symmetrization allows to restrict the search of a minimizer to the subset of symmetric functions. Similarly we show here that on certain critical levels, there is a critical point which is symmetric. Let us first recall a general minimax principle.

Theorem 3.1 (Willem [15]). *Let X be a Banach space. Let M_0 be a closed subspace of the metric space M and $\Gamma_0 \subset C(M_0, X)$. Define*

$$\Gamma = \{\gamma \in C(M, X) \ : \ \gamma|_{M_0} \in \Gamma_0\}.$$

If $\varphi \in C^1(X, \mathbf{R})$ satisfies

$$\infty > c = \inf_{\gamma \in \Gamma} \sup_{t \in M} \varphi(\gamma(t)) > a = \sup_{\gamma_0 \in \Gamma_0} \sup_{t \in M_0} \varphi(\gamma_0(t))$$

then for every $\varepsilon \in]0, (c-a)/2[$, $\delta > 0$ and $\gamma \in \Gamma$ such that

$$\sup_M \varphi \circ \gamma \leqslant c + \varepsilon,$$

there exists $u \in X$ such that

a) $c - 2\varepsilon \leqslant \varphi(u) \leqslant c + 2\varepsilon$,
b) $\operatorname{dist}(u, \gamma(M)) \leqslant 2\delta$,
c) $\|\varphi'(u)\| \leqslant 8\varepsilon/\delta$.

Remark 3.2. A slight modification of the proof gives the better estimate

$$\operatorname{dist}(u, \gamma(M) \cap \varphi^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \leqslant 2\delta.$$

Theorem 3.1 yields a Palais–Smale sequence $(u_n)_{n \geqslant 1}$, i.e. such that $\varphi'(u_n) \to 0$ and $\varphi(u_n) \to c$. This is an important step in order to prove that c is a critical value of φ. This is the case if φ satisfies the $(PS)_c$ condition: any sequence (u_n) such that $\varphi'(u_n) \to 0$ and $\varphi(u_n) \to c$ contains a subsequence that converges strongly.

It should be possible to have more information on the symmetry of u under Assumption 2.25 provided $\varphi(u^*) \leqslant \varphi(u)$. A naive idea consists in replacing the path γ by is its symmetrization $\gamma^* : t \in M \mapsto \gamma(t)^*$. Then

u given by Theorem 3.1 would be near of the set $\gamma^*(M)$. Unfortunately, when $N > 1$, $X = \mathrm{W}^{1,p}(\mathbf{R}^N)$ and $*$ is the Schwarz symmetrization, $*$ is not continuous on X [1] so that the symmetrized path γ^* could be discontinuous.

This idea works if the symmetrization is approximated uniformly by continuous transformations. The convergence of the approximation scheme of the symmetrization $*$ by polarizations of Theorem 2.32 is not uniform; it becomes uniform by an appropriate change of variable.

Proposition 3.3. *Suppose M is a metric space, M_0 and M_1 are disjoint closed sets of M and $\gamma \in C(M, X)$. Suppose that X, V, $*$ and $\mathcal{H}_*$ satisfy Assumption 2.25, $H_0 \in \mathcal{H}_*$ and $\gamma(M) \subset S$. For any $\varepsilon > 0$, there exists $\tilde{\gamma} \in C(M, X)$ such that*

$$\tag{3.1}
\begin{aligned}
&\tilde{\gamma}(t) = \gamma(t)^{H_1 \dots H_{\lfloor \theta \rfloor} H_\theta} && \forall t \in M, \textit{with } \theta \geqslant 0\\
& && \quad \textit{and } H_\tau \in \mathcal{H}_* \textit{ for } \tau \geqslant 0,\\
&\tilde{\gamma}(t) = \gamma(t)^{H_0} && \forall t \in M_0,\\
&\|\tilde{\gamma}(t) - \gamma(t)^*\|_V \leqslant \varepsilon && \forall t \in M_1.
\end{aligned}$$

Proof. For any $t \in M_1$, let δ_t be such that $\mathrm{B}(t, \delta_t) \cap M_0 = \phi$ and such that for all $s \in \mathrm{B}(t, \delta_t)$, $\|\gamma(s) - \gamma(t)\|_V \leqslant \varepsilon/3$ (this is possible because γ is continuous and X is continuously embedded in V). For every $t \in M_1$, there exists θ_t such that $\|\gamma(t)^\theta - \gamma(t)^*\|_V \leqslant \varepsilon/3$ for $\theta \geqslant \theta_t$, with the notation of Proposition 2.32. The collection $\mathcal{O} = \{M \setminus M_1\} \cup \{\mathrm{B}(t, \delta_t)\}_{t \in M_1}$ forms an open covering of the metric space M. There exists thus a partition of the unity $(\rho_i)_{i \in \mathcal{O}}$ subordinate to this covering [10, Theorems (T2, XXII, 5; 1) and (T2, XXII, 5; 5)]. Let $\Theta(t) = \sum_{s \in M_1} \rho_s(t)\theta_s$. The function Θ is continuous. Let $\tilde{\gamma}(t) = \gamma(t)^{\Theta(t)}$. If $t \in M_0$, then $\Theta(t) = 0$ and $\tilde{\gamma}(t) = \gamma(t)^{H_0}$. If $t \in M_1$, there exists $s \in M$ such that $\rho_s(t) > 0$ and $\theta_s \leqslant \Theta(t)$; hence by Proposition 2.31

$$\begin{aligned}
&\|\tilde{\gamma}(t) - \gamma(t)^*\|_V\\
&\qquad \leqslant \|\tilde{\gamma}(t) - \tilde{\gamma}(s)\|_V + \|\tilde{\gamma}(s) - \gamma(s)^*\|_V + \|\gamma(s)^* - \gamma(t)^*\|_V\\
&\qquad\qquad \leqslant \|\gamma(s)^{\Theta(t)} - \gamma(s)^*\|_V + 2\|\gamma(s) - \gamma(t)\|_V \leqslant \varepsilon
\end{aligned}$$

since $t \in \mathrm{B}(s, \delta_s)$ implies $\|\gamma(s) - \gamma(t)\|_V \leqslant \varepsilon/3$. □

Corollary 3.4 (Uniform approximation of symmetrization). *For any $\varepsilon > 0$, there exists a continuous mapping $T : S \to S$ such that $\|Tu - u^*\|_V < \varepsilon$ for each $u \in S$.*

Proof. Apply Proposition 3.3 with $M_0 = \phi$, $M = M_1 = S$ and $\gamma(u) = u$. Let $Tu = \tilde{\gamma}(u)$. □

We can now state and prove a symmetric variational principle.

Theorem 3.5 (Symmetric variational principle). *Suppose X, V, $*$ and $\mathcal{H}_*$ satisfy Assumption 2.25. Let M_0 be a closed subspace of the metric space M and $\Gamma_0 \subset C(M_0, X)$. Define*

$$\Gamma = \{\gamma \in C(M, X) \,:\, \gamma|_{M_0} \in \Gamma_0\}\,.$$

If $\varphi \in C^1(X, \mathbf{R})$ satisfies

$$\infty > c = \inf_{\gamma \in \Gamma} \sup_{t \in M} \varphi(\gamma(t)) > a = \sup_{\gamma_0 \in \Gamma_0} \sup_{t \in M_0} \varphi(\gamma_0(t))$$

and if for any $H \in \mathcal{H}_*$ and $u \in S$, $\varphi(u^H) \leqslant \varphi(u)$, then for every $\varepsilon \in]0, (c-a)/2[$, $\delta > 0$ and $\gamma \in \Gamma$ such that

(i) $\sup_M \varphi \circ \gamma \leqslant c + \varepsilon$,
(ii) $\gamma(M) \subset S$,
(iii) there exists $H_0 \in \mathcal{H}_*$ such that ${\gamma|_{M_0}}^{H_0} \in \Gamma_0$,

there exists $u \in X$ such that

a) $c - 2\varepsilon \leqslant \varphi(u) \leqslant c + 2\varepsilon$,
b) $\|u - u^*\|_V \leqslant 2(2K+1)\delta$,
c) $\|\varphi'(u)\|_{X'} \leqslant 8\varepsilon/\delta$,

where K is the norm of the injection of X into V.

Proof. Without loss of generality, we can assume that $c - 2\varepsilon > a$. Let $M_1 = (\varphi \circ \gamma)^{-1}([c - 2\varepsilon, c + \varepsilon])$. This set is clearly closed. Proposition 3.3, yields a path $\tilde\gamma \in C(M, X)$ such that (3.1) holds with δ in place of ε. Theorem 3.1 with $\tilde\gamma$ in place of γ gives u such that

a) $c - 2\varepsilon \leqslant \varphi(u) \leqslant c + 2\varepsilon$,
b) $\operatorname{dist}(u, \tilde\gamma(M_1)) \leqslant \operatorname{dist}(u, \tilde\gamma(M) \cap \varphi^{-1}([c - 2\varepsilon, c + 2\varepsilon])) \leqslant 2\delta$,
c) $\|\varphi'(u)\| \leqslant 8\varepsilon/\delta$.

Since the symmetrization $*$ is a contraction in V by Proposition 2.31,

$$\begin{aligned}\|u - u^*\|_V &\\ &\leqslant \inf_{t \in M_1} \bigl(\|u - \tilde\gamma(t)\|_V + \|\tilde\gamma(t) - \gamma(t)^*\|_V + \|\gamma(t)^* - u^*\|_V\bigr)\\ &\leqslant 2(2K+1)\delta. \qquad \square\end{aligned}$$

Informally Theorem 3.5 says that when a functional does not increase by any polarization and if the minimax construction is invariant by one polarization (existence of H_0 that preserves Γ_0), then there exists an almost symmetric Palais–Smale sequence.

It is not equivalent for a functional to decrease by symmetrization and to decrease by polarizations. In fact, many symmetrization inequalities can be proved by polarization inequalities [4]; but some inequalities, e.g. the Riesz–Sobolev inequality, hold for the symmetrization, but they do not hold for polarizations [12].

The condition $\gamma|_{M_0}^{H_0} \in \Gamma_0$ on the paths may seem weak, since it does not require invariance by symmetrization. In applications, finding such a polarizer can be impossible because of the highly noninjective character of the polarization. This imposes some kind of minimality to the energy levels on which it is possible to ensure the existence of symmetric critical points.

4. Applications

4.1. Symmetric critical points

We first investigate the symmetry properties of solutions of the semilinear elliptic problem

$$\begin{cases} -\Delta u + a(x)u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where Ω is a ball or an annulus and $f(x,u) = \tilde{f}(|x|,u)$ and $a(x) = \tilde{a}(|x|)$ are continuous. Those are critical points of the functional

$$\varphi(u) = \int_\Omega \frac{|\nabla u|^2}{2} + \frac{a(x)u^2}{2} - F(x,u)\,dx,$$

defined on $H_0^1(\Omega)$, where $F(x,t) = \int_0^t f(x,s)\,ds$ if $t \geqslant 0$ and $F(x,t) = 0$ if $t \leqslant 0$.

Here we assume

(a_1) $a \in \mathrm{L}^{N/2}(\Omega)$ if $N \geqslant 3$, $a \in \mathrm{L}^q(\Omega)$ for $q > 1$ if $N = 2$ and $a \in \mathrm{L}^1(\Omega)$ if $N = 1$.

Under assumption (a_1), the operator $u \mapsto -\Delta u + a(x)u$ has a nondecreasing sequence of eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_i \leqslant \ldots$, repeated according to their multiplicity and with associated orthonormal eigenfunctions $(e_i)_{i\geqslant 1}$ in $\mathrm{L}^2(\Omega)$ [15].

We also assume

(f_1) $f \in C(\Omega \times \mathbf{R})$ and for some $1 < p < 2^* = 2N/(N-2)$ and $C > 0$,
$$|f(x,u)| \leqslant C(1 + |u|^{p-1}),$$

(f_2) there exists $\alpha > 2$ and $R > 0$ such that
$$|u| \geqslant R \Rightarrow 0 < \alpha F(x,u) \leqslant u f(x,u),$$

(f_3) $|f(x,u)| = o(|u|)$, $|u| \to 0$, uniformly on Ω.

Under assumption (f_1), the functional φ is of class $C^1(H_0^1(\Omega), \mathbf{R})$. Under assumptions (f_1) and (f_2), the functional φ satisfies the Palais–Smale condition: Any sequence $(u_n)_{n\in\mathbf{N}} \subset H_0^1(\Omega)$ such that

$$d = \sup_n \varphi(u_n) < \infty$$

and $\varphi'(u_n) \to 0$ contains a convergent subsequence [15].

Consider the class

$$\Gamma = \left\{\gamma \in C([0,1], H_0^1(\Omega) \ : \ \gamma(0) = 0 \text{ and } \varphi(\gamma(1)) < 0\right\},$$

and let

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \varphi(\gamma(t)).$$

By the mountain pass Theorem, there is a critical point such that $\varphi(u) = c$ [15]. Under symmetry assumptions, we obtain slightly more symmetry.

Theorem 4.1. *Under assumptions (a_1) and (f_{123}), if $\lambda_1 > 0$, Ω is a ball, $a(x) \leqslant a(y)$ and $-f(x,-s) = f(x,s) \geqslant f(y,s)$ for $x, y \in \Omega$ with $|x|_2 \leqslant |y|_2$ and $s \in \mathbf{R}^+$, then there exists a nonnegative critical point u invariant by Schwarz symmetrization such that $\varphi(u) = c$.*

Proof. For each $n \geqslant 1$, let $\gamma \in \Gamma$ be such that

$$\max_{t\in[0,1]} \varphi(\gamma(t)) \leqslant c + 1/n.$$

Since $\varphi(u_+) \leqslant \varphi(u)$, we can assume $\gamma(t) \geqslant 0$ for each $t \in [0,1]$. Theorem 3.5 with $\delta = 1/n^{1/2}$ and $\varepsilon = 1/n$ yields $u_n \in H_0^1(\Omega)$ such that

$$|\varphi(u_n) - c| \leqslant 2/n,$$
$$\|\varphi'(u_n)\|_{H_0^{-1}(\Omega)} \leqslant 8/n^{1/2}$$

and

$$\|u_n - u_n^*\|_{\mathrm{L}^2(\Omega)} \leqslant 2(2K+1)/n^{1/2},$$

where $*$ denotes the Schwarz symmetrization. Since φ satisfies the Palais–Smale condition, up to a subsequence, $u_n \to u$ in $H_0^1(\Omega)$, with $\varphi(u) = c$, $\varphi'(u) = 0$ and $u = u^*$. □

Remark 4.2. The method of proof is robust with respect to changes in the minimax principle. If Γ was defined as

$$\Gamma = \{\gamma \in C([0,1] \ : \ \gamma(0) = 0 \text{ and } \gamma(1) = e\},$$

where $e \in H_0^1(\Omega)$ is a fixed nonnegative function and $\varphi(e) < 0$, then the conclusions of Theorem 4.1 would remain valid.

If a and f are slightly more regular, the moving plane method proves that any nonnegative critical point is invariant by Schwarz symmetrization. Theorem 4.1 sheds some light on the limit case where a and f are merely continuous functions.

If Ω is not a ball, a and f are not both monotone, or $f(x,\cdot)$ is not even anymore, then the moving plane method fails, but there is still some symmetry in the solutions.

Theorem 4.3. *Under assumptions (a_1) and (f_{123}), if $\lambda_1 > 0$, Ω is a ball or an annulus, $a(x) = a(y)$ and $f(x,s) = f(y,s)$ if $x, y \in \Omega$ with $|x|_2 = |y|_2$ and $s \in \mathbf{R}$, then there exists a nonnegative critical point u invariant by spherical cap symmetrization such that $\varphi(u) = c$.*

Proof. The proof is similar to the proof of Theorem 4.1. □

Remark 4.4. The method of proof is robust with respect to changes in the minimax principle. Assume $e \in H_0^1(\Omega)$ is a fixed function, $\varphi(e) < 0$ and that there exists a polarizer H_0 with $0 \in \partial H_0$ and $u^{H_0} = u$. If Γ is defined as

$$\Gamma = \{\gamma \in C([0,1] \;:\; \gamma(0) = 0 \text{ and } \gamma(1) = e\},$$

then the conclusions of Theorem 4.3 remain valid.

Theorem 4.3 generalizes the symmetry result of Smets and Willem for homogeneous problems [11].

If $\lambda_1 \leqslant 0$, it is not possible anymore to obtain solutions by the mountain pass Theorem. Let k be such that $\lambda_k \leqslant 0 < \lambda_{k+1}$. Solutions of (4.1) can be obtained by Rabinowitz's linking Theorem.

Theorem 4.5 (Rabinowitz). *Let $X = Y \oplus Z$ be a Banach space with $\dim Y < \infty$. Let $\rho > r > 0$ and let $z \in Z$ be such that $\|z\| = r$. Define*

$$\begin{aligned} M &= \{u = y + \lambda z \;:\; \|u\| \leqslant \rho, \lambda \geqslant 0, y \in Y\}, \\ M_0 &= \{u = y + \lambda z \;:\; y \in Y, \|u\| = \rho \textit{ and } \lambda \geqslant 0 \\ &\qquad\qquad\qquad\qquad \textit{or } \|u\| \leqslant \rho \textit{ and } \lambda = 0\}, \\ N &= \{u \in Z \;:\; \|u\| = r\}. \end{aligned}$$

Let $\varphi \in C^1(X, \mathbf{R})$ be such that

$$b = \inf_N \varphi > a = \max_{M_0} \varphi.$$

If φ satisfies the $(PS)_c$ condition with

$$c = \inf_{\gamma \in \Gamma} \max_{u \in M} \varphi(\gamma(u)) \tag{4.2}$$

$$\Gamma = \{\gamma \in C(M, X) \;:\; \gamma|_{M_0} = \mathrm{id}\}, \tag{4.3}$$

then c is a critical value of φ.

It is a particular case of the general minimax Theorem 3.1 [15]. In order to find solutions of (4.1) assume

(f_4) $\lambda_k \frac{u^2}{2} \leqslant F(x,u)$ for $u \in \mathbf{R}$.

Let

$$
\begin{aligned}
Y &= \operatorname{span}(e_1, e_2, \dots, e_k),\\
Z &= \left\{ u \in X \ : \ \forall v \in Y, \int_\Omega uv = 0 \right\},\\
z &= e_{k+1}.
\end{aligned}
$$

For some $0 < r < \rho$, c defined by (4.2) is a critical value under assumptions (a_1) and (f_{1234}) [15].

Theorem 4.6. *Under assumptions (a_1) and (f_{1234}), suppose that Ω is a ball or an annulus, a is Hölder-continuous and that $a(x) = a(y)$ and $f(x,s) = f(y,s)$, for each $x, y \in \Omega$ with $|x|_2 = |y|_2$ and $s \in \mathbf{R}$. If $e_1, \dots, e_k$ are radial functions, then there exists a nonnegative critical point u invariant by spherical cap symmetrization u such that $\varphi(u) = c$.*

Proof. For each $n \geqslant 1$, let $\gamma \in \Gamma$ be such that

$$\max_{t \in M} \varphi(\gamma(t)) \leqslant c + 1/n.$$

Since $e_1, \dots, e_k$ are radial and since by Lemma 4.7 e_{k+1} is invariant with respect to a spherical cap symmetrization, there exists H_0 such that for any $u \in (Y + \mathbf{R}^+ e_{k+1})$, $u^{H_0} = u$. Hence if $\gamma_0 \in \Gamma_0$, then $\gamma_0^{H_0} \in \Gamma_0$. Theorem 3.5 with $\delta = 1/n^{1/2}$ and $\varepsilon = 1/n$ yields $u_n \in H_0^1(\Omega)$ such that

$$
\begin{gathered}
|\varphi(u_n) - c| \leqslant 2/n,\\
\|\varphi'(u_n)\|_{H_0^{-1}(\Omega)} \leqslant 8/n^{1/2}
\end{gathered}
$$

and

$$\|u_n - u_n^*\|_{\mathrm{L}^2(\Omega)} \leqslant K/n^{1/2},$$

where $*$ denotes a spherical cap symmetrization. Since φ satisfies the Palais–Smale condition, up to a subsequence $u_n \to u$ in $H_0^1(\Omega)$, with $\varphi(u) = c$, $\varphi'(u) = 0$ and $u = u^*$. $\square$

Lemma 4.7. *If a is Hölder-continuous and radial, and e_i is radial for each $1 \leqslant i \leqslant k$, there exists $P \in S^{N-1}$ such that e_{k+1} is invariant under the spherical cap symmetrization with respect to P.*

Proof. The proof is a slight variation on a proof of Bartsch, Weth and Willem [2]. Recall that e_{k+1} minimizes

$$R(u) = \int_\Omega |\nabla u|^2 + a(x)$$

on the set

$$V = \left\{ u \in H_0^1(\Omega) \;:\; \|u\|_{\mathrm{L}^2(\Omega)} = 1 \text{ and } \int_\Omega u e_i = 0 \text{ for } 1 \leqslant i \leqslant k \right\}.$$

Any minimizer u of R on V satisfies the equation

$$-\Delta u + a(x)u = \lambda_{k+1} u, \tag{4.4}$$

with $R(u) = \lambda_{k+1}$. Since a is Hölder continuous, by standard regularity estimates, u is twice differentiable and is continuous up to the boundary.

Let H be a polarizer such that $0 \in \partial H$. One checks that $e_{k+1}^H \in V$ since the eigenfunctions e_i are radial for $1 \leqslant i \leqslant k$ and that $R(e_{k+1}^H) = R(e_{k+1})$. Therefore e_{k+1}^H is a minimizer of R on V, and it satisfies the equation (4.4). The function e_{k+1} and e_{k+1}^H are thus both twice continuously differentiable and continuous up to the boundary. For $x \in H \cap \Omega$, $|e_{k+1}(x) - e_{k+1}(x_H)| = 2{e_{k+1}}^H - (e_{k+1}(x) + e_{k+1}(x_H))$. Therefore $|e_{k+1} - {e_{k+1}}^H| \in C^2(H \cap \Omega) \cap C_0(\overline{H \cap \Omega})$. Since $e_{k+1}(\cdot_H)$ also solves (4.4),

$$\begin{aligned} - \Delta |e_{k+1}(x) - e_{k+1}(x_H)| &+ (a_H(x) - \lambda_{k+1})_+ |e_{k+1}(x) - e_{k+1}(x_H)| \\ &= (a(x) - \lambda_{k+1})_- |e_{k+1}(x) - e_{k+1}(x_H)| \geqslant 0. \end{aligned}$$

By the strong maximum principle either $|u - u_H| = 0$ on Ω, or $|u - u_H| > 0$ on the interior of $H \cap \Omega$.

Now take x_0 in the interior of Ω such that

$$e_{k+1}(x_0) = \max \{ u(x) \;:\; x \in \Omega, |x| = |x_0| \}.$$

For any polarizer such that x_0 is in the interior of H and $0 \in \partial H$, by the preceding reasoning, $u^H = u$. Hence u is invariant by spherical cap symmetrization with respect to $P = x_0/|x_0|$. □

It is not possible to go further in the analysis of symmetry breaking. In fact, if Ω is a ball and e_i is not radial for some $1 \leqslant i \leqslant k$ but is spherical cap symmetric, then Theorem 3.5 is not applicable anymore since e_i and all its rotations can not be invariant under the same spherical cap symmetrization. This obstruction remains even when the $(N-1)$–spherical cap symmetrization is replaced by any k–spherical cap symmetrization. This is not surprising when compared with the situation of spherical harmonics: the first eigenfunction of the Laplace-Beltrami operator on the sphere is the constant function. Then the eigenfunctions associated to the second eigenvalue are restrictions of linear functions, and depend on only one variable. For the third eigenvalue, the eigenfunctions are restrictions of harmonic polynomials of degree two: among these some depend up to rotation on only one variable (the zonal harmonics), but some others depend on all the variables (since spherical harmonics of degree n are restrictions of homogeneous harmonic polynomials of degree n, this follows from Proposition 4.8). This explains why it is not possible to prove any symmetry properties of eigenfunctions of $-\Delta + a(x)$ for eigenvalues above the first nonradial eigenfunction. This suggests that when e_i, for some $1 \leqslant i \leqslant k$ is not radial, a critical point of a nonlinear problem could be noninvariant with respect to any nontrivial rotation group.

Proposition 4.8. *There exists a homogeneous harmonic polynomial h of degree two such that the group G of linear isometries T of $\mathbf{R}^N$ that satisfy $h \circ T = h$ is generated by the reflections with respect to N orthogonal hyperplanes. In particular, G is finite.*

Proof. In general if a function $f \in C^1(\mathbf{R}^N)$ is invariant with respect to a linear isometry T if and only if for any $x \in \mathbf{R}^N$, $\nabla f = T^* \nabla f(Tx)$, where T^* denotes the adjoint of T. If h is a second order harmonic polynomial, it can be written as $h(x) = x \cdot Ax$, where $A : \mathbf{R}^N \to \mathbf{R}^N$ is linear and selfadjoint. The polynomial h is invariant with respect to T if and only if for each $x \in \mathbf{R}^N$, $2Ax = 2T^*ATx$, i.e. $TA = AT$. Choose A with eigenvalues of multiplicity one and vanishing trace. Since A and T commute, the eigenvectors of A must be eigenvectors of T. Since T is an isometry, $Tv = v$ or $Tv = -v$ for each eigenvector v of A. Therefore if h is invariant with respect to T, then T is in the group generated by reflections with respect to hyperplanes orthogonal to the eigenvectors of A. $\square$

The method of this section is also adapted to Neumann boundary conditions. If the functional φ is defined on the set $H^1(\Omega)$ in place of $H^1_0(\Omega)$, then the critical points of φ are weak solutions of

$$\begin{cases} -\Delta u + a(x)u = f(x,u) & \text{in } \Omega, \\ \partial u/\partial n = 0 & \text{on } \partial\Omega. \end{cases}$$

We are in the setting of Example 2.29; Theorems 4.3 and 4.6 remain valid for the new functional φ.

4.2. Noncompact problems

Consider the following semilinear partial differential equation:

$$\begin{cases} -\Delta u + V(x)u = f(x,u) & \text{in } \mathbf{R}^N, \\ u \geqslant 0, \end{cases}$$

where $f \in C(\mathbf{R}^N \times \mathbf{R}^+)$. Solutions are critical points of

$$\varphi : H^1(\mathbf{R}^N) \to \mathbf{R} : u \mapsto \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2 + V(x)|u|^2 \, dx - \int_{\mathbf{R}^N} F(x,u)\, dx,$$

where $F(x,t) = \int_0^t f(x,s)\, ds$ if $t \geqslant 0$ and $F(x,t) = 0$ if $t \leqslant 0$.

Such problems were treated by Rabinowitz [8] without symmetry assumptions. When the problem is invariant by rotations, solutions may be found in the space of radial functions by Palais's symmetric criticality principle [15]. But then global minimizing properties are lost. In our approach, we consider the minimax principle for the unrestricted functional

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)),$$

where

$$\Gamma = \left\{ \gamma \in C([0,1], H^1(\mathbf{R}^N)) \;:\; \gamma(0) = 0 \text{ and } \varphi(\gamma(1)) < 0 \right\}.$$

Therefrom, we construct an almost symmetric Palais–Smale sequence. This proves that c is a symmetric critical level, i.e. there exists a symmetric $u \in H^1(\mathbf{R}^N)$ such that $\varphi'(u) = 0$ and $\varphi(u) = c$. This provides an alternative in some cases to concentration-compactness.

Our assumptions are:

(f_1) There exist $C \geqslant 0$, $2 < p < 2^* = 2N/(N-2)$, such that for all $s \in \mathbf{R}^+$ and $x \in \mathbf{R}^N$

$$f(x,s) \leqslant C(|s| + |s|^{p-1})$$

(f_2) there exists $x \in \mathbf{R}^N$ and $s > 0$ such that $F(x,s) > 0$,

(f_3) there exists $\alpha > 2$ such that for each $x \in \mathbf{R}^N$ and $s \in \mathbf{R}^+$,

$$\alpha F(x,s) \leqslant s f(x,s),$$

(f_4) for $x, y \in \mathbf{R}^N$, if $|x| \leqslant |y|$ then for all $s \in \mathbf{R}^+$, $f(x,s) \geqslant f(y,s)$,

(f_5) $f(x,s) = o(|s|)$, as $|s| \to 0$, uniformly in $x \in \mathbf{R}^N$,

(V_1) there exists $m, M \in \mathbf{R}$ such that for any $x \in \mathbf{R}^N$, $0 < m \leqslant V(x) \leqslant M$,

(V_2) for $x, y \in \mathbf{R}^N$, if $|x| \leqslant |y|$ then $V(x) \leqslant V(y)$.

Remark 4.9. The condition $V(x) \leqslant M$ can be dropped provided that the functional φ is defined on the subset of functions u of $H_0^1(\Omega)$ such that $\int_\Omega V(x)u^2\, dx < \infty$.

Theorem 4.10. *Under the preceding assumptions, c is a critical value and there is a radial symmetric decreasing critical point u such that $\varphi(u) = c$.*

Lemma 4.11. *Under assumptions (f_{12345}) and (V_{12}), there exists a sequence $(u_n)_{n\geqslant 1} \subset H^1(\mathbf{R}^N)$ such that*

$$\begin{aligned} \varphi(u_n) &\to c, \\ \varphi'(u_n) &\to 0 && \text{strongly in } H^{-1}(\mathbf{R}^N), \\ u_n - u_n^* &\to 0 && \text{in } (\mathrm{L}^2 \cap \mathrm{L}^{2^*})(\Omega), \end{aligned}$$

where $*$ denotes the Schwarz symmetrization.

Proof. Note first that the set

$$\Gamma = \left\{\gamma \in C([0,1], H^1(\mathbf{R}^N)) \ : \ \gamma(0) = 0 \text{ and } \varphi(\gamma(1)) < 0\right\}$$

is not empty. From assumptions (f_2) and (f_3), there exists K_1 and an open set $U \subset \mathbf{R}^N$ such that for $x \in U$ and $s \in \mathbf{R}^+$,

$$F(x,s) \geqslant K_1(|s|^\alpha - 1),$$

Let $u \in \mathcal{D}_+(U)$ be nonzero. For any $\tau \geqslant 0$

$$\varphi(\tau u) \leqslant \frac{\tau^2}{2}\int_{\mathbf{R}^N} |\nabla u|^2 + V(x)u^2\,dx - \tau^\alpha K_1\|u\|_\alpha^\alpha + K_1\mathcal{L}^N(\operatorname{supp} u).$$

Since $\alpha > p$, there exists $\bar{\tau} > 0$ such that $\varphi(\bar{\tau}u) < 0$. Let $\gamma(t) = ut/\bar{\tau}$. It is clear that $\gamma \in \Gamma$.

By assumptions (f_4) and (f_5), there is $C' > 0$ such that $|f(x,s)| \leqslant m|s|/2 + C'|s|^{p-1}$. That implies

$$\begin{aligned} \varphi(u) \geqslant \min(1,m)\frac{\|u\|_{H^1}^2}{2} - \frac{\|u\|_{\mathrm{L}^2}^2}{2} - C'\frac{\|u\|_{\mathrm{L}^p}^p}{p} \\ \geqslant (\min(1,m) - m/2)\frac{\|u\|_{H^1}^2}{2} - C''\frac{\|u\|_{H^1}^p}{p}. \end{aligned}$$

Therefore, there exists $\rho > 0$ such that $\varphi(u) \geqslant 0$ if $\|u\|_{H^1} \leqslant \rho$ and $\varphi(u) \geqslant \mu > 0$ if $\|u\|_{H^1} = \rho$. Hence if $\gamma \in \Gamma$, $\|\gamma(1)\|_{H^1} > \rho$ and so $\max_{t\in[0,1]}\varphi(\gamma(t)) \geqslant \mu > 0$. This shows that $c > a$ in Theorem 3.5. For any polarizer, by Proposition 2.19, $\varphi(u^H) = \varphi(u)$. Let H_0 be any fixed polarizer. Then $\gamma(0)^{H_0} = 0 = \gamma(0)$ and $\varphi(\gamma(1)^{H_0}) \leqslant \varphi(\gamma(1))$. The conclusions follow from the symmetric minimax principle (Theorem 3.5). ∎

Proof of Theorem 4.10. Let $(u_n)_{n\geqslant 1}$ be the sequence given by Lemma 4.11. For sufficiently large n, we have

$$\begin{aligned} 1+c+\|u_n\|_{H^1(\mathbf{R}^N)} &\geqslant \varphi(u_n)-\frac{1}{\alpha}\langle\varphi'(u_n),u_n\rangle \\ &\geqslant \left(\frac{1}{2}-\frac{1}{\alpha}\right)\|u_n\|^2_{H^1(\mathbf{R}^N)}; \end{aligned}$$

since $\alpha>2$, the sequence (u_n) is bounded in $H^1(\mathbf{R}^N)$.

The sequence (u_n^*) is also bounded in $H^1(\mathbf{R}^N)$ by the Pólya–Szegő inequality (see e.g. [4]) and by Strauss' Theorem [15], (u_n^*) is compact in $\mathrm{L}^p(\mathbf{R}^N)$. Finally, since $\|u_n-u_n^*\|_p\to 0$, the sequence (u_n) is also compact in $\mathrm{L}^p(\mathbf{R}^N)$. We can thus suppose that $u_n\to u$ weakly in $H^1(\mathbf{R}^N)$ and strongly in $\mathrm{L}^p(\mathbf{R}^N)$.

Finally, we need to prove that

$$\int_{\mathbf{R}^N}(f(x,u_n)-f(x,u))(u_n-u)\,dx\to 0$$

as $n\to\infty$. By (f_4) and (f_5), for any $\varepsilon>0$, there is c_ε such that

$$|f(x,s)|\leqslant\varepsilon|s|+c_\varepsilon|s|^{p-1}.$$

Then

$$\begin{aligned} \limsup_{n\to\infty}&\left|\int_{\mathbf{R}^N}(f(x,u_n)-f(x,u))(u_n-u)\,dx\right| \\ &\leqslant 2\varepsilon\|u\|^2_{H^1(\mathbf{R}^N)}+2c_\varepsilon\|u\|^{p-1}_{\mathrm{L}^p(\mathbf{R}^N)}\limsup_{n\to\infty}\|u_n-u\|_{\mathrm{L}^p(\mathbf{R}^N)}, \end{aligned}$$

and our claim is proved since u is in $H^1(\mathbf{R}^N)$ and converges in $\mathrm{L}^p(\mathbf{R}^N)$. Since the sequence (u_n) is Palais–Smale, by standard arguments, $u_n\to u$ in $H^1(\mathbf{R}^N)$ and thus u is a critical point of φ and $\varphi(u)=c$. Furthermore $u^*=u$. □

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CHAPTER III

Approximation of symmetrizations and symmetry of critical points

1. Introduction

A symmetrization by rearrangement transforms a set or a function into a more symmetric one, while some quantities remain under control. For example, for each $u \in \mathrm{W}_0^{1,p}(\mathrm{B}(0,R))$ with $1 \leqslant p < \infty$ and $u \geqslant 0$, one can construct a radial and radially decreasing function u^* such that for every Borel-measurable function $f : \mathbf{R} \to \mathbf{R}^+$,

$$\int_{\mathrm{B}(0,R)} f(u^*)\, dx = \int_{\mathrm{B}(0,R)} f(u)\, dx\,.$$

In particular, $u^* \in \mathrm{L}^p(\mathrm{B}(0,R))$ and $\|u^*\|_p = \|u\|_p$. While the map $u \mapsto u^*$ is nonlinear, it is still non-expansive in $\mathrm{L}^p(\mathrm{B}(0,R))$. Furthermore, $u^* \in \mathrm{W}_0^{1,p}(\mathrm{B}(0,R))$ and one has the Pólya–Szegő inequality:

$$\int_{\mathrm{B}(0,R)} |\nabla u^*|^p \; dx \leqslant \int_{\mathrm{B}(0,R)} |\nabla u|^p \; dx\,.$$

Other useful inequalities, such as the Riesz–Sobolev rearrangement inequality hold. For symmetrization inequalities, we refer to [12,16]. Symmetrizations were defined for sets in the nineteenth century by Steiner and Schwarz. Symmetrizations of functions go back to Hardy, Littlewood and Pólya [11] and to Pólya and Szegő [19].

Applications of symmetrization by rearrangement are multiple. Symmetrizations were used by Talenti and Aubin to compute the optimal constants for the Sobolev inequality [2, 27]. They can be used to obtain estimates on the first eigenvalue of the Laplacian with Dirichlet boundary conditions (Faber-Krahn inequality [19, 28, 33]). By symmetrization techniques, it is also possible to prove that solutions of problems in the

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calculus of variations are symmetric functions [23]. In some cases they provide also an alternative to concentration-compactness [8].

Since symmetrizations and symmetrization inequalities are useful, it would be nice to have general, simple and elegant methods to construct symmetrizations and prove the associated inequalities. The main difficulty is that symmetrizations are nonlinear and nonlocal transformations. One way to manage these problems is the level-sets method. The functional for which an inequality is needed is decomposed in integrals on level sets. For example, if $u : \Omega \to \mathbf{R}^+$ is nonnegative and measurable and $f \in C^1(\mathbf{R}^+, \mathbf{R}^+)$, one has

$$\int_\Omega f(u)\,dx = \int_{\mathbf{R}^+} \mathcal{L}^N(\{x \in \Omega \ : \ f(x) \leqslant t\}) f'(t)\,dt.$$

This can be thought as localizing the functional with respect to the u variable. As long as the functionals in consideration do not involve gradients or convolution products, the inequalities are proved trivially. — For example, the proof of the Hardy-Littlewood inequality becomes very elegant [10, 33]. — When it is not the case any more, the set inequalities become nontrivial geometric inequalities. For example, the Pólya–Szegő inequality follows from the classical isoperimetric inequality [18], and the Riesz–Sobolev rearrangement inequality is a consequence of the same inequality for characteristic functions of sets [16]. In those cases the level set method does not essentially simplify the proof. The method of level-sets is used extensively by Mossino [18].

Another method to study symmetrization is to approximate a symmetrization by a sequence of simpler symmmetrizations — which are more localized than more elaborated symmetrizations. This goes back to the original definition of the Steiner symmetrization as a tool to prove the classical isoperimetric Theorem. Later, inequalities for capacitors were proved by approximation of Steiner and cap symmetrizations by lower-order Steiner and cap symmetrizations [21]; the Riesz–Sobolev inequality was proved by approximation of a Steiner symmetrization by lower-order Steiner symmetrizations [5]; Recently, a still simpler transformation, the polarization, was used to approximate many symmetrizations in order to obtain simple proofs of the isoperimetric inequality, the Pólya–Szegő inequality and a weak form of the Riesz–Sobolev rearrangement inequality [3, 6, 23, 31].

In a recent work [30], we used approximation of symmetrization in order to investigate the symmetry properties of critical points obtained by minimax methods. The key point was the use of polarizations to obtain a continuous approximation of a Steiner or cap symmetrization which is not continuous in general in Sobolev spaces [1].

In this paper, we investigate further the approximation of symmetrizations by simpler symmetrizations. We study which sequences of symmetrizations approximate a given symmetrization, and we give a simple sufficient condition. Since almost every sequence of symmetrizations in a well-chosen set satisfies this condition, we solve by the way a conjecture of Mani-Levitska concerning random sequences of Steiner symmetrizations [17]. This sufficient condition allows us to obtain some information about the symmetry of critical points of symmetric functionals obtained by minimax methods using the Krasnoselskii genus.

The paper begins by reviewing in section 2 the main facts about symmetrizations used in the sequel. We define in section 2.1 the Steiner with respect to an affine subspace and cap symmetrizations with respect to a closed affine half subspace. The set of affine subspaces and closed affine half subspaces is denoted by $\mathcal{S}$, and the symmetrization of u with respect to $S \in \mathcal{S}$ is denoted by u^S. The simplest cap symmetrizations are the polarizations; they are symmetrizations with respect to $H \in \mathcal{H}$, where $H \subset \mathcal{H}$ is the set of closed affine halfspaces. Many of their properties are easy to prove (section 2.2). We introduce a partial order $\prec$, such that $S \prec T$ if the symmetrization with respect to T can be used to approximate the symmetrization with respect to S (Definition 2.19 and Proposition 2.20). For $S \in \mathcal{S}$, the set of $T \in \mathcal{S}$ (resp. $\in \mathcal{H}$) such that $S \prec T$ is denoted by $\mathcal{S}_S$ (resp. $\mathcal{H}_S$). With these notations, we restate in a common framework all the approximation results of [31]:

Theorem 2.28. *Let $S \in \mathcal{S}$ and $\mathcal{T} \subset \mathcal{S}_S$. If for every $H \in \mathcal{H}_S$, there exists $T \in \mathcal{T}$ such that $T \prec H$, then there exists a sequence $(T_n)_{n\geqslant 1} \subset \mathcal{T}$ such that if $\Omega \subset \mathbf{R}^N$ is open, $u \in \mathcal{K}(\Omega)$ and (u, S) is admissible, then*

$$\|u^{T_1\dots T_n} - u^S\|_\infty \to 0.$$

The condition "(u, S) is admissible" simply means that the symmetrization u^S is defined. In order to state a sufficient condition for a sequence of symmetrizations to approximate a symmetrization, we define a metric d on $\mathcal{S}$ for which the mapping $(u, S) \mapsto u^S$ is continuous (Definition 2.35, Proposition 2.38 and Corollary 2.39).

With all the machinery of section 2, we can state and prove the main result of Section 3,

Theorem 3.2. *Let $S \in \mathcal{S}$, $\mathcal{T} \subset \mathcal{S}_S$ and $(T_n)_{n\geqslant 1} \subset \mathcal{S}_S$ be such that*

a) for every $H \in \mathcal{H}_S$, there exists $T \in \mathcal{T}$ such that $T \prec H$, *b) for each* $m \geqslant 1$ *and* $S_1, \ldots, S_m \in \mathcal{T}$*, there exists* $k \geqslant 0$ *such that for every* $1 \leqslant i \leqslant m$*,* $\mathrm{d}(S_i, T_{k+i}) \leqslant \delta$*,*

Then for each open set $\Omega \subset \mathbf{R}^N$ *and* $u \in \mathcal{K}(\Omega)$ *such that* (u, S) *is admissible,*

$$\|u^{T_1 \ldots T_n} - u^S\|_\infty \to 0, \qquad \text{as } n \to \infty.$$

The proof relies on the fact that for every $m \geqslant 1$ and $\delta > 0$, the m first terms of the sequence of Theorem 2.28 are contained up to an error δ in the sequence $(T_n)_{n \geqslant 1}$.

Given $\mathcal{T}$, it is easy to construct sequences satisfying the hypotheses of Theorem 3.2. In fact, if the approximating symmetrizations are symmetrization with respect to random variables that are distributed throughout the whole of $\mathcal{T}$, then the convergence occurs almost surely (Theorem 3.4).

All the preceding results can be extended to the approximation of the symmetrization of compact sets in Hausdorff distance d_H (Proposition 3.10). For example, if $\mathfrak{K}(\mathbf{R}^N)$ denotes the set of compact sets of $\mathbf{R}^N$, one has:

Theorem 3.13. *Let* $S \in \mathcal{S}$ *with* $\partial S = \phi$ *and let* (E, Σ, P) *be a probability space. Let* $\ell > \dim S$ *and*

$$\mathcal{T}_S^\ell = \{T \in \mathcal{S}_S \ : \ \partial T = \phi \ \text{and} \ \dim T = \ell\}.$$

If $(T_n)_{n \geqslant 1}$ *are independent random variables with values in* $\mathcal{T}_S^\ell$ *whose distribution functions are invariant under isometries that preserve* S*, then*

$$P\Big(sete \in E \ : \ \forall K \in \mathfrak{K}(\mathbf{R}^N), \lim_{n \to \infty} \mathrm{d}_H(K^{T_1(e) \ldots T_n(e)}, K^S) = 0\Big) = 1.$$

Finally, in section 4, Theorem 3.2 is applied to the proof of symmetry properties of critical points obtained by minimax methods using the Krasnoselskii genus. If A is a symmetric (i.e. $A = -A$) set in a Banach space V, its Krasnoselkii genus $\gamma(A)$ is the least integer k such that there is an odd mapping in $C(A, S^{k-1})$. The properties of γ are developed in section 4.1. For $\varphi : M \subset V \to \mathbf{R}$, let

$$\beta_\ell = \inf_{\substack{A \subset M \\ A \text{ is closed} \\ \gamma(A) \geqslant \ell}} \sup_{u \in A} \varphi(u).$$

Theorem 3.2 allows us to construct, given a set of small Krasnoselskii genus, a set of more symmetric functions that has not a smaller Krasnoselskii genus (Propositions 4.7).

The main result is that when the functional φ satisfies some symmetry assumptions, then there are symmetric critical points on the levels β_ℓ for small ℓ:

Theorem 4.8. *Let $\Omega = \Omega' \times \Omega'' \subset \mathbf{R}^N$ be open, with $\Omega' \subset \mathbf{R}^k$ invariant under $O(k)$. Let $M \subset \mathrm{W}^{1,p}(\Omega) \setminus \{0\}$ be a complete symmetric $C^{1,1}$-manifold. Suppose $\varphi \in C^1(M)$ is an even functional that satisfies the Palais–Smale condition, and is bounded from below on M. Also suppose that if $H \in \mathcal{H}$, $\{0\} \times \mathbf{R}^{N-k} \subset \partial H$ and $u \in M$, then $u^H \in M$ and $\varphi(u^H) \leqslant \varphi(u)$. If $\ell \leqslant k$, then there is a critical point $u \in M$ and $x \in S^{k-1}$ such that $\varphi(u) = \beta_\ell$ and $u^{S_x} = u$.*

Here S_x denotes the cap symmetrization with respect to $\mathbf{R}x \times \mathbf{R}^{N-k}$. We end with simple applications of this result. The method applies to Dirichlet and Neumann problems (Theorems 4.9 and 4.10).

2. Symmetrizations

2.1. Definitions

In the following, $\mathcal{H}^k$ denotes the k-dimensional outer Hausdorff measure, while for $x \in \mathbf{R}^N$ and $0 \leqslant r \leqslant \infty$,

$$\mathrm{B}(x,r) = \left\{ y \in \mathbf{R}^N \ : \ |x-y| < r \right\}.$$

The extended set of real numbers is denoted by $\bar{\mathbf{R}} = \mathbf{R} \cup \{-\infty, +\infty\}$. The set of compactly supported continuous functions on the open set Ω is denoted by $\mathcal{K}(\Omega)$ and the modulus of continuity of a function $u \in \mathcal{K}(\Omega)$ is the function $\omega_u : \mathbf{R}^+ \to \mathbf{R}^+$ defined by

$$\omega_u(\delta) = \sup \left\{ |u(x) - u(y)| \ : \ x, y \in \Omega \text{ and } |x-y| \leqslant \delta \right\}.$$

We define the Steiner and spherical cap symmetrizations according to Sarvas [21]. In contrast with Sarvas, our definition does not make difference between compact and open sets, but is valid for any set, possibly non-measurable. This ensures a good pointwize definition of the symmetrization of measurable sets and functions.

Definition 2.1 (Steiner symmetrization). Let S be a k-dimensional affine subspace of $\mathbf{R}^N$, $0 \leqslant k \leqslant N-1$. The symmetrization of a set $A \subset \mathbf{R}^N$ with respect to S is the unique set A^S such that for any $x \in S$, if L is the $(N-k)$-dimensional hyperplane orthogonal to S that contains x,

$$A^S \cap L = \mathrm{B}(x,r) \cap L,$$

where $0 \leqslant r \leqslant \infty$ is defined by $\mathcal{H}^{N-k}(\mathrm{B}(x,r) \cap L) = \mathcal{H}^{N-k}(A \cap L)$.

Remark 2.2. The symmetrization with respect to a 0-dimensional plane is called point symmetrization or Schwarz symmetrization. (Some authors call Schwarz symmetrization a symmetrization with respect to a 1-dimensional plane and Steiner symmetrization a symmetrization with respect to a $(N-1)$-dimensional plane [16].)

Definition 2.3 (Cap symmetrization). Let S be a k-dimensional closed affine half subspace of $\mathbf{R}^N$, $1 \leqslant k \leqslant N$ and let ∂S be the boundary of S inside the affine plane generated by S. The symmetrization of a set $A \subset \mathbf{R}^N$ with respect to S is the unique set A^S such that $A^S \cap \partial S = A \cap \partial S$ and for each $x \in \partial S$, if L is the $(N-k+1)$-dimensional hyperplane orthogonal to ∂S that contains x and y is the unique point of the intersection $\partial \mathrm{B}(x,\varrho) \cap S$, then for every $\varrho > 0$

$$A^S \cap \partial \mathrm{B}(x,\varrho) \cap L = \mathrm{B}(y,r) \cap \partial \mathrm{B}(x,\varrho) \cap L,$$

where $r \geqslant 0$ is defined by

$$\mathcal{H}^{N-k}(\mathrm{B}(y,r) \cap \partial \mathrm{B}(x,\varrho) \cap L) = \mathcal{H}^{N-k}(A \cap \partial \mathrm{B}(x,\varrho) \cap L).$$

Remark 2.4. The symmetrization with respect to a one dimensional closed affine subspace is also called foliated Schwarz symmetrization [23].

Definition 2.5. The set of all the k-dimensional affine subspaces of $\mathbf{R}^N$ for $0 \leqslant k \leqslant N-1$, and of all the k-dimensional closed affine half subspaces of $\mathbf{R}^N$ for $1 \leqslant k \leqslant N$ is denoted by $\mathcal{S}$.

Symmetrizations have the following basic properties:

Proposition 2.6. *Let A, $B \subset \mathbf{R}^N$ and $S \in \mathcal{S}$. If $A \subset B$, then $A^S \subset B^S$.*
If A is measurable, then A^S is measurable and $\mathcal{L}^N(A^S) = \mathcal{L}^N(A)$.
If A is open, then A^S is open.

We need some condition to ensure that the symmetrization of a function is meaningful.

Definition 2.7. Let $\Omega \subset \mathbf{R}^N$, $u : \Omega \to \bar{\mathbf{R}}$ and $S \in \mathcal{S}$. The pair (u, S) is *admissible* if $\Omega^S = \Omega$, and, for every $c > 0$,

$$\mathcal{L}^N(\{x \in \Omega \;:\; |u(x)| > c\}) < \infty$$

and either $u \geqslant 0$, or $\partial S \neq \phi$ and $(\mathbf{R}^N \setminus \Omega)^S = \mathbf{R}^N \setminus \Omega$.

Definition 2.8. Let $\Omega \subset \mathbf{R}^N$, $u : \Omega \to \bar{\mathbf{R}}$ and $S \in \mathcal{S}$. Suppose that (u, S) is admissible. The *symmetrization* of u with respect to S is the unique function u^S such that for each $c \in \bar{\mathbf{R}}$,

$$\left\{x \in \Omega \;:\; u^S(x) > c\right\} = \{x \in \Omega \;:\; u(x) > c\}^S.$$

Remark 2.9. The function u^S can be defined as

$$u^S(x) = \sup \left\{ c \in \mathbf{R} \,:\, x \in \{y \in \Omega \,:\, u(y) > c\}^S \right\}.$$

The definitions with open balls of symmetrization of sets are of crucial importance in order to obtain the existence of u^S satisfying Definition 2.8 (see [29]).

The symmetrization of a function does not essentially depend on the domain:

Proposition 2.10. *Let $u : \Omega \to \bar{\mathbf{R}}$, $\tilde{u} : \mathbf{R}^N \to \bar{\mathbf{R}}$ be defined by $\tilde{u}|_\Omega = u$ and $\tilde{u}|_{\mathbf{R}^N \setminus \Omega} = 0$ and $S \in \mathcal{S}$. If (u, S) is admissible, then $(\tilde{u}, S)$ is admissible and $\tilde{u}^S|_\Omega = u^S$.*

The symmetrization of functions in L^p is a non-expansive nonlinear mapping that preserves the norm:

Proposition 2.11 (L^p properties of symmetrizations). *Let $1 \leqslant p \leqslant \infty$, $\Omega \subset \mathbf{R}^N$ be measurable and $u, v \in \mathrm{L}^p(\Omega)$. If (u, S) and (v, S) are admissible, then $u^S, v^S \in \mathrm{L}^p(\Omega)$, $\|u^S\|_p = \|u\|_p$, $\|v^S\|_p = \|v\|_p$ and $\|u^S - v^S\|_p \leqslant \|u - v\|_p$.*

Proof. See e.g. [10, 32]. □

Remark 2.12. If $u \in \mathrm{W}^{1,p}(\Omega)$ then $u^S \in \mathrm{W}^{1,p}(\Omega)$ and $\|\nabla u^S\|_p \leqslant \|\nabla u\|_p$, but if $\partial S = \phi$, the mapping $u \mapsto u^S$ is continuous in $\mathrm{W}^{1,p}(\Omega)$ if and only if $\dim S = N - 1$ [1, 7, 9]. If $\partial S \neq \phi$, $u \mapsto u^S$ is continuous if $\dim S = N$ (see [30] and Corollary 2.40 below). If $\dim S < N - 1$, then a reasoning in the spirit of Lemma 2.33 and the results of Almgren and Lieb [1] shows that $u \mapsto u^S$ is not continuous. The case $\dim S = N - 1$ remains open, but it is likely that the method of Burchard would show that the cap symmetrization is then continuous.

We introduce the complementary of a affine half subspace.

Definition 2.13. Let $u \in \mathcal{S}$ and $S \in \mathcal{S}$ with $\partial S \neq \phi$. The *complementary* of S is the reflexion of S with respect to ∂S. It is denoted by S^*.

As a straightforward consequence of the definitions, one has

Proposition 2.14. *Let $S \in \mathcal{S}$ and $u : \Omega \to \bar{\mathbf{R}}$. If (u, S) and $(-u, S^*)$ are admissible, then*

$$(-u)^{S^*} = -(u^S).$$

2.2. Polarizations

We recall briefly some facts about the simplest symmetrizations, the polarizations.

Definition 2.15. The symmetrization with respect to $H \in \mathcal{S}$ is a *polarization* if ∂H is a hyperplane, or, equivalently, $\dim H = N$. The reflexion of $x \in \mathbf{R}^N$ with respect to ∂H is denoted by x_H. The set of $H \in \mathcal{S}$ such that $\dim H = N$ is denoted by $\mathcal{H}$.

Proposition 2.16. *Let $H \in \mathcal{H}$, $\Omega \subset \mathbf{R}^N$ and $u : \Omega \to \bar{\mathbf{R}}$. If (u, H) is admissible, then*

$$u^H(x) = \begin{cases} \max(u(x), u(x_H)) & \text{if } x \in H\,, \\ \min(u(x), u(x_H)) & \text{if } x \notin H\,. \end{cases}$$

Remark 2.17. The characterization of Proposition 2.16 is the classical definition of the polarization of a function [6].

Proposition 2.18. *Let $H \in \mathcal{H}$, $\Omega \subset \mathbf{R}^N$ be open and $u : \Omega \to \bar{\mathbf{R}}$ be measurable. If (u, H) is admissible, $f : \Omega \times \bar{\mathbf{R}} \to \mathbf{R}^+$ is a Borel measurable function, and for every $t \in \bar{\mathbf{R}}$ and $x \in \Omega$ such that $x_H \in \Omega$, $f(x_H, t) = f(x, t)$, then*

$$\int_\Omega f(x, u^H(x))\, dx = \int_\Omega f(x, u(x))\, dx\,.$$

Furthermore, if $1 \leqslant p < \infty$, $u \in \mathrm{W}_0^{1,p}(\Omega)$ (resp. $(-u, H)$ is admissible and $u \in \mathrm{W}^{1,p}(\Omega)$) then $u^H \in \mathrm{W}_0^{1,p}(\Omega)$ (resp. $u^H \in \mathrm{W}^{1,p}(\Omega)$) and

$$\int_\Omega |\nabla u^H|^p\, dx = \int_\Omega |\nabla u|^p\, dx\,.$$

If $u \in \mathcal{K}(\Omega)$, then $u^H \in \mathcal{K}(\Omega)$ and for any $\delta > 0$,

$$\omega_{u^H}(\delta) \leqslant \omega_u(\delta)\,.$$

Proof. See [6, 30]. □

2.3. Approximating symmetrization

In order to study the approximations of a symmetrization by other symmetrizations we introduce a partial order $\prec$ on the symmetrizations such that $S \prec T$ if the symmetrization with respect to T can be used to approximate the symmetrization with respect to S.

Definition 2.19. Let $S, T \in \mathcal{S}$. We write $S \prec T$ if $S \subseteq T$ and $\partial S \subseteq \partial T$. For $S \in \mathcal{S}$, let

$$\mathcal{S}_S = \{T \in \mathcal{S} \;:\; S \prec T\}$$

and

$$\mathcal{H}_S = \{H \in \mathcal{H} \,:\, S \prec H\}\,.$$

This definition is justified by the next proposition.

Proposition 2.20. *Let $S, T \in \mathcal{S}$ and suppose $S \prec T$. If A is Borel measurable, then $A^{ST} = A^{TS} = A^S$.*

If $\Omega \subset \mathbf{R}^N$ and $u : \Omega \to \bar{\mathbf{R}}$ are Borel measurable, and (u, S) is admissible, then (u, T), (u^T, S) and (u^S, T) are admissible and $u^{ST} = u^{TS} = u^S$.

Proof. The definitions yields $A^{TS} = A^{ST} = A^S$ for any Borel measurable set $A \subset \mathbf{R}^N$. The conclusion follows from the definitions of the admissibility and of the symmetrization of a function. □

Remark 2.21. By Proposition 2.11, if $S \prec T$, then

$$\|u^T - u^S\|_p \leqslant \|u - u^S\|_p,$$

i.e. T does not increase the distance between u and u^S and T 0can be used to approximate S.

Remark 2.22. If A is merely measurable, its intersection with some affine subspace could be $\mathcal{H}^k$-non-measurable, resulting in $A^{TS} \supsetneq A^S = A^{ST}$. However, one can still conclude that $A^S \subset A^{TS}$ and that $\mathcal{L}^N(A^{TS} \backslash A^S) = 0$.

Many properties of the symmetrizations can be deduced from the next

Theorem 2.23. *Let $S \in \mathcal{S}$. There exists a sequence $(H_n)_{n\geqslant 1} \subset \mathcal{H}_S$ such that if $\Omega \subset \mathbf{R}^N$ is open, $u \in \mathcal{K}(\Omega)$ and (u, S) is admissible, then*

$$\|u^{H_1\ldots H_n} - u^S\|_\infty \to 0.$$

Proof. See [31]. □

Remark 2.24. Weaker forms of Theorem 2.23, where the sequence could depend on the function to symmetrize were proved by Brock and Solynin [6] and by Smets and Willem [23].

Corollary 2.25. *Let $S \in \mathcal{S}$ and $u \in \mathcal{K}(\Omega)$. If (u, S) is admissible, then $u^S \in \mathcal{K}(\Omega)$ and for any $\delta > 0$,*

$$\omega_{u^S}(\delta) \leqslant \omega_u(\delta).$$

Proof. This follows from Proposition 2.18 and Theorem 2.23. □

Among the consequences, there is the compactness of the set of functions obtained by symmetrizations compatible with a given symmetrization:

Proposition 2.26. *Let $S \in \mathcal{S}$, $\Omega \subset \mathbf{R}^N$ and $u \in \mathcal{K}(\Omega)$. If (u, S) is admissible, then*

$$\mathcal{U} = \left\{ u^{T_1 \dots T_n} \;:\; n \geqslant 1,\, T_i \in \mathcal{S}_S \text{ for each } 1 \leqslant i \leqslant n \right\}$$

is totally bounded in $\mathrm{L}^\infty(\Omega)$.

Proof. By Proposition 2.11, if $v \in \mathcal{U}$, then $\|v\|_\infty = \|u\|_\infty$. Since u is compactly supported, there exists $x \in \partial S$ ($x \in S$ if $u \geqslant 0$) and $r \geqslant 0$ such that $\operatorname{supp} u \subset \mathrm{B}(x, r)$. Since $S \prec T$, $\mathrm{B}(x, r)^T = \mathrm{B}(x, r)^{ST} = \mathrm{B}(x, r)^S = \mathrm{B}(x, r)$. By Proposition 2.6, for each $v \in \mathcal{U}$, one has $\operatorname{supp} v \subset \mathrm{B}(x, r)$. Finally, by Corollary 2.25, for every $v \in \mathcal{U}$, we have $v \in \mathcal{K}(\Omega)$ and

$$\omega_v(\delta) \leqslant \omega_u(\delta).$$

The conclusion comes from the Ascoli–Arzelá Theorem. □

Remark 2.27. In fact, $\mathcal{U}$ is totally bounded in $\mathrm{L}^p(\mathbf{R}^N)$ for every $1 \leqslant p \leqslant \infty$.

Proposition 2.26 is one of the ingredients of

Theorem 2.28. *Let $S \in \mathcal{S}$ and $\mathcal{T} \subset \mathcal{S}_S$. If for every $H \in \mathcal{H}_S$, there exists $T \in \mathcal{T}$ such that $T \prec H$, then there exists a sequence $(T_n)_{n \geqslant 1} \subset \mathcal{T}$ such that if $\Omega \subset \mathbf{R}^N$ is open, $u \in \mathcal{K}(\Omega)$ and (u, S) is admissible, then*

$$\|u^{T_1 \dots T_n} - u^S\|_\infty \to 0.$$

Proof. See [31]. □

Remark 2.29. For every $1 \leqslant p < \infty$, the convergence happens for any $u \in \mathrm{L}^p(\Omega)$ such that (u, S) is admissible.

2.4. The metric structure of $\mathcal{S}$

In order to construct other sequences of symmetrizations approximating a symmetrization by some kind of perturbation, we give a metric structure to the set $\mathcal{S}$. Since the definition of the metric on $\mathcal{S}$ relies on isometries of $\mathbf{R}^N$, we briefly investigate the relationship between symmetrizations and isometries. We call $i : \mathbf{R}^N \to \mathbf{R}^N$ an isometry provided that for every $x, y \in \mathbf{R}^N$, one has $|i(x) - i(y)| = |x - y|$.

Proposition 2.30. *Let $i : \mathbf{R}^N \to \mathbf{R}^N$ be an isometry and $S \in \mathcal{S}$. If $A \subset \mathbf{R}^N$, then $i(A^S) = i(A)^{i(S)}$. If $(u, i(S))$ is admissible, then $(u \circ i, S)$ is admissible, and $u^{i(S)} \circ i = (u \circ i)^S$.*

Proof. Since the definitions of the symmetrizations are invariant by isometry, this is straightforward. □

Remark 2.31. The isometries is the largest class of transformations of $\mathbf{R}^N$ for which Proposition 2.30 holds for every $S \in \mathcal{S}$.

We need also some information about elements of $\mathcal{S}$ which are identical in a ball.

Proposition 2.32. *There exist constants $K_1 > 1$ and $K_2 > 0$ that depend only on the dimension of the space N such that the following holds: Let $r \geqslant 0$, $R \geqslant K_1 r$, $S, T \in \mathcal{S}$, $x \in S$, and $u \in \mathcal{K}_+(\Omega)$. If (u, S) and (u, T) are admissible,* $\operatorname{supp} u \subset \mathrm{B}(x, r)$ *and* $\mathrm{B}(x, R) \cap S = \mathrm{B}(x, R) \cap T$, *then*

$$\|u^S - u^T\|_\infty \leqslant \omega_u(K_2 r^2/R)\,.$$

Proof. This follows from the next Lemma applied to $u|_{\mathrm{B}(x,r)}$ and from Proposition 2.10, since u^S and u^T are the extensions by 0 outside of $\mathrm{B}(x, r)$ of $(u|_{\mathrm{B}(x,r)})^S$ and $(u|_{\mathrm{B}(x,r)})^T$. □

Lemma 2.33. *There exist constants $K_1 > 1$ and $K_2 > 0$ that depend only on the dimension of the space N such that the following holds: Let $r \geqslant 0$, $R \geqslant K_1 r$, $S, T \in \mathcal{S}$, and $x \in S$. If $\mathrm{B}(x, R) \cap S = \mathrm{B}(x, R) \cap T$ then there exists an injective map $g : \mathrm{B}(0, r) \to \mathbf{R}^N$ such that for each $x \in \mathrm{B}(x, r)$, $|g(x) - x| \leqslant K_2 r^2/R$. Furthermore, for any $A \subset \mathrm{B}(x, r)$, $g(A^S) = g(A)^T$ and if $\Omega \subset \mathrm{B}(x, r)$, $u : \Omega \to \mathbf{R}$ and (u, T) is admissible, then $(u \circ g, S)$ is admissible and $u^T \circ g = (u \circ g)^S$.*

Remark 2.34. This was proved by Sarvas when $\dim S = N - 1$ [21].

Proof. If $\partial S \cap \mathrm{B}(x, R) = \partial T \cap \mathrm{B}(x, R) \neq \phi$ the proposition is trivial. The result is also trivial when $\dim S = \dim T = N$. Assume thus $\partial S \cap \mathrm{B}(x, R) = \partial T \cap \mathrm{B}(x, R) = \phi$ and $\dim S < N$. For any y, let C_{Sy} denote the circle that contains y, whose center is in ∂S and that is contained in an affine (two-dimensional) plane perpendicular to ∂S. If $\partial S = \phi$, define C_{Sy} to be the straight line perpendicular to S that contains y. Define C_{Ty} analogously.

The mapping g is the unique mapping such that if $y \in S \cap \mathrm{B}(x, r)$, $g(C_{Sy} \cap \mathrm{B}(x, r)) \subset C_{Tx}$, and if $A \subset C_{Sy} \cap \mathrm{B}(x, r)$ is Borel measurable, then $\mathcal{H}^{N-k}(A) = \mathcal{H}^{N-k}(g(A))$, where k is the dimension of S and of T. A direct computation shows that for sufficiently large K_1 and K_2, the map g has the required properties. □

Now we define a distance on $\mathcal{S}$.

Definition 2.35. Let $S, T \in \mathcal{S}$ and

$$\varrho(S,T) = \inf\Big\{\ln\Big(1 + \sup_{x\in\mathbf{R}^N} \frac{|x-i(x)|}{1+|x|} + \sup_{x\in i(S)\,\Delta\,T} \frac{1}{1+|x|}\Big) \ : \\ i : \mathbf{R}^N \to \mathbf{R}^N \text{ is an isometry}\Big\}.$$

The *distance* between S, T is

$$\mathrm{d}(S,T) = \varrho(S,T) + \varrho(T,S).$$

Proposition 2.36. *The pair $(\mathcal{S}, \mathrm{d})$ is a separable metric space.*

Remark 2.37. The metric space $(\mathcal{S}, \mathrm{d})$ is not complete, but it is locally compact.

The symmetrization is continuous with respect to this distance. More precisely,

Proposition 2.38. *Let $\Omega \subset \mathbf{R}^N$ be open. The mapping*

$$\{(u,S) \in (\mathcal{K}(\Omega), \|\cdot\|_\infty) \times (\mathcal{S}, \mathrm{d}) \ : \ (u,S) \text{ is admissible}\} \\ \to (\mathcal{K}(\Omega), \|\cdot\|_\infty) : (u,S) \mapsto u^S$$

is continuous.

Proof. Let $(u,S) \in (\mathcal{K}(\Omega), \|\cdot\|_\infty) \times (\mathcal{S}, \mathrm{d})$ be admissible, and let $\varepsilon > 0$. By Proposition 2.10, we can assume $\Omega = \mathbf{R}^N$.

First suppose $u \geqslant 0$. Let $(u,S) \in \mathcal{K}_+(\mathbf{R}^N) \times \mathcal{S}$ be admissible. Let K_1 and K_2 be given by Proposition 2.32. Fix $x \in S$ and $r \geqslant \varepsilon K_1/K_2$ such that $\operatorname{supp} u \in \mathrm{B}(x,r)$. There exists $\delta > 0$, depending only on ε, x and r, such that if $T \in \mathcal{S}$ and $\mathrm{d}(S,T) \leqslant \delta$, then there is an isometry $i : \mathbf{R}^N \to \mathbf{R}^N$ with $|y - i(y)| \leqslant \varepsilon$ for each $y \in \mathrm{B}(x,r)$ and $i(T) \cap \mathrm{B}(x, K_2 r^2/\varepsilon) = S \cap \mathrm{B}(x, K_2 r^2/\varepsilon)$. By Proposition 2.32, since $K_2 r^2/\varepsilon \geqslant K_1 r$, $\|u^S - u^{i(T)}\|_\infty \leqslant \omega_u(\varepsilon)$. Moreover, since by Proposition 2.30, $u^{i(T)} \circ i = (u \circ i)^T$,

$$\|u^{i(T)} - u^T\|_\infty = \|u^{i(T)} \circ i - u^T \circ i\|_\infty = \|(u\circ i)^T - u^T \circ i\|_\infty \\ \leqslant \|(u\circ i)^T - u^T\|_\infty + \|u^T - u^T \circ i\|_\infty.$$

Since the symmetrization is non-expansive in $\mathrm{L}^\infty(\mathbf{R}^N)$ by Proposition 2.11,

$$\|(u\circ i)^T - u^T\|_\infty \leqslant \|u \circ i - u\|_\infty \leqslant \omega_u(\varepsilon).$$

By Corollary 2.25, the modulus of continuity does not increase by symmetrization:

$$\|u^T - u^T \circ i\|_\infty \leqslant \omega_{Tu}(\varepsilon) \leqslant \omega_u(\varepsilon).$$

For any $(v, T) \in \mathcal{K}_+(\mathbf{R}^N) \times \mathcal{S}$, if $\mathrm{d}(T, S) \leqslant \delta$ and $\|u - v\|_\infty \leqslant \varepsilon$, then, by the non-expansiveness of the symmetrizations,

$$\|u^S - v^T\|_\infty \leqslant \|u^S - u^T\|_\infty + \|u^T - v^T\|_\infty \leqslant 3\omega_u(\varepsilon) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, our claim is proved.

If $u \not\geqslant 0$, then by definition of admissibility, $\partial S \neq \phi$. Let $x \in \partial S$ and choose $r > 0$ such that $\operatorname{supp} u \subset \mathrm{B}(x, r)$. By definition of d, there is $\delta > 0$ such that if $d(S, T) \leqslant \delta$, there exists an isometry $i : \mathbf{R}^N \to \mathbf{R}^N$ such that $|y - i(y)| \leqslant \varepsilon$ for $y \in \mathrm{B}(x, r)$ and $i(T) \cap \mathrm{B}(x, r) = S \cap \mathrm{B}(x, r)$. Since $x \in \partial S$, S and T are closed affine half subspaces, and i is an isometry, $i(T) = S$. By Proposition 2.30,

$$\|u^S - u^T\|_\infty = \|u^{i(T)} \circ i - u^T \circ i\|_\infty = \|(u \circ i)^T - u^T \circ i\|_\infty.$$

The end of the proof is similar to the case when $u \geqslant 0$. □

Corollary 2.39. *Let $\Omega \subset \mathbf{R}^N$ be open and $1 \leqslant p < \infty$. The mapping*

$$\Big\{(u, S) \in (\mathrm{L}^p(\Omega), \|\cdot\|_p) \times (\mathcal{S}, \mathrm{d}) \ :\ (u, S) \text{ is admissible}\Big\} \to (\mathrm{L}^p(\Omega), \|\cdot\|_p) : (u, S) \mapsto u^S$$

is continuous.
This remains true if $p = \infty$, provided $\mathrm{L}^p(\Omega)$ is replaced by $C_0(\Omega)$.

As in [30], we can obtain the

Corollary 2.40. *Let $\Omega \subset \mathbf{R}^N$ be open and $1 < p < \infty$. The mapping*

$$\Big\{(u, H) \in \mathrm{W}^{1,p}(\Omega) \times (\mathcal{H}, \mathrm{d}) \ :\ (u, H) \text{ and } (-u, H) \text{ are admissible}\Big\} \to \mathrm{W}^{1,p}(\Omega) : (u, H) \mapsto u^H$$

is continuous.

Proof. This is a consequence of Proposition 2.18, of Corollary 2.39 and of the uniform convexity of the norm $\|\nabla u\|_p$. □

3. Constructing approximating sequences

3.1. A sufficient condition

Since the result of a symmetrization is stable under small perturbations on the symmetrization (Proposition 2.38), we can prove that some perturbations of an approximating sequence are approximating sequences.

Proposition 3.1. *Let $S \in \mathcal{S}$, $(S_n)_{n\geqslant 1} \subset \mathcal{S}_S$ and $(T_n)_{n\geqslant 1} \subset \mathcal{S}_S$. If for each open set $\Omega \subset \mathbf{R}^N$ and $u \in \mathcal{K}(\Omega)$ such that (u, S) is admissible,*

$$\|u^{S_1 \dots S_n} - u^S\| \to 0, \qquad \text{as } n \to \infty,$$

and if for every $\delta > 0$ and $m \geqslant 1$, there exists $k \geqslant 0$ such that for each $1 \leqslant i \leqslant m$,

$$\mathrm{d}(S_i, T_{k+i}) \leqslant \delta,$$

then for each open set $\Omega \subset \mathbf{R}^N$ and $u \in \mathcal{K}(\Omega)$ such that (u, S) is admissible,

$$\|u^{T_1 \dots T_n} - u^S\| \to 0, \qquad \text{as } n \to \infty.$$

Proof. Let $u \in \mathcal{K}(\Omega)$ and $\varepsilon > 0$. Since by Proposition 2.26, the sequence $(u^{T_1 \dots T_n})_{n\geqslant 1}$ is totally bounded in $\mathrm{L}^\infty(\Omega)$ and since by hypothesis

$$u^{T_1 \dots T_n S_1 \dots S_m} \to u^S, \qquad \text{as } m \to \infty,$$

there exists $m \geqslant 1$ such that for every $n \geqslant 0$,

$$\|u^{T_1 \dots T_n S_1 \dots S_m} - u^S\|_\infty \leqslant \varepsilon.$$

By the continuity of symmetrization (Proposition 2.38) and the fact that $(u^{T_1 \dots T_n})_{n\geqslant 1}$ is totally bounded, there exists $\delta > 0$ such that for each $1 \leqslant i \leqslant m$, for each $n \geqslant 0$ and for each $T \in \mathcal{S}_S$, if $\mathrm{d}(S_i, T) \leqslant \delta$, then

$$\|u^{T_1 \dots T_n S_i} - u^{T_1 \dots T_n T}\|_\infty \leqslant \varepsilon / m.$$

By hypothesis, there is $k \geqslant 0$ such that for each $1 \leqslant i \leqslant m$, $\mathrm{d}(S_i, T_{k+i}) \leqslant \delta$. We can then use the non-expansiveness of symmetrizations (Proposition 2.11) and the preceding estimates to obtain, for every $\ell \geqslant m + k$,

$$\begin{aligned}
&\|u^S - u^{T_1 \dots T_\ell}\|_\infty \leqslant \|u^S - u^{T_1 \dots T_{m+k}}\|_\infty \\
&\leqslant \|u^S - u^{T_1 \dots T_k S_1 \dots S_m}\|_\infty + \sum_{i=1}^m \|u^{T_1 \dots T_{k+i-1} S_i \dots S_m} - u^{T_1 \dots T_{k+i} S_{i+1} \dots S_m}\| \\
&\leqslant \|u^S - u^{T_1 \dots T_k S_1 \dots S_m}\|_\infty + \sum_{i=1}^m \|u^{T_1 \dots T_{k+i-1} S_i} - u^{T_1 \dots T_{k+i}}\| \leqslant 2\varepsilon. \quad \square
\end{aligned}$$

Theorem 3.2. *Let $S \in \mathcal{S}$, $\mathcal{T} \subset \mathcal{S}_S$ and $(T_n)_{n\geqslant 1} \subset \mathcal{S}_S$ be such that*

a) for every $H \in \mathcal{H}_S$, there exists $T \in \mathcal{T}$ such that $T \prec H$,

b) for each $m \geqslant 1$ and $S_1, \dots, S_m \in \mathcal{T}$, there exists $k \geqslant 0$ such that for every $1 \leqslant i \leqslant m$, $\mathrm{d}(S_i, T_{k+i}) \leqslant \delta$,

Then for each open set $\Omega \subset \mathbf{R}^N$ and $u \in \mathcal{K}(\Omega)$ such that (u, S) is admissible,

$$\|u^{T_1 \dots T_n} - u^S\|_\infty \to 0, \qquad \text{as } n \to \infty.$$

Remark 3.3. Since $(\mathcal{S}, \mathrm{d})$ is separable, $(\mathcal{T}, \mathrm{d})$ is also separable so that given a countable dense set of $\mathcal{T}$ it is possible to construct explicitly a sequence $(T_n)_{n\geqslant 1}$ satisfying the hypotheses of Theorem 3.2.

Proof. This follows from Theorem 2.28 and Proposition 3.1. □

3.2. Random sequences of symmetrizations

As a first application of Theorem 3.2, we prove that symmetrizations can be approximated by random sequences of symmetrizations.

Recall that if (E, Σ, P) is a probability space, (M, d) is a metric space and $X : E \to M$ is measurable, then X is called a random variable. The sequence $(X_n)_{n\geqslant 1}$ is a sequence of independent random variables if for any $n \geqslant 1$ and for any open sets $U_1, \dots, U_n \subset M$,

$$P(\{e \in E \;:\; (X_1(e), \dots, X_n(e)) \in U_1 \times \cdots \times U_n\}) \\ = \prod_{i=1}^{n} P(\{e \in E \;:\; X_i(e) \in U_i\}).$$

(See e.g. Stromberg [24].)

Theorem 3.4. *Let $S \in \mathcal{S}$, $\mathcal{T} \subset \mathcal{S}_S$, (E, Σ, P) be probability space and $T_n : E \to \mathcal{T}$, $n \geqslant 1$, be independent random variables. If for every $H \in \mathcal{H}_S$, there exists $T \in \mathcal{T}$ such that $T \prec H$ and if for each $T \in \mathcal{T}$ and $\delta > 0$,*

$$\varliminf_{n\to\infty} P\bigl(\{e \in E \;:\; \mathrm{d}(T_n(e), T) \leqslant \delta\,\}\bigr) > 0,$$

then

$$P\Bigl(\Bigl\{e \in E \;:\; \forall \text{ open set } \Omega \subset \mathbf{R}^N, \\ \forall u \in \mathcal{K}(\Omega) \text{ such that } (u, S) \text{ is admissible}, \\ \lim_{n\to\infty} \|u^{T_1(e)\dots T_n(e)} - u^S\| = 0\Bigr\}\Bigr) = 1.$$

Proof. This follows from Theorem 3.2 and from the next Lemma, since $(\mathcal{T}, \mathrm{d})$ is a separable metric spaces by Proposition 2.36. □

Lemma 3.5. *Let (E, Σ, P) be a probability space, (M, d) be a separable metric space and $X_n : E \to M$, $n \geqslant 1$, be independent random variables. If for each $x \in M$ and $\delta > 0$,*

$$\varliminf_{n\to\infty} P(\{e \in E \;:\; \mathrm{d}(X_n(e), x) \leqslant \delta\}) > 0,$$

then

$$P\big(\big\{e \in E \,:\, \forall m \geqslant 1, \forall r \geqslant 1, \forall x_1, \ldots, x_m \in M,\\ \exists k \geqslant 0, \forall 1 \leqslant i \leqslant m, \mathrm{d}(X_{k+i}(e), x_i) \leqslant 1/r\big\}\big) = 1.$$

Proof. Since M is separable, there exists a countable dense subset $D \subset M$. Since D is dense,

$$\begin{aligned} &P\big(\big\{e \in E \,:\, \forall m \geqslant 1, \forall r \geqslant 1, \forall x_1, \ldots, x_m \in M,\\ &\qquad \exists k \geqslant 0, \forall 1 \leqslant i \leqslant m, \mathrm{d}(X_{k+i}(e), x_i) \leqslant 1/r\big\}\big)\\ &= P\big(\big\{e \in E \,:\, \forall m \geqslant 1, \forall r \geqslant 1, \forall x_1, \ldots, x_m \in D,\\ &\qquad \exists k \geqslant 0, \forall 1 \leqslant i \leqslant m, \mathrm{d}(X_{k+i}(e), x_i) \leqslant 1/r\big\}\big)\\ &= 1 - P\big(\big\{e \in E \,:\, \exists m \geqslant 1, \exists r \geqslant 1, \exists x_1, \ldots, x_m \in D,\\ &\qquad \forall k \geqslant 0, \exists 1 \leqslant i \leqslant m, \mathrm{d}(X_{k+i}(e), x_i) > 1/r\big\}\big). \end{aligned}$$

Since D is countable,

$$\begin{aligned} &P\big(\big\{e \in E \,:\, \exists m \geqslant 1, \exists r \geqslant 1, \exists x_1, \ldots, x_m \in D,\\ &\qquad \forall k \geqslant 0, \exists 1 \leqslant i \leqslant m, \mathrm{d}(X_{k+i}(e), x_i) > 1/r\big\}\big)\\ &\leqslant \sum_{\substack{m \geqslant 1\\ r \geqslant 1}} \sum_{x_1, \ldots, x_n \in D} P(\{e \in E \,:\, \forall k \geqslant 0, \exists 1 \leqslant i \leqslant m,\\ &\qquad \mathrm{d}(X_{k+i}(e), x_i) > 1/r\}). \end{aligned}$$

Let now m, r and $x_1, \ldots, x_m \in D$ be fixed. Since the random variables $(X_n)_{n \geqslant 1}$ are independent,

$$\begin{aligned} &P(\{e \in E \,:\, \forall k \geqslant 0, \exists 1 \leqslant i \leqslant m, \mathrm{d}(X_{k+i}(e), x_i) > 1/r\})\\ &\leqslant P(\{e \in E \,:\, \forall \ell \geqslant 0, \exists 1 \leqslant i \leqslant m, \mathrm{d}(X_{\ell m+i}(e), x_i) > 1/r\})\\ &= \prod_{\ell \geqslant 0} P(\{e \in E \,:\, \exists 1 \leqslant i \leqslant m, \mathrm{d}(X_{\ell m+i}(e), x_i) > 1/r\}). \end{aligned}$$

Since by hypothesis

$$\begin{aligned}
&\varlimsup_{\ell\to\infty} P(\{e \in E \,:\, \exists 1 \leqslant i \leqslant m, \mathrm{d}(X_{\ell m+i}(e), x_i) > 1/r\}) \\
&\qquad = 1 - \varliminf_{\ell\to\infty} \prod_{i=1}^{m} P(\{e \in E \,:\, \mathrm{d}(X_{\ell m+i}(e), x_i) \leqslant 1/r\}) \\
&\qquad \leqslant 1 - \prod_{i=1}^{m} \varliminf_{\ell\to\infty} P(\{e \in E \,:\, \mathrm{d}(X_{\ell m+i}(e), x_i) \leqslant 1/r\}) \\
&\qquad \leqslant 1 - \prod_{i=1}^{m} \varliminf_{n\to\infty} P(\{e \in E \,:\, \mathrm{d}(X_n(e), x_i) \leqslant 1/r\}) < 1,
\end{aligned}$$

the conclusion follows. □

3.3. Approximation of the symmetrization of sets

Proposition 3.6. *Let u, $v \in C(\Omega)$, $S \in \mathcal{S}$, $c > 0$. If (u, S) and (v, S) are admissible and*

$$\{x \in \Omega \,:\, u(x) \geqslant c\} = \{x \in \Omega \,:\, v(x) \geqslant c\}\,,$$

then

$$\left\{x \in \Omega \,:\, u^S(x) \geqslant c\right\} = \left\{x \in \Omega \,:\, v^S(x) \geqslant c\right\}.$$

Definition 3.7. Let $K \subset \mathbf{R}^N$ be compact and $\mathcal{S}$. The *compact symmetrization* of K with respect to S is the set

$$\{x \,:\, u(x) \geqslant 1\}$$

for any function $u \in \mathcal{K}(\mathbf{R}^N)$, such that $u \leqslant 1$ and $u(x) = 1$ if and only if $x \in K$.

This definition is equivalent to the classical definitions of symmetrization of compact sets [6, 19]. By an abuse of notation, throughout this section, if K is compact, then K^S denotes the compact symmetrization of K. We recall some basic facts about the Hausdorff distance [14, 15].

Definition 3.8. Let K_1, $K_2 \subset \mathbf{R}^N$ be compact sets. The Hausdorff distance between K_1 and K_2 is

$$\begin{aligned}
&\mathrm{d}_H(K_1, K_2) \\
&\qquad = \inf\{r > 0 \,:\, K_1 \subseteq K_2 + \mathrm{B}(0, r) \text{ and } K_2 \subseteq K_1 + \mathrm{B}(0, r)\}\,.
\end{aligned}$$

The set of compact subsets of $\mathbf{R}^N$ is denoted by $\mathfrak{K}(\mathbf{R}^N)$. The metric space $(\mathfrak{K}(\mathbf{R}^N), \mathrm{d}_H)$ is complete. One has

Proposition 3.9. *Let $\mathfrak{A} \subset \mathfrak{K}(\mathbf{R}^N)$. The following are equivalent*

(1) $\mathfrak{A}$ *is totally bounded,*
(2) $\cup_{K \in \mathfrak{A}} K$ *is bounded,*
(3) $\mathfrak{A}$ *is bounded.*

We are now in measure to prove how approximation of symmetrizations of functions yields approximations of the symmetrizations of sets.

Proposition 3.10. *Let* $S \in \mathcal{S}$, $(T_n)_{n \geqslant 1} \subset \mathcal{S}_S$, $u \in \mathcal{K}_+(\mathbf{R}^N)$ *such that* $\|u\|_\infty = 1$ *and* $K = \{x \in \mathbf{R}^N \,:\, u(x) = 1\}$. *If* $\|u^{T_1 \dots T_n} - u^S\|_\infty \to 0$ *as* $n \to \infty$, *then*

$$\mathrm{d}_H(K^{T_1 \dots T_n}, K^S) \to 0, \qquad \text{as } n \to \infty.$$

Remark 3.11. By Tietze's extension Theorem, for every $K \in \mathfrak{K}(\mathbf{R}^N)$, there exists $u \in \mathcal{K}_+(\mathbf{R}^N)$ such that $\|u\|_\infty = 1$ and

$$K = \left\{x \in \mathbf{R}^N \,:\, u(x) = 1\right\}.$$

Proof. Since u is compactly supported, there exists $x \in S$ and $r \geqslant 0$ such that $\operatorname{supp} u \subset \mathrm{B}(x, r)$. Hence $K^{T_1 \dots T_n} \subset \operatorname{supp} u^{T_1 \dots T_n} \subset \mathrm{B}(x, r)$. By Proposition 3.9 the sequence $(K^{T_1 \dots T_n})_{n \geqslant 1}$ is conditionally compact in $(\mathfrak{K}(\mathbf{R}^N), \mathrm{d}_H)$.

Let $\tilde{K}$ be an accumulation point of the sequence $(K^{T_1 \dots T_n})_{n \geqslant 1}$, let $(K_m)_{m \geqslant 1}$ be a subsequence of $(K^{T_1 \dots T_n})_{n \geqslant 1}$ converging to $\tilde{K}$ and let $(u_m)_{m \geqslant 1}$ denote the corresponding subsequence of $(u^{T_1 \dots T_n})_{n \geqslant 1}$. We are going to show that $\tilde{K} = K^S$.

Let $\varrho > 0$. Since by Corollary 2.25, $u^S \in \mathcal{K}(\mathbf{R}^N)$, there exists $\varepsilon > 0$ such that if $u^S(x) \geqslant 1 - \varepsilon$, there is $y \in K^S$ with $|x - y| < \varrho$. Since $u_m \to u^S$ in $\mathrm{L}^\infty(\mathbf{R}^N)$, for sufficiently large m, $\|u_m - u^S\| \leqslant \varepsilon$. By definition of K_m, one has $K_m \subset K^S + \mathrm{B}(0, \varrho)$. Since this is valid for any $\varrho > 0$, we conclude that $\tilde{K} \subseteq K^S$.

For every $x \in S \setminus \partial S$, let C_x denote the $(N-k)$–dimensional sphere that has its center on ∂S, is contained in an affine plane orthogonal to ∂S and contains the point x. (If $\partial S = \phi$, then C_x is the $(N-k)$–dimensional plane orthogonal to S that contains the point x.) If $K \cap C_x = \phi$, then $K^S \cap C_x = \phi \subset \tilde{K} \cap C_x$. If $K \cap C_x \neq \phi$, then $\tilde{K} \cap C_x \neq \phi$, the set $K^S \cap C_x$ is a closed geodesic ball (possibly degenerate to a point), and, since the $N-k$-dimensional Hausdorff measure restricted to C_x is a Radon measure, it is upper semicontinuous with respect to the Hausdorff distance [4]

$$\begin{aligned}\mathcal{H}^{N-k}(\tilde{K} \cap C_x) \geqslant \varlimsup_{m \to \infty} \mathcal{H}^{N-k}(K_m \cap C_x) \\ = \mathcal{H}^{N-k}(K \cap C_x) = \mathcal{H}^{N-k}(K^S \cap C_x).\end{aligned}$$

Since $\tilde{K} \cap C_x \subseteq K^S \cap C_x$, one concludes that $\tilde{K} \cap C_x = K^S \cap C_x$.

Since $K_m \cap \partial S = K \cap \partial S = K^S \cap \partial S$, one has $K^S \cap \partial S \subseteq \tilde{K} \cap \partial S$. In view of $\mathbf{R}^N = \partial S \cap \cup_{x \in S \setminus \partial S} C_x$, one has

$$\tilde{K} = K^S.$$

This proves that the set K^S is the unique accumulation point of the sequence $(K^{T_1 \dots T_n})_{n \geqslant 1}$. □

Remark 3.12. The proof of Proposition 3.10 is a simplification of a proof of Brock and Solynin [6], who did not use the compactness of the sequence $(K^{T_1 \dots T_n})_{n \geqslant 1}$ in $\mathfrak{K}(\mathbf{R}^N)$. In particular, the proof of the inclusion $\tilde{K} \subset K^S$ is directly inspired by their proof.

As an easy consequence of Theorem 3.4 and Proposition 3.10, we have

Theorem 3.13. *Let $S \in \mathcal{S}$ with $\partial S = \phi$ and let (E, Σ, P) be a probability space. Let $\ell > \dim S$ and*

$$\mathcal{T}_S^\ell = \{T \in \mathcal{S}_S \;:\; \partial T = \phi \textit{ and } \dim T = \ell\}.$$

If $(T_n)_{n \geqslant 1}$ are independent random variables with values in $\mathcal{T}_S^\ell$ whose distribution functions are invariant under isometries that preserve S, then

$$P\big(\{e \in E \;:\; \forall K \in \mathfrak{K}(\mathbf{R}^N),\ \lim_{n \to \infty} \mathrm{d}_H(K^{T_1(e) \dots T_n(e)}, K^S) = 0\}\big) = 1.$$

This solves a conjecture of Mani-Levitska. He proved Theorem 3.13 under the additional assumptions that K should be convex, $S = \{0\}$ and $\ell = N - 1$ [17].

One can obtain similar theorems for the approximation by polarizations or spherical cap symmetrizations.

4. Symmetry of critical points

This section is devoted to the proof of a symmetry result concerning critical points obtained by a minimax theorem of Struwe based on the Krasnoselskii genus [26]. First we recall the definition and basic properties of the Krasnoselskii genus (section 4.1). Then we symmetrize approximately sets of small Krasnoselskii genus (section 4.2) before going on to a minimax theorem with symmetry information and an application (section 4.3).

4.1. Krasnoselskii genus

Let V be a Banach space. Define

$$\mathcal{A} = \{A \subset V \ : \ A \text{ is closed}, \ A = -A\}\,.$$

Definition 4.1. For $A \in \mathcal{A}, A \neq \phi$, let

$$\gamma(A) = \inf\left\{m \ : \ \text{there exists } h \in C(A, S^{m-1}) \ : \ h(-u) = h(u)\right\},$$

with $\gamma(A) = \infty$ if the set on the right-hand side is empty and $\gamma(\phi) = 0$.

The genus has the following properties

Proposition 4.2 (Krasnoselskii [13]). *Let A, A_1, $A_2 \in \mathcal{A}$, and let $h \in C(V, V)$ be an odd map. Then the following hold*

(1) $\gamma(A) \geqslant 0$, $\gamma(A) = 0$ *if and only if* $A = \phi$,
(2) *if* $A_1 \subset A_2$, *then* $\gamma(A_1) \leqslant \gamma(A_2)$,
(3) $\gamma(A_1 \cup A_2) \leqslant \gamma(A_1) + \gamma(A_2)$,
(4) $\gamma(A) \leqslant \gamma(\overline{h(A)})$,
(5) *if* $A \in \mathcal{A}$ *is compact and* $0 \notin A$, *then* $\gamma(A) < \infty$ *and there is a neighborhood* N *of* A *such that* $\bar{N} \in \mathcal{A}$ *and* $\gamma(A) = \gamma(\bar{N})$.

It will be only possible to symmetrize sets with a small Krasnoselskii genus. In the following proposition it is shown that any set contains a subset of lower Krasnoselskii genus that contains some prescribed points.

Lemma 4.3. *If $A \in \mathcal{A}$ and if $Y \subset A$ is finite, there exists $A' \in \mathcal{A}$ such that $Y \subset A' \subset A$ and $\gamma(A') = \gamma(A) - 1$.*

Proof. Let $k = \gamma(A)$. By definition of $\gamma(A)$, there exists an odd mapping $h \in C(A, S^{k-1})$. Take $m \in S^{k-1} \setminus h(Y)$ and let $\eta = \max_{y \in Y} |m \cdot h(y)|$. Since $m \notin h(Y)$, one has $\eta < 1$. Define

$$A' = \{x \in A \ : \ |m \cdot h(x)| \leqslant \eta\}\,.$$

Since h is odd and continuous, $A' \in \mathcal{A}$. For $x \in A'$, let $\sigma(x) = h(x) - (m \cdot h(x))m$ and $\hat{h}(x) = \sigma(x)/|\sigma(x)|$. It is clear that $\hat{h}$ is odd and continuous on A' and that $\hat{h}(A') \subset S^{k-2}$. Hence, $\gamma(A') \leqslant \gamma(A) - 1$.

Let $l = \gamma(A')$. By definition of $\gamma(A')$, there exists an even mapping $h' \in C(A', S^{l-1})$. For $x \in A$, let

$$\tilde{h}(x) = \begin{cases} ((\eta - |m \cdot h(x)|)h'(x), m \cdot h(x)) & \text{if } x \in A', \\ (0, m \cdot h(x)) & \text{if } x \notin A'. \end{cases}$$

Then $\tilde{h} : A \to \mathbf{R}^{l+1}$ is continuous and odd on A. The function $\bar{h} = \tilde{h}/|\tilde{h}| : A \to S^l$ is also continuous and odd. Hence $\gamma(A) \leqslant \gamma(A') + 1$. $\square$

4.2. Almost-symmetrization of sets

Throughout this section we assume that $\Omega = \Omega' \times \Omega''$, where $\Omega' \subset \mathbf{R}^k$ is invariant under the action of the group of isometries $O(k)$. To every any $x \in S^{k-1}$, we associate the closed affine half subspace $S_x = \mathbf{R}x \times \mathbf{R}^{N-k}$ and a closed affine halfspace $\zeta(x) = \big\{y \in \mathbf{R}^N \ : \ x \cdot y \geqslant 0\big\}$.

Proposition 4.4. *The map*

$$\zeta : \mathcal{S}^{k-1} \to \Big\{H \in \mathcal{H} \ : \ \{0\} \times \mathbf{R}^{N-k} \subset \partial H\Big\}$$

is a homeomorphism.
For every $x,\, y \in \mathcal{S}^{k-1}$, $\zeta(x) \in \mathcal{H}_{S_x}$ if and only if $x \cdot y \geqslant 0$.

Lemma 4.5. *There exists $\bar\sigma \in C(\mathrm{W}^{1,p}(\Omega) \times S^{k-1} \times \mathbf{R}^+; \mathrm{W}^{1,p}(\Omega))$ such that*

1. *(1) for every $u \in \mathrm{W}^{1,p}(\Omega)$, $\bar\sigma(u,x,t) \to u^{S_x}$ in $\mathrm{L}^p(\Omega)$ as $t \to \infty$, uniformly in $x \in S^{k-1}$,*
2. *(2) for every $(x,t) \in S^{k-1} \times \mathbf{R}^+$, there exists $H_1, \dots, H_{\lfloor t\rfloor+1} \in \mathcal{H}_{S_x}$ such that, for each $u \in \mathrm{W}^{1,p}(\Omega)$,*
$$\bar\sigma(u,x,t) = u^{H_1 \dots H_{\lfloor t\rfloor+1}},$$
3. *(3) for every $(u,x,t) \in \mathrm{W}^{1,p}(\Omega) \times S^{k-1} \times \mathbf{R}^+$,*
$$\bar\sigma(-u,-x,t) = -\bar\sigma(u,x,t).$$

Proof. Let $\mathcal{R} = \big\{R \in SO(k) \ : \ \forall x \in R^k,\, x \cdot R(x) \geqslant 0\big\}$. With the operator norm, $\mathcal{R}$ is a separable metric space. Consider a sequence $(R_n)_{n\geqslant 1} \subset \mathcal{R}$ such that for every $\delta > 0$, $m \geqslant 1$ and $Q_1, \dots, Q_m \in \mathcal{R}$, there exists $k \geqslant 0$ such that for each $1 \leqslant i \leqslant m$,

$$\|Q_i - R_{k+i}\| \leqslant \delta.$$

This construction is possible because $\mathcal{R}$ is separable. Since $\mathcal{R}$ is path-connected it is possible to extend the definition of R_t for $t \in \mathbf{R}^+$ so that $t \mapsto R_t$ is continuous. For $(u,x,t) \in \mathrm{W}^{1,p}(\Omega) \times S^{k-1} \times \mathbf{R}^+$, let

$$\bar\sigma(u,x,t) = u^{\zeta(R_1(x)) \dots \zeta(R_{\lfloor t\rfloor}(x))\zeta(R_t(x))}.$$

The map $\bar\sigma$ is continuous by construction of R_t, by Proposition 4.4 and by Corollary 2.40.

Fix $x \in S^{k-1}$. Let $\delta > 0$, $m \geqslant 1$ and $y_1, \dots, y_m \in S^{k-1}$ such that $x \cdot y_i \geqslant 0$ for each $1 \leqslant i \leqslant m$. For every $1 \leqslant i \leqslant m$, there exists $Q_i \in \mathcal{R}$ such that $Q_i(x) = y_i$. By construction of the sequence $(R_n)_{n\geqslant 1}$ there is $k \geqslant 0$ such that for every $1 \leqslant i \leqslant m$,

$$|y_i - R_{k+i}(x)| \leqslant \|Q_i - R_{k+i}\| \leqslant \delta.$$

Since ζ is continuous and $\zeta(R_n(x)) \in \mathcal{S}_{S_x}$, Theorem 3.2 is applicable and for every $(u,x) \in \mathrm{W}^{1,p}(\Omega) \times S^k$, we obtain

$$\|\bar{\sigma}(u,x,n) - u^{S_x}\|_p \to 0, \qquad \text{as } n \to \infty.$$

Since $\|\bar{\sigma}(u,x,n) - u^{S_x}\|_p$ is decreasing with respect to n (Remark 2.21), $\|\bar{\sigma}(u,x,n) - u^{S_x}\|_p$ is continuous with respect to x (Corollary 2.39) and S^{k-1} is compact, by Dini's Lemma [25], for every $u \in \mathrm{W}^{1,p}(\Omega)$, we obtain

$$\|\bar{\sigma}(u,x,n) - u^{S_x}\|_p \to 0, \qquad \text{as } n \to \infty, \text{ uniformly in } x \in S^{k-1}.$$

Finally by Proposition 2.11, we conclude

$$\|\bar{\sigma}(u,x,t) - u^{S_x}\|_p \leqslant \|\bar{\sigma}(u,x,\lfloor t \rfloor) - u^{S_x}\|_p \to 0,$$

as $t \to \infty$, uniformly in $x \in S^{k-1}$.

The last conclusion is a consequence of Proposition 2.14. □

Lemma 4.6. *For every $\varepsilon > 0$, there exists*

$$\tilde{\sigma} \in C(\mathrm{W}^{1,p}(\Omega) \times S^{k-1}; \mathrm{W}^{1,p}(\Omega))$$

such that for every $(u,x) \in \mathrm{W}^{1,p}(\Omega) \times S^{k-1}$

(1) $\|\tilde{\sigma}(u,x) - u^{S_x}\| < \varepsilon$,

(2) there exists $m \geqslant 1$ and $H_1, \dots, H_m \in \mathcal{H}_{S_x}$ such that

$$\tilde{\sigma}(u,x) = u^{H_1 \dots H_m},$$

(3) $\tilde{\sigma}(-u,-x) = -\tilde{\sigma}(u,x)$.

Proof. By the previous lemma, for any $u \in \mathrm{W}^{1,p}(\Omega)$, there exists $t_u \geqslant 0$ such that for every $t \geqslant t_u$ and $x \in S^{k-1}$,

$$\|\bar{\sigma}(u,t,x) - u^{S_x}\| \leqslant \varepsilon/3.$$

The space $\mathrm{W}^{1,p}(\Omega)$ with the norm of $\mathrm{L}^p(\Omega)$ is a metric space. It is thus paracompact and there is a locally finite partition of the unity $(\varrho_v)_{v \in \mathrm{W}^{1,p}}$ subordinate to the covering $\{\mathrm{B}(u,\varepsilon/3)\}_{u \in \mathrm{W}^{1,p}(\Omega)}$ [22]. For every $u \in \mathrm{W}^{1,p}(\Omega)$, let

$$\theta(u) = \frac{1}{2} \sum_{v \in \mathrm{W}^{1,p}(\Omega)} (\varrho_v(u) + \varrho_v(-u)) t_v.$$

It is clear that θ is continuous and even. For $(u,x) \in \mathrm{W}^{1,p}(\Omega) \times S^k$, let

$$\tilde{\sigma}(u,x) = \bar{\sigma}(u,x,\theta(u)).$$

For every $u \in \mathrm{W}^{1,p}(\Omega)$, there exists $v \in \mathrm{W}^{1,p}$ such that $t_v \leqslant \theta(u)$ and either $\|v-u\|_p \leqslant \varepsilon/3$, or $\|v-(-u)\|_p \leqslant \varepsilon/3$. If $\|v-(-u)\|_p \leqslant \varepsilon/3$, then using successively Proposition 2.14, Proposition 2.11 and the properties of v, we obtain

$$\begin{aligned}
&\|\tilde{\sigma}(u,x) - u^{S_x}\|_p \\
&\qquad = \|\bar{\sigma}(u,x,\theta(u)) - u^{S_x}\|_p = \|\bar{\sigma}(-u,-x,\theta(u)) - (-u)^{S_{-x}}\|_p \\
&\qquad\qquad \leqslant \|\bar{\sigma}(-u,-x,\theta(u)) - \bar{\sigma}(v,-x,\theta(u))\|_p \\
&\qquad\qquad\qquad + \|\bar{\sigma}(v,-x,\theta(u)) - v^{S_{-x}}\|_p + \|v^{S_{-x}} - (-u)^{S_{-x}})\|_p \leqslant \varepsilon.
\end{aligned}$$

Similarly $\|\tilde{\sigma}(u,x) - u^{S_x}\|_p \leqslant \varepsilon$ whenever $\|v-u\|_p \leqslant \varepsilon/3$.

The other conclusions follow easily from the properties of $\bar{\sigma}$. □

Proposition 4.7. *Let $A \subset \mathrm{W}^{1,p}(\Omega)$. If there exists an odd mapping $h \in C(A, S^{k-1})$, then for every $\varepsilon > 0$, there exists $\sigma \in C(A, \mathrm{W}^{1,p}(\Omega))$ such that for every $u \in A$*

(1) $\|\sigma(u) - u^{S_{h(x)}}\| < \varepsilon$,

(2) there exists $m \geqslant 1$ and $H_1, \ldots, H_m \in \mathcal{H}_{S_x}$ such that

$$\sigma(u) = u^{H_1 \ldots H_m},$$

(3) $\sigma(-u) = -\sigma(u)$.

Proof. For every $u \in A$, let $\sigma(u) = \tilde{\sigma}(u, h(u))$, where $\tilde{\sigma}$ is given by the previous lemma. The properties of σ follow from the properties of $\tilde{\sigma}$ and h. □

4.3. Minimax theorem with symmetry information

If φ is an even functional of class C^1 on a closed symmetric $C^{1,1}$-submanifold M of the Banach space V. For any $\ell \leqslant \gamma(M)$,

$$\mathcal{F}_\ell = \{A \in \mathcal{A} \ : \ A \subset M, \gamma(A) \geqslant \ell\}.$$

Consider the values

$$\beta_\ell = \inf_{A \in \mathcal{F}_\ell} \sup_{u \in A} \varphi(u).$$

If the functional φ satisfies the Palais–Smale condition at the level β_ℓ and

$$1 \leqslant \ell \leqslant \hat{\gamma}(M) = \sup\{\gamma(K) \ : \ K \subset M \text{ is compact and symmetric}\}$$

then there is a critical point $u \in M$ such that $\varphi(u) = \beta_\ell$ [26].

Theorem 4.8. *Let $\Omega = \Omega' \times \Omega'' \subset \mathbf{R}^N$ be open, with $\Omega' \subset \mathbf{R}^k$ invariant under $O(k)$. Let $\ell \leqslant k$. Let $M \subset \mathrm{W}^{1,p}(\Omega) \setminus \{0\}$ be a complete symmetric $C^{1,1}$-manifold. Suppose $\varphi \in C^1(M)$ is an even functional that satisfies the Palais–Smale condition at the level β_ℓ, and is bounded from below on M. Also suppose that if $H \in \mathcal{H}$, $\{0\} \times \mathbf{R}^{N-k} \subset \partial H$ and $u \in M$,* *then* $u^H \in M$ *and* $\varphi(u^H) \leqslant \varphi(u)$. *If* $\ell \leqslant k$, *then there is a critical point* $u \in M$ *and* $x \in S^{k-1}$ *such that* $\varphi(u) = \beta_\ell$ *and* $u^{S_x} = u$.

Proof. The theorem is proved by Struwe without the conclusion $u^{S_x} = u$ [26]. By a close inspection of his proof, for each sequence $(A_n)_{n\geqslant 1}$ of $\mathcal{F}_\ell$ such that $\sup_{u\in A_n} \varphi(u) \to \beta_\ell$, up to a subsequence of the sequence $(A_n)_{n\geqslant 1}$, there exists a sequence $(u_n)_{n\geqslant 1}$ in M such that $u_n \in A_n$, $u_n \to \bar{u}$, $\varphi(u_n) \to \beta_\ell$ and $\bar{u}$ is a critical point.

By Proposition 4.3, we can find a sequence $(A_n)_{n\geqslant 1} \subset \mathcal{F}_\ell$ such that $\gamma(A_n) = \ell$ and $\sup_{u\in A_n} \varphi(u) \to \beta_\ell$. Since φ decreases by polarization, by Proposition 4.7, we can take $A'_n = \sigma(A_n)$ with $\varepsilon = 1/n$, so that for each $u \in A'_n$, there exists $x_n \in \mathcal{S}^{k-1}$ such that $\|u - u^{S_{x_n}}\|_p < 1/n$. Since $\sup_{u\in A'_n} \varphi(u) \leqslant \sup_{u\in A_n} \varphi(u)$ and $\gamma(A'_n) \geqslant \gamma(A_n)$, there exists a sequence $(u_n)_{n\geqslant 1}$ such that $u_n \in A'_n$, $u_n \to u$, $\varphi(u_n) \to \beta_\ell$ and u is a critical point of φ. Moreover, for each n there exists x_n such that$\|u_n - u^{S_{x_n}}\|_p < 1/n$. Up to a subsequence, $x_n \to x \in S^{k-1}$, so that $\|u - u^{S_x}\|_p = 0$. □

For an application, let $f \in C(\Omega \times \mathbf{R})$ such that

- (f_1) there is $C > 0$ and $1 \leqslant p \leqslant (N+2)/(N-2)$ such that for every $(x,s) \in \Omega \times \mathbf{R}$, $f(x,s) \leqslant C(1 + |s|^p)$,
- (f_2) for every $(x,t) \in \Omega \times \mathbf{R}$, $f(x,s)s < 0$,
- (f_3) for every $(x,t) \in \Omega \times \mathbf{R}$, $f(x,-s) = -f(x,s)$.

Let $F(x,s) = \int_0^s f(x,\sigma)\, d\sigma$.

First consider the functional

$$\varphi : \mathrm{W}_0^{1,2}(\Omega) \to \mathbf{R} : u \mapsto \frac{1}{2}\int_\Omega F(x,u)\, dx$$

restricted to the set $M = \{u \in \mathrm{W}_0^{1,2}(\Omega) \ : \ \|\nabla u\|_2^2 + \lambda \|u\|_2^2 = 1\}$. Let λ_0 denote the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions.

Theorem 4.9. *Let* Ω *be as before. For* $0 \leqslant \ell \leqslant k$ *and* $\lambda > -\lambda_0(\Omega)$, *the functional* φ *has a critical point* u_ℓ *such that* $\varphi(u_\ell) = \beta_\ell$ *and* u_ℓ *is invariant by the symmetrization with respect to* S_x, *for some* $x \in S^{k-1}$.

Proof. Since $\lambda > -\lambda_0(\Omega)$, M is a $C^{1,1}$ manifold in $\mathrm{W}_0^{1,2}(\Omega)$. The functional φ is even, satisfies the Palais–Smale condition at any level $c \neq 0$ and is bounded from below (see Rabinowitz [20]). Since by (f_3), $\varphi(u) < 0$ for $u \neq 0$, then $\beta_\ell < 0$. Furthermore, if $u \in M$, then $u^H \in \mathrm{W}_0^{1,2}(\Omega)$ and $\|u^H\|_{\mathrm{L}^p(\Omega)} = \|u\|_{\mathrm{L}^p(\Omega)} = 1$. Therefore, the conclusion follows from Theorem 4.8. □

Since $u^{S_x} = u$ for some $x \in S^{k-1}$, the function u depends on $N-k+2$, variables: $u(y,z) = u(|y|, x\cdot y, z)$. In particular, when $k = N$, Ω is a ball or an annulus, u depends on two variables. (Similar results were proved by Smets and Willem [23].)

Similarly we can consider the functional associated to a Neumann problem

$$\varphi : \mathrm{W}^{1,2}(\Omega) \to \mathbf{R} : u \mapsto \int_\Omega F(x,u)\, dx$$

restricted to the set $M = \{u \in \mathrm{W}^{1,2}(\Omega) \ : \ \|\nabla u\|_2^2 + \lambda \|u\|_2^2 = 1\}$.

Theorem 4.10. *Let Ω be as before. For $0 \leqslant \ell \leqslant k$ and $\lambda > 0$, the functional φ has a critical point $u_\ell \in M$ such that $\varphi(u_\ell) = \beta_\ell$ and u_ℓ is invariant by the symmetrization with respect to S_x, for $x \in S^{k-1}$.*

The restriction $\ell \leqslant k$ of Theorems 4.9, and 4.10 seems natural when one considers the particular case $f(x,s) = -s$. If Ω is a sufficiently thin annulus, then the critical points associated to β_{N+1} are of the form $u(|x|)H(x/|x|)$, where u is a fixed function and H is a spherical harmonic of order two. Among the spherical harmonics, there are the zonal harmonics, which are invariant under $O(N-1)$, but there is also the function $H(x) = \sum_{i=1}^{N-1} ix_i^2 - N(N-1)x_N^2/2$. The latter has a discrete symmetry group. Since some of the critical points associated to β_{N+1} are nonsymmetric in the linear case, it is quite possible that for some nonlinear problems the critical points at the level β_{N+1} are not invariant under any $N-1$-dimensional spherical cap symmetrization. The same kind of heuristic arguments can be developed for β_{k+1} when $k < N$. (The analysis of the symmetry of critical points obtained by the linking theorem lead to similar considerations [30].)

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CHAPTER IV

Anisotropic Symmetrization

1. Introduction

A symmetrization transforms functions into more symmetrical functions. This transformation preserves or decreases some integral functionals. This is useful to enquire about minimizers of a functional, which can then be sought among symmetrical functions. For example, consider the functional I, defined on $\mathrm{W}_0^{1,p}(\Omega)$ by

$$I(u) = \frac{\int_\Omega |\nabla u(x)|_2^p \, dx}{\left(\int_\Omega |u(x)|^q \, dx\right)^{\frac{p}{q}}}.$$

If $\Omega = \mathrm{B}(0,1)$, then the Schwarz symmetrization $\cdot^\star$ maps any nonnegative function u to a radial function $u^\star(x) = v(|x|_2)$, where $v : \mathbf{R}^+ \to \mathbf{R}^+$ is a nonincreasing function, such that $I(u^\star) \leqslant I(u)$. Therefore one can search for a minimizer among radial functions.

The symmetrization of functions remains possible whenever Ω has less symmetry. For example, if $\Omega = \mathrm{B}(0,1) \times \mathbf{R}^{N-k} \subset \mathbf{R}^N$, then the Steiner symmetrization of u, also denoted by $\cdot^\star$ is a function $u^\star$ such that $I(u^\star) \leqslant I(u)$ and $u(x', x'') = v(|x|_2, x'')$ for some function $v : \mathbf{R}^+ \times \mathbf{R}^{N-k} \to \mathbf{R}$ that does not increase with respect to its first argument.

The anisotropic symmetrization is a symmetrization adapted to anisotropic variational problems. In those problems, the function of the gradient in the functional does not depend on the euclidian norm, but on another positively homogeneous function $H : \mathbf{R}^N \to \mathbf{R}^+$. Anisotropic problems have in general too small symmetry groups to obtain symmetry from uniqueness arguments as it is possible in the isotropic case. Therefore symmetrization seems to be the most natural way to prove symmetry of minimizers of anisotropic functionals.

Anisotropic problems arose at the beginning of the twentieth century in Wulff's work on crystal shapes and minimization of anisotropic surface tensions. He considered the minimization problem

$$\min_{\mathcal{L}^N(\Omega)=1} \int_{\partial\Omega} H(\nu)\, d\sigma,$$

among sufficiently regular domains Ω, and computed the solution whose optimality was proved by Dinghas and Taylor:

$$\Omega = \left\{ x \in \mathbf{R}^N \ : \ H^\circ(-x) \leqslant 1 \right\} \ , \quad H^\circ(x) = \sup_{t \in \mathbf{R}^N \setminus \{0\}} \frac{\langle t, x \rangle}{H(t)} \ .$$

This model explains the polyhedral shape of many crystals. The structure of the functional and of the solution are not the same, but are dual. A Schwarz anisotropic symmetrization was constructed for nonlinear variational problems by Alvino, Ferone, Lions and Trombetti [3] with the same duality relation. They proved Pólya–Szegő and Hardy-Littlewood inequalities for non-partial anisotropic symmetrizations.

In this paper, we study anisotropic symmetrization associated to a homogeneous convex function G and its associated inequalities. Such hypotheses appear in other frameworks in the works of Taylor, Busemann, and Dacorogna and Pfister [9, 12, 24]. Our main objective is to define and understand partial anisotropic symmetrizations that generalize the convex symmetrization of Alvino, Ferone, Lions and Trombetti [3].

For any nonnegative measurable function $u : \mathbf{R}^N \to \bar{\mathbf{R}}^+$ whose positive sublevel sets have finite measure and for any convex function $G \in \mathcal{H}(\mathbf{R}^k)$, a unique function $u^* : \mathbf{R}^N \to \bar{\mathbf{R}}^+$ is defined such that $u^*(x', x'') = v(G(-x'), x'')$, for any $x' \in \mathbf{R}^k$ and $x'' \in \mathbf{R}^{N-k}$, where the function $v : \mathbf{R}^+ \times \mathbf{R}^{N-k} \to \bar{\mathbf{R}}^+$ decreases with respect to its first argument and such that for any $c > 0$ and $x'' \in \mathbf{R}^{N-k}$,

$$\mathcal{L}^{N-k}(\left\{x' \ : \ u^*(x', x'') > c\right\}) = \mathcal{L}^{N-k}(\left\{x' \ : \ u(x', x'') > c\right\})$$

(See the beginning of section 2 for precision on the notations.) This function u^* is the anisotropic symmetrization of u with respect to G. This transformation is a rearrangement in the sense of [6, 26]. Therefore all classical integral inequalities follow easily, e.g. for any Borel measurable function $f : \mathbf{R} \times \mathbf{R}^{N-k} \to \mathbf{R}^+$ such that $f(0, \cdot) = 0$,

$$\int_{\mathbf{R}^N} f(u^*, x'')\, dx = \int_{\mathbf{R}^N} f(u, x'')\, dx.$$

Similarly, $\cdot^*$ is a contraction in L^p spaces (and many other spaces, see Proposition 2.29). The definitions and basic properties of the anisotropic symmetrizations are the object of section 2.

Section 4 is devoted to convolution inequalities for the anisotropic symmetrization of the form

$$\int_{\mathbf{R}^N}\int_{\mathbf{R}^N} u(x)\,v(y)\,w(x-y)\,dx\,dy \leqslant \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} u^*(x)\,v^*(y)\,w^*(x-y)\,dx\,dy.$$

The conclusion is that such inequalities can occur only when the rearrangement is made with respect to an euclidian gauge (Propositions 4.2 and 4.4). The same arguments show that the full Riesz–Sobolev rearrangement inequality does not hold for the spherical cap symmetrizations and the polarizations (Corollary 4.3). The proof of Proposition 4.4 uses dual characterizations of symmetrized functions studied in section 3: For example if, for any $\varphi \in \mathcal{K}_+(\mathbf{R}^N)$,

$$\int_{\mathbf{R}^N} \varphi u\,dx \leqslant \int_{\mathbf{R}^N} \varphi^* u\,dx\,,$$

then $u = u^*$ almost everywhere (Lemma 3.1). The case where u is a measure is also investigated (Lemma 3.3).

Even if convolution inequalities do not hold for the anisotropic symmetrization, if $\varphi_* = -(-\varphi)^*$ and $\overline{\varphi}$ is the Fenchel transform of φ (Definition 5.1), there are Klimov inequalities of the form

$$\int_\Omega \overline{\varphi_*}(x'', u^*(x), \nabla u^*(x))\,dx \leqslant \int_\Omega \overline{\varphi}(x'', u(x), \nabla u(x))\,dx,$$

(Theorem 6.9). In this inequality only the gauge G appears, but the dual of Wulff's crystal is embedded inside the Fenchel transform.

The Pólya–Szegő inequality for anisotropic symmetrization can be stated as

$$\int_\Omega J(x'', u^*, H(\nabla' u^*), \nabla'' u^*)\,dx \leqslant \int_\Omega J(x'', u, H(\nabla' u), \nabla'' u)\,dx.$$

where J is convex with respect to its last two variables and $G = H^\circ$ (Theorem 6.8). The left-hand side is not necessarily convex in ∇u, since H is not convex in general, but it is convex on the subset of gradients of symmetrized functions.

These results emphasize the local character of symmetries of crystals in contrast with the long-range of isotropic symmetry. Physically, this could be the fact that we observe anisotropic symmetries for crystals, whose energy is mainly an interface energy, but not for stars, whose energy depends of long-distance (gravitational) interaction terms.

The proof of these inequalities consists a generalization of an anisotropic inequality for Steiner symmetrization of Klimov [17] (section 5), followed by a change of variable in order to return to anisotropic functionals (section 6). The high degree of generality of the proof makes it appealing even for the isotropic symmetrization.

Applications of the previous inequalities are given as an anisotropic isoperimetric inequality (Theorem 7.2) and optimal constants for Sobolev and Hardy-Sobolev inequalities (Propositions 7.3 and 7.5). Finally, the existence and symmetry of solutions of two model anisotropic variational problems is showed (Propositions 7.6 and 7.9).

The definition of symmetrization is interesting also in the isotropic case because of its good pointwise behavior. As symmetrizations of sets were originally defined from compact sets to compact sets, they were nonexpansive mappings with respect to the Hausdorff distance. The symmetrization inequalities of Sarvas [22] for condensers or capacitors, defined by a compact set and an open set, required the extension of symmetrizations to open sets. Any extension to measurable sets, was only defined on almost every hyperplane. The use of Lebesgue's outer measure gives a precise definition of the symmetrized set which was needed in this paper since the Fenchel transform is sensitive to modifications on sets of measure zero.

This paper also clarifies the relationship between the different symmetrization inequalities by giving examples of symmetrizations for which Pólya-Segő and Klimov inequalities hold but Riesz–Sobolev rearrangement inequalities do not hold.

2. Definition and properties of symmetrizations

In this section, the symmetrization with respect to a gauge is defined. Its basic properties, similar to those of the classical rearrangements, are studied.

Notation 2.1. For $f : X \to \bar{\mathbf{R}}$ and $c \in \bar{\mathbf{R}}$, let

$$\{f < c\} = \{x \in X \ : \ f(x) < c\} .$$

The characteristic function of a set A is denoted χ_A. The N*–dimensional outer Lebesgue measure* is denoted $\mathcal{L}^N$. The extended set of real numbers is $\bar{\mathbf{R}} = \mathbf{R} \cup \{+\infty, -\infty\}$. The set of compactly supported continuous functions is denoted $\mathcal{K}(\mathbf{R}^N)$, while $\mathcal{D}(\mathbf{R}^N)$ is the set of smooth functions with compact support. The subscript $_+$ denotes the subset of nonnegative functions of a function space. For $0 \leqslant k \leqslant N$ and

$x = (x_1, \dots, x_N) \in \mathbf{R}^N$, let $x' = (x_1, \dots, x_k)$ and $x'' = (x_{k+1}, \dots, x_N)$. Similarly, let $\nabla' u = (\nabla u)'$ and $\nabla'' u = (\nabla u)''$.

Definition 2.2. Let X be a vector space. The function $H : X \to \mathbf{R}$ belongs to $\mathcal{H}(X)$ if

(1) if $x \in X$ and $\lambda \geqslant 0$, then $H(\lambda x) = \lambda H(x)$,
(2) if $x \in X$ and $x \neq 0$, then $H(x) > 0$,
(3) H is lower semi-continuous.

Definition 2.3. The *polar transform* of $H \in \mathcal{H}(\mathbf{R}^k)$ is

$$H^\circ : \mathbf{R}^k \to \mathbf{R} : H^\circ(t) = \sup_{x \in \mathbf{R}^k} \frac{\langle t, x \rangle}{H(x)} ,$$

where $\langle t, x \rangle = \sum_{i=1}^k t_i x_i$.

Definition 2.4. The function $G : \mathbf{R}^k \to \mathbf{R}$ is a *gauge* if $G \in \mathcal{H}(\mathbf{R}^k)$ and G is convex. For any gauge G, let

$$K_G = \sqrt[k]{\frac{\mathcal{L}^k(\{G(-\cdot) < 1\})}{\omega_k}},$$

where $\omega_k = \mathcal{L}^k(\mathrm{B}(0,1))$ is the volume of the unit ball in $\mathbf{R}^k$.

Remark 2.5. Any gauge G is a continuous function, and $0 < K_G < \infty$.

Example 2.6. If $H \in \mathcal{H}(\mathbf{R}^k)$, its polar transform H° is a gauge.

Definition 2.7. Let $G : \mathbf{R}^k \to \mathbf{R}^+$ be a gauge. The *anisotropic symmetrization* (called convex symmetrization in [3]) of the set $A \subseteq \mathbf{R}^k$ with respect to G is the set

$$A^* = \left\{ x \in \mathbf{R}^k \ : \ G(-x) < K_G^{-1} \Big(\frac{\mathcal{L}^k(A)}{\omega_k} \Big)^{1/k} \right\}.$$

Remark 2.8. The set A^* is chosen among the sets $(\{G(-x) < r\})_{0 \leqslant r \leqslant \infty}$ so that $\mathcal{L}^k(A^*) = \mathcal{L}^k(A)$. The set A does not have to be measurable.

Definition 2.9. Given a decomposition of $\mathbf{R}^N = L \times T$, and a gauge $G : L \to \mathbf{R}^+$, the (G, L, T)*–anisotropic symmetrization* of the set $A \subset \mathbf{R}^N$, is the unique set A^* such that, for all $x'' \in T$,

$$[A^*]_{x''} = [A]_{x''}{}^*,$$

where $[B]_{x''} = \{x' \in L \ : \ (x', x'') \in B\}$ and the symmetrization on the right-hand side comes from Definition 2.7.

The symmetrization of a finite-measure set $A \subset \mathbf{R}^N$ with respect to a gauge $G : \mathbf{R}^k \to \mathbf{R}^+$ is the $(G, \mathbf{R}^k \times \{0\}, \{0\} \times \mathbf{R}^{N-k})$–anisotropic symmetrization of A.

Remark 2.10. Even when A is measurable, $[A]_{x''}$ is not measurable in general. Therefore Definition 2.7 embraces nonmeasurable sets.

Remark 2.11. The result A^* of the anisotropic symmetrization with respect to the gauge G has a cylindrical geometry:

$$A^* = \left\{ (x', x'') \in \mathbf{R}^N \;:\; G(-x') < K_G^{-1} \Big(\frac{\mathcal{L}^k([A]_{x''})}{\omega_k} \Big)^{1/k} \right\}. \tag{2.1}$$

Example 2.12. The (k,N)–*Steiner symmetrization* with respect to the subspace $T \subseteq \mathbf{R}^N$, is the $(G, T^\perp, T)$–anisotropic symmetrization, where $G : x'' \mapsto |x''|_2$. The Steiner symmetrization with respect to $\mathbf{R}^N$ is the *Schwarz symmetrization* (see e.g. [4, 6, 22]).

Example 2.13. If $G : \mathbf{R}^N \to \mathbf{R}^+$ is a gauge and is an even function, then the anisotropic symmetrization with respect to G is the convex symmetrization with respect to G of Alvino, Ferone, Lions and Trombetti [3].

The following proposition summarizes the properties of anisotropic symmetrization of sets.

Proposition 2.14. *Let $\cdot^*$ be a (G, L, T)-anisotropic symmetrization on $\mathbf{R}^N$.*

(1) (Monotonicity) *If $A \subseteq B \subseteq \mathbf{R}^N$, then $A^* \subseteq B^*$.*
(2) (Interior continuity) *If $(A_n)_{n\in\mathbf{N}}$ is an increasing sequence of subsets of $\mathbf{R}^N$ (i.e. $A_n \subseteq A_{n+1}$), then*
$$\Big(\bigcup_{n\in\mathbf{N}} A_n \Big)^* = \bigcup_{n\in\mathbf{N}} A_n^*.$$
(3) (Preservation of measure) *If $A \subset \mathbf{R}^N$ is measurable, then A^* is measurable and $\mathcal{L}^N(A) = \mathcal{L}^N(A^*)$.*
(4) *If $A \subset \mathbf{R}^N$ is open, then A^* is open.*

Remark 2.15. Whereas the continuity of symmetrization held in previous works only up to sets of zero measure [6, 22], this definition ensures interior continuity. The exterior continuity (Property (2) with reversed inclusions) still holds up to sets of zero measure, but it is not used in the sequel. This property was true already for the definitions of the Schwarz symmetrization which mapped all sets to open sets (see e.g. [18]).

Remark 2.16. Part (3) remains true when A is not measurable if $T = \{0\}$, by definition of the anisotropic symmetrization. If $T \neq \{0\}$, then part (3) implies $\mathcal{L}^N(A^*) \leqslant \mathcal{L}^N(A)$, and for some nonmeasurable sets $A \subset \mathbf{R}^N$, the inequality is strict.

Proof. The notations of Definition 2.9 are used throughout the proof. Part (1) comes from the monotonicity of outer measures: for any $x'' \in T$, since $[A]_{x''} \subset [B]_{x''}$, $\mathcal{L}^k([A]_{x''}) \leqslant \mathcal{L}^k([B]_{x''})$, and $[A^*]_{x''} \subset [B^*]_{x''}$. It is then clear that $A^* \subset B^*$.

If (A_n) is any sequence satisfying the hypotheses of (2), for any $x'' \in T$, by an elementary property of Lebesgue's outer measure,

$$\lim_{n\to\infty} \mathcal{L}^k([A_n]_{x''}) = \mathcal{L}^k(\cup_{n\in\mathbf{N}}[A_n]_{x''}) = \mathcal{L}^k([A]_{x''}).$$

(This is a consequence of monotonicity and countable subadditivity of outer measures, see e.g. [14, 2.1.5] for a proof.) Since the sequence $([A_n]_{x''})_{n\in\mathbf{N}}$ is increasing, by Definition 2.7, $\cup_{n\in\mathbf{N}}([A_n]_{x''}{}^*) = [A]_{x''}{}^*$. By Definition 2.9, $\cup_{n\in\mathbf{N}} A_n{}^* = A^*$.

Part (3) is a consequence of Remark 2.11 and of Fubini's Theorem (see e.g. [14, 2.6.2]):

$$\mathcal{L}^N(A^*) = \int_T \mathcal{L}^k([A]_{x''}{}^*)\,dx'' = \int_T \mathcal{L}^k([A]_{x''})\,dx'' = \mathcal{L}^N(A).$$

Part (4) relies on Remark 2.11. If the set A is open, the right-hand side of the inequality inside (2.1) is lower semi-continuous, whence the symmetrized set A^* is open. □

Following [6, 26, 27], the symmetrization is extended from sets to functions.

Definition 2.17. The (G, L, T)*–anisotropic (decreasing) symmetrization* $\cdot^*$ of a function $u : \mathbf{R}^N \to \bar{\mathbf{R}}$ is

$$u^* : \mathbf{R}^N \to \bar{\mathbf{R}} : x \mapsto u^*(x) = \sup\left\{c \in \bar{\mathbf{R}} \;:\; x \in \{u > c\}^*\right\}.$$

Remark 2.18. Since $\chi_{(A^*)} = (\chi_A)^*$, the symmetrization of functions is an extension of the symmetrization of sets.

Notation 2.19. For a function u and a sequence of functions $(u_n)_{n\in\mathbf{N}}$ from a set X to $\bar{\mathbf{R}}$, we write $u_n \nearrow u$ if for all $x \in X$, $\lim_{n\to\infty} u_n(x) = u(x)$ and for all $n \in \mathbf{N}$, $u_n(x) \leqslant u_{n+1}(x)$. Similarly, $\varphi_n \searrow \varphi$ if $-\varphi_n \nearrow -\varphi$.

The simplest properties of symmetrization of functions are consequence of the corresponding properties of symmetrization of sets [6, 26].

Proposition 2.20. *Let* $\cdot^*$ *be a* (G, L, T)*-anisotropic symmetrization.*

(1) *For any* $u : \mathbf{R}^N \to \bar{\mathbf{R}}$*,*

$$u^*(x) = \sup\left\{c \in \mathbf{R} \;:\; x \in \{u \geqslant c\}^*\right\}.$$

(2) *For any* $c \in \bar{\mathbf{R}}$ *and* $u : \mathbf{R}^N \to \bar{\mathbf{R}}$,

$$\{u > c\}^* = \{u^* > c\}.$$

(3) *Let* $u, v : \mathbf{R}^N \to \bar{\mathbf{R}}$. *If* $u \leqslant v$, *then* $u^* \leqslant v^*$.
(4) *If* $(u_n)_{n \in \mathbf{N}}$ *is a sequence of functions from* $\mathbf{R}^N$ *to* $\bar{\mathbf{R}}$, *and* $u_n \nearrow u$, *then*

$$u_n{}^* \nearrow u^*.$$

Remark 2.21. Part (1) is Hildén's definition of the Schwarz symmetrization of a function.

Remark 2.22. Part (2) means that if $\cdot^*$ is the (G, L, T)-symmetrization, the hypograph of the symmetrization is the symmetrization of the hypograph:

$$\left\{(x, c) \in \mathbf{R}^N \times \mathbf{R} \ : \ u^*(x) > c\right\} = \left\{(x, c) \in \mathbf{R}^N \times \mathbf{R} \ : \ u(x) > c\right\}^*.$$

(The symmetrization on the right-hand side is the $(G, L \times \{0\}, T \times \mathbf{R})$-anisotropic symmetrization in $\mathbf{R}^N \times \mathbf{R}$.) This is essentially Pólya and Szegő's definition of the symmetrization of a function [21].

Remark 2.23. Part (2) implies in particular that if $u(x) > c$ for almost every $x \in \mathbf{R}^N$, then $u^*(x) > c$ for all $x \in \mathbf{R}^N$. If the function u does not take the value $-\infty$, neither does its symmetrization u^*.

Remark 2.24. The equality of sets in part (2) holds pointwise. This is not the case for most of the usual definitions of rearrangements, for which the equality in part (2) holds only up to a set of zero measure. This comes from the fact our definition of symmetrization ensures interior continuity in a pointwise sense.

Part (2) holds with the strict inequality sign, but not with the nonstrict inequality (if u^* is a nonconstant continuous function, then for some c the set $\{u^* \geqslant c\}$ is closed, while $\{u \geqslant c\}^*$ is not closed).

Remark 2.25. In part (4), $\lim_{n \to \infty} u_n(x) = u(x)$ holds everywhere. That is crucial in section 5, since the Fenchel transform is continuous for increasing sequences converging everywhere.

Proof of Proposition 2.20. Part (3) is a consequence of the monotonicity of the anisotropic symmetrization.

For part (1), let $\tilde{u}(y)$ denote the right-hand side of the inequality. It is clear from monotonicity of the symmetrization that for any $\varepsilon > 0$, $u^*(y) \leqslant \tilde{u}(y) \leqslant (u + \varepsilon)^*(y) = u^*(y) + \varepsilon$. The conclusion follows as $\varepsilon \to 0$.

By the interior continuity of the symmetrization of sets (Proposition 2.14), for any $c \in \mathbf{R}$,

$$\begin{aligned}\{u > c\}^* &\subseteq \{u^* > c\} \\ &\subseteq \bigcup_{n\in\mathbf{N}} \{u > c + \tfrac{1}{n}\}^* = \Big(\bigcup_{n\in\mathbf{N}} \{u > c + \tfrac{1}{n}\}\Big)^* = \{u > c\}^*.\end{aligned}$$

The proof is the same for $c = -\infty$, provided $c + 1/n$ is replaced by $-n$, while for $c = +\infty$, both sides are the empty set. This proves part (2).

Part (4) is a consequence of the interior continuity of symmetrization of sets (Proposition 2.14, part (2)) and of the description of the symmetrization of a function of part (2). □

Remark 2.26. The proof of Proposition 2.20 only relies on the corresponding properties for the symmetrization of sets (Proposition 2.14, parts (1) and (2)).

The preservation of measure for the symmetrization of sets has as counterpart integral equalities and inequalities for the symmetrization of functions. A natural class for studying integrals is the class of functions vanishing at the infinity (see Lieb and Loss [18]). This class contains the functions spaces $\mathrm{L}^p(\mathbf{R}^N)$ and $C_0(\mathbf{R}^N)$ (continuous functions such that $\lim_{|x|_2\to\infty} u(x) = 0$).

Definition 2.27. A measurable function $u : \mathbf{R}^N \to \bar{\mathbf{R}}$ *vanishes at the infinity* with respect to a k–dimensional linear subspace $L \subset \mathbf{R}^N$ if for all $c > 0$ and $x \in \mathbf{R}^N$, $\mathcal{L}^k(\{u > c\} \cap (L + x)) < \infty$. We also say that u vanishes at the infinity with respect to the (G, L, T)–anisotropic symmetrization.

Proposition 2.28. (Cavalieri principle) *Let $\cdot^*$ be the anisotropic symmetrization with respect to a gauge $G : \mathbf{R}^k \to \mathbf{R}^+$.*
If $u : \mathbf{R}^N \to \bar{\mathbf{R}}^+$ vanishes at the infinity with respect to $\cdot^*$ and $f : \mathbf{R} \times \mathbf{R}^{N-k} \to \bar{\mathbf{R}}^+$ is a Borel measurable function such that $f(0, x'') = 0$ for almost every $x'' \in \mathbf{R}^{N-k}$, then

$$\int_{\mathbf{R}^N} f(u^*, x'')\,dx = \int_{\mathbf{R}^N} f(u, x'')\,dx.$$

(Hardy-Littlewood inequality)*Let $F : \mathbf{R} \times \mathbf{R} \times \mathbf{R}^{N-k} \to \mathbf{R}$ be a function such that*

(i) $F(s, t, \cdot)$ is measurable for every $(s, t) \in \mathbf{R} \times \mathbf{R}$,
(ii) $F(\cdot, \cdot, x'')$ is continuous for almost every $x'' \in T$,

(iii) for almost every $x'' \in \mathbf{R}^{N-k}$*, and for any* $a, b, c, d \in \mathbf{R}$*, if* $a \leqslant b$*,* $c \leqslant d$*,*

$$F(a, c, x'') + F(b, d, x'') \geqslant F(a, d, x'') + F(b, c, x''),$$

If u, v *are nonnegative measurable functions defined on* $\mathbf{R}^N$ *and if the functions* $F(u, 0, x'')$*,* $F(0, v, x'')$ *and* $F(u, v, x'')$ *are summable, then*

$$\int_{\mathbf{R}^N} F(u, v, x'')\, dx \leqslant \int_{\mathbf{R}^N} F(u^*, v^*, x'')\, dx.$$

Proof. The result is true without any dependence on x'' (see [20, 26]). For any x'', there holds

$$\int_{\mathbf{R}^{N-k}} f(u^*(x', x''), x'')\, dx' = \int_{\mathbf{R}^{N-k}} f(u(x', x''), x'')\, dx'.$$

Since f is a Borel measurable function, $f(u(\cdot), \cdot)$ is measurable and is almost everywhere equal to a Borel measurable function. The result comes from the application of Fubini's Theorem.

For the second inequality uses the fact that F is a Carathédory function and that such functions are almost everywhere equal to a Borel measurable function. The conclusion comes from the corresponding inequality in [10, 27] and Fubini's Theorem. □

Proposition 2.29. *Let* $\cdot^*$ *be the anisotropic symmetrization with respect to* G *and* $g : \mathbf{R} \times \mathbf{R}^{N-k} \to \bar{\mathbf{R}}^+$*. Suppose that for almost every* $x'' \in \mathbf{R}^{N-k}$*,* $g(\cdot, x'')$ *is a convex and lower semi-continuous function and* $g(0, x'') = 0$*, and that for all* $s \in \mathbf{R}$*,* $g(s, x'')$ *is measurable. If* u *and* v *are measurable functions, then*

$$\int_{\mathbf{R}^N} g(u^*(x) - v^*(x), x'')\, dx \leqslant \int_{\mathbf{R}^N} g(u(x) - v(x), x'')\, dx.$$

For any $1 \leqslant p \leqslant +\infty$ *and for any measurable functions* u *and* v*,*

$$\|u^* - v^*\|_p \leqslant \|u - v\|_p\,.$$

Remark 2.30. There are no integrability assumptions in this Proposition.

Proof. By Fubini's Theorem, we have to prove

$$\int_{\mathbf{R}^k} \int_{\mathbf{R}^{N-k}} g(u^* - v^*, x'')\, dx \leqslant \int_{\mathbf{R}^k} \int_{\mathbf{R}^{N-k}} g(u - v, x'')\, dx.$$

The inequality will be proved for the interior integral. Without loss of generality we can thus assume $k = N$ and $g(s, x'') = g(s)$. Let

$$c = \inf \left\{ s \;:\; \mathcal{L}^N(\{u > s\}) < \infty \right\} \quad d = \inf \left\{ s \;:\; \mathcal{L}^N(\{v > s\}) < \infty \right\}$$

If $g(c-d) > 0$, then, without loss of generality, we can assume that $c > d$. By definition of c and d, for any $\varepsilon > 0$, $\mathcal{L}^N(\{v > d + \varepsilon\}) < +\infty$ and $\mathcal{L}^N(\{u > c - \varepsilon\}) = +\infty$, hence the measure of the set $\Omega_\varepsilon = \{u > c - \varepsilon\} \cap \{v \leqslant d + \varepsilon\}$ is infinite. Furthermore, for any $x \in \Omega_\varepsilon$, $u(x) - v(x) > c - d - 2\varepsilon$. Since g is convex and $g(0) = 0$, it is increasing on $\mathbf{R}^+$. Hence, for $\varepsilon \leqslant \frac{c-d}{2}$,

$$g(u(x) - v(x)) \geqslant g(c - d - 2\varepsilon). \tag{2.2}$$

Since g is lower semi-continuous, the right-hand side of (2.2) is positive for sufficiently small ε. This means

$$\int_{\mathbf{R}^N} g(u(x) - v(x))\, dx = +\infty.$$

The inequality is then trivial.

If $g(c-d) = 0$, without loss of generality we can assume that $c = d = 0$. If g is continuous, the function $F : (s,t) \mapsto -g(s-t)$ verifies the hypotheses of the last part of Proposition 2.28. If $u_n = \min(n, \max(u - 1/n, 0))$, and $v_n = \min(n, \max(v - 1/n, 0))$, then $g(u_n)$, $g(-v_n)$ and $g(u_n - v_n)$ are summable and, by Proposition 2.28,

$$\int_{\mathbf{R}^N} g(u_n^*(x) - v_n^*(x))\, dx \leqslant \int_{\mathbf{R}^N} g(u_n(x) - v_n(x))\, dx.$$

Furthermore, $u_n \nearrow u^+$, $v_n \nearrow v^+$ and $g(u_n - v_n) \nearrow g(u^+ - v^+) \leqslant g(u-v)$, whence ${u_n}^* \nearrow u^*$, and ${v_n}^* \nearrow v^*$. Moreover, $\liminf_{n\to\infty} g({u_n}^*(x) - {v_n}^*(x)) \geqslant g(u^*(x) - v^*(x))$ since g is continuous. Fatou's Lemma and Levi's monotone convergence Theorem bring the conclusion. If g is not continuous, it can be approximated by an increasing sequence of continuous convex functions and Levi's monotone convergence Theorem gives the conclusion.

The second part is a consequence of the first part, with $g = |t|^p$ if $1 \leqslant p < +\infty$. If $p = +\infty$, let $g(t) = 0$ for $t \leqslant \|u - v\|_\infty$ and $g(t) = +\infty$ else. $\square$

The anisotropic symmetrizations can be defined for functions which are defined on totally invariant sets.

Definition 2.31. A set $\Omega \subset \mathbf{R}^N$ is *totally invariant with respect to a hyperplane* L if $\Omega + L = \Omega$. The set Ω is totally *invariant with respect to the* (G, L, T)*-anisotropic symmetrization* if it is totally invariant with respect to L.

A function $f : \mathbf{R}^N \to \bar{\mathbf{R}}$ is *totally invariant with respect to a hyperplane* L if $f(x + l) = f(x)$ for any $l \in L$.

If Ω is totally invariant with respect to the (G, L, T)-anisotropic symmetrization $\cdot^*$, then for any $A \subset \mathbf{R}^N$, $(A \cap \Omega)^* = A^* \cap \Omega$.

Definition 2.32. If $\cdot^*$ is an anisotropic symmetrization, Ω is totally invariant with respect to $\cdot^*$, then the symmetrization of $u : \Omega \to \bar{\mathbf{R}}$ is $u^* = \tilde{u}^*|_\Omega$, where $\tilde{u}$ denotes an extension of u to $\mathbf{R}^N$.

The definition of u^* does not depend on the extension $\tilde{u}$. All the previous results remain valid with Ω in place of $\mathbf{R}^N$, provided the set Ω is measurable for the integral inequalities.

The increasing symmetrization is a natural counterpart to the decreasing symmetrization.

Definition 2.33. The (G, L, T)*–anisotropic increasing symmetrization of a function* $\varphi : \mathbf{R}^N \to \bar{\mathbf{R}}$ is

$$\varphi_*(y) = \inf \{ s \in \mathbf{R} \;:\; y \in \{\varphi < s\}^* \} .$$

The increasing and decreasing anisotropic symmetrization are essentially the same transformation.

Proposition 2.34. *For any function* $\varphi : \mathbf{R}^N \to \bar{\mathbf{R}}$*,*

$$\varphi_*(y) = -(-\varphi)^*(y).$$

Remark 2.35. This means that all the properties of the decreasing symmetrization are true for the increasing symmetrization up to obvious modifications.

Proof. For any $y \in \mathbf{R}^N$,

$$\begin{aligned} \varphi_*(y) &= \inf \{ s \in \mathbf{R} \;:\; y \in \{\varphi < s\}^* \} \\ &= -\sup \{ c \in \mathbf{R} \;:\; y \in \{\varphi < -c\}^* \} \\ &= -\sup \{ c \in \mathbf{R} \;:\; y \in \{-\varphi > c\}^* \} = -(-\varphi)^*(y). \end{aligned}$$ □

3. Dual characterization of symmetrized functions

This section is devoted to dual characterizations of symmetrized functions that are used in the proof of Proposition 4.4.

Lemma 3.1. *Let* $\cdot^*$ *be an anisotropic symmetrization and* $u \in \mathrm{L}^1_+(\mathbf{R}^N)$*. If for any compact set* $K \subset \mathbf{R}^N$*,*

$$\int_K u \, dx \leqslant \int_{K^*} u \, dx, \tag{3.1}$$

then $u = u^*$ *almost everywhere.*

If, for any $\varphi \in \mathcal{K}_+(\mathbf{R}^N)$*,*

$$\int_{\mathbf{R}^N} \varphi u \, dx \leqslant \int_{\mathbf{R}^N} \varphi^* u \, dx, \tag{3.2}$$

then $u = u^*$ almost everywhere.

Remark 3.2. This Lemma is reminiscent of the bathtub principle and of the necessary condition of equality of the Hardy-Littlewood inequality (see [18]).

Proof. If the inequality (3.1) holds for any compact set K, then it holds also for any measurable set B by interior continuity of the anisotropic symmetrization and by Levi's monotone convergence Theorem. The inequality (3.1) is equivalent to

$$\int_{B \setminus B^*} u\, dx \leqslant \int_{B^* \setminus B} u\, dx, \tag{3.3}$$

for any measurable set B. For $c \in \mathbf{R}$, let $B = \{u > c\}$. Then the inequality (3.3) holds if and only if $\mathcal{L}^N(B \setminus B^*) = 0$. The function u is then almost everywhere equal to a function $\tilde{u}$ such that $\{\tilde{u} > c\} = \{u > c\}^* = \{u^* > c\}$. By the characterization of the symmetrization by sublevel sets (Proposition 2.20), $\tilde{u} = u^*$.

Suppose now that the inequality (3.2) holds for any $\varphi \in \mathcal{K}_+(\mathbf{R}^N)$. Let $K \subset \mathbf{R}^N$ be compact. Let $\varphi_n = \chi_{\mathrm{B}(0,1/n)} * \chi_K$, where $*$ denotes the convolution product. Then φ_n is continuous, $\varphi_n \to \chi_K$ in $\mathrm{L}^1(\mathbf{R}^N)$. Hence, by Proposition 2.29, ${\varphi_n}^* \to \chi_{K^*}$ in $\mathrm{L}^1(\mathbf{R}^N)$. Up to a subsequence ${\varphi_n}^*(x) \to \chi_{B^*}(x)$ and ${\varphi_n}^*(x) \leqslant 1$ for almost every $x \in \mathbf{R}^N$. Hence by Lebesgue's dominated convergence Theorem, the inequality (3.1) holds, and the conclusion comes from the first part of the proof. □

Lemma 3.3. *Let $G : \mathbf{R}^k \to \mathbf{R}^+$ be a gauge, $\cdot^*$ be the anisotropic symmetrization with respect to G. Suppose μ is a nonnegative Radon measure such that for any $\varphi \in \mathcal{K}_+(\mathbf{R}^k)$,*

$$\int_{\mathbf{R}^k} \varphi\, d\mu \leqslant \int_{\mathbf{R}^k} \varphi^*\, d\mu.$$

Then there exists $w \in \mathrm{L}^1_+(\mathbf{R}^k)$ and $a \geqslant 0$ such that $w = w^*$ and

$$\int_{\mathbf{R}^k} \varphi\, d\mu = a\varphi(0) + \int_{\mathbf{R}^k} w\, \varphi\, dx.$$

Proof. Let $\psi \in \mathcal{D}_+(\mathrm{B}(0,1))$ and $\psi_\rho(x) = \psi^*(\rho^{-1}x)$. Fix $x \in \mathbf{R}^k \setminus \{0\}$. For any $\rho > 0$ and $\varepsilon > 0$, if $\rho + \varepsilon < |x|_2$, then $(\psi_\rho(\cdot - x) + \psi_\varepsilon(\cdot))^* = \psi_{(\rho^k+\varepsilon^k)^{1/k}}$. The function $f(\rho) = \int_{\mathbf{R}^N} \psi_\rho\, d\mu$ is smooth on $(0, +\infty)$. The hypothesis applied to the function $\psi_\rho(\cdot - x) + \psi_\varepsilon(\cdot)$ gives

$$\begin{aligned}\int_{\mathbf{R}^k} \psi_\varepsilon(\cdot - x)\, d\mu &\leqslant \int_{\mathbf{R}^k} \psi_{(\rho^k+\varepsilon^k)^{1/k}}\, d\mu - \int_{\mathbf{R}^k} \psi_\rho\, d\mu \\ &\leqslant \sup_{0\leqslant \varepsilon' \leqslant \varepsilon} \left| f'(\sqrt[k]{\rho^k+\varepsilon^k})\right| \frac{{\varepsilon'}^{k-1}}{\sqrt[k]{\rho^k+{\varepsilon'}^k}}\varepsilon \leqslant C_x \varepsilon^k,\end{aligned}$$

where C_x is a constant that depends only on x and ρ for ε in a neighborhood of 0. Hence μ is absolutely continuous on $\mathbf{R}^k \setminus \{0\}$ and by the Radon-Nikodym Theorem of decomposition of a measure, the support of the singular part of μ lies in the set $\{0\}$. Therefore, there exists $a \in \mathbf{R}$ and $w \in \mathrm{L}^1(\mathbf{R}^k)$ such that

$$\int_{\mathbf{R}^k} \psi\, d\mu = a\psi(0) + \int_{\mathbf{R}^k} w\, dx$$

Since $w \in \mathrm{L}^1(\mathbf{R}^k)$, for any fixed $x \neq 0$,

$$0 = \lim_{\rho\to 0} \int_{\mathbf{R}^k} \psi_\rho(\cdot - x)\, d\mu \leqslant \lim_{\rho \to 0} \int_{\mathbf{R}^k} \psi_\rho\, d\mu = a\psi(0).$$

Because $\psi(0) > 0$, $a \geqslant 0$.

Now, let $\varphi \in \mathcal{K}(\mathbf{R}^k \setminus \{0\})$. For sufficiently small $\varepsilon > 0$, it is clear that

$$\left(\varphi + \varphi^*(\frac{\cdot}{\varepsilon})\right)^* = \varphi^*\left(\frac{\cdot}{(1+\varepsilon^k)^{1/k}}\right).$$

Hence,

$$\begin{aligned}\int_{\mathbf{R}^k} \left(\varphi(x) + \varphi^*(\tfrac{x}{\varepsilon})\right) w(x)\, dx + a\varphi^*(0) &= \int_{\mathbf{R}^k} \varphi(x) + \varphi^*(\tfrac{x}{\varepsilon})\, d\mu \\ \leqslant \int_{\mathbf{R}^k} \varphi^*\left(\tfrac{x}{(1+\varepsilon^k)^{1/k}}\right) d\mu &= \int_{\mathbf{R}^k} \varphi^*\left(\tfrac{x}{(1+\varepsilon^k)^{1/k}}\right) w(x)\, dx + a\varphi^*(0).\end{aligned}$$

If $\varepsilon \to 0$, inequality (3.2) follows. For a general $\varphi \in \mathcal{K}_+(\mathbf{R}^k)$, there exists a sequence $(\varphi_m)_{m\in\mathbf{N}}$ in $\mathcal{K}_+(\mathbf{R}^k \setminus \{0\})$ such that $\varphi_m \nearrow \varphi$ almost everywhere. As $m \to \infty$, the inequality (3.2) follows. The conclusion comes from Lemma 3.1. □

4. Riesz-Sobolev rearrangement inequalities

In this section we prove that the Riesz-Sobolev rearrangement inequalities do not hold for an anisotropic symmetrizations unless it is the classical Steiner symmetrization. That is the crucial difference between Steiner and anisotropic symmetrizations. This justifies the approach of the following sections for the Pólya-Szegő inequalities.

If $\cdot^\star$ denotes a Steiner symmetrization, the Riesz–Sobolev inequality (4.1)

$$\int_{\mathbf{R}^N}\int_{\mathbf{R}^N} u(x)\, v(y)\, w(x-y)\, dx\, dy \leqslant \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} u^\star(x)\, v^\star(y)\, w^\star(x-y)\, dx\, dy$$

holds for any nonnegative functions vanishing at infinity u, v and w (see Brascamp, Lieb and Luttinger [5], and Lieb and Loss [18]).

Lemma 4.1. *Let $A \subset \mathbf{R}^N$ and $E \subset \mathbf{R}^N$. If E is an ellipsoid, A is measurable, $\mathcal{L}^N(A) = \mathcal{L}^N(E)$, and*

$$\int_{\mathbf{R}^{2N}} \chi_E(x)\chi_E(y)\chi_E(x-y)\, dx\, dy \leqslant \int_{\mathbf{R}^{2N}} \chi_A(x)\chi_A(y)\chi_A(x-y)\, dx\, dy,$$

then A is an ellipsoid centered qround the origin up to a set of measure 0.

Proof. Since the inequality remains invariant through affine change of variables, suppose $E = \mathrm{B}(0,1)$ without loss of generality. Let $\cdot^\star$ be the Schwarz symmetrization. Then $A^\star = \mathrm{B}(0,1) = E^\star = E$. Since the inequality (4.1) holds for the Schwarz symmetrization,

$$\begin{aligned}\int_{\mathbf{R}^N}\int_{\mathbf{R}^N} &\chi_E(x)\, \chi_E(y)\, \chi_E(x-y)\, dx\, dy \\ &\leqslant \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} \chi_A(x)\, \chi_A(y)\, \chi_A(x-y)\, dx\, dy \\ &\qquad\leqslant \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} \chi_{A^\star}(x)\, \chi_{A^\star}(y)\, \chi_{A^\star}(x-y)\, dx\, dy.\end{aligned}$$

The first and the last term of the inequality are equal. By the work of Burchard on the necessary conditions for equality in the Riesz–Sobolev inequality, A is an ellipsoid centered around the origin up to a set of measure zero [7]. $\square$

Proposition 4.2. *If $\cdot^*$ is an anisotropic symmetrization and the inequality* (4.1) *holds for any $u, v, w \in \mathcal{K}_+(\mathbf{R}^N)$, then G is an euclidian norm on $\mathbf{R}^k$.*

Proof. By standard arguments, the inequality (4.1) holds also for characteristic functions of open sets. Lemma 4.1 with $A = \mathrm{B}(0,1)^*$ brings the conclusion. $\square$

The same arguments shows also that the Riesz-Sobolev rearrangement inequality does not hold for the spherical cap symmetrization and for the polarization (see [4, 22, 26] for definitions).

Corollary 4.3. *If $\cdot^*$ denotes the spherical cap symmetrization or the polarization, the inequality* (4.1) *does not hold for any $u, v, w \in \mathcal{K}_+(\mathbf{R}^N)$.*

Proof. Suppose by contradiction that the inequality (4.1) holds for any $u, v, w \in \mathcal{K}_+(\mathbf{R}^N)$. By standard arguments, the inequality (4.1) holds also for characteristic functions of open sets. By Lemma 4.1, the set E^* should be an ellipsoid when E is an ellipsoid. This is not the case: for the spherical cap symmetrization, take e.g. the ellipsoid

$$\left\{x \in \mathbf{R}^N \ : \ \sum_{i=1}^N i x_i^2\right\} < 1$$

, and for a polarization take an ellipsoid centered on the polarization plane and which is not symmetric with respect to it. This is possible for $N > 1$.

If $N = 1$ and the boundary of the polarizing halfspace is $\{c\}$, then the inequality fails for $u = \chi_{[2c-1,2c+1]}$, $v = \chi_{[-c-1,-c+1]}$ and $w = \chi_{[c-1,c+1]}$. $\square$

The Riesz-Sobolev rearrangement inequality is a strong inequality, which requires good properties with respect to the convolution product. For the spherical cap symmetrization or the polarization the weaker inequality

$$\int_{\mathbf{R}^N}\int_{\mathbf{R}^N} u(x)v(y)w(|x-y|)\,dx\,dy \leqslant \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} u^*(x)v^*(y)w(|x-y|)\,dx\,dy$$

holds for any $u, v \in \mathcal{K}(\mathbf{R}^N)$ and for any decreasing function $w : \mathbf{R}^+ \to \mathbf{R}^+$ [4, 6, 26]. This is not the case for nontrivial anisotropic symmetrization.

Proposition 4.4. *Let $G : \mathbf{R}^k \to \mathbf{R}^+$ be a gauge, and $\cdot^*$ be the anisotropic symmetrization with respect to G. Let μ be a nonnegative Radon measure such that $\int_{\mathbf{R}^N} |x|_2^2\,d\mu < +\infty$. If the inequality*

$$\int_{\mathbf{R}^N}\int_{\mathbf{R}^N} u(x-y)v(y)\,d\mu(x)\,dy \leqslant \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} u^*(x-y)v^*(y)\,d\mu(x)\,dy. \tag{4.2}$$

holds for any $u, v \in \mathcal{K}_+(\mathbf{R}^N)$, then either μ is concentrated on $\mathbf{R}^k \times \{0\}$ or $G(x') = \sqrt{x'^t A x'}$ for some positive definite symmetric matrix $A \in \mathbf{R}^{k\times k}$. Furthermore, for any $u \in \mathcal{K}_+(\mathbf{R}^N)$,

$$\int_{\mathbf{R}^N} u\,d\mu \leqslant \int_{\mathbf{R}^N} u^*\,d\mu. \tag{4.3}$$

If $k = N$, then $\mu = w^* + a\delta_0$, where $w \in L^1_+(\mathbf{R}^N)$, $a \geqslant 0$ and δ_0 is Dirac's measure.

Remark 4.5. If $\mu = w \in L^1_+(\mathbf{R}^N)$, the condition (4.3) and Lemma 3.1 ensure $w = w^*$.

Remark 4.6. The inequality (4.2) always hold for measures concentrated on T. This is a consequence of the Hardy-Littlewood inequality (Proposition 2.28).

Proof. First suppose $k = N$. Without loss of generality, $\mu(\mathbf{R}^N) = 1$. If (4.2) holds for any $u, v \in \mathcal{K}_+(\mathbf{R}^N)$, by density it holds also for any $u \in L^1_+(\mathbf{R}^N)$ and for any $v \in \mathcal{K}_+(\mathbf{R}^N)$. Take $u_1 \in \mathcal{K}_+(\mathbf{R}^N)$ such that $\int_{\mathbf{R}^N} u_1\,dx = 1$, $\int_{\mathbf{R}^N} u_1\,|x|_2^2\,dx < \infty$, and ${u_1}^* = u_1$. For $n \geqslant 1$, let $u_{n+1}(x) = \int_{\mathbf{R}^N} u_n(x-y)\,d\mu(y)$. By Fubini's Theorem, $\int_{\mathbf{R}^N} u_{n+1}\,dx = 1$. Furthermore, since $u_n \in L^1(\mathbf{R}^N)$, inequality (4.2) holds with u_n in place of u, and then, by Lemma 3.3, ${u_{n+1}}^* = u_{n+1}$. Let

$$\bar{x} = \int_{\mathbf{R}^N} x\,d\mu.$$

Consider the sequence of independent identically distributed random variables $(X_n)_{n\geqslant 2}$ with probability law μ and the independent random variable X_1 with distribution law u_1. All X_i have mean $\bar{x}$ for $i \geqslant 2$. The function u_n is the probability distribution of $\sum_{i=1}^n X_i$. By Lindeberg and Lévy's central-limit Theorem (see Stromberg [23]), the sequence $n^{-1/2}\sum_{i=1}^n (X_i - \bar{x})$ converges in law to a normal distribution ν with mean 0, i.e. for any bounded continuous function $\varphi : \mathbf{R}^N \to \mathbf{R}$

$$\int_{\mathbf{R}^N} \varphi(n^{-1/2}(x - n\bar{x}))u_n(y)\,dx \to \int_{\mathbf{R}^N} \varphi\,d\nu, \tag{4.4}$$

where ν is characterized by

$$\int_{\mathbf{R}^N} \varphi(x)\,d\nu = (2\pi)^{-N/2}\int_{\mathbf{R}^N} \varphi(Mx)e^{-x^2/2}\,dx$$

for some fixed linear operator $M : \mathbf{R}^N \to \mathbf{R}^N$. (When M is the identity one recovers the standard normal distribution. The operator M is not necessarily invertible.)

For every $\varphi \in \mathcal{K}_+(\mathbf{R}^N)$, since ${u_n}^* = u_n$, one has

$$\int_{\mathbf{R}^N} \varphi(n^{-1/2}(x - n\bar{x}))u_n(y)\,dx \leqslant \int_{\mathbf{R}^N} \varphi^*(n^{-1/2}x)u_n(y)\,dx. \tag{4.5}$$

Since $\nu(\mathbf{R}^N) = 1$, there is $\varphi \in \mathcal{K}_+(\mathbf{R}^N)$ such that $\varphi \leqslant 1$ and

$$\int_{\mathbf{R}^N} \varphi\,d\nu > \frac{1}{2}.$$

Therefore, for large n both sides of the inequality (4.5) must be strictly greater than 1/2. Since

$$\int_{\mathbf{R}^N} u_n\,dx = 1,$$

this implies that the supports of $\varphi(n^{-1/2}(\cdot - n\bar{x}))$ and $\varphi(n^{-1/2}\cdot)$ have a nonempty intersection for large n. This is only possible if $\bar{x} = 0$.

Since $\bar{x} = 0$, letting $n \to \infty$ in (4.5) yields, by (4.4),

$$\int_{\mathbf{R}^N} \varphi\,d\nu \leqslant \int_{\mathbf{R}^N} \varphi^*\,d\nu.$$

In view of Lemma 3.3, either the normal distribution ν is concentrated at zero, or $\nu = w \in L^1_+(\mathbf{R}^N)$, with $w(x) = e^{-x^t A x}$, $w = w^*$ and $A \in \mathbf{R}^{N\times N}$ is a positive-definite symmetric matrix. In the first case, this implies that $\int_{\mathbf{R}^N} |x|_2^2\,d\mu = \int_{\mathbf{R}^N} |x|_2^2\,d\nu = 0$, and thus that the support of μ lies in T. In the second case, $w = w^*$ means $G(x) = \lambda\sqrt{x^t A x}$ for some $\lambda > 0$.

Now, if $k < N$, let $\bar{\mu}$ denote the projection of μ on $\mathbf{R}^k$, i.e., if $\varphi \in \mathcal{K}(\mathbf{R}^k)$, $\int_{\mathbf{R}^k} \varphi\,d\bar{\mu} = \int_{\mathbf{R}^N} \varphi(x')\,d\mu$. Then the inequality (4.2) holds for $\bar{\mu}$. By the first part of the proof, either G is euclidian or $\bar{\mu}$ is concentrated at 0, whence μ is concentrated on $\{0\} \times \mathbf{R}^k = T$.

For (4.3), take $\rho \in \mathcal{K}_+(\mathbf{R}^N)$ with $\int_{\mathbf{R}^N} \rho\,dx = 1$ and $\rho_\varepsilon = \varepsilon^{-N}\rho(\frac{\cdot}{\varepsilon})$. Inequality (4.2) with $v = \rho_\varepsilon$ gives (4.3) as $\varepsilon \to 0$. □

The Riesz–Sobolev type inequalities (4.1) and (4.2) are useful to prove Pólya–Szegő inequalities (see [4, 18]). Propositions 4.2 and 4.4 show that this is not a valid method for the anisotropic symmetrization.

5. Anisotropic inequalities for Steiner symmetrizations

The objective of this section is to prove that for any Steiner symmetrization $\cdot^\star$, any suitable weakly differentiable u and any function $\varphi : \mathbf{R}^N \to \bar{\mathbf{R}}^+$ such that $\varphi(0) = 0$, the inequality

$$\int_{\mathbf{R}^N} \overline{\varphi_\star}(u^\star)\,dx \leqslant \int_{\mathbf{R}^N} \overline{\varphi}(u)\,dx.$$

holds. Recall that $\varphi_\star$ denotes the increasing Steiner symmetrization of φ, i.e. $\varphi_\star = -\varphi(-\cdot)^\star$ (see Definition 2.33 and Proposition 2.34). Klimov proved this inequality for the Steiner symmetrization with respect to a hyperplane when $\varphi : \mathbf{R}^N \to \mathbf{R}$ is convex and even [17]. He suggested the inequality for a general Steiner symmetrization. We first prove the inequality for the Steiner symmetrization with respect to a hyperplane and then extend it to general Steiner symmetrizations.

Definition 5.1. The Fenchel transform of $\varphi : \mathbf{R}^N \to \bar{\mathbf{R}}$ is

$$\overline{\varphi} : \mathbf{R}^N \to \bar{\mathbf{R}} : t \mapsto \overline{\varphi}(t) = \sup_{x \in \mathbf{R}^N} \langle t, x \rangle - \varphi(x).$$

Remark 5.2. By an abuse of notation, when φ comes from a functional of the form

$$\int_\Omega \varphi(x, u, \nabla u)\, dx,$$

then $\overline{\varphi}$ denotes the Fenchel transform and the symmetrization with respect to the gradient coordinates: $\overline{\varphi}(x, s, \cdot) = \overline{\varphi(x, s, \cdot)}$. The same abuse of notation is made for the symmetrization: $\varphi_\star(x, s, \cdot) = (\varphi(x, s, \cdot))_\star$.

Proposition 5.3. *Let φ and ψ be functions from $\mathbf{R}^N$ to $\bar{\mathbf{R}}$. If $\varphi \leqslant \psi$, then $\overline{\varphi} \geqslant \overline{\psi}$.*
Let $(\varphi_n)_{n \geqslant 1}$ and φ be functions from $\mathbf{R}^N$ to $\bar{\mathbf{R}}$. If $\varphi_n \searrow \varphi$, then $\overline{\varphi_n} \nearrow \overline{\varphi}$.

Proof. Immediate. □

Definition 5.4. An open set Ω is an *extension domain* if there exists a bounded linear operator $E_\Omega : \mathrm{W}^{1,1}(\Omega) \to \mathrm{W}^{1,1}(\mathbf{R}^N)$, such that, for each $u \in \mathrm{W}^{1,1}(\Omega)$, $(E_\Omega u)|_\Omega = u$.

Example 5.5. A Lipschitz domain is an extension domains [1].

Proposition 5.6 (1–dimensional Steiner symmetrization inequality for anisotropic functionals). *Let $\varphi : \mathbf{R}^N \to \bar{\mathbf{R}}^+$ with $\varphi(0) = 0$. If T is a $(N-1)$–dimensional vector subspace of $\mathbf{R}^N$, $\cdot^\star$ is the Steiner symmetrization with respect to T, Ω is an extension domain, Ω is totally invariant with respect to T and $u \in \mathrm{W}^{1,1}_+(\mathbf{R}^N)$, then*

$$\int_\Omega \overline{\varphi_\star}(\nabla u^\star)\, dx \leqslant \int_\Omega \overline{\varphi}(\nabla u)\, dx. \tag{5.1}$$

Remark 5.7. The hypothesis $\varphi(0) = 0$ ensures $\overline{\varphi}(t) \geqslant 0$ for $t \in \mathbf{R}^N$, while $\varphi \geqslant 0$ implies $\overline{\varphi}(0) \leqslant 0$.

Definition 5.8. A function $u : \Omega \to \mathbf{R}$ is simplicial if it is continuous, it has a bounded support, and if there exists a finite collection of open sets $(S_i)_{1 \leqslant i \leqslant n}$, such that $\tilde{u}|_{S_i}$ is an affine function for each $1 \leqslant i \leqslant n$ and u vanishes outside the closure of $\cup_{i=1}^n S_i$.

It is standard that if Ω is an extension domain, then simplicial functions are dense in $\mathrm{W}^{1,1}(\Omega)$ [13, Chapter X, section 2.1]. Furthermore, simplicial functions with $\frac{\partial u}{\partial x_1} \neq 0$ in $\cup_{i=1}^n S_i$ are also dense in $\mathrm{W}^{1,1}(\Omega)$.

Proof of Proposition 5.6. The proof is an adaptation from the proof of Klimov [17], with modifications allowing more general functions φ and other domains than $\mathbf{R}^N$. For simple functions, it relies on the geometric results in Lemma 5.12 and a coarea formula. The result is extended by density to non-simplicial functions with some restriction on φ and is finally generalized to any function φ by Levi's monotone convergence Theorem.

Step 1: u is a simplicial function. Without loss of generality, let $T = \{0\}\times\mathbf{R}^{N-1}$, and $\Omega = \mathbf{R}\times\Omega''$. Suppose $u \in \mathrm{W}^{1,1}(\Omega)$ is a nonnegative simplicial function such that $\partial_1 u \neq 0$ on $\bigcup_{i=1}^n S_i$. On each set S_i, ∇u is constant. We have thus

$$\begin{aligned}\int_{S_i} \overline{\varphi}(\nabla u)dx &= \int_{\Omega''}\int_{\{x_1\in\mathbf{R}\,:\,(x_1,x'')\in S_i\}} \overline{\varphi}(\nabla u)\,dx_1\,dx'' \\ &= \int_{\Omega''}\int_{\left\{s>0\,:\,(u|_{S_i\cap\{(x'',t)\,:\,t\in\mathbf{R}\}})^{-1}(\{s\})\neq\phi\right\}} \frac{\overline{\varphi}(\nabla u)}{|\partial_1 u|}\,ds\,dx'' \\ &= \int_{\Omega''}\int_{s>0}\sum_{(x_1,x'')\in(u^{-1}(\{s\})\cap S_i)} \frac{\overline{\varphi}(\nabla u(x_1,x''))}{|\partial_1 u(x_1,x'')|}\,ds\,dx_1,\end{aligned}$$

where the sum contains zero or one term. Summing over i gives, since $\nabla u(x) = 0$ for almost all $x \notin \bigcup_{i=1}^n S_i$, and $\overline{\varphi}(0) = 0$ by Remark 5.7,

$$\int_{\Omega} \overline{\varphi}(\nabla u)dx = \int_{\Omega''}\int_{s>0}\sum_{(x_1,x'')\in u^{-1}(\{s\})} \frac{\overline{\varphi}(\nabla u(x_1,x''))}{|\partial_1 u(x_1,x'')|}\,ds\,dx''.$$

For each $x'' \in \Omega''$, the sum contains always a finite number of terms (at most n). Furthermore, the number of terms for which $\partial_1 u(x_1,x'') > 0$ is equal to the number of terms for which $\partial_1 u(x_1,x'') < 0$. Similarly,

$$\begin{aligned}\int_{\Omega}\overline{\varphi_\star}(\nabla u^\star)\,dx &= \int_{\Omega''}\int_{s>0}\sum_{(x_1,x'')\in u^{-1}(s)} \frac{\overline{\varphi_\star}(\nabla u^\star(x_1,x''))}{|\partial_1 u^\star(x_1,x'')|}\;ds\;dx'' \\ &= 2\int_{\Omega''}\int_{s>0} \frac{\overline{\varphi_\star}(\nabla u^\star)}{|\partial_1 u^\star|}\left((u^\star|_{\Omega''\times\mathbf{R}^+})^{-1}(\{s\})\right)\;ds\;dx''.\end{aligned}$$

For all $x'' \in \Omega''$, the definition of the Steiner symmetrization gives, for all but a finite number of $s \in (0, \sup u(\cdot, x''))$,

$$2\frac{\nabla'' u^\star}{|\partial_1 u^\star|}\big((u^\star|_{\mathbf{R}^+\times\{x''\}})^{-1}(s)\big) = \sum_{(x_1,x'')\in u^{-1}(s)} \frac{\nabla'' u(x_1,x'')}{|\partial_1 u(x_1,x'')|},$$

$$2\frac{1}{|\partial_1 u^\star|}\big((u^\star|_{\mathbf{R}^+\times\{x''\}})^{-1}(s)\big) = \sum_{(x_1,x'')\in u^{-1}(s)} \frac{1}{|\partial_1 u(x_1,x'')|}.$$

Then by Lemma 5.12, inequality (5.1) holds for u.

Step 2: extension to $u \in \mathrm{W}^{1,1}_+(\Omega)$. Suppose $u \in \mathrm{W}^{1,1}_+(\Omega)$ and that there exists $R \in \mathbf{R}$ such that $\varphi(x) = \infty$ if $|x|_2 > R$. Then $\overline{\varphi}(t) \leqslant |t|_2/R$ and $\overline{\varphi_\star}(t) \leqslant |t|_2/R$ (since the right-hand side of the inequality is symmetrization-invariant). The functionals on both sides of (5.1) are continuous in $\mathrm{W}^{1,1}(\Omega)$. Since simplicial functions are dense in $\mathrm{W}^{1,1}(\Omega)$ and the Steiner symmetrization is continuous in $\mathrm{W}^{1,1}(\Omega)$ [8], the inequality follows.

(Alternatively, the nonexpansiveness of symmetrization in $\mathrm{L}^1(\Omega)$ and the classical Pólya–Szegő inequality can be used to prove that if $u_n \to u$ in $\mathrm{W}^{1,1}(\Omega)$, then ${u_n}^\star \rightharpoonup u^\star$ in $\mathrm{W}^{1,1}(\Omega)$. The inequality comes then from the weak lower semi-continuity of the left-hand side with respect to $u \in \mathrm{W}^{1,1}(\Omega)$.)

Step 3: general φ. For general $u \in \mathrm{W}^{1,1}_+(\Omega)$ and $\varphi : \mathbf{R}^N \to \bar{\mathbf{R}}^+$, let

$$\varphi_n(x) = \begin{cases} \varphi(x) & \text{if } |x|_2 \leqslant n, \\ +\infty & \text{if } |x|_2 > n \end{cases}$$

for $n \geqslant 1$. Since $\varphi_n \searrow \varphi$ for $x \in \Omega$, Propositions 2.20 and 5.3 imply $\varphi_{n\star} \searrow \varphi_\star$, $\overline{\varphi_n} \nearrow \overline{\varphi}$ and $\overline{\varphi_{n\star}} \nearrow \overline{\varphi_\star}$. The inequality follows by Levi's monotone convergence Theorem. □

We now prove Klimov's geometric Lemma used in the proof of the inequality for the 1–dimensional Steiner symmetrization. The simplest case is when φ is the indicator function of some set A:

$$\varphi(x) = \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{if } x \notin A. \end{cases}$$

Definition 5.9. The function

$$\delta_A(t) = \overline{\varphi}(t) = \sup_{x\in A} \langle t, x\rangle$$

is the *support function of A*.

Lemma 5.10. *Let $\cdot^\star$ be the Steiner symmetrization with respect to $\{0\}\times\mathbf{R}^{N-1}$ and $A\subseteq\mathbf{R}^N$. For any $a,b\in\mathbf{R}^{N-1}$,*

$$\delta_A(1,a)+\delta_A(-1,b)\geqslant\delta_{A^\star}(2,a+b).$$

Remark 5.11. Klimov proves this result for a convex compact set A. His representation of the set $A=\{(x_1,x'')\ :\ h(x'')\leqslant x_1\leqslant g(x'')\}$ is not anymore valid, but his arguments can be adapted to sets which are not bounded, measurable and convex.

Proof. For $x''\in\mathbf{R}^{N-1}$, let

$$\begin{aligned}m_A(x'')&=\inf\left\{x_1\ :\ (x_1,x'')\in A\right\},\\ M_A(x'')&=\sup\left\{x_1\ :\ (x_1,x'')\in A\right\}.\end{aligned}$$

Then

$$\begin{aligned}\delta_A(1,a)&=\sup_{x''\in\mathbf{R}^{N-1}}\langle a,x''\rangle+M_A(x''),\\ \delta_A(-1,b)&=\sup_{x''\in\mathbf{R}^{N-1}}\langle b,x''\rangle-m_A(x'').\end{aligned}$$

The identities hold with $A^\star$ in place of A. The inequality comes from the fact that that for all $x''\in\mathbf{R}^{N-1}$, $2M_{A^\star}(x'')=2m_{A^\star}(x'')\leqslant M_A(x'')-m_A(x'')$ by definition of the Steiner symmetrization. □

Lemma 5.12 (Klimov). *Let $\varphi:\mathbf{R}^N\to\bar{\mathbf{R}}$, $a_i,b_i\in\mathbf{R}^{N-1}$, $\alpha_i,\beta_i\in\mathbf{R}^+$, for all $1\leqslant i\leqslant m$. If $\alpha=\sum_{i=1}^m\alpha_i$, $\beta=\sum_{i=1}^m\beta_i$, $a=\sum_{i=1}^m a_i$ and $b=\sum_{i=1}^m b_i$, then*

(5.2)
$$\sum_{i=1}^m\overline{\varphi}\left(\frac{1}{\alpha_i},\frac{a_i}{\alpha_i}\right)\alpha_i+\overline{\varphi}\left(-\frac{1}{\beta_i},\frac{b_i}{\beta_i}\right)\beta_i\geqslant\overline{\varphi_\star}\left(\frac{2}{\alpha+\beta},\frac{a+b}{\alpha+\beta}\right)(\alpha+\beta).$$

Proof. Without loss of generality, the left hand side of (5.2) is finite. By definition of the Fenchel transform,

$$\begin{aligned}\overline{\varphi}(t)=\sup_{x\in\mathbf{R}^N}\langle t,x\rangle-\varphi(x)&=\sup_{\substack{x\in\{\varphi<\lambda\}\\ \lambda\in\mathbf{R}}}\langle t,x\rangle-\lambda\\ =\sup_{\lambda\in\mathbf{R}}\left[\sup_{x\in\{\varphi<\lambda\}}\langle t,x\rangle-\lambda\right]&=\sup_{\lambda\in\mathbf{R}}\left(\delta_{\{\varphi<\lambda\}}(t)-\lambda\right).\end{aligned}$$

Thus for any $\lambda \in \mathbf{R}$,

$$\begin{aligned}
(5.3)\quad & \sum_{i=1}^{m} \left[\overline{\varphi}\left(\frac{1}{\alpha_i}, \frac{a_i}{\alpha_i}\right) \alpha_i + \overline{\varphi}\left(-\frac{1}{\beta_i}, \frac{b_i}{\beta_i}\right) \beta_i \right] \\
& \geqslant \sum_{i=1}^{m} \left[\delta_{\{\varphi<\lambda\}}\left(\frac{1}{\alpha_i}, \frac{a_i}{\alpha_i}\right) - \lambda \right] \alpha_i + \left[\delta_{\{\varphi<\lambda\}}\left(-\frac{1}{\beta_i}, \frac{b_i}{\beta_i}\right) - \lambda \right] \beta_i \\
& = \left[\sum_{i=1}^{m} \delta_{\{\varphi<\lambda\}}(1, a_i) + \delta_{\{\varphi<\lambda\}}(-1, b_i) \right] - \lambda(\alpha+\beta).
\end{aligned}$$

By Lemma 5.10, and since the function $\delta_{\{\varphi\leqslant\lambda\}^\star}(t_1, t'')$ increases with respect to t_1 when $t_1 \geqslant 0$,

$$\begin{aligned}
\sum_{i=1}^{m} & \left[\overline{\varphi}\left(\frac{1}{\alpha_i}, \frac{a_i}{\alpha_i}\right) \alpha_i + \overline{\varphi}\left(-\frac{1}{\beta_i}, \frac{b_i}{\beta_i}\right) \beta_i \right] \\
& \geqslant \delta_{\{\varphi<\lambda\}^\star}(2m, a+b) - \lambda(\alpha+\beta) \\
& \geqslant \delta_{\{\varphi<\lambda\}^\star}(2, a+b) - \lambda(\alpha+\beta).
\end{aligned}$$

This implies the inequality since, from the first part of the proof and from Proposition 2.20,

$$\overline{\varphi_\star}(t) = \sup_{\lambda\in\mathbf{R}} \left[\delta_{\{\varphi_\star<\lambda\}}(t) - \lambda \right] = \sup_{\lambda\in\mathbf{R}} \left[\delta_{\{\varphi<\lambda\}^\star}(t) - \lambda \right]. \qquad \square$$

We extend the inequality to Steiner symmetrizations with respect to higher dimensional subspaces by approximation, as suggested by Klimov [17]. Since the approximation procedure in Hausdorff distance of open sublevel sets is unusual, we give a complete proof.

Proposition 5.13 (Steiner symmetrization inequality for anisotropic functionals). *Let $T \subset \mathbf{R}^N$ be a vector space, Ω a totally invariant extension domain, $\cdot^\star$ denote the Steiner symmetrization with respect to T, $u \in \mathrm{W}^{1,1}_+(\Omega)$ and $\varphi : \mathbf{R}^N \to \bar{\mathbf{R}}^+$ such that $\varphi(0) = 0$. If u vanishes at the infinity with respect to $\cdot^\star$, and for any $M > 0$,*

$$\sup_{\mathcal{L}^N(A)=M} \int_A |\nabla u|_2 + |u| \, dx < +\infty, \tag{5.4}$$

then $u^\star \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega \overline{\varphi_\star}(\nabla u^\star) \, dx \leqslant \int_\Omega \overline{\varphi}(\nabla u) \, dx. \tag{5.5}$$

Remark 5.14. In general, when $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega)$, it is not true that $u^* \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega)$. The condition (5.4) is a slightly stronger than $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega)$ and guarantees that $u^* \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega)$.

Proof. The inequality is established by approximation of the symmetrization for $u \in \mathrm{W}^{1,1}_{+}(\Omega)$ and φ with a finite image, and then extended to general φ and then general u.

Step 1: φ has a finite image. First suppose $u \in \mathrm{W}^{1,1}_{+}(\Omega)$, φ is lower semi-continuous and coercive, and φ has a finite image, i.e. the set $\varphi(\mathbf{R}^N)$ is finite. The conclusion is contained in Proposition 5.6 if $\dim T = N-1$. From now on, $\dim T < N-1$. By classical approximation results in symmetrization theory [5,6,18,25], there exists a sequence of $(N-1)$–dimensional hyperplanes $(T_n)_{n\geqslant 1}$ such that, if $\cdot^{T_n}$ denotes the Steiner symmetrization with respect to T_n,

(i) for any $n \geqslant 1$, $T \subset T_n$,

(ii) the iterated sequence of symmetrizations $u_n = u^{T_1\cdots T_n}$ converges to $u^\star$:

$$\begin{aligned} u_n &\rightharpoonup u^\star \quad \text{in } \mathrm{W}^{1,1}(\Omega),\\ u_n &\to u^\star \quad \text{in } \mathrm{L}^1(\Omega); \end{aligned}$$

(iii) if A is measurable and $A^\star$ is bounded and open, then the sequence of iterated symmetrizations $A_n = A^{T_1\cdots T_n}$, converges to $A^\star$ in the sense that small neighborhoods of A_n contain $A^\star$:

$$\lim_{n\to\infty} \sup_{x\in A^\star} \operatorname{dist}(x, A_n) = 0. \tag{5.6}$$

This last assertion is proved in Lemma 5.16.

Since the function φ is lower semi-continuous and $\lim_{|t|_2\to\infty} \varphi(t) = +\infty$, the set $\{\varphi < \lambda\}$ is open and bounded for each $\lambda \geqslant 0$. Hence the set $\{\varphi < \lambda\}^*$ is open and bounded. Since φ has a finite image, the convergence of sublevel sets is uniform with respect to levels in (5.6):

$$\lim_{n\to\infty} \sup_{\substack{\lambda>0\\ x\in\{\varphi_\star<\lambda\}}} \operatorname{dist}(x, \{\varphi<\lambda\}^{T_1\ldots T_n}) = 0.$$

Thus for any $\varepsilon > 0$, there exists $n_0 \geqslant 0$ such that for all $n \geqslant n_0$, $x \in \mathbf{R}^N$, there exists $y \in \mathbf{R}^N$ such that $|x-y|_2 \leqslant \varepsilon$ and $\varphi_\star(x) \geqslant$

$\varphi_n(y) = \varphi_{T_1 \cdots T_n}(y)$. Therefore

$$\overline{\varphi_\star}(t) = \sup_{x \in \mathbf{R}^N} \langle t, x\rangle - \varphi_\star(x) \leqslant \varepsilon\, |t|_2 + \sup_{y \in \mathbf{R}^N} \langle t, y\rangle - \varphi_n(y) = \varepsilon\, |t|_2 + \overline{\varphi_n}(t).$$

and

$$\int_\Omega \overline{\varphi_\star}(\nabla u_n) \leqslant \varepsilon\, \|\nabla u_n\|_1 + \int_\Omega \overline{\varphi_n}(\nabla u_n).$$

Since $\overline{\varphi_\star}$ is convex and lower semi-continuous in $\mathbf{R}^N$, the left-hand side is lower semi-continuous in $\mathrm{W}^{1,1}(\Omega)$. By induction on Proposition 5.6 and by letting $n \to \infty$,

$$\begin{aligned}
\int_\Omega \overline{\varphi_\star}(\nabla u^\star)\, dx &\leqslant \liminf_{m\to\infty} \int_\Omega \overline{\varphi_\star}(\nabla u_n)\, dx \\
&\leqslant \liminf_{m\to\infty} \varepsilon\, \|\nabla u_n\|_1 + \liminf_{m\to\infty} \int_\Omega \overline{\varphi_n}(\nabla u_n)\, dx \\
&\leqslant \varepsilon\, \|\nabla u\|_1 + \int_\Omega \overline{\varphi}(\nabla u)\, dx.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the result follows.

Step 2: general φ. If φ does not have a finite image or is not coercive, but is lower semi-continuous, then it can be approximated by a decreasing sequence of coercive and lower semi-continuous functions with a finite image $\varphi_n \searrow \varphi$. Because $\overline{\varphi_n} \nearrow \overline{\varphi}$, the result follows by Levi's monotone convergence Theorem. If φ is not lower semi-continuous, let $\psi(x) = \liminf_{y\to x} \varphi(y)$. Then $\overline{\psi} = \overline{\varphi}$ and $\overline{\psi_\star} \geqslant \overline{\varphi_\star}$, whence the inequality for φ follows from the inequality for ψ.

Step 3: general u. Let $u_m = \max(u - 1/m, 0)$. It is clear that u_m converges uniformly to u, so that ${u_m}^\star$ converges to $u^\star$ uniformly and in $\mathrm{L}^1_{\mathrm{loc}}(\Omega)$. Furthermore u_m converges to u in $\mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega)$ by Lebesgue's dominated convergence Theorem. The sequence of functions $|\nabla {u_m}^\star|_2$ in $\mathrm{L}^1_{\mathrm{loc}}(\Omega)$ is also nondecreasing. By a result of Alvino, Ferone and Lions [2], for any compact subset K of Ω,

$$\int_K |\nabla {u_m}^\star|_2 \leqslant \sup_{\mathcal{L}^N(L) = \mathcal{L}^N(K)} \int_L |\nabla u_m|_2\, .$$

Hence, if $g(x) = \lim_{n\to\infty} \nabla u_m(x)$, by Levi's monotone convergence theorem,

$$\int_K |g|_2 < +\infty.$$

This implies, by Lebesgue's dominated convergence Theorem, that

$$\nabla {u_m}^\star \to g \qquad\qquad \text{in } \mathrm{L}^1_{\mathrm{loc}}(\Omega).$$

Thus $u^\star \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega)$. The inequality follows by Levi's monotone convergence Theorem. □

We have to prove the convergence result of iterated symmetrizations of sets. Since our definition of symmetrization is different from the classical one that maps compact sets into compact sets, we cannot use the classical results on approximation in Hausdorff distance. Lemma 5.15 is a general measure-theoretic convergence result.

Lemma 5.15. *Suppose $(A_n)_{n\geqslant 1}$ is a sequence of measurable sets in $\mathbf{R}^N$, $G \subset \mathbf{R}^N$ is open, bounded, and nonempty. If*

$$\lim_{n\to\infty} \mathcal{L}^N(G \setminus A_n) = 0, \tag{5.7}$$

then

$$\lim_{n\to\infty} \sup_{x\in G} \operatorname{dist}(x, A_n) = 0,$$

where

$$\operatorname{dist}(x, A) = \inf_{y\in A} \operatorname{dist}(x, y).$$

Proof. Suppose the conclusion is false. Then there exists an increasing sequence $(n_k)_{k\in\mathbf{N}}$ in $\mathbf{N}$, a sequence $x_k \in G$ and $\delta > 0$ such that, for each $k \in \mathbf{N}$,

$$d(x_k, A_{n_k}) > \delta.$$

Since G is a bounded subset of $\mathbf{R}^N$, the sequence $(x_k)_{k\in\mathbf{N}}$ has a subsequence $(x_{k_\ell})_{\ell\in\mathbf{N}}$ that converges to $\tilde{x} \in \bar{G}$. If ℓ is sufficiently large,

$$\phi \neq \mathrm{B}(\tilde{x}, \tfrac{\delta}{2}) \cap G \subset \mathrm{B}(x_{k_\ell}, \delta) \cap G \subseteq G \setminus A_{n_{k_\ell}}.$$

Because $\mathrm{B}(\tilde{x}, \frac{\delta}{2}) \cap G$ is open and not empty,

$$\mathcal{L}^N(G \setminus A_{n_{k_\ell}}) \geqslant \mathcal{L}^N(\mathrm{B}(\tilde{x}, \tfrac{\delta}{2}) \cap G) > 0,$$

in contradiction with (5.7). □

Lemma 5.16. *Suppose $G \in \mathbf{R}^N$ is a bounded open set and $(G_n)_{n\geqslant 1}$ is the sequence of sets obtained by iterated Steiner symmetrizations of G. Then*

$$\lim_{n\to\infty} \sup_{x\in G^\star} \operatorname{dist}(x, G_n) = 0.$$

Proof. This comes from the fact that $G^\star$ is open, the convergence in measure of G_n to $G^\star$ and Lemma 5.15. □

Proposition 5.17. *Let Ω be a totally invariant domain, $\varphi : \Omega \times \mathbf{R}^+ \times \mathbf{R}^N \to \bar{\mathbf{R}}^+$ and $u \in \mathrm{W}^{1,1}_{\mathrm{loc},+}(\Omega)$ vanishing at the infinity with respect to $\cdot^*$. If $\varphi(\cdot, s, \xi)$ is totally invariant with respect to $\cdot^\star$ for each $(s,\xi) \in \mathbf{R}^+ \times \mathbf{R}^N$, $\varphi(\cdot,\cdot,0) = 0$, $\varphi(\cdot,\cdot,\xi)$ is lower semi-continuous for each $\xi \in \mathbf{R}^N$ and if each $x \in \Omega$ has a totally invariant neighborhood $\mathcal{N}_x \subset \Omega$ such that for any $M > 0$,*

$$\sup_{\substack{A \subset \mathcal{N}_x \\ \mathcal{L}^N(A) = M}} \int_A \lvert\nabla u\rvert_2 + \lvert u\rvert \, dx < +\infty, \tag{5.8}$$

then

$$\int_\Omega \overline{\varphi_\star}(x, u^\star(x), \nabla u^\star(x))\, dx \leqslant \int_\Omega \overline{\varphi}(x, u(x), \nabla u(x))\, dx.$$

Proof. Without loss of generality, $T = \mathbf{R}^k$ and $\Omega = \mathbf{R}^{N-k} \times \Omega''$. Let

$$\begin{aligned}\mathcal{B}_m = \Big\{\beta = \mathbf{R}^{N-k} \times \Big(\frac{\ell_{N-k+1}}{2^m}, \frac{\ell_{N-k+1}+1}{2^m}\Big) \times \cdots \\ \cdots \times \Big(\frac{\ell_N}{2^m}, \frac{\ell_N+1}{2^m}\Big) \times \Big(\frac{n}{2^m}, \frac{n+1}{2^m}\Big) \;: \\ \ell_i \in \mathbf{Z},\ n \in \mathbf{N},\ \beta \subset \Omega \times \mathbf{R}^+ \\ \text{and } u \text{ verifies (5.8) with } \mathcal{N}_x = P(\beta)\Big\}.\end{aligned}$$

where P is the projection $P : \Omega \times \mathbf{R}^+ \to \Omega : (x,s) \mapsto x$. Let $\Omega_m = P(\cup_{\beta\in\mathcal{B}_m}\beta)$. For $(x,s) \in \Omega_m \times \mathbf{R}^+$, let $\beta_m(x)$ denote the unique $\beta \in \mathcal{B}_m$ such that $(x,s) \in \beta$. Let $\omega_m(x) = P(\beta_m(x))$. For any $(x,s) \in \Omega_m \times \mathbf{R}^+$, let

$$\varphi_m(x,s,\xi) = \sup_{(y,t)\in\beta_m(x,s)} \varphi(y,t,\xi).$$

Fix $\ell > 0$. From Proposition 5.13, it is clear that for any $\omega_m(x) \subset \Omega_m$, and $u_{n,m} = \max(\frac{n}{2^m}, \min(u, \frac{n+1}{2^m}))$,

$$\int_{\omega_m(x)} \overline{\varphi_{m\star}}(x, {u_{n,m}}^\star, \nabla {u_{n,m}}^\star) \leqslant \int_{\omega_m(x)} \overline{\varphi_m}(x, u_{n,m}, \nabla u_{n,m})$$

since $\varphi_m(\cdot,\cdot,\xi)$ is constant on β for any $\xi \in \mathbf{R}^N$. Since $\Omega_\ell \subset \Omega_m$ up to a set of measure zero, the sum with fixed m for all $\omega_m(x)$ with $\omega_m(x) \subset \Omega_\ell$ and $n \geqslant 0$ is

$$\int_{\Omega_\ell} \overline{\varphi_{m\star}}(x, u^\star, \nabla u^\star) \leqslant \int_{\Omega_\ell} \overline{\varphi_m}(x, u, \nabla u).$$

Since $\varphi(\cdot,\cdot,\xi)$ is lower semi-continuous, $\varphi_m(\cdot,\cdot,\xi) \searrow \varphi(\cdot,\cdot,\xi)$ as $k \to \infty$. Therefore $\overline{\varphi_m}(x,s,\cdot) \nearrow \overline{\varphi}(x,s,\cdot)$ and $\overline{\varphi_{m\star}}(x,s,\cdot) \nearrow \overline{\varphi_\star}(x,s,\cdot)$, so that by Levi's monotone convergence Theorem,

$$\int_{\Omega_\ell} \overline{\varphi_\star}(x, u^\star, \nabla u^\star) \leqslant \int_{\Omega_\ell} \overline{\varphi}(x, u, \nabla u).$$

The conclusion comes from Levi's monotone convergence Theorem for $\ell \to \infty$ and from the fact that $\cup_{\ell \in \mathbf{N}} \Omega_\ell = \Omega$ up to a set of zero measure. □

Proposition 5.18. *Let $I : \Omega \times \mathbf{R}^+ \times \mathbf{R}^N \to \bar{\mathbf{R}}^+$ be lower semi-continuous, suppose $I(\cdot, s, \xi)$ is totally invariant with respect to L and, for any $x \in \Omega$, $I(x, \cdot)$ is convex and $\lim_{|s|+|\xi|_2 \to \infty} I(x, s, \xi) = +\infty$. If*

$$\int_\Omega I(x, u, \nabla u)\, dx < \infty,$$

then for any $x \in \Omega$, there exists a neighborhood $\mathcal{N}_x$ totally invariant with respect to L such that for any $M > 0$,

$$\sup_{\substack{A \subset \mathcal{N}_x \\ \mathcal{L}^N(A) = M}} \int_A |\nabla u|_2 + |u|\ dx < +\infty.$$

Proof. Let $x \in \mathbf{R}^N$. Since $I(x, \cdot)$ is coercive, there exists $R > 0$ such that $I(x, s, \xi) > 1$ if $|s| + |\xi|_2 = R$. Since I is lower semi-continuous and the set $\{x\} \times \{(s, \xi)\ :\ |s| + |\xi|_2 = R\}$ is compact, this remains true in a neighborhood of this set. Hence, there is a neighborhood $\mathcal{N}_x$ of x such that $I(y, s, \xi) > 1$ if $|s| + |\xi|_2 = R$ and $y \in \mathcal{N}_x$. Since $I(\cdot, s, \xi)$ is totally invariant, without loss of generality, $\mathcal{N}_x$ is totally invariant. By convexity of $I(x, \cdot)$, $I(y, s, \xi) > (|s| + |\xi|_2)/R$ if $(|s| + |\xi|_2) \geqslant R$. Therefore $I(y, s, \xi) > (|s| + |\xi|_2)/R - 1$. If $A \subset \mathcal{N}_x$ and $\mathcal{L}^N(A) = M$, then

$$\begin{aligned}
\int_A |(u, \nabla u)|_2\ dx &\leqslant R \int_A I(x, u, \nabla u) + 1\, dx \\
&\leqslant R \int_A I(x, u, \nabla u)\, dx + R\mathcal{L}^N(A) \\
&\leqslant R \int_\Omega I(x, u, \nabla u) + RM < +\infty.
\end{aligned}$$

Since the right-hand side does not depend on A, the proof is complete. □

6. Inequalities for anisotropic symmetrization

Definition 6.1. The vector $t \in \mathbf{R}^N$ is a subdifferential of $f : \mathbf{R}^N \to \overline{\mathbf{R}}$ at $x \in \mathbf{R}^N$ if for all $y \in \mathbf{R}^N$,

$$f(y) \geqslant f(x) + \langle t, y - x \rangle.$$

The set of the subdifferentials of f at x is denoted $\partial f(x)$.

It is standard in the theory of convex functions that if $\varphi : \mathbf{R}^N \to \mathbf{R}$ is convex, then $\partial\varphi(x)$ is nonempty for every $x \in \mathbf{R}^N$. If φ is also Gateaux-differentiable at $x \in \mathbf{R}^N$, then $\partial\varphi(x) = \{\nabla\varphi(x)\}$.

Proposition 6.2. *If $H \in \mathcal{H}(\mathbf{R}^k)$, then for all $t \in \mathbf{R}^k$, there exists $x \in \partial H^\circ(t)$ such that $H(x) = 1$. In particular, if H° is differentiable at t, $H(\nabla H^\circ(t)) = 1$.*

Proof. Let $t \in \mathbf{R}^k$. By definition of H° and by positive homogeneity of H,

$$H^\circ(t) = \sup_{\substack{y \in \mathbf{R}^N \\ y \neq 0}} \frac{\langle t, y \rangle}{H(y)} = \sup_{|y|_2 = 1} \frac{\langle t, y \rangle}{H(y)}.$$

Since the function on right-hand side is upper semi-continuous, its least upper bound is attained for some $x \in \mathbf{R}^N$. Since H is positively homogeneous, without loss of generality, $H(x) = 1$ and $H^\circ(t) = \langle t, x \rangle$. For any $s \in \mathbf{R}^k$

$$H^\circ(s) \geqslant \langle s, x \rangle = H^\circ(t) + \langle s - t, x \rangle. \qquad \square$$

Lemma 6.3. *If $G_1, G_2 : \mathbf{R}^N \to \mathbf{R}^+$ are gauges, then the function*

$$\Psi : \mathbf{R}^N \to \mathbf{R}^N : x \mapsto x \frac{G_1(x)}{G_2(x)}$$

is Lipschitz-continuous.

Proof. Since $|G_i(x) - G_i(y)| \leqslant G_i(x - y)$, there holds

$$\begin{aligned}
&|\Psi(x) - \Psi(y)|_2 \\
&\quad \leqslant \left| x\tfrac{G_1(x)}{G_2(x)} - x\tfrac{G_1(y)}{G_2(x)} \right|_2 + \left| x\tfrac{G_1(y)}{G_2(x)} - x\tfrac{G_1(y)}{G_2(y)} \right|_2 + \left| x\tfrac{G_1(y)}{G_2(y)} - y\tfrac{G_1(y)}{G_2(y)} \right|_2 \\
&\quad\quad \leqslant |x - y|_2 \Bigg[\sup_{z \in \mathbf{R}^N} \tfrac{G_1(z)}{|z|_2} \cdot \sup_{x \in \mathbf{R}^N} \tfrac{|x|_2}{G_2(x)} \\
&\quad\quad\quad + \sup_{x \in \mathbf{R}^N} \tfrac{|x|_2}{G_2(x)} \cdot \sup_{y \in \mathbf{R}^N} \tfrac{G_1(y)}{G_2(y)} \cdot \sup_{z \in \mathbf{R}^N} \tfrac{G_2(z)}{|z|_2} + \sup_{z \in \mathbf{R}^N} \tfrac{G_1(z)}{G_2(z)} \Bigg].
\end{aligned}$$

Since all least upper bounds can be restricted to the unit sphere in $\mathbf{R}^N$ and G_1, G_2 and $|\cdot|_2$ are continuous and do not vanish on the unit sphere, the function Ψ is Lipschitz-continuous. $\square$

Proposition 6.4 (Gauge change of variable). *Let* $\Omega'' \subset \mathbf{R}^{N-k}$, $\Omega = \mathbf{R}^k \times \Omega''$, $w : \mathbf{R}^+ \times \Omega'' \to \mathbf{R}$, $H \in \mathcal{H}(\mathbf{R}^k)$, $u(x) = w(K_{H^\circ}^{-1} |x'|_2, x'')$ *and* $v(x) = (H^\circ(-x'), x'')$*. Then,* $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega)$ *if and only if* $v \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega)$*. Furthermore, for any* $f : \Omega'' \times \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^{N-k} \to \mathbf{R}$,

$$\int_\Omega f(x'', v, H(\nabla' v), \nabla'' v)\, dx = \int_{\mathbf{R}^N} f(x'', u, K_{H^\circ} \left|\nabla' u\right|_2, \nabla'' u)\, dx,$$

provided one of the integral exists.

Remark 6.5. Recall that for any gauge $G : \mathbf{R}^k \to \mathbf{R}$, K_G is the positive constant given by Definition 2.4.

Proof. Since v is a obtained by a bi-Lipschitzian mapping from weakly differentiable u, it is also weakly differentiable [28, Theorem 2.2.2, p. 52].

Since H° is Lipschitz, it is almost everywhere differentiable, its weak derivative coincides almost everywhere with its unique subgradient, and, by Proposition 6.2, $H(-\nabla H^\circ(-x')) = 1$ almost everywhere. Hence

$$\begin{aligned} f(x'', u, K_{H^\circ} &\left|\nabla' u\right|_2, \nabla'' u)(x', x'') \\ &= f(x'', w, -\partial_r w, \nabla'' w)(K_{H^\circ}^{-1} \left|x'\right|_2, x'') \end{aligned}$$

and

$$\begin{aligned} f(x'', v, &H(\nabla' v), \nabla'' v)(x', x'') \\ &= f(x'', w(H^\circ(-x'), x''), H(\partial_r w, \nabla H^\circ(-x')), \nabla'' w(H^\circ(-x'), x'')) \\ &= f(x'', w, -\partial_r w, \nabla'' w)(H^\circ(-x'), x''). \end{aligned}$$

Since $\mathcal{L}^k(\{H^\circ(-x') < \lambda\}) = \mathcal{L}^k(\{|x'|_2 < \lambda K_{H^\circ}\})$ for all $\lambda \in \mathbf{R}$, the equality follows. $\square$

Corollary 6.6. *Let* $H \in \mathcal{H}(\mathbf{R}^k)$, $\cdot^*$ *denote the* (G, L, T)*-anisotropic symmetrization and* $\cdot^\star$ *denote the Steiner symmetrization with respect to* T*. If* Ω *is totally invariant with respect to* $\cdot^*$, $f : \mathbf{R}^+ \times \mathbf{R}^N \times \Omega \to \mathbf{R}$, $f(\cdot, s, \eta, \xi)$ *is totally invariant for each* $(s, \eta, \xi) \in \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^{N-k}$, u *vanishes at the infinity with respect to* $\cdot^*$ *and* $u^* \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega)$*, then*

$$\int_{\mathbf{R}^N} f(x, u^*, H(\nabla' u^*), \nabla'' u^*)\, dx = \int_{\mathbf{R}^N} f(u^\star, K_{H^\circ} \left|\nabla' u^\star\right|_2, \nabla'' u^\star)\, dx,$$

provided one of the integrals exist.

Lemma 6.7. *Let* $J : \mathbf{R}^+ \times \mathbf{R}^{N-k} \to \bar{\mathbf{R}}^+$, $H \in \mathcal{H}(\mathbf{R}^k)$, *and*

$$\varphi(x) = J(H^\circ(x'), x'').$$

If $J(0) = 0$, *then*

$$\overline{\varphi}(t) = \overline{J}(H^{\circ\circ}(t'), t'') \leqslant \overline{J}(H(t'), t'').$$

Proof. By definition of the Fenchel transform,

$$\begin{aligned}
\overline{\varphi}(t) &= \sup_{x \in \mathbf{R}^N} \langle t, x\rangle - J(H^\circ(x'), x'') \\
&= \sup_{\substack{\lambda \in \mathbf{R}^+ \\ x \in \mathbf{R}^N, H^\circ(x')=\lambda}} \langle t', x'\rangle + \langle t'', x''\rangle - J(\lambda, x'') \\
&= \sup_{\substack{\lambda \in \mathbf{R}^+ \\ x'' \in \mathbf{R}^{N-k}}} H^{\circ\circ}(t')\lambda + \langle t'', x''\rangle - J(\lambda, x'') \\
&= \overline{J}(H^{\circ\circ}(t'), t'') \leqslant \overline{J}(H(t'), t''),
\end{aligned}$$

the last inequality coming from the convexity of $\overline{J}$, and the fact $\overline{J}(0) = 0$ and $\overline{J} \geqslant 0$. $\square$

Theorem 6.8. *Let* $H \in \mathcal{H}(\mathbf{R}^k)$, $\cdot^*$ *be the anisotropic symmetrization with respect to* H°, *let* $\Omega \subset \mathbf{R}^N$ *be an open set totally invariant with respect to* $\cdot^*$ *and* $J : \Omega \times \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^{N-k} \to \bar{\mathbf{R}}^+$. *If* $J(\cdot, s, \eta, \xi)$ *is totally invariant for each* $(s, \eta, \xi) \in \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^{N-k}$, $J(x, s, \cdot, \cdot)$ *is convex and lower semi-continuous for each* $(x, s) \in \Omega \times \mathbf{R}^+$, $\overline{J}(\cdot, \cdot, \eta, \xi)$ *is lower semi-continuous for each* $(\eta, \xi) \in \mathbf{R}^+ \times \mathbf{R}^{N-k}$ *and* $u \in \mathrm{W}^{1,1}_{\mathrm{loc},+}(\Omega)$, *and there exists* $I : \Omega \times \mathbf{R}^+ \times \mathbf{R}^N \to \bar{\mathbf{R}}^+$ *such that* I *is lower semi-continuous,* $I(\cdot, s, \xi)$ *is totally invariant for each* $(s, \xi) \in \mathbf{R}^+ \times \mathbf{R}^N$, $I(x, \cdot)$ *is convex and* $\lim_{|s|+|\xi|_2 \to \infty} I(x, s, \xi) = +\infty$ *for each* $x \in \Omega$ *and*

$$\int_\Omega I(x, u, \nabla u)\, dx < \infty,$$

then $u^* \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega)$ *and*

$$\int_\Omega J(x, u^*, H(\nabla' u^*), \nabla'' u^*)\, dx \leqslant \int_\Omega J(x, u, H(\nabla' u), \nabla'' u)\, dx.$$

Proof. Let $\varphi(x, s, t) = \overline{J}(x, s, H^\circ(t'), t'')$. Then, by Lemma 6.7,

$$\begin{aligned}
\overline{\varphi}(x, s, \theta) &= J(x, s, H^{\circ\circ}(\theta'), \theta'') \leqslant J(x, s, H(\theta'), \theta''), \\
\varphi^\star(x, s, t) &= \overline{J}(x, s, \left|t'\right|_2 / K_{H^\circ}, t''), \\
\overline{\varphi^\star}(x, s, \theta) &= J(x, s, K_{H^\circ} \left|\theta'\right|_2, \theta'').
\end{aligned}$$

The function u verifies the hypotheses of Proposition 5.17 by Proposition 5.18. Hence, by Proposition 5.17

$$\int_\Omega J(x, u^\star, K_{H^\circ} \left|\nabla' u^\star\right|_2, \nabla'' u^\star)\, dx \leqslant \int_\Omega J(x, u, H(\nabla' u), \nabla'' u)\, dx.$$

The conclusion comes from Proposition 6.4. □

Theorem 6.9 (Anisotropic symmetrization inequality for anisotropic functional). *Let $\cdot^*$ be the anisotropic symmetrization with respect to a gauge $G : \mathbf{R}^k \to \mathbf{R}^+$, let Ω be a totally invariant open set, $\varphi : \Omega \times \mathbf{R}^+ \times \mathbf{R}^N \to \bar{\mathbf{R}}^+$ and $u \in \mathrm{W}^{1,1}_{\mathrm{loc},+}(\Omega)$. If $\varphi(\cdot,\cdot,0) = 0$, $\varphi(\cdot, s, \xi)$ is totally invariant with respect to $\cdot^\star$ for each $(s, \xi) \in \mathbf{R}^+ \times \mathbf{R}^N$ and $\varphi(\cdot,\cdot,\xi)$ is lower semi-continuous for each $\xi \in \mathbf{R}^N$, and there exists $I : \Omega \times \mathbf{R}^+ \times \mathbf{R}^N \to \bar{\mathbf{R}}^+$ such that I is lower semi-continuous, $I(\cdot, s, \xi)$ is totally invariant for each $(s, \xi) \in \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^{N-k}$, $I(x, \cdot)$ is convex and $\lim_{|s|+|\xi|_2 \to \infty} I(x, s, \xi) = +\infty$ for each $x \in \Omega$, and*

$$\int_\Omega I(x, u, \nabla u)\, dx < \infty,$$

then $u^* \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega \overline{\varphi_*}(x, u^*(x), \nabla u^*(x))\, dx \leqslant \int_\Omega \overline{\varphi}(x, u(x), \nabla u(x))\, dx,$$

where the symmetrization and the Fenchel transform of φ are taken with respect to the last variable.

Proof. There exists a unique function $J : \Omega \times \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^{N-k} \to \bar{\mathbf{R}}^+$ such that for each $(x, s, \xi) \in \Omega \times \mathbf{R}^+ \times \mathbf{R}^N$,

$$\begin{aligned}\varphi_\star(x, s, \xi', \xi'') &= J(x, s, K_G^{-1} \left|\xi'\right|_2, \xi''),\\ \varphi_*(x, s, \xi', \xi'') &= J(x, s, G(-\xi'), \xi'');\end{aligned}$$

Lemma 6.7 implies

$$\begin{aligned}\overline{\varphi_\star}(x, s, t', t'') &= \overline{J}(x, s, K_G \left|t'\right|_2, t''),\\ \overline{\varphi_*}(x, s, t', t'') &= \overline{J}(x, s, G^\circ(-t'), t'');\end{aligned}$$

with Proposition 6.4 we have

$$\int_\Omega \overline{\varphi_*}(x, s, \nabla u^*)\, dx = \int_\Omega \overline{\varphi_\star}(x, s, \nabla u^\star)\, dx;$$

and the conclusion follows from Proposition 5.6. □

Definition 6.10. For $u \in \mathrm{W}^{1,1}_{\mathrm{loc},+}(\Omega)$, $F \in \mathcal{H}(\mathbf{R}^k)$ and a Borel measurable function $J : \Omega \times \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^{N-k} \to \mathbf{R}^+$ such that $J(x, \cdot)$ is convex for each $x \in \Omega$, let

$$\|u\|_{J,F,\Omega} = \inf\Bigl\{\lambda > 0 \ : \ \int_\Omega J\bigl(x, \tfrac{u}{\lambda}, F(\tfrac{\nabla' u}{\lambda}), \tfrac{\nabla'' u}{\lambda}\bigr)\,dx < +\infty\Bigr\}$$

and let

$$\mathrm{W}^{1,J,F}_+(\Omega) = \bigl\{u \in \mathrm{W}^{1,1}_{\mathrm{loc},+}(\Omega) \ : \ \|u\|_{J,F,\Omega} < +\infty\bigr\}.$$

Corollary 6.11. *Suppose $F : \mathbf{R}^k \to \mathbf{R}$ is a gauge, $\cdot^*$ is the anisotropic symmetrization with respect to F°, $\Omega \subset \mathbf{R}^N$ is a totally invariant open set and $J : \Omega \times \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^{N-k} \to \mathbf{R}^+$ is lower semi-continuous, $J(x, s, 0, 0) = 0$ for each $(x, s) \in \Omega \times \mathbf{R}^+$, $J(x, \cdot)$ is convex for each $x \in \Omega$, $\overline{J}(\cdot, \cdot, \eta, \xi)$ is lower semi-continuous for each $(\eta, \xi) \in \mathbf{R}^+ \times \mathbf{R}^{N-k}$, and for each $x \in \Omega$,*

$$\lim_{|s|+|\eta|+|\xi|_2 \to \infty} J(x, s, \eta, \xi) = \infty.$$

If $u \in \mathrm{W}^{1,J,F}_+(\Omega)$ vanishes at the infinity with respect to $\cdot^*$, then $u^* \in \mathrm{W}^{1,J,F}_+(\Omega)$ and

$$\|u^*\|_{J,F,\Omega} \leqslant \|u\|_{J,F,\Omega}\,.$$

7. Applications

7.1. Anisotropic isoperimetric inequalities

The results can be extended to $BV(\mathbf{R}^N)$ and to isoperimetric inequalities. Since our approach uses perimeters in the sense of Cacciopoli defined by duality, it cannot prove anything for non-convex perimeter functions like the ones arising in Wulff's theory of crystals.

Definition 7.1. For any $u \in \mathrm{L}^1(\mathbf{R}^N)$, let

$$P_H(u) = \sup\Bigl\{\sum_{i=1}^N \int_{\mathbf{R}^N} u \frac{\partial h_i}{\partial x_i} \ : \ h \in \mathcal{D}(\mathbf{R}^N),\ \forall x \in \mathbf{R}^N, \\ H^\circ\bigl(-h(x)\bigr) \leqslant 1\Bigr\}.$$

Theorem 7.2 (Anisotropic isoperimetric inequality in $BV(\mathbf{R}^N)$). *Let $H : \mathbf{R}^N \to \mathbf{R}$ be a gauge. Let $\cdot^*$ denote the anisotropic symmetrization with respect to H°. If $u \in \mathrm{L}^1(\mathbf{R}^N)$, then*

$$P_H(u^*) \leqslant P_H(u).$$

Proof. This will be deduced as a corollary of Theorem 6.8. Note that P_H is convex and lower semi-continuous in $\mathrm{L}^1(\mathbf{R}^N)$. If $u \in \mathcal{D}(\mathbf{R}^N)$, then

$$P_H(u) = \sup \left\{ \int_{\mathbf{R}^N} \langle \nabla u, h \rangle \; : \; H^\circ(h) \leqslant 1 \right\} = \int_{\mathbf{R}^N} H^\circ(\nabla u).$$

Let $(\rho_n)_{n \in \mathbf{N}}$ be a sequence of nonnegative smooth mollifiers. By lower semi-continuity of P_H,

$$\liminf_{n \to \infty} P_H(\rho_n * u) \geqslant P_H(u).$$

Conversely, for $h \in \mathcal{D}(\mathbf{R}^N)$, if $H(-h) \leqslant 1$, then the convexity of H implies $H(-\rho_n * h) \leqslant 1$, whence

$$P_H(\rho_n * u) \leqslant P_H(u).$$

We have thus $\lim_{n \to \infty} P_H(\rho_n * u) = P_H(u)$. Since $\rho_n * u \to u$ in $\mathrm{L}^1(\mathbf{R}^N)$, then $(\rho_n * u)^* \to u^*$ in $\mathrm{L}^1(\mathbf{R}^N)$. Since P_H is lower semi-continuous in $\mathrm{L}^1(\mathbf{R}^N)$, by Theorem 6.8,

$$P_H(u^*) \leqslant \liminf_{n \to \infty} P_H(\rho_n * u^*) \leqslant \liminf_{n \to \infty} P_H(\rho_n * u) = P_H(u). \qquad \square$$

A consequence of this proposition is the following isoperimetric inequality: For any measurable set $A \subset \mathbf{R}^N$,

$$P_H(A^*) = P_H(\chi_{A^*}) \leqslant P_H(A) = P_H(\chi_A).$$

7.2. Anisotropic Sobolev and Hardy-Sobolev inequalities

Proposition 7.3. *Suppose $F \in \mathcal{H}(\mathbf{R}^k)$, $\cdot^*$ is the anisotropic symmetrization with respect to F°, $\Omega \subset \mathbf{R}^N$ is a totally invariant open set and $J : \Omega \times \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^{N-k} \to \mathbf{R}^+$ is lower semi-continuous, $J(x, \cdot)$ is convex for each $x \in \Omega$, $J(x, s, 0, 0) = 0$ for each $(x, s) \in \Omega \times \mathbf{R}^+$, $\overline{J}(\cdot, \cdot, \eta, \xi)$ is lower semi-continuous for each $(\eta, \xi) \in \mathbf{R}^+ \times \mathbf{R}^{N-k}$, and for each $x \in \Omega$, $\lim_{|s|+|\eta|+|\xi|_2 \to \infty} J(x, s, \eta, \xi) = \infty$. Suppose $\|\cdot\|_X$ is a norm that is invariant with respect to $\cdot^*$. Suppose for any $u \in \mathrm{W}^{1,J,F}_+(\Omega)$ that vanishes at the infinity with respect to $\cdot^*$,*

$$\|u\|_X \leqslant \|u\|_{J,F,\Omega}\,, \tag{7.1}$$

then for any $E \in \mathcal{H}(\mathbf{R}^k)$ such that $K_{E^\circ} = K_{F^\circ}$,

$$\|u\|_X \leqslant \|u\|_{J,E,\Omega}\,.$$

If there exists $u \in \mathrm{W}^{1,J,F}_+(\Omega)$ that vanishes at the infinity with respect to $\cdot^*$ such that $\|u\|_X = \|u\|_{J,F,\Omega}$, then $\|u^\star\|_X = \|u^\star\|_{J,E,\Omega}$, where $\cdot^\star$ denotes the anisotropic symmetrization with respect to E°.

Proof. First note that $\|\cdot\|_X$ is invariant with respect to $\cdot^\star$. In fact, for any u, $\|u\|_X = \|u^*\|_X = \|u^\star\|_X$, since $u^{\star *} = u^*$ and $\|\cdot\|_X$ is invariant with respect to $\cdot^*$.

It is then clear that

$$\|u\|_X = \|u^*\|_X \leqslant \|u^*\|_{F,J,\Omega} = \|u^\star\|_{E,J,\Omega} \leqslant \|u\|_{E,J,\Omega}\,,$$

where the first inequality comes from the hypothesis (7.1) and the second from Corollary 6.11. The conclusions follow. □

Remark 7.4. Sobolev inequalities thus do not rely essentially neither on the convexity nor on the evenness of the euclidian norm. It is not surprising that such inequalities are possible since for any $F \in \mathcal{H}(\mathbf{R}^k)$, there exists $\alpha > 0$ such that $F(y) \geqslant \alpha\,|y|_2$. The striking fact is that optimal Sobolev-Orlicz constants depend on $F \in \mathcal{H}(\mathbf{R}^F)$ only through K_{H°.

Proposition 7.5. *For any $H \in \mathcal{H}(\mathbf{R}^N)$ and any $u \in \mathcal{D}(\mathbf{R}^N)$ and $1 < p < N$,*

$$\int_{\mathbf{R}^N} H(\nabla u)^p\,dx \geqslant \left(\frac{N-p}{p}\right)^p \int_{\mathbf{R}^N} \frac{|u|^p}{H^\circ(x)^p}\,dx.$$

The constant is optimal for any fixed $H \in \mathcal{H}(\mathbf{R}^k)$.

Proof. Without loss of generality, $K_{H^\circ} = 1$. Let $\cdot^\star$ denote the Schwarz symmetrization. Then, by Theorem 6.8, the classical Hardy-Sobolev inequality [27] and by Proposition 2.28,

$$\begin{aligned}\int_{\mathbf{R}^N} H(\nabla u)^p\,dx &\geqslant \int_{\mathbf{R}^N} |\nabla u^\star|_2^p\,dx \\ &\geqslant \left(\frac{N-p}{p}\right)^p \int_{\mathbf{R}^N} \frac{|u^\star|^p}{|x|_2^p}\,dx \geqslant \left(\frac{N-p}{p}\right)^p \int_{\mathbf{R}^N} \frac{|u|^p}{H^\circ(x)^p}\,dx.\end{aligned}$$

The fact that the constant is optimal comes from the same reasoning with the symmetrization $\cdot^*$ with respect to H° and the fact that the constant is optimal in the isotropic case [27]. □

7.3. Recovering continuity and compactness

Proposition 7.6. *Suppose $H \in \mathcal{H}(\mathbf{R}^k)$, $J : \mathbf{R}^+ \times \mathbf{R}^{N-k} \to \mathbf{R}^+$, $J(t,\cdot)$ is a gauge for each $t \in \mathbf{R}^+$ and $1 < p < N$. Let $\cdot^*$ denote the anisotropic symmetrization with respect to H° and Ω be totally invariant with respect to $\cdot^*$. If $f \in \mathrm{L}^q(\Omega)$, $q^{-1} + p^{-1} = 1 + N^{-1}$ and $f^* = f$, then the function $E : \mathcal{D}^{1,p}_{0,+}(\Omega) \to \mathbf{R}$ (where $\mathcal{D}^{1,p}_0$ is the completion of $\mathcal{D}(\Omega)$ with respect to* *the norm* $\|u\| = \left(\int_\Omega |\nabla u|^p\right)^{1/p}$*),*

$$u \mapsto E(u) = \int_{\mathbf{R}^N} J(H(\nabla' u), \nabla'' u)^p - fu\, dx$$

has a minimizer $u = u^*$*.*

Remark 7.7. If (u_n) is a minimizing sequence for E, it is bounded in $\mathcal{D}^{1,p}(\Omega)$, hence up to a subsequence, $u_n \rightharpoonup u$ in $\mathcal{D}^{1,p}(\Omega)$. Then

$$\int_\Omega f u_n\, dx \to \int_\Omega f u\, dx.$$

But if H is not convex it is not true in general that

$$\int_\Omega J(H(\nabla' u), \nabla'' u)^p\, dx \leqslant \liminf_{n\to\infty} \int_\Omega J(H(\nabla' u_n), \nabla'' u_n)^p\, dx$$

(see [11]).

In a different setting, the symmetry of the domain helps to recover the existence for some nonconvex problems [15]. On the other hand, some crystalline problems close to Wulff's problem do not have any solution (except in the varifold sense) when the energy is not convex [24].

Remark 7.8. Even if the minimizer is unique (if e.g. J is strictly convex), that does not guarantee the symmetry of the minimizer since, except in the radial case, the problem is not invariant under the action of a continuous group.

Proof. Let u_n be a minimizing sequence. Then $(v_n) = ({u_n}^*)$ is also a minimizing sequence. Up to a subsequence $v_n \rightharpoonup v$ in $\mathcal{D}_0^{1,p}(\mathbf{R}^N)$, hence

$$\int_\Omega f v_n\, dx \to \int_\Omega f v\, dx.$$

Furthermore

$$\int_\Omega J(H(\nabla' v), \nabla'' v)^p\, dx \leqslant \liminf_{n\to\infty} \int_\Omega J(H(\nabla' v_n), \nabla'' v_n)^p\, dx,$$

since the functional $u \mapsto \int_\Omega J(H(\nabla' u), \nabla'' u)^p\, dx$ is convex on the image set of $\cdot^*$. Then v is a minimizer for the functional E. Hence $v = v^*$ is a minimizer of E. $\square$

Proposition 7.9. *Suppose* $H \in \mathcal{H}(\mathbf{R}^k)$*,* $\cdot^*$ *is the anisotropic symmetrization with respect to* H°*,* Ω *is totally invariant with respect to* $\cdot^*$*,* $J : \Omega \times \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^{N-k} \to \mathbf{R}^+$ *is such that* $\overline{J}(\cdot, \cdot, \eta, \xi)$ *is lower semi-continuous and totally invariant with respect to* $\cdot^*$ *for each* $(\eta, \xi) \in \mathbf{R}^+ \times \mathbf{R}^{N-k}$*, and* $J(x, \cdot)$ *is convex for each* $x \in \Omega$*. Let* $f : \Omega \times \mathbf{R}^+ \to \mathbf{R}^+$ *be such that for almost every* $x \in \Omega$*,* $f(x, 0) = 0$ *and* $f(x, \cdot)$ *is continuous* *for each* $x \in \Omega$ *and* $f(\cdot, s)$ *is measurable and totally invariant for each* $s \in \mathbf{R}^+$.

For any $u \in \mathrm{W}^{1,p}_{0,+}(\Omega)$, *let*

$$E(u) = \int_\Omega J(x, u(x), H(\nabla' u(x)), \nabla'' u(x))\, dx,$$

and

$$\mathcal{M} = \Big\{ u \in \mathrm{W}^{1,p}_{0,+}(\Omega) \ :\ \int_\Omega f(x, u(x))\, dx = 1 \Big\}.$$

If there exists $1 < p < +\infty$, $0 < \alpha < \beta$, $p < q < \frac{Np}{N-p}$ *and* $\gamma > 0$ *such that*

$$\alpha(|s| + |\eta| + |\xi|_2)^p \leqslant J(x, s, \eta, \xi) \leqslant \beta(|s| + |\eta| + |\xi|_2)^p$$

and

$$|f(x, s)| \leqslant \gamma(|s|^p + |s|^q),$$

and if $\lim_{s\to 0} \sup_{x\in\Omega} \frac{|f(x,s)|}{|s|^p} = 0$, *then there exists* $u \in \mathcal{M}$ *such that* $E(u) = \inf_{v\in\mathcal{M}} E(v)$. *Furthermore,* $u^* = u$.

Proof. Let $(u_n)_{n\in\mathbf{N}}$ be a minimizing sequence of E in $\mathcal{M}$. By Theorem 6.8 with $I(x, s, \xi) = \alpha(|s| + |\xi'|_2 + |\xi''|_2)^p$, $E({u_n}^*) \leqslant E(u_n)$ and by Proposition 2.28, ${u_n}^* \in \mathcal{M}$. Without loss of generality, ${u_n}^* = u_n$ for each $n \in \mathbf{N}$. Since $E(u_n)$ is bounded, the sequence (u_n) is bounded in $\mathrm{W}^{1,p}_0(\Omega)$. Hence up to a subsequence it converges weakly to $u \in \mathrm{W}^{1,p}_0(\Omega)$. By Lemma 3.1, $u^* = u$. Since the functional E is convex on the set of symmetrized functions and is strongly lower semi-continuous on $\mathrm{W}^{1,p}_0(\Omega)$, it is weakly lower semi-continuous on the set of symmetrized functions, whence

$$E(u) \leqslant \liminf_{n\to\infty} E(u_n) = \inf_{v\in\mathcal{M}} E(v).$$

Let $\cdot^\star$ denote the Steiner symmetrization with respect to $\{0\} \times \mathbf{R}^{N-k}$. By a theorem of Lions on compact embeddings of sets of symmetric functions [19] the sequence $({u_n}^\star)$ is compact in $\mathrm{L}^q(\Omega)$. Therefore, the sequence $(u_n) = ({u_n}^{\star *})$ is compact in $\mathrm{L}^q(\Omega)$.

Up to a subsequence, $u_n \to u$ almost everywhere. Since

$$\lim_{s\to 0} \sup_{x\in\Omega} \frac{|f(x, s)|}{|s|^p} = 0,$$

for any $\varepsilon > 0$, there exists $\gamma_\varepsilon > 0$ such that

$$|f(x, s)| \leqslant \varepsilon\, |s|^p + \gamma_\varepsilon\, |s|^q\, .$$

By Fatou's Lemma and by the weak convergence of the sequence (u_n) in $\mathrm{L}^p(\Omega)$,

$$\begin{aligned}\varepsilon \|u\|_p^p + \gamma_\varepsilon \|u\|_q^q + \int_\Omega f(x,u)\,dx \\ \leqslant \liminf_{n\to\infty} \int_\Omega \varepsilon\,|u_n| + \gamma_\varepsilon\,|v_n|^q - f(x,u_n)\,dx \\ \leqslant \varepsilon \liminf_{n\to\infty} \|u_n\|_p^p + \gamma_\varepsilon \|u\|_q^q + \liminf_{n\to\infty} \int_\Omega f(x,u_n)\,dx.\end{aligned}$$

As $\varepsilon \to 0$ this becomes

$$\int_\Omega f(x,u)\,dx \leqslant \liminf_{n\to\infty} \int_\Omega f(x,u_n)\,dx.$$

The same argument holds also for $-f$, hence

$$\int_\Omega f(x,u)\,dx = \lim_{n\to\infty} \int_\Omega f(x,u_n)\,dx = 1,$$

$u \in \mathcal{M}$ and $E(u) = \inf_{v\in\mathcal{M}} E(v)$. □

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Part 2

L^1 estimates

Introduction

1. The critical Sobolev embedding

If $p < N$, the Sobolev space $\mathrm{W}^{1,p}(\mathbf{R}^N)$ is continuously embedded in $\mathrm{L}^{p^*}(\mathbf{R}^N)$ for $p^* = Np/(N-p)$, while if $p > N$, it is embedded in the space of Hölder continuous functions $\mathrm{C}^{0,\alpha}(\mathbf{R}^N)$ for $\alpha = 1 - N/p$. In the limit case $p = N$, these embeddings suggest an embedding of the space $\mathrm{W}^{1,N}(\mathbf{R}^N)$ in the space of bounded continuous functions $\mathrm{C}(\mathbf{R}^N)$. There is no such embedding when $N > 1$, but some natural consequences of this embedding still hold. For example, there is a degree theory for maps in $\mathrm{W}^{1,N}(S^N; S^N)$ [3].

Another surprising property of $\mathrm{W}^{1,N}(\mathbf{R}^N)$ was discovered recently by Bourgain, Brezis and Mironescu: For every $u \in \mathrm{W}^{1,N}(\mathbf{R}^N; \mathbf{R}^N)$ and every rectifiable closed curve Γ of length $|\Gamma|$ and unit tangent vector t, the inequality

$$\int_\Gamma u \cdot t \leqslant C_N |\Gamma| \, \|Du\|_N \tag{1.1}$$

holds, where the constant C_N is independent of u and Γ. The proof was based on a Paley–Littlewood decomposition [2]. In contrast with the existence of the degree which comes from the embedding of $\mathrm{W}^{1,N}(\mathbf{R}^N)$ in $\mathrm{VMO}(\mathbf{R}^N)$, there is no such estimate for $u \in \mathrm{VMO}(\mathbf{R}^N; \mathbf{R}^N)$. In chapter V, we provide an elementary proof of this inequality, which relies on the Morrey–Sobolev embedding in $\mathbf{R}^{N-1}$ and the Hölder inequality.

Bourgain and Brezis generalized this inequality as follows [1]: Let $f \in \mathrm{L}^1(\mathbf{R}^N; \mathbf{R}^N)$, and $u \in \mathrm{W}^{1,N}(\mathbf{R}^N; \mathbf{R}^N)$. If $\operatorname{div} f = 0$ in the sense of distributions, then

$$\int_{\mathbf{R}^N} f \cdot u \, dx \leqslant C_N \|f\|_1 \, \|Du\|_N,$$

where the constant C_N is independent of f and u. Using a Paley–Littewood decomposition, they proved a stronger inequality, where $\|f\|_1$ is replaced by $\|f\|_{\mathrm{L}^1 + \mathrm{W}^{-1,N/(N-1)}}$.

In chapter VI we prove, that if $f \in \mathrm{L}^1(\mathbf{R}^N;\mathbf{R}^N)$, $\operatorname{div} f \in \mathrm{L}^1(\mathbf{R}^N)$ and $u \in (\mathrm{W}^{1,N} \cap \mathrm{L}^\infty)(\mathbf{R}^N;\mathbf{R}^N)$, then

$$\int_{\mathbf{R}^N} f \cdot u \, dx \leqslant C_N(\|f\|_1 \|Du\|_N + \|\operatorname{div} f\|_1 \|u\|_N). \tag{1.2}$$

In chapter VII, we prove that if $f_{ij} \in \mathrm{L}^1(\mathbf{R}^N)$ and $g_i \in \mathrm{L}^1(\mathbf{R}^N)$ for $1 \leqslant i \leqslant 2$ and $i \leqslant j \leqslant N$ satisfy the condition

$$\sum_{i=1}^{2} \sum_{j=i}^{N} \frac{\partial^2 f_{ij}}{\partial x_i \partial x_j} = \sum_{i=1}^{2} \frac{\partial g_i}{\partial x_i},$$

and if $u \in \mathrm{W}^{1,N}(\mathbf{R}^N)$, then for every $1 \leqslant i \leqslant 2$ and $i \leqslant j \leqslant N$,

$$\int_{\mathbf{R}^N} f_{ij} u \, dx \leqslant C_N(\|f\|_1 \|Du\|_N + \|g\|_1 \|u\|_N). \tag{1.3}$$

This kind of inequality was suggested by Haïm Brezis.

2. Regularity estimates

These estimates can be reformulated by a simple application of the Hahn-Banach Theorem and the Riesz representation Theorem. If $f \in \mathrm{L}^1(\mathbf{R}^N;\mathbf{R}^N)$ with $\operatorname{div} f \in \mathrm{L}^1(\mathbf{R}^N)$, then by (1.2) the linear functional

$$\langle \ell, u \rangle = \int_{\mathbf{R}^N} f \cdot u \, dx$$

is a linear and continuous on $\mathrm{W}^{1,N}(\mathbf{R}^N;\mathbf{R}^N)$ which can be considered as a closed linear subspace of $\mathrm{L}^N(\mathbf{R}^N;\mathbf{R}^{N+N^2})$ under the injection $u \mapsto (u, Du)$. By the Hahn–Banach Theorem, there is an extension $\bar{\ell}$ of ℓ to $\mathrm{L}^N(\mathbf{R}^N;\mathbf{R}^{N+N^2})$ such that

$$\langle \ell, (u, v) \rangle \leqslant C_N(\|f\|_1 \|v\|_N + \|\operatorname{div} f\|_1 \|u\|_N).$$

From the Riesz representation Theorem, there is $f_0 \in \mathrm{L}^{N/(N-1)}(\mathbf{R}^N)^N$ and $F \in \mathrm{L}^{N/(N-1)}(\mathbf{R}^N)^{N^2}$ such that

$$\langle \bar{\ell}, u \rangle = \int_{\mathbf{R}^N} f_0 \cdot u + F \cdot v \, dx.$$

Moreover, f_0 and F satisfy the estimates

$$\begin{aligned} \|f_0\|_{N/(N-1)} &\leqslant C_N \|\operatorname{div} f\|_1 \\ \|F\|_{N/(N-1)} &\leqslant C_N \|f\|_1. \end{aligned}$$

Since $\bar{\ell}$ is the extension of ℓ, for every $u \in \mathcal{D}(\mathbf{R}^N;\mathbf{R}^N)$

$$\langle \ell, u \rangle = \int_{\mathbf{R}^N} f \cdot u \, dx = \int_{\mathbf{R}^N} f_0 \cdot u + F \cdot Du \, dx.$$

Therefore

$$f = f_0 + \operatorname{div} F$$

in the sense of distributions.

New regularity estimates for elliptic systems can be obtained as an application. For example, if $N \geqslant 4$, $f \in \mathrm{L}^1(\mathbf{R}^N;\mathbf{R}^N)$ and $\operatorname{div} f \in \mathrm{L}^1(\mathbf{R}^N)$, then the fundamental solution of

$$-\Delta u = f$$

is given by $u = K_N * f$, where

$$K_N = \frac{(N/2)!}{(N-2)N\pi^{N/2}} \frac{1}{|x|^{N-2}}$$

is Newton's kernel. Then u can be decomposed as

$$u = K_N * f_0 + K_N * (\operatorname{div} F) = K_N * f_0 + \operatorname{div}(K_N * F).$$

By the classical Calderón–Zygmund theory of singular integrals (see e.g. [4]), if $1 < p < \infty$, and $g \in \mathrm{L}^p(\mathbf{R}^N)$, then $D^2(K_N * g) \in \mathrm{L}^p(\mathbf{R}^N)$, and

$$\|D^2(K_N * g)\|_p \leqslant C_{N,p}\|g\|_p,$$

where the constant $C_{N,p}$ is independent of g and $D^2 g$ denotes the Hessian of g in the weak sense. From this and the classical Sobolev embedding Theorem it follows that

$$u_1 \in \mathrm{L}^{N/(N-2)}(\mathbf{R}^N;\mathbf{R}^N)$$
$$u_2 \in \mathrm{L}^{N/(N-3)}(\mathbf{R}^N;\mathbf{R}^N).$$

Moreover, one has the following estimates

$$\|D^1 u_1\|_{N/(N-1)} \leqslant C_N \|f\|_1,$$
$$\|D^2 u_2\|_{N/(N-1)} \leqslant C_N \|\operatorname{div} f\|_1.$$

Other applications to $\operatorname{div} - \operatorname{curl}$ systems which given by Bourgain and Brezis in [1], are presented in chapter VI.

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CHAPTER V

A simple proof of an inequality of Bourgain, Brezis and Mironescu

1. Introduction

Bourgain, Brezis and Mironescu proved in [1] the following inequality.

Proposition 1.1. *Let* Γ *be a closed, oriented, rectifiable curve of* $\mathbf{R}^3$*, and denote by* $\vec{t}$ *the unit tangent vector along* Γ*; let* $\vec{k} \in \mathrm{W}^{1,3}(\mathbf{R}^3; \mathbf{R}^3)$*. Then*

$$\left| \int_\Gamma \vec{k} \cdot \vec{t} \right| \leqslant C \left\| k \right\|_{\mathrm{W}^{1,3}} |\Gamma| .$$

The proof of proposition 1.1 in [1] is technically involved. We provide an elementary proof and a generalization to k dimensional surfaces and N-dimensional space. For simplicity, we begin with the case of a curve in $\mathbf{R}^N$.

Proposition 1.2. *Let* Γ *be an oriented, compact and closed Lipschitz curve of* $\mathbf{R}^N$*,* $N \geqslant 2$*; let* $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbf{R}^N)$*. If* $\nabla u \in \mathrm{L}^N(\mathbf{R}^N)$*, then*

$$\left| \int_\Gamma u \, d\gamma \right| \leqslant C_N \left\| \nabla u \right\|_{\mathrm{L}^N(\mathbf{R}^N)} |\Gamma| , \tag{1.1}$$

where $|\Gamma|$ *denotes the length of curve* Γ*.*

Remark 1.3. When $N = 1$, the left-hand side of (1.1) is 0; when $N = 2$ and γ is a Jordan curve, proposition 1.2 is a simple consequence of Green theorem and the isoperimetric inequality; and when $N = 3$ it is equivalent to proposition 1.1.

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2. Proof of Proposition 1.2

Proof. Without loss of generality, the curve Γ is connected and is the image of S^1 by a Lipschitz map γ. We assume first that u and $\gamma : S^1 \to \mathbf{R}^N$ are of class C^1. We start with the same strategy as [1], bounding

$$e \cdot \int_{S^1} u(\gamma(x))\dot{\gamma}(x)\,dx\ ,$$

for an arbitrary unit-norm vector $e \in \mathbf{R}^N$.

Let

$$\Gamma_t = \left\{ x \in S^1 \ : \ e \cdot \gamma(x) = t \right\}\ .$$

Since $e \cdot \gamma(s)$ is of class C^1, Sard's lemma implies that for almost every $t \in \mathbf{R}$, Γ_t is finite and $e \cdot \dot{\gamma}(s) \neq 0$ if $\gamma(s) \in \Gamma_t$. We have then

$$\begin{aligned} e \cdot \int_{S^1} u(\gamma(x))\dot{\gamma}(x)\,dx &= \int_{S^1} u(\gamma(x))\, e \cdot \dot{\gamma}(x)\,dx \\ &= \int_{\mathbf{R}} \sum_{x \in \Gamma_t} \sigma(x) u(x)\,dt\ , \end{aligned} \tag{2.1}$$

where $\sigma(x) = \operatorname{sign}(e \cdot \dot{\gamma}(x))$. Since Γ is closed, for almost every $t \in \mathbf{R}$ we can write $\Gamma_t = \left\{P_1, \dots, P_{r(t)}\right\} \cup \left\{N_1, \dots, N_{r(t)}\right\}$ so that $\sigma(P_i) = 1$, $\sigma(N_i) = -1$ and $\sum_{i=1}^r |\gamma(P_i) - \gamma(N_i)|$ is minimal (in particular, it is bounded by $|\Gamma|$).

In order to estimate $\sum_{x \in \Gamma_t} \sigma(x)u(x)$, we proceed differently from [1]. They used a Littlewood–Paley decomposition. Instead, we apply Morrey's inequality (see e.g. [2, theorem IX.12]) in $\mathbf{R}^{N-1}$ for $u_t = u|_{\{y \in \mathbf{R}^N \,:\, e \cdot y = t\}}$ before applying the discrete Hölder inequality to the sum

$$\begin{aligned} \sum_{x \in \Gamma_t} \sigma(x)u(x) &\leqslant C_N \|\nabla u_t\|_N \sum_{i=1}^{r(t)} |\gamma(P_i) - \gamma(N_i)|^{\frac{1}{N}} \\ &\leqslant C_N \|\nabla u_t\|_p |\Gamma|^{\frac{1}{N}}\, r(t)^{\frac{N-1}{N}}\ . \end{aligned}$$

We are now ready to estimate the integral of (2.1):

$$\begin{aligned} \int_{\mathbf{R}} \sum_{x \in \Gamma_t} \sigma(x)u(x)\,dt &\leqslant C_N |\Gamma|^{\frac{1}{N}} \int_{\mathbf{R}} \|\nabla u_t\|_N\, r(t)^{\frac{N-1}{N}}\,dt \\ &\leqslant C_N |\Gamma|^{\frac{1}{N}} \left(\int_{\mathbf{R}} \|\nabla u_t\|_N^N\,dt \right)^{\frac{1}{N}} \left(\int_{\mathbf{R}} r(t)\,dt \right)^{\frac{N-1}{N}} \\ &\leqslant C'_{p,N} |\Gamma|\, \|\nabla u\|_N \end{aligned}$$

since $2\int_{\mathbf{R}} r(t)\,dt = \int_{S^1} |e \cdot \gamma'(x)|\ dx \leqslant \int_{S^1} |\gamma'(x)|\ dx = |\Gamma|$.

The result is extended to general Γ and u by standard smoothing arguments. □

3. Generalization to surfaces

Proposition 1.2 generalizes straightforwardly to k-dimensional surfaces defined as follows.

Definition 3.1. A pair $\Gamma = (M, \gamma)$ is a C^r k-dimensional Lipschitz surface of $\mathbf{R}^N$ if

(1) M is a compact oriented k-dimensional C^r manifold without boundary,
(2) $\gamma : M \to \mathbf{R}^N$ is a Lipschitz function.

When $\Gamma = (M, \gamma)$ is a k-dimensional Lipschitz surface of $\mathbf{R}^N$, it is possible, since M is oriented, to define the integral of Borel function $u : \mathbf{R}^N \to \mathbf{R}$ as the k-vector $\int_\Gamma u\, d\gamma(x)$, where $d\gamma(x)[a_1, \ldots, a_k] = \gamma'(x)a_1 \wedge \ldots \wedge \gamma'(x)a_k$, and the mass of Γ as $|\Gamma| = \int_M |d\gamma|$, where $|\cdot|$ denotes the euclidean norm of a k-vector.

Proposition 3.2. *Let Γ be a C^k k-dimensional surface. Then*

$$\left| \int_M u\, d\gamma \right| \leqslant C_N \left\| \nabla u \right\|_N |\Gamma| \ , \tag{3.1}$$

where the norm on the left is the comass-norm (see [3]).

Proof. Since the proof is similar to the proof of proposition 1.2, we only give a sketch. For an arbitrary simple unit covector $e = e_1 \wedge \cdots \wedge e_k$, we write

$$e \cdot \int_\Gamma u\ d\gamma = \int_{\mathbf{R}^k} \sum_{x \in \Gamma_y} \sigma(x) u(x)\, dy,$$

where

$$\Gamma_y = \{x \in M \ : \ e_i \cdot \gamma(x) = y_i, 1 \leqslant i \leqslant k\}$$

and

$$\sigma(x) = \ \mathrm{sign}\,(e_1\gamma'(x) \wedge \ldots \wedge e_k\gamma'(x)).$$

This formula is valid by Sard's lemma because M is a C^k manifold. Then, using Morrey's and Hölder's inequalities, with the notations of the proof of Proposition 1.2,

$$\int_{\mathbf{R}^k} \sum_{x\in\Gamma_y} \sigma(x)u(x)\,dy$$
$$\leqslant \int_{\mathbf{R}} \|\nabla u_{y_1}\|_N \Big(\int_{\mathbf{R}^{k-1}} \sum_{i=1}^{r(y)} |\gamma(P_i)-\gamma(N_i)|\,dy''\Big)^{1/(N-1)}$$
$$\Big(\int_{\mathbf{R}^{k-1}} r(y)\,dy''\Big)^{N/(N-1)}\,dy_1,$$

where $y''=(y_2,\dots,y_k)$ and one concludes using

$$\int_{\mathbf{R}^{k-1}} \sum_{i=1}^{r(y)} |\gamma(P_i)-\gamma(N_i)|\; dy'' \leqslant |\Gamma|\,,$$

Hölder's inequality and $2\int_{\mathbf{R}^k} r(y)\,dy \leqslant |\Gamma|$. □

Remark 3.3. Proposition 3.2 can also be proved by induction on k. The case $k=1$ is proposition 1.2 and for $k>1$, Γ is cut into slices of dimension $k-1$, for which the estimate of proposition 3.2 holds. The integration of this estimate, with Hölder's inequality gives the conclusion.

Remark 3.4. The inequality (3.1) is the limit case of

$$\left|\int_\Gamma u\,d\gamma\right| \leqslant C_{p,N}\delta(\Gamma)^{1-\frac{N}{p}}\,|\Gamma|\,\|\nabla u\|_p\,,$$

where $p>N$, $\nabla u\in \mathrm{L}^p(\mathbf{R}^N)$ and $\delta(\Gamma)$ denotes the diameter of Γ. It is a simple consequence of Morrey's inequality.

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CHAPTER VI

Estimates for L^1-vector fields

1. Introduction

In a recent work [2], Bourgain and Brezis considered in $\mathbf{R}^3$ the system

$$\begin{cases} \operatorname{curl} Z = f, \\ \operatorname{div} Z = 0. \end{cases}$$

for a given divergence-free vector field f. The vector field

$$Z = (-\Delta)^{-1} \operatorname{curl} f$$

is a solution of this system. If $f \in \mathrm{L}^p_{\#}(\mathbf{R}^3; \mathbf{R}^3)$ (here and in the sequel, the subscript $\#$ denotes the subspace of vector fields whose divergence vanishes in the sense of distributions), then by the standard Calderón–Zygmund estimates and Sobolev's imbedding, $\|Z\|_{p^*} \leqslant C_p \|f\|_p$, where $p^* = 3p/(3-p)$ and $1 < p < 3$. The Calderón–Zygmund theory does not hold when $p = 1$, but surprisingly, when $p = 1$ one still has

Theorem 1.1 (Bourgain and Brezis [2]). *There exists C, such that for any $f \in \mathrm{L}^1_{\#}(\mathbf{R}^3; \mathbf{R}^3)$,*

$$\|Z\|_{3/2} \leqslant C \|f\|_1 .$$

Bourgain and Brezis gave two proofs of this result. The first one relies on the following

Theorem 1.2. *Given $g \in \mathrm{L}^3_{\#}(\mathbf{R}^3; \mathbf{R}^3)$, there exists $Y \in \mathrm{L}^\infty(\mathbf{R}^3; \mathbf{R}^3)$ with $\nabla Y \in \mathrm{L}^3(\mathbf{R}^3; \mathbf{R}^3)$ satisfying*

$$\operatorname{curl} Y = g$$

and the estimate

$$\|Y\|_\infty + \|\nabla Y\|_3 \leqslant C \|g\|_3 .$$

The proof of Theorem 1.2 is rather involved and uses a Littlewood-Paley decomposition; no simple proof has been found so far.

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Proof of Theorem 1.1 using Theorem 1.2. It suffices to show that

$$\left|\int_{\mathbf{R}^3} Z \cdot h\right| \leqslant C \|f\|_1 \|h\|_3$$

for every $h \in \mathrm{L}^3(\mathbf{R}^3; \mathbf{R}^3)$ with some universal constant C. Given any $h \in \mathrm{L}^3(\mathbf{R}^3; \mathbf{R}^3)$ consider its Hodge decomposition $h = g + \operatorname{grad} p$ with $g \in \mathrm{L}^3_\#(\mathbf{R}^3; \mathbf{R}^3)$ and $\|g\|_3 \leqslant C \|h\|_3$. Then

$$\int_{\mathbf{R}^3} Z \cdot h = \int_{\mathbf{R}^3} Z \cdot (g + \operatorname{grad} p) = \int_{\mathbf{R}^3} Z \cdot g.$$

Next, by Theorem 1.2, we may write $g = \operatorname{curl} Y$ for some Y with $\|Y\|_\infty \leqslant C \|g\|_3$ (here one uses only part of Theorem 1.2) and then

$$\begin{aligned}\left|\int_{\mathbf{R}^3} Z \cdot h\right| = \left|\int_{\mathbf{R}^3} \operatorname{curl} Z \cdot Y\right| = \left|\int_{\mathbf{R}^3} f \cdot Y\right| &\leqslant \|f\|_1 \|Y\|_\infty \\ &\leqslant C \|f\|_1 \|g\|_3 \leqslant C \|f\|_1 \|h\|_3 . \quad \square\end{aligned}$$

The second proof in [1] uses two ingredients. The first one is

Theorem 1.3 (Bourgain, Brezis and Mironescu [1]). *Let Γ be a closed rectifiable curve in $\mathbf{R}^N$ with unit tangent vector t and let $u \in \mathrm{C}(\mathbf{R}^N; \mathbf{R}^N)$. If $\nabla u \in \mathrm{L}^N(\mathbf{R}^N)$, then*

$$\left|\int_\Gamma u \cdot t\right| \leqslant C |\Gamma| \|\nabla u\|_N ,$$

where $|\Gamma|$ denotes the length of Γ. The constant C is independent of the curve Γ and the vector-field u.

See [4] for a simple proof of Theorem 1.3.

The next ingredient is Smirnov's theorem which asserts roughly speaking that any divergence free vector field is a limit of convex combinations of the form $\sum_i \alpha_i \mathcal{H}^1_{\llcorner \Gamma_i} t_i$, where $\mathcal{H}^1_{\llcorner \Gamma}$ is Hausdorff's one-dimensional measure restricted to Γ and t_i is the tangent vector to Γ_i.

Combining this with Theorem 1.3, Bourgain and Brezis obtain the following

Corollary 1.4. *There exists a constant C_N such that for each $f \in \mathrm{L}^1_\#(\mathbf{R}^N; \mathbf{R}^N)$ and $u \in \mathrm{L}^\infty(\mathbf{R}^N; \mathbf{R}^N)$ such that $\nabla u \in \mathrm{L}^N(\mathbf{R}^N)$,*

$$\left|\int_{\mathbf{R}^N} f \cdot u \, dx\right| \leqslant C_N \|f\|_1 \|\nabla u\|_N .$$

Note that Theorem 1.3 is a special case of Corollary 1.4.

Proof of Theorem 1.1 using Corollary 1.4. Indeed we start as in the first proof of Theorem 1.1. Write as above

$$\int_{\mathbf{R}^3} Z\cdot h = \int_{\mathbf{R}^3} Z\cdot g.$$

Next, by standard L^p estimates we may solve $\operatorname{curl}\tilde{Y} = g$ and $\operatorname{div}\tilde{Y} = 0$; and then $\|\nabla\tilde{Y}\|_3 \leqslant \|g\|_3$. (Here we do not use the difficult part in Theorem 1.2 which gives some $Y \in \mathrm{L}^\infty(\mathbf{R}^3;\mathbf{R}^3)$.) Thus

$$\int_{\mathbf{R}^3} Z\cdot h = \int_{\mathbf{R}^3} Z\cdot \operatorname{curl}\tilde{Y} = -\int_{\mathbf{R}^3} f\cdot\tilde{Y}.$$

so that by Corollary 1.4,

$$\left|\int_{\mathbf{R}^3} Z\cdot h\right| \leqslant C\,\|f\|_1\,\|\nabla\tilde{Y}\|_3 \leqslant C\,\|f\|_1\,\|g\|_3 \leqslant C\,\|f\|_1\,\|h\|_3\,. \qquad \square$$

Remark 1.5. Note that Corollary 1.4 is an easy consequence of Theorem 1.1. Indeed write, using Theorem 1.1,

$$\begin{aligned}\left|\int_{\mathbf{R}^3} f\cdot u\right| = \left|\int_{\mathbf{R}^3} (\operatorname{curl} Z)\cdot u\right| &= \left|\int_{\mathbf{R}^3} Z\cdot\operatorname{curl} u\right| \\ &\leqslant \|Z\|_{3/2}\,\|\operatorname{curl} u\|_3 \leqslant C\,\|f\|_1\,\|\nabla u\|_3\,.\end{aligned}$$

Remark 1.6. Another consequence of Theorem 1.7 already mentioned in [2] is that $\Delta\, u = f \in \mathrm{L}^1_\#(\mathbf{R}^N)$ implies $\|\nabla u\|_{3/2} \leqslant C\,\|f\|_1$. This is proved as follows. Let Z solve $\operatorname{curl} Z = f$ and $\operatorname{div} Z = 0$. Then $\Delta Z = \operatorname{curl} f$, so that, by Theorem 1.1, $\|\Delta^{-1}\operatorname{curl} f\|_{3/2} \leqslant C\,\|f\|_1$. Therefore $\|\operatorname{curl} u\|_{3/2} = \|\Delta^{-1}\operatorname{curl}\Delta u\|_{3/2} \leqslant C\,\|f\|_1$. Finally, since $\operatorname{div} u = 0$,

$$\|\nabla u\|_{3/2} \leqslant C(\|\operatorname{curl} u\|_{3/2} + \|\operatorname{div} u\|_{3/2}) \leqslant C\,\|f\|_1\,.$$

The goal of this note is to give a direct and elementary proof of Corollary 1.4. In fact we present a slightly more general version.

Theorem 1.7. *There exists a constant C_N such that for each $f \in \mathrm{L}^1(\mathbf{R}^N;\mathbf{R}^N)$ such that $\operatorname{div} f \in \mathrm{L}^1$ and $u \in (\mathrm{L}^\infty \cap \mathrm{W}^{1,N})(\mathbf{R}^N;\mathbf{R}^N)$,*

$$\left|\int_{\mathbf{R}^N} f\cdot u\,dx\right| \leqslant C_N(\|f\|_1\,\|\nabla u\|_N + \|\operatorname{div} f\|_1\,\|u\|_N).$$

In a work in preparation we prove an extension of Corollary 1.4 in which the condition $\operatorname{div} f = 0$ is replaced by a weaker second order condition [5].

2. Proof of Theorem 1.7

First the estimate will be made under the additional assumptions that f and u are in $C^1(\mathbf{R}^N;\mathbf{R}^N)$. The first term in the scalar product is

$$\int_{\mathbf{R}^N} f_1 u_1\,dx = \int_{\mathbf{R}} \int_{\mathbf{R}^{N-1}} f_1 u_1\,dy\,dx_1.$$

The inside integral is estimated as follows. Let $\rho \in \mathrm{L}^1(\mathrm{B}(0,1) \cap \mathbf{R}^{N-1})$ be such that $\int_{\mathbf{R}^{N-1}} \rho = 1$. Let $\rho_\varepsilon(\cdot) = \varepsilon^{1-N}\rho(\frac{\cdot}{\varepsilon})$, $u^{x_1}(y) = u(x_1, y)$ and $f^{x_1}(y) = f(x_1, y)$. The integral can be decomposed as

$$\int_{\mathbf{R}^{N-1}} f_1^{x_1} u_1^{x_1}\,dy \\ = \int_{\mathbf{R}^{N-1}} f_1^{x_1}(u_1^{x_1} - \rho_\varepsilon * u_1^{x_1})\,dy + \int_{\mathbf{R}^{N-1}} f_1^{x_1}(\rho_\varepsilon * u_1^{x_1})\,dy.$$

By the Morrey–Sobolev imbedding in $\mathbf{R}^{N-1}$ (see e.g. [3, theorem IX.12]),

$$\int_{\mathbf{R}^{N-1}} f_1^{x_1}(u_1^{x_1} - \rho_\varepsilon * u_1^{x_1})\,dy \leqslant C'_N \varepsilon^{1/N} \left\|f^{x_1}\right\|_1 \left\|\nabla u_1^{x_1}\right\|_N.$$

On the other hand,

$$\begin{aligned}
&\int_{\mathbf{R}^{N-1}} f_1^{x_1}(\rho_\varepsilon * u_1^{x_1})\,dy \\
&\quad = \int_{\mathbf{R}^{N-1}} \int_{-\infty}^{x_1} \frac{\partial}{\partial x_1}\big(f(t,y)\,(\rho_\varepsilon * u_1^{x_1})(y)\big)\,dt\,dy \\
&\quad = \int_{(-\infty,x_1)\times\mathbf{R}^{N-1}} \operatorname{div}\big(f(t,y)\,(\rho_\varepsilon * u_1^{x_1})(y)\big)\,dt\,dy \\
&\quad = \int_{(-\infty,x_1)\times\mathbf{R}^{N-1}} f(t,y)\cdot(0, \nabla(\rho_\varepsilon * u_1^{x_1})(y)) \\
&\qquad\qquad + (\operatorname{div} f(t,y))(\rho_\varepsilon * u_1^{x_1})(y)\,dt\,dy \\
&\quad \leqslant C''_N \varepsilon^{(1/N)-1}(\|f\|_1 \left\|\nabla u_1^{x_1}\right\|_N + \|\operatorname{div} f\|_1 \left\|u_1^{x_1}\right\|_N)
\end{aligned}$$

where C'_N et C''_N are constants which depend only on the dimension N (and of ρ). (The third equality relies on the vector calculus identity $\operatorname{div}(Zf) = (\operatorname{div} f)Z + f\cdot\nabla Z$, and the last inequality comes from Hölder's inequality.) For each $x_1 \in \mathbf{R}$ such that $\left\|f^{x_1}\right\|_1 \left\|\nabla u_1^{x_1}\right\|_N \neq 0$, let $\varepsilon =$

$(\|f\|_1 \|\nabla u_1^{x_1}\|_N + \|\operatorname{div} f\|_1 \|u_1^{x_1}\|_N)/(\|f^{x_1}\|_1 \|\nabla u_1^{x_1}\|_N)$, so that

$$\int_{\mathbf{R}^{N-1}} f_1^{x_1} u_1^{x_1}\, dy \leqslant C_N'''(\|f^{x_1}\|_1 \|\nabla u_1^{x_1}\|_N)^{(N-1)/N} (\|f\|_1 \|\nabla u_1^{x_1}\|_N + \|\operatorname{div} f\|_1 \|u_1^{x_1}\|_N)^{1/N}.$$

If $\|f^{x_1}\|_1 \|\nabla u_1^{x_1}\|_N \neq 0$, choosing $\varepsilon \to \infty$ gives the same inequality; hence the inequality is true for any $x_1 \in \mathbf{R}$.

Finally, Hölder's inequality yields

$$\begin{aligned}
\int_{\mathbf{R}^N} f_1 u_1\, dx &\leqslant \int_{\mathbf{R}} C_N'''(\|f^{x_1}\|_1 \|\nabla u_1^{x_1}\|_N)^{(N-1)/N} \\
&\qquad (\|f\|_1 \|\nabla u_1^{x_1}\|_N + \|\operatorname{div} f\|_1 \|u_1^{x_1}\|_N)^{1/N}\, dx_1 \\
&\leqslant C_N''' \|f\|_1^{\frac{1}{N}} \left(\int_{\mathbf{R}} \|f^{x_1}\|_1\, dx_1\right)^{\frac{N-1}{N}} \left(\int_{\mathbf{R}} \|\nabla u^{x_1}\|_N^N\, dx_1\right)^{\frac{N-1}{N^2}} \\
&\qquad \left(\int_{\mathbf{R}} (\|f\|_1 \|\nabla u_1^{x_1}\|_N + \|\operatorname{div} f\|_1 \|u_1^{x_1}\|_N)^N\, dx_1\right)^{\frac{1}{N^2}} \\
&\leqslant C_N(\|f\|_1 \|\nabla u\|_N)^{(N-1)/N}(\|f\|_1 \|\nabla u\|_N + \|\operatorname{div} f\|_1 \|u\|_N)^{1/N}.
\end{aligned} \tag{2.1}$$

The same estimate holds for $\int_{\mathbf{R}^N} u_i f_i$, $1 \leqslant i \leqslant N$. By classical approximation arguments, the inequality is true for any $f \in \mathrm{L}^1_{\#}(\mathbf{R}^N;\mathbf{R}^N)$ and $u \in (\mathrm{L}^\infty \cap \mathrm{W}^{1,N})(\mathbf{R}^N;\mathbf{R}^N)$. □

Remark 2.1. In fact the proof yields a slighly stronger inequality where $\|\nabla u\|_N$ is replaced by $\sum_{i\neq j} \|\partial_i u_j\|_N$, from which the inequality (2.1) can be recovered by a scaling argument.

Remark 2.2. The same arguments show that Theorem 1.7 remains true when f is a measure whose divergence is a measure.

Remark 2.3. As Bourgain and Brezis pointed out for Theorem 1.3 in [2], the proof works also when $\|\nabla u\|_N$ is replaced by any fractional Sobolev semi-norm $|\cdot|_{s,p}$, with $1 < p < \infty$, $0 < s < 1$, $sp = N$ and

$$|u|_{s,p}^p = \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{2N}}\, dx\, dy.$$

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CHAPTER VII

Estimates for L^1 vector fields with a second order condition

1. Introduction

This note originates in the inequality proved by the author in [3].

Theorem 1.1. *There exists a constant C_N such that for each $f \in \mathrm{L}^1(\mathbf{R}^N;\mathbf{R}^N)$ such that* $\operatorname{div} f \in \mathrm{L}^1$ *and* $u \in (\mathrm{L}^\infty \cap \mathrm{W}^{1,N})(\mathbf{R}^N;\mathbf{R}^N)$,

$$\left|\int_{\mathbf{R}^N} f \cdot u \, dx\right| \leqslant C_N(\|f\|_1 \|\nabla u\|_N + \|\operatorname{div} f\|_1 \|u\|_N).$$

Theorem 1.1 was proved when $\operatorname{div} f = 0$ by Bourgain and Brezis [1]. In this note, a variant of Theorem 1.1 is proved where the divergence is replaced by a second order operator.

Theorem 1.2. *Let $u \in (\mathrm{L}^\infty \cap \mathrm{W}^{1,N})(\mathbf{R}^N)$ and $f_{ij} \in \mathrm{L}^1(\mathbf{R}^N)$, $g_i \in \mathrm{L}^1(\mathbf{R}^N)$ for $N-1 \leqslant i \leqslant N$ and $1 \leqslant j \leqslant i$. If*

$$\sum_{\substack{N-1\leqslant i\leqslant N\\ 1\leqslant j\leqslant i}} \frac{\partial^2 f_{ij}}{\partial x_i \partial x_j} = \sum_{N-1\leqslant i\leqslant N} \frac{\partial g_i}{\partial x_i}, \tag{1.1}$$

in the sense of distributions, then for each $N-1 \leqslant i \leqslant N$ and $1 \leqslant j \leqslant i$

$$\left|\int_{\mathbf{R}^N} f_{ij} u\right| \leqslant C_N(\|f\|_1 \|\nabla u\|_N + \|g\|_1 \|u\|_N),$$

where

$$\|f\|_1 = \sum_{\substack{N-1\leqslant i\leqslant N\\ 1\leqslant j\leqslant i}} \|f_{ij}\|_1$$

and

$$\|g\|_1 = \sum_{N-1\leqslant i\leqslant N} \|g_i\|_1 .$$

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Remark 1.3. Theorem 1.2 implies Theorem 1.1. Indeed, suppose f satisfies the hypotheses of Theorem 1.1. If $f_{Nj} = f_j$, $f_{N-1j} = 0$ for each j, $g_N = \operatorname{div} f$ and $g_{N-1} = 0$, then f and g satisfy the hypotheses of Theorem 1.2. The conclusion of Theorem 1.2 implies the conclusion of Theorem 1.1.

The restriction $N-1 \leqslant i \leqslant N$ does not seem natural when $N \geqslant 3$. In particular, Theorem 1.2 does not answer the question whether

$$\left| \int_{\mathbf{R}^3} f \cdot u \, dx \right| \leqslant C_N \left\| f \right\|_1 \left\| \nabla u \right\|_3 .$$

for each $u \in (\mathrm{L}^\infty \cap \mathrm{W}^{1,3})(\mathbf{R}^3; \mathbf{R}^3)$ and $f \in \mathrm{L}^1(\mathbf{R}^3; \mathbf{R}^3)$ such that

$$\sum_{i=1}^{3} \partial_i^2 f_i = 0$$

excepted when one of the components f_i vanishes. More generally one can ask whether Theorem 1.2 is true under more natural assumptions:

Open Problem 1. Let $u \in (\mathrm{L}^\infty \cap \mathrm{W}^{1,N})(\mathbf{R}^N)$, $f_{ij} \in \mathrm{L}^1(\mathbf{R}^N)$ and $g_i \in \mathrm{L}^1(\mathbf{R}^N)$ for $1 \leqslant i \leqslant N$ and $1 \leqslant j \leqslant i$. If

$$\sum_{\substack{1 \leqslant i \leqslant N \\ 1 \leqslant j \leqslant i}} \frac{\partial^2 f_{ij}}{\partial x_i \partial x_j} = \sum_{1 \leqslant i \leqslant N} \frac{\partial g_i}{\partial x_i},$$

in the sense of distributions, then is it true that for each $1 \leqslant i \leqslant N$ and $1 \leqslant j \leqslant i$,

$$\left| \int_{\mathbf{R}^N} f_{ij} u \right| \leqslant C_N (\left\| f \right\|_1 \left\| \nabla u \right\|_N + \left\| g \right\|_1 \left\| u \right\|_N),$$

where

$$\left\| f \right\|_1 = \sum_{\substack{1 \leqslant i \leqslant N \\ 1 \leqslant j \leqslant i}} \left\| f_{ij} \right\|_1$$

and

$$\left\| g \right\|_1 = \sum_{1 \leqslant i \leqslant N} \left\| g_i \right\|_1 ?$$

The problem is open even in the simple case where $g_i = 0$ for all i and $f_{ij} = 0$ for $i \neq j$.

Open Problem 2. Suppose $u \in (\mathrm{L}^\infty \cap \mathrm{W}^{1,N})(\mathbf{R}^N; \mathbf{R}^N)$ and $f \in \mathrm{L}^1(\mathbf{R}^N; \mathbf{R}^N)$. If

$$\sum_{i=1}^{N} \frac{\partial^2 f_i}{\partial {x_i}^2} = 0,$$

in the sense of distributions, then is it true that

$$\left| \int_{\mathbf{R}^N} f \cdot u \right| \leqslant C_N \|f\|_1 \|\nabla u\|_N ?$$

2. Proof of Theorem 1.2

The key estimate is in the following

Lemma 2.1. *Let $u \in C^1(\mathbf{R}^{N-1})$. Let $f_{ij} \in \mathrm{L}^1(\mathbf{R}^N)$ and $g_i \in \mathrm{L}^1(\mathbf{R}^N)$ for $N-1 \leqslant i \leqslant N$ and $1 \leqslant j \leqslant i$. If (1.1) holds in the sense of distributions, then for each $t \in \mathbf{R}$,*

$$\begin{aligned} \left| \int_{\mathbf{R}^{N-1}} f_{NN}(x,t) u(x)\, dx \right| \leqslant \frac{1}{2} \Big(& \|f_{NN}\|_1 \|\partial_{N-1} u\|_\infty \\ & + \sum_{\substack{N-1 \leqslant i \leqslant N \\ 1 \leqslant j \leqslant N-1}} \|f_{ij}\|_1 \|\partial_j u\|_\infty + \sum_{N-1 \leqslant i \leqslant N} \|g_i\|_1 \|u\|_\infty \Big). \end{aligned}$$

Proof. Let $y \in \mathbf{R}^{N-2}$ and $z \in \mathbf{R}$. Write the integrand as

$$\begin{aligned} f_{NN}(y,z,t) = \frac{1}{2} \int_{-\infty}^0 & \left(\frac{\partial}{\partial x_{N-1}} + \frac{\partial}{\partial x_N} \right) f_{NN}(y, z+s, t+s) \\ & + \left(\frac{\partial}{\partial x_{N-1}} - \frac{\partial}{\partial x_N} \right) f_{NN}(y, z+s, t-s)\, ds. \end{aligned}$$

This gives

$$\begin{aligned} (2.1) \quad 2 & \int_{\mathbf{R}^{N-1}} f_{NN}(y,z,t) u(y,z)\, dz\, dy \\ &= \int_{\mathbf{R}^{N-1}} \int_{-\infty}^0 u(y,z) \Big(\frac{\partial f_{NN}}{\partial x_{N-1}}(y, z+s, t+s) \\ & \qquad\qquad + \frac{\partial f_{NN}}{\partial x_{N-1}}(y, z+s, t-s) \Big)\, ds\, dz\, dy \\ & \quad + \int_{\mathbf{R}^{N-1}} \int_{-\infty}^0 u(y,z) \Big(\frac{\partial f_{NN}}{\partial x_N}(y, z+s, t+s) \\ & \qquad\qquad - \frac{\partial f_{NN}}{\partial x_N}(y, z+s, t-s) \Big)\, ds\, dz\, dy. \end{aligned}$$

The first term is estimated by integration by parts

$$
\begin{aligned}
(2.2)\quad & \int_{\mathbf{R}^{N-2}} \int_{-\infty}^{0} \int_{\mathbf{R}} u(y,z) \Big(\frac{\partial f_{NN}}{\partial x_{N-1}}(y, z+s, t+s) \\
&\qquad\qquad + \frac{\partial f_{NN}}{\partial x_{N-1}}(y, z+s, t-s) \Big)\, dz\, ds\, dy \\
&= - \int_{\mathbf{R}^{N-2}} \int_{-\infty}^{0} \int_{\mathbf{R}} \frac{\partial u}{\partial x_{N-1}}(y,z) \Big(f_{NN}(y, z+s, t+s) \\
&\qquad\qquad + f_{NN}(y, z+s, t-s) \Big)\, dz\, ds\, dy \\
&= - \int_{\mathbf{R}^{N-2}} \int_{-\infty}^{0} \int_{\mathbf{R}} \frac{\partial u}{\partial x_{N-1}}(y, z'-s) \Big(f_{NN}(y, z', t+s) \\
&\qquad\qquad + f_{NN}(y, z', t-s) \Big)\, dz'\, ds\, dy \\
&\leqslant \left\| \frac{\partial u}{\partial x_{N-1}} \right\|_{\infty} \int_{\mathbf{R}^N} |f_{NN}| \, .
\end{aligned}
$$

For any y, z, t and s, the integrand of the second term of (2.1) can be written as

$$
\begin{aligned}
(2.3)\quad & \frac{\partial f_{NN}}{\partial x_N}(y, z+s, t+s) - \frac{\partial f_{NN}}{\partial x_N}(y, z+s, t-s) \\
&\qquad = \int_{-s}^{s} \frac{\partial^2 f_{NN}}{\partial {x_N}^2}(y, z+s, t+\tau)\, d\tau.
\end{aligned}
$$

Bringing (2.3) and (1.1) together yields

$$
\begin{aligned}
& \int_{\mathbf{R}^{N-2}} \int_{\mathbf{R}} \int_{-\infty}^{0} u(y,z) \Big(\frac{\partial f_{NN}}{\partial x_N}(y, z+s, t+s) \\
&\qquad\qquad - \frac{\partial f_{NN}}{\partial x_N}(y, z+s, t-s) \Big)\, ds\, dz\, dy \\
&= \int_{\mathbf{R}^{N-2}} \int_{\mathbf{R}} \int_{-\infty}^{0} u(y,z) \int_{-s}^{s} \frac{\partial^2 f_{NN}}{\partial {x_N}^2}(y, z+s, t+\tau)\, d\tau\, ds\, dz\, dy
\end{aligned}
$$

$$= -\sum_{\substack{N-1\leqslant i\leqslant N\\ 1\leqslant j\leqslant N-1}} \int_{\mathbf{R}^{N-2}} \int_{\mathbf{R}} \int_{-\infty}^{0} u(y,z) \int_{-s}^{s} \frac{\partial^2 f_{ij}}{\partial x_i \partial x_j}(y, z+s, t+\tau)\, d\tau\, ds\, dz\, dy$$
$$+ \sum_{N-1\leqslant i\leqslant N} \int_{\mathbf{R}^{N-2}} \int_{\mathbf{R}} \int_{-\infty}^{0} u(y,z) \int_{-s}^{s} \frac{\partial g_i}{\partial x_i}(y, z+s, t+\tau)\, d\tau\, ds\, dz\, dy.$$

Each term of the sum will now be bounded separately. For $i = N$ and $1 \leqslant j \leqslant N-1$, one has

$$\begin{aligned}
&\int_{\mathbf{R}^{N-2}} \int_{\mathbf{R}} \int_{-\infty}^{0} u(y,z) \int_{-s}^{s} \frac{\partial^2 f_{Nj}}{\partial x_N \partial x_j}(y, z+s, t+\tau)\, d\tau\, ds\, dz\, dy \\
&= \int_{-\infty}^{0} \int_{\mathbf{R}^{N-2}} \int_{\mathbf{R}} u(y,z) \Big(\frac{\partial f_{Nj}}{\partial x_j}(y, z+s, t+s) - \frac{\partial f_{Nj}}{\partial x_j}(y, z+s, t-s) \Big)\, dz\, dy\, ds \\
&= -\int_{-\infty}^{0} \int_{\mathbf{R}^{N-2}} \int_{\mathbf{R}} \frac{\partial u}{\partial x_j}(y,z) \Big(f_{Nj}(y, z+s, t+s) - f_{Nj}(y, z+s, t-s) \Big)\, dz\, dy\, ds \\
&\leqslant \left\| \frac{\partial u}{\partial x_j} \right\|_\infty \int_{\mathbf{R}^N} |f_{Nj}|\,.
\end{aligned} \tag{2.4}$$

If $i = N-1$ and $1 \leqslant j \leqslant N-1$, then

$$\begin{aligned}
&\int_{\mathbf{R}^{N-2}} \int_{\mathbf{R}} \int_{-\infty}^{0} \int_{-s}^{s} u(y,z) \frac{\partial^2 f_{N-1j}}{\partial x_{N-1} \partial x_j}(y, z+s, t+\tau)\, d\tau\, ds\, dz\, dy \\
&= \int_{\mathbf{R}^{N-2}} \int_{\mathbf{R}} \int_{\mathbf{R}} \int_{-\infty}^{-|\tau|} u(y,z) \frac{\partial^2 f_{N-1j}}{\partial x_{N-1} \partial x_j}(y, z+s, t+\tau)\, ds\, d\tau\, dz\, dy \\
&= \int_{\mathbf{R}^{N-2}} \int_{\mathbf{R}} \int_{\mathbf{R}} u(y,z) \frac{\partial f_{N-1j}}{\partial x_j}(y, z-|\tau|, t+\tau)\, d\tau\, dz\, dy \\
&= \int_{\mathbf{R}} \int_{\mathbf{R}^{N-2}} \int_{\mathbf{R}} u(y,z) \frac{\partial f_{N-1j}}{\partial x_j}(y, z-|\tau|, t+\tau)\, dz\, dy\, d\tau
\end{aligned} \tag{2.5}$$

$$= \int_{\mathbf{R}} \int_{\mathbf{R}^{N-2}} \int_{\mathbf{R}} \frac{\partial u}{\partial x_j}(y,z) f_{N-1j}(y, z - |\tau|, t+\tau)\, dz\, dy\, d\tau$$
$$\leqslant \left\| \frac{\partial u}{\partial x_j} \right\|_\infty \int_{\mathbf{R}^N} |f_{N-1j}| \,.$$

The reasoning is similar for the terms with g_i. The sum of all these inequalities yields the result. □

Next lemma proves an estimate for $u \in \mathrm{W}^{1,N}(\mathbf{R}^N)$ when there is an estimate of the type of Lemma 2.1 for $v \in \mathrm{C}^1(\mathbf{R}^{N-1})$.

Lemma 2.2. *Let $f \in \mathrm{L}^1(\mathbf{R}^N)$ and $a, b \in \mathbf{R}^+$ such that, for any function $v \in \mathrm{C}^1(\mathbf{R}^{N-1})$ and for any $t \in \mathbf{R}$,*

$$\left| \int_{\mathbf{R}^{N-1}} f(x,t) v(x)\, dx \right| \leqslant a \|\nabla v\|_\infty + b \|v\|_\infty \,,$$

then, for any $u \in (\mathrm{L}^\infty \cap \mathrm{W}^{1,N})(\mathbf{R}^N)$,

$$\left| \int_{\mathbf{R}^N} fu \right| \leqslant C_N (\|f\|_1 \|\nabla u\|_N)^{1-(1/N)} (a \|\nabla u\|_N + b \|u\|_N)^{1/N}.$$

Proof. Let $\rho : \mathbf{R}^{N-1} \to \mathbf{R}$ be a measurable bounded function with compact support such that $\int_{\mathbf{R}^{N-1}} \rho = 1$ and let $\rho_\varepsilon(\cdot) = \varepsilon^{1-N} \rho(\frac{\cdot}{\varepsilon})$. If $u^t(y) = u(t,y)$ and $f^t(y) = f(t,y)$, then

$$\int_{\mathbf{R}^{N-1}} f^t u^t\, dy = \int_{\mathbf{R}^{N-1}} f^t (u^t - \rho_\varepsilon * u^t)\, dy + \int_{\mathbf{R}^{N-1}} f^t (\rho_\varepsilon * u^t)\, dy.$$

The Morrey–Sobolev embedding in $\mathbf{R}^{N-1}$ gives

$$\left| \int_{\mathbf{R}^{N-1}} f^t (u^t - \rho_\varepsilon * u^t)\, dy \right| \leqslant C'_N \varepsilon^{1/N} \left\|f^t\right\|_1 \left\|\nabla u^t\right\|_N \cdot$$

On the other hand

$$\left| \int_{\mathbf{R}^{N-1}} f^t (\rho_\varepsilon * u^t)\, dy \right| \leqslant C''_N \varepsilon^{(1/N)-1} (a \left\|\nabla u^t\right\|_N + b \left\|u^t\right\|_N).$$

The constants C'_N and C''_N depend only on the dimension N (and of ρ). For each $t \in \mathbf{R}$, if $\left\|f^t\right\|_1 \left\|\nabla u^t\right\|_N \neq 0$, the choice $\varepsilon = (a \left\|\nabla u^t\right\|_N + b \left\|u^t\right\|_N) / (\left\|f^t\right\|_1 \left\|\nabla u^t\right\|_N)$ yields

$$\left| \int_{\mathbf{R}^{N-1}} f^t u^t\, dy \right|$$
$$\leqslant C'''_N (\left\|f^t\right\|_1 \left\|\nabla u^t\right\|_N)^{1-(1/N)} (a \left\|\nabla u^t\right\|_N + b \left\|u^t\right\|_N)^{1/N}.$$

If $\left\|f^t\right\|_1 \left\|\nabla u^t\right\|_N \neq 0$, let $\varepsilon \to \infty$ to obtain the same inequality. The inequality is thus valid for any $t \in \mathbf{R}$.

Finally, by Hölder's inequality

$$
\begin{aligned}
&\left|\int_{\mathbf{R}^N} f u\,dx\right| \\
&\quad \leqslant \int_{\mathbf{R}} C_N'''(\|f^t\|_1 \|\nabla u^t\|_N)^{1-(1/N)}(a\|\nabla u^t\|_N + b\|u^t\|_N)^{1/N}\,dt \\
&\quad \leqslant C_N''' \left(\int_{\mathbf{R}} \|f^t\|_1\,dt\right)^{(N-1)/N} \left(\int_{\mathbf{R}} \|\nabla u^t\|_N^N\,dt\right)^{(N-1)/N^2} \\
&\qquad\qquad \left(\int_{\mathbf{R}} (a\|\nabla u_1^t\|_N + b\|u_1^t\|_N)^N\,dt\right)^{1/N^2} \\
&\quad \leqslant C_N(\|f\|_1 \|\nabla u\|_N)^{(N-1)/N}(a\|\nabla u\|_N + b\|u\|_N)^{1/N}. \quad \square
\end{aligned}
$$

The combination of Lemmas 2.1 and 2.2 yields a special case of Theorem 1.2.

Lemma 2.3. *Under the hypotheses of Theorem 1.2,*

$$\left|\int_{\mathbf{R}^N} f_{NN} u\right| \leqslant C_N(\|f\|_1 \|\nabla u\|_N + \|g\|_1 \|u\|_N),$$

where C_N is a constant independent of f, g and u.

Proof. This is a direct consequence of Lemmas 2.1 and 2.2. $\square$

By appropriate changes of variable, Theorem 1.2 can be deduced from Lemma 2.3.

Proof of Theorem 1.2. By Lemma 2.3, the result is true for $i = j = N$. It is also true for $i = j = N - 1$, by interverting N and $N - 1$ in the hypotheses.

If $j < N - 1$ and $i = N$, define new variables by $x_j' = x_j - x_N$ and $x_k' = x_k$ if $k \neq j$ and a new vector field f', defined by $f_{NN}' = f_{NN} + f_{Nj}$ and $f_{NN-1}' = f_{NN-1} + f_{N-1j}$ (for the other components, let $f_{ij}' = f_{ij}$). Let $g' = g$. One checks that f' verifies the same hypotheses as f and that $\|f'\|_1 \leqslant 2\|f\|_1$. Since the inequality is true for f_{NN}' and for f_{NN}, it is true for $f_{NN}' - f_{NN} = f_{Nj}$.

The situation is somewhat more tedious when $j = N - 1$. Define new variables by $x_{N-1}' = x_{N-1} - x_N$ and $x_k' = x_k$ for $k \neq N - 1$. Let

$$f_{Nk}' = \begin{cases} f_{Nk} + f_{N-1k} & \text{if } k < N-1, \\ f_{NN-1} + 2f_{N-1N-1} & \text{if } k = N-1, \\ f_{NN} + f_{NN-1} - f_{N-1N-1} & \text{if } k = N, \end{cases}$$

and $f'_{N-1k} = f_{N-1k}$. Let $g'_{N-1} = g_{N-1}$ and $g'_N = g_N + g_{N-1}$. The condition (1.1) is checked by f' and g'. Since the inequality holds for f_{NN}, f_{N-1N-1} and f'_{NN}, it holds for f_{NN-1}. □

3. Relationship with a Korn-Sobolev inequality

The Sobolev-Gagliardo-Nirenberg inequality

$$\|u\|_{N/(N-1)} \leqslant C\,\|\nabla u\|_1$$

can be obtained by a combination of Theorem 1.1 and the classical Calderón–Zygmund estimates.

In a similar way a Korn–Sobolev inequality of Strauss results from Theorem 1.2 and the classical Calderón–Zygmund estimates.

Theorem 3.1 (Strauss [2]). *For any $u \in \mathcal{D}(\mathbf{R}^N; \mathbf{R}^N)$,*

$$\|u\|_{\frac{N}{N-1}} \leqslant K_N \sum_{1\leqslant i\leqslant j\leqslant N} \|\partial_i u_j + \partial_j u_i\|_1 \,.$$

Sketch of the proof of Theorem 3.1 using Theorem 1.2. Let

$$H \in \mathcal{D}(\mathbf{R}^N; \mathbf{R}^N).$$

Let A be the differential operator defined for $\mathbf{R}^N$-valued functions by

$$(Au)_{ij} = (\partial_i u_j + \partial_j u_i).$$

Its formal adjoint is defined for $\mathbf{R}^{N^2}$-valued functions by

$$(A^* v)_i = -\sum_{j=1}^{N} (\partial_j v_{ij} + \partial_j v_{ji}).$$

Consider the system $A^*Ap = H$. It is equivalent to

$$\Delta p_i + \partial_i \sum_{j=1}^{N} \partial_j p_j = -\frac{H_i}{2}. \tag{3.1}$$

This system is elliptic and has a solution

$$p \in (\mathrm{W}^{1,\infty} \cap \mathrm{W}^{2,1}_{\mathrm{loc}})(\mathbf{R}^N; \mathbf{R}^{N^2}).$$

Furthermore, there exists a constant B_N independent of H such that

$$\left\|D^2 p\right\|_N \leqslant B_N \,\|H\|_N \,.$$

Since p solves (3.1),

$$\int_{\mathbf{R}^N} uH = \int_{\mathbf{R}^N} u\, A^*Ap = \int_{\mathbf{R}^N} Au\, Ap.$$

Recalling $\partial_i^2(Au)_{jj}+\partial_j^2(Au)_{ii} = 2\partial_i\partial_j(Au)_{ij}$, the application of Theorem 1.2 to each 2×2 submatrix of Au gives

$$\left|\int_{\mathbf{R}^N} uH\right| = \left|\int_{\mathbf{R}^N} Au\, Ap\right| \leqslant C_N \|Au\|_1 \|\nabla Ap\|_N$$
$$\leqslant B_N C_N \|Au\|_1 \|H\|_N .$$

Since H is arbitrary, the result follows. □

Remark 3.2. The proof of Theorem 3.1 needs only a weak version of Theorem 1.2 where $f_{ij} = 0$ and $g_i = 0$ for $j < N-1$ and $N-1 \leqslant i \leqslant N$.

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