PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 138, Number 1, January 2010, Pages 235–240
S 0002-9939(09)10005-9
Article electronically published on September 3, 2009

LIMITING FRACTIONAL AND LORENTZ SPACE ESTIMATES OF DIFFERENTIAL FORMS

JEAN VAN SCHAFTINGEN

(Communicated by Michael T. Lacey)

ABSTRACT. We obtain estimates in Besov, Triebel–Lizorkin and Lorentz spaces of differential forms on $\mathbf{R}^n$ in terms of their L^1 norm.

1. INTRODUCTION

The classical Hodge theory states that if $u \in C_c^\infty(\mathbf{R}^N; \bigwedge^\ell \mathbf{R}^n)$ and if $1 < p < \infty$, one has

$$\|Du\|_{L^p} \le C(\|du\|_{L^p} + \|\delta u\|_{L^p}). \tag{1}$$

Here Du is the derivative of u, du is its exterior differential of u, δu is its exterior codifferential, and $\bigwedge^\ell \mathbf{R}^n$ is the ℓth exterior product of $\mathbf{R}^n$. This estimate is known to fail when $p = 1$ or $p = \infty$.

When $p = 1$, J. Bourgain and H. Brezis [2, 3] and L. Lanzani and E. Stein [5] obtained for $2 \le \ell \le n - 2$ the estimate

$$\|u\|_{L^{n/(n-1)}} \le C(\|du\|_{L^1} + \|\delta u\|_{L^1}),$$

which would be the consequence by the Sobolev embedding of (1) with $p = 1$. When $\ell = 1$ or $\ell = n - 1$ one has to assume that du or δu vanishes.

I. Mitrea and M. Mitrea [6] have in a recent work extended these estimates to homogeneous Besov spaces. Using interpolation theory, they could replace the norm $\|u\|_{L^{n/(n-1)}}$ by $\|u\|_{\dot{B}^s_{p,q}}$ with $\frac{1}{p} - \frac{s}{n} = 1 - \frac{1}{n}$ and $q = \frac{2}{1-s}$. The goal of the present paper is to improve the assumption on q by relying on previous results and methods.

The first result is the estimate for the Besov spaces $\dot{B}^s_{p,q}(\mathbf{R}^n)$. We follow H. Triebel [10] for the definitions of the function spaces.

Theorem 1. *For every $s \in (0, 1)$, $p > 1$ and $q > 1$, if*

$$\frac{1}{p} - \frac{s}{n} = 1 - \frac{1}{n}, \tag{2}$$

Received by the editors March 12, 2009, and, in revised form, April 20, 2009.
2000 *Mathematics Subject Classification.* Primary 35B65; Secondary 26D10, 35F05, 42B20, 46E30, 46E35, 58A10.
Key words and phrases. Differential forms, div-curl system, Hodge decomposition, exterior differential, Besov spaces, Triebel–Lizorkin spaces, Lorentz-Sobolev spaces, regularity, limiting embedding.
The author is supported by the Fonds de la Recherche Scientifique–FNRS.

then there exists $C > 0$ such that for every $u \in C_c^\infty(\mathbf{R}^n; \bigwedge^\ell \mathbf{R}^n)$ with, moreover, $\delta u = 0$ if $\ell = 1$ and $du = 0$ if $\ell = n-1$, one has

$$\|u\|_{\dot{B}^s_{p,q}} \le C(\|du\|_{L^1} + \|\delta u\|_{L^1}).$$

In particular, since $\|u\|_{\dot{W}^{s,p}} = \|u\|_{\dot{B}^s_{p,p}}$, one has the estimate

$$\|u\|_{\dot{W}^{s,p}} \le C(\|du\|_{L^1} + \|\delta u\|_{L^1}). \tag{3}$$

In Theorem 1, we assume that $q > 1$. If it held for some $q \in (0,1]$, then the embedding of $F^1_{1,2}(\mathbf{R}^n) \subset \dot{B}^0_{n/(n-1),q}(\mathbf{R}^n)$ would hold. This can only be the case when $q \ge 1$. Therefore, the only possible improvement of Theorem 1 would be the limiting case $q = 1$.

Open Problem 1. Does Theorem 1 hold for $q = 1$?

The estimate of Theorem 1 follows from the corresponding estimate for homogeneous Triebel–Lizorkin spaces $\dot{F}^s_{p,q}(\mathbf{R}^n)$.

Theorem 2. *For every $s \in (0,1)$, $p > 1$ and $q > 0$, if (2) holds, then there exists $C > 0$ such that for every $u \in C_c^\infty(\mathbf{R}^n; \bigwedge^\ell \mathbf{R}^n)$ with, moreover, $\delta u = 0$ if $\ell = 1$ and $du = 0$ if $\ell = n-1$, one has*

$$\|u\|_{\dot{F}^s_{p,q}} \le C(\|du\|_{L^1} + \|\delta u\|_{L^1}).$$

Note that here there is no restriction on $q > 0$. Finally, the latter estimate has an interesting consequence for Lorentz spaces.

Theorem 3. *For every $q > 1$, there exists $C > 0$ such that for every $u \in C_c^\infty(\mathbf{R}^n; \bigwedge^\ell \mathbf{R}^n)$ with, moreover, $\delta u = 0$ if $\ell = 1$ and $du = 0$ if $\ell = n-1$, one has*

$$\|u\|_{L^{\frac{n}{n-1},q}} \le C(\|du\|_{L^1} + \|\delta u\|_{L^1}).$$

In Theorem 3 the case $q = 1$ and $\ell = 0$ is equivalent to the embedding of $W^{1,1}(\mathbf{R}^n)$ in $L^{\frac{n}{n-1},1}(\mathbf{R}^n)$, which was obtained by J. Peetre [8] (see also [15]). This raises the following question:

Open Problem 2. Does Theorem 3 hold for $q = 1$ and $\ell \ge 1$?

The proofs of the theorems rely on the techniques developed by the author [11, 12] and on classical embeddings and regularity theory in fractional spaces.

2. The main tool

Our main tool is a generalization of an estimate for divergence-free L^1 vector fields by the author [12].

Proposition 2.1. *For every $s \in (0,1)$, $p > 1$ and $q > 0$ with $sp = n$, there exists $C > 0$ such that for every $f \in (C_c^\infty \cap L^1)(\mathbf{R}^n; \bigwedge^{n-1} \mathbf{R}^n)$ and $\varphi \in C_c^\infty(\mathbf{R}^n; \bigwedge^1 \mathbf{R}^n)$, if $df = 0$,*

$$\int_{\mathbf{R}^n} f \wedge \varphi \le C\|f\|_{L^1}\|\varphi\|_{\dot{F}^s_{p,q}}.$$

Here and in the sequel, $\wedge$ denotes the exterior product of forms. The proof of this proposition follows the method introduced by the author [4, 11, 12, 14] and followed subsequently by L. Lanzani and E. Stein [5] and by I. Mitrea and M. Mitrea [6]. The extension to the case $q = p$ is suggested in a previous work of the author [12, Remark 5] (see also [2, Remark 1], [3, Remark 11], [14, Remark 2], and [4]); the proposition can be deduced therefrom by following a remark in a subsequent paper [13, Remark 4.2].

Proof. Write $\varphi = \varphi^1 dx_1 + \cdots + \varphi^n dx_n$ and $f = f_1 dx_2 \wedge \cdots \wedge dx_n - f_2 dx_1 \wedge dx_3 \wedge \cdots \wedge dx_n + \cdots + (-1)^n f_n dx_1 \wedge \cdots \wedge dx_{n-1}$. Without loss of generality, we shall estimate

$$\int_{\mathbf{R}^n} f_1 \varphi^1.$$

Fix $t \in \mathbf{R}$, and consider the function $\psi : \mathbf{R}^{n-1} \to \mathbf{R}$ defined by $\psi(y) = \varphi^1(t, y)$. Choose $\rho \in C_c^\infty(\mathbf{R}^n)$ such that $\int_{\mathbf{R}^{n-1}} \rho = 1$, and set for $\varepsilon > 0$ and $y \in \mathbf{R}^{n-1}$ $\rho_\varepsilon(y) = \frac{1}{\varepsilon^{n-1}} \rho(\frac{y}{\varepsilon})$. For every $\alpha \in (0, 1)$, there is a constant $C > 0$ that only depends on ρ and α such that

$$\|\nabla \rho_\varepsilon * \psi\|_{L^\infty} \le C \varepsilon^{\alpha - 1} |\psi|_{C^{0,\alpha}(\mathbf{R}^{n-1})}$$

and

$$\|\psi - \rho_\varepsilon * \psi\|_{L^\infty} \le C \varepsilon^{\alpha} |\psi|_{C^{0,\alpha}(\mathbf{R}^{n-1})},$$

where $|\psi|_{C^{0,\alpha}(\mathbf{R}^{n-1})}$ is the $C^{0,\alpha}$ seminorm of ψ, i.e.,

$$|\psi|_{C^{0,\alpha}(\mathbf{R}^{n-1})} = \sup_{y, z \in \mathbf{R}^{n-1}} \frac{|\psi(z) - \psi(y)|}{|z - y|^\alpha}.$$

One has on the one hand

$$\int_{\mathbf{R}^{n-1}} f_1(t, \cdot)(\psi - \rho_\varepsilon * v) \le C \|f_1(t, \cdot)\|_{L^1(\mathbf{R}^{n-1})} \varepsilon^\alpha |\psi|_{C^{0,\alpha}(\mathbf{R}^{n-1})}.$$

On the other hand, by integration by parts, and since $\sum_{i=1}^n \partial_i f_i = 0$,

$$\begin{aligned} \int_{\mathbf{R}^{n-1}} f_1(t, \cdot) \rho_\varepsilon * \psi &= \sum_{i=2}^n (-1)^i \int_{\mathbf{R}^{n-1}} \int_{\mathbf{R}^+} f_i(t, y) \partial_i (\rho_\varepsilon * \psi)(y) \, dt \, dy \\ &\le C \|f_1(t, \cdot)\|_{L^1(\mathbf{R}^{n-1})} \varepsilon^{\alpha - 1} |\psi|_{C^{0,\alpha}(\mathbf{R}^{n-1})}. \end{aligned}$$

Taking $\varepsilon = \|f\|_{L^1(\mathbf{R}^n)} / \|f(t, \cdot)\|_{L^1(\mathbf{R}^{n-1})}$, one obtains

$$\int_{\mathbf{R}^{n-1}} f_1 \psi \le C \|f\|_{L^1(\mathbf{R}^n)}^\alpha \|f_1(t, \cdot)\|_{L^1(\mathbf{R}^{n-1})}^{1-\alpha} |\psi|_{C^{0,\alpha}(\mathbf{R}^{n-1})}. \tag{4}$$

Now, by the embedding theorem for Triebel–Lizorkin spaces, one has the estimate

$$|\psi|_{C^{0,\alpha}} \le C \|\psi\|_{\dot{F}^s_{p,q}(\mathbf{R}^{n-1})}$$

with $\alpha = \frac{1}{p}$; hence from (4) we deduce the inequality

$$\int_{\mathbf{R}^{n-1}} f_1 \psi \le C \|f\|_{L^1}^{\frac{1}{p}} \|f_1(\cdot, t)\|_{L^1}^{1 - \frac{1}{p}} \|\psi\|_{\dot{F}^s_{p,q}(\mathbf{R}^{n-1})}.$$

Now, recalling that as a direct consequence of the Fubini property of Triebel–Lizorkin spaces ([10, Theorem 2.5.13], [1, Théorème 2], [9, Theorem 2.3.4/2])

$$\Big(\int_{\mathbf{R}} \|\varphi(t, \cdot)\|_{\dot{F}^s_{p,q}(\mathbf{R}^{n-1})}^p \, dt \Big) \le C \|\varphi\|_{\dot{F}^s_{p,q}(\mathbf{R}^n)}^p,$$

one concludes, using Hölder's inequality, that

$$\int_{\mathbf{R}^n} f_1 u^1 \le C\|f\|_{L^1}^{\frac{1}{p}} \int_{\mathbf{R}} \big(\|f_1(\cdot,t)\|_{L^1}^{1-\frac{1}{p}} \|\varphi(t,\cdot)\|_{\dot{F}^s_{p,q}(\mathbf{R}^{n-1})}\big)\,dt \le C'\|f\|_{L^1}\|\varphi\|_{\dot{F}^s_{p,q}(\mathbf{R}^n)}.$$

□

Proposition 2.2. *For every $s \in (0,1)$, $p > 1$ with $\frac{1}{p} + \frac{s}{n} = 1$, $q > 1$ and $1 \le \ell \le n-1$, there exists $C > 0$ such that for every $f \in C_c^\infty(\mathbf{R}^n; \bigwedge^\ell \mathbf{R}^n)$ with $df = 0$, one has*

$$\|f\|_{\dot{F}^{-s}_{p,q}} \le C\|f\|_{L^1}.$$

Proof. The proposition will be proved by downward induction. The proposition is true for $\ell = n-1$ by Proposition 2.1. Assume now that it holds for $\ell + 1$, and let $f \in C_c^\infty(\mathbf{R}^n; \bigwedge^\ell \mathbf{R}^n)$. Since $d(f \wedge dx_i) = 0$, Proposition 2.1 is applicable and

$$\|f\|_{\dot{F}^{-s}_{\frac{n}{n-s},q}} \le \sum_{i=1}^n \|f \wedge dx_i\|_{\dot{F}^{-s}_{\frac{n}{n-s},q}} \le C\sum_{i=1}^n \|f\|_{L^1} = Cn\|f\|_{L^1}.$$

□

A useful corollary of the previous proposition is

Corollary 2.3. *For every $s \in (0,1)$, $p > 1$ with $\frac{1}{p}+\frac{s}{n} = 1$, $q > 1$ and $1 \le \ell \le n-1$, there exists $C > 0$ such that for every $f \in C_c^\infty(\mathbf{R}^n; \bigwedge^\ell \mathbf{R}^n)$ with $df = 0$, one has*

$$\|f\|_{\dot{B}_q^{-s,p}} \le C\|f\|_{L^1}.$$

Proof. This follows from classical embeddings between Besov and Triebel–Lizorkin spaces; see the proof of Theorem 1 below. □

3. Proofs of the main results

We begin by proving Theorem 2.

Proof of Theorem 2. To fix ideas, assume that $2 \le \ell \le n-1$. Recall that one has

$$u = d\big(K * (\delta u)\big) + \delta\big(K * (du)\big),$$

where the Newton kernel is defined for $n \ge 3$ by $K(x) = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}|x|^{n-2}}$ and for $n = 2$ by $K(x) = -\frac{1}{2\pi}\log|x|$. By the classical elliptic estimates for Triebel–Lizorkin spaces,

$$\|K * (\delta u)\|_{\dot{F}^{s+1}_{p,q}} \le C\|\delta u\|_{\dot{F}^{s-1}_{p,q}} \qquad \text{and} \qquad \|K * (\delta u)\|_{\dot{F}^{s+1}_{p,q}} \le C\|\delta u\|_{\dot{F}^{s-1}_{p,q}}.$$

Now, since $d(du) = 0$, Proposition 2.2 is applicable and yields

$$\|K * (du)\|_{\dot{F}^{s+1}_{p,q}} \le C\|\delta u\|_{L^1}.$$

Since $\delta(\delta u) = 0$, one can, by the Hodge duality between d and δ, treat $\|K*(du)\|_{\dot{F}^{s+1}_{p,q}}$ similarly. □

We can now deduce Theorem 1 from Theorem 2.

Proof of Theorem 1. First assume that $q \ge p$. Then one has

$$\|u\|_{\dot{B}^s_{p,q}} \le C\|u\|_{\dot{F}^s_{p,q}},$$

and Theorem 1 follows from Theorem 2. Otherwise, if $q < p$, then by the embedding theorems of Besov spaces,

$$\|u\|_{\dot{B}^s_{p,q}} \le C\|u\|_{\dot{F}^r_{q,q}},$$

with $r = s + n(\frac{1}{q} - \frac{1}{p})$, and Theorem 1 also follows from Theorem 2. □

We finish with the proof of Theorem 3. It relies on

Lemma 3.1. *For every $s > 0$, $p > 1$ and $q > 1$ with $sq < n$ and*

$$\frac{1}{p} = \frac{1}{q} - \frac{s}{n}, \tag{5}$$

there exists $C > 0$ such that for every $u \in C_c^\infty(\mathbf{R}^n)$,

$$\|u\|_{L^{p,q}} \le C\|u\|_{\dot{F}^s_{q,2}}.$$

Proof. One has

$$u = I_s * ((-\Delta)^{\frac{s}{2}} u),$$

where the Riesz kernel I_s is defined for $x \in \mathbf{R}^n$ by

$$I_s(x) = \frac{\Gamma(\frac{n-\alpha}{2})}{\pi^{\frac{n}{2}} 2^s \Gamma(\frac{s}{2}) |x|^{n-s}}.$$

One then has, by the Sobolev inequality for Riesz potentials in Lorentz spaces of R. O'Neil [7] (see also e.g. [15, Theorem 2.10.2]),

$$\|u\|_{L^{r,p}} \le C\|(-\Delta)^{\frac{s}{2}} u\|_{L^p}.$$

One concludes by noting that $\|(-\Delta)^{\frac{s}{2}} u\|_{L^p}$ and $\|u\|_{\dot{F}^s_{p,2}}$ are equivalent norms [10, Theorem 2.3.8 and section 5.2.3]. □

Proof of Theorem 3. Choose $s > 0$ so that (5) holds with $p = \frac{n}{n-1}$. Since $\frac{1}{q} - \frac{s}{n} = 1 - \frac{1}{n}$, one can combine Theorem 2 and Lemma 3.1 to obtain the conclusion. □

References

[1] G. Bourdaud, *Calcul fonctionnel dans certains espaces de Lizorkin-Triebel*, Arch. Math. (Basel) **64** (1995), no. 1, 42–47. MR1305659 (96a:46064)

[2] J. Bourgain and H. Brezis, *New estimates for the Laplacian, the div-curl, and related Hodge systems*, C. R. Math. Acad. Sci. Paris **338** (2004), no. 7, 539–543. MR2057026 (2004m:26018)

[3] ———, *New estimates for elliptic equations and Hodge type systems*, J. Eur. Math. Soc. (JEMS) **9** (2007), no. 2, 277–315. MR2293957

[4] S. Chanillo and J. Van Schaftingen, *Subelliptic Bourgain-Brezis estimates on groups*, Math. Res. Lett. **16** (2009), no. 3, 487-501. MR2511628

[5] L. Lanzani and E. M. Stein, *A note on div curl inequalities*, Math. Res. Lett. **12** (2005), no. 1, 57–61. MR2122730 (2005m:58001)

[6] I. Mitrea and M. Mitrea, *A remark on the regularity of the div-curl system*, Proc. Amer. Math. Soc. **137** (2009), 1729–1733. MR2470831

[7] R. O'Neil, *Convolution operators and $L(p, q)$ spaces*, Duke Math. J. **30** (1963), 129–142. MR0146673 (26:4193)

[8] J. Peetre, *Espaces d'interpolation et théorème de Soboleff*, Ann. Inst. Fourier (Grenoble) **16** (1966), fasc. 1, 279–317. MR0221282 (36:4334)

[9] T. Runst and W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, de Gruyter Series in Nonlinear Analysis and Applications, vol. 3, Walter de Gruyter & Co., Berlin, 1996. MR1419319 (98a:47071)

[10] H. Triebel, *Theory of function spaces*, Monographs in Mathematics, vol. 78, Birkhäuser Verlag, Basel, 1983. MR781540 (86j:46026)

[11] J. Van Schaftingen, *A simple proof of an inequality of Bourgain, Brezis and Mironescu*, C. R. Math. Acad. Sci. Paris **338** (2004), no. 1, 23–26. MR2038078 (2004k:26033)

[12] ———, *Estimates for L^1-vector fields*, C. R. Math. Acad. Sci. Paris **339** (2004), no. 3, 181–186. MR2078071 (2005b:35018)

[13] ———, *Function spaces between BMO and critical Sobolev spaces*, J. Funct. Anal. **236** (2006), no. 2, 490–516. MR2240172 (2007e:46028)

[14] ———, *Estimates for L^1 vector fields under higher-order differential conditions*, J. Eur. Math. Soc. (JEMS) **10** (2008), no. 4, 867–882. MR2443922

[15] W. P. Ziemer, *Weakly differentiable functions*, Sobolev spaces and functions of bounded variation, Graduate Texts in Mathematics, vol. 120, Springer-Verlag, New York, 1989. MR1014685 (91e:46046)

Département de Mathématique, Université Catholique de Louvain, Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium

E-mail address: Jean.VanSchaftingen@uclouvain.be