

Reiterated homogenization for elliptic operators

Nicolas Meunier^a Jean Van Schaftingen^b

^aLaboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, 175 rue du Chevaleret, Paris 75013, France

^bDépartement de Mathématique, Université catholique de Louvain, 2 chemin du Cyclotron, Louvain-la-Neuve, 1348, Belgique

Abstract

In this Note, using the periodic unfolding method, see [4], we study reiterated homogenization for equations of the form $-\operatorname{div}(a_\epsilon(x, Du_\epsilon)) = f$, where a_ϵ is Carathéodory and satisfies some monotone and growth conditions. We show that if we assume that $\mathcal{T}'_{\delta(\epsilon)}(\mathcal{T}_\epsilon(a_\epsilon))(x, y, z, \xi)$ converges, for almost all $(x, y, z) \in \Omega \times Y \times Z$, to a Carathéodory operator, then the sequences u_ϵ and Du_ϵ converge in a certain sense to the solution $(u_0, \hat{u}, \tilde{u})$ of a limit variational problem, as $\epsilon \rightarrow 0$. In particular this contains the case $a_\epsilon(x, \xi) = a(x, \frac{x}{\epsilon}, \frac{\{\frac{x}{\epsilon}\}_Y}{\delta(\epsilon)}, \xi)$, where a is periodic in the second and third arguments, and continuous in each argument. *To cite this article: N. Meunier, J. Van Schaftingen, C. R. Acad. Sci. Paris, Ser. I.*

Résumé

Homogénéisation réitérée pour des opérateurs elliptiques Dans cette note, on étudie, en utilisant la méthode d'éclatement périodique, voir [4], l'homogénéisation réitérée pour des équations de la forme $-\operatorname{div}(a_\epsilon(x, Du_\epsilon)) = f$, où a_ϵ est de Carathéodory et satisfait des conditions de monotonie et de croissance. On montre que si l'on suppose la convergence de $\mathcal{T}'_{\delta(\epsilon)}(\mathcal{T}_\epsilon(a_\epsilon))(x, y, z, \xi)$, pour presque tout $(x, y, z) \in \Omega \times Y \times Z$, vers un opérateur de Carathéodory, alors les suites u_ϵ et Du_ϵ convergent dans un certain sens vers la solution $(u_0, \hat{u}, \tilde{u})$ d'un problème variationnel limite, quand $\epsilon \rightarrow 0$. Ce résultat s'applique en particulier au cas $a_\epsilon(x, \xi) = a(x, \frac{x}{\epsilon}, \frac{\{\frac{x}{\epsilon}\}_Y}{\delta(\epsilon)}, \xi)$, où a est périodique par rapport aux deuxième et troisième variables, et continue par rapport à chaque variable. *Pour citer cet article : N. Meunier, J. Van Schaftingen, C. R. Acad. Sci. Paris, Ser. I.*

Version française abrégée

La notion d'homogénéisation réitérée a été introduite dans Bensoussan, Lions et Papanicolaou [2] et Sanchez-Palencia [8] pour les opérateurs linéaires et périodiques. Le cas non linéaire convexe a été traité en utilisant la Γ -convergence [3]. Le cas des opérateurs monotones périodiques non linéaires a été étudié par

Email addresses: meunier@ann.jussieu.fr (Nicolas Meunier), vanschaftingen@math.ucl.ac.be (Jean Van Schaftingen).

Lions, Lukkassen, Persson et Wall [6] en utilisant une méthode d'énergie et une convergence multi-échelle, voir [1] pour cette théorie. Des applications de ces résultats pour des matériaux non linéaires peuvent être trouvés dans [3]. La généralisation de l'homogénéisation au cas linéaire non périodique elliptique a été faite par Tartar [9] et Murat et Tartar [7] en utilisant la H -convergence.

Dans cette note, nous étudions l'homogénéisation réitérée pour des opérateurs non-linéaires monotones elliptiques en utilisant la méthode d'éclatement périodique introduite par Cioranescu, Damlamian et Griso [4].

Nous considérons la classe d'équations aux dérivées partielles de la forme :

$$\begin{cases} -\operatorname{div}(a_\epsilon(x, Du_\epsilon)) = f & \text{sur } \Omega, \\ u_\epsilon \in W_0^{1,p}(\Omega), \end{cases} \quad (1)$$

où Ω est un ouvert borné lipschitzien de \mathbf{R}^N , $1 < p < \infty$, $p^{-1} + q^{-1} = 1$ et $f \in W^{-1,q}(\Omega)$. On suppose que $a_\epsilon(x, \xi)$ est de Carathéodory et satisfait des conditions de monotonie (3) et de croissance (4).

Le Théorème 1 assure que si pour presque tout $(x, y) \in \Omega \times Y$, la suite $\left(\mathcal{T}_\epsilon(a_\epsilon)(x, y, \xi)\right)_\epsilon$ converge simplement vers un opérateur de Carathéodory $a_{hom}(x, y, \xi)$, alors la suite $(u_\epsilon)_\epsilon$ converge faiblement dans $W_0^{1,p}(\Omega)$ vers u_0 où (u_0, \hat{u}) est l'unique solution du problème variationnel (7) et $\mathcal{T}_\epsilon(u)(x, y) = u\left(\epsilon\left[\frac{x}{\epsilon}\right]_Y + \epsilon y\right)$. On obtient en outre la convergence forte dans $L^p(\Omega \times Y)$ de la suite $\left(\mathcal{T}_\epsilon(Du_\epsilon)\right)_\epsilon$ vers $D_x u_0 + D_y \hat{u}$ ainsi que la convergence forte des correcteurs dans le Théorème 2.

Remarque 1 *Le Théorème 1 contient en particulier le cas où $a_\epsilon(x, \xi) = a(x, \frac{x}{\epsilon}, \xi)$, $a(x, y, \xi)$ étant une fonction Y -périodique dans la seconde variable et continue par rapport à chacune des variables.*

Dans le Théorème 5 nous étudions l'homogénéisation réitérée. Nous introduisons l'opérateur d'éclatement réitéré, voir [4], pour toute fonction u , étendue par 0 en dehors de Ω , $u \in L^p(\Omega) : \mathcal{T}'_{\delta(\epsilon)}(\mathcal{T}_\epsilon(u))(x, y, z) = u\left(\epsilon\left[\frac{x}{\epsilon}\right]_Y + \epsilon\delta(\epsilon)\left[\frac{y}{\delta(\epsilon)}\right]_Z + \epsilon\delta(\epsilon)z\right)$, où $\lim_{\epsilon \rightarrow 0} \delta(\epsilon) = 0$.

Si la suite $\left(\mathcal{T}_{\delta(\epsilon)}(\mathcal{T}_\epsilon(a_\epsilon))(x, y, z, \xi)\right)_\epsilon$ converge vers un opérateur de carathéodory $a_{homrei}(x, y, z, \xi)$, pour presque tout $(x, y, z) \in \Omega \times Y \times Z$, alors la suite $(u_\epsilon)_\epsilon$ converge faiblement dans $W_0^{1,p}(\Omega)$ vers u_0 où $(u_0, \hat{u}, \tilde{u})$ est l'unique solution du problème variationnel (9) ainsi que la convergence forte dans $L^p(\Omega \times Y \times Z; \mathbf{R}^N)$ de la suite

$\left(\mathcal{T}_{\delta(\epsilon)}(\mathcal{T}_\epsilon(Du_\epsilon))\right)_\epsilon$ vers $D_x u_0 + D_y \hat{u} + D_z \tilde{u}$. La convergence forte des correcteurs est assurée par le Théorème 6.

Remarque 2 *Le Théorème 5 contient le cas où $a_\epsilon(x, \xi) = a(x, \frac{x}{\epsilon}, \frac{\{\frac{x}{\epsilon}\}_Y}{\delta(\epsilon)}, \xi)$, $a(x, y, z, \xi)$ étant une fonction Y -périodique dans la seconde variable, Z -périodique dans la troisième variable et continue par rapport à chacune des variables. De plus, si $Y = Z$ et $\frac{1}{\delta(\epsilon)} \in \mathbf{N}^*$ alors on a $a_\epsilon(x, \xi) = a(x, \frac{x}{\epsilon}, \frac{\{\frac{x}{\epsilon}\}_Y}{\delta(\epsilon)}, \xi) = a(x, \frac{x}{\epsilon}, \frac{x}{\epsilon\delta(\epsilon)}, \xi)$. Cela généralise ainsi les résultats obtenus dans [6] où il était nécessaire de supposer une hypothèse plus forte sur la fonction $a(x, \cdot, z, \xi)$.*

Remarque 3 *On peut généraliser le Théorème 5 au cas n fois réitéré, si on suppose que pour presque tout $(x, y_1, \dots, y_n) \in \Omega \times Y_1 \times \dots \times Y_n$, la suite $\left(\left(\mathcal{T}'_{\delta_n(\epsilon)} \circ \dots \circ \mathcal{T}'_{\delta_1(\epsilon)} \circ \mathcal{T}_\epsilon\right)a_\epsilon(x, y_1, \dots, y_n, \xi)\right)_\epsilon$ converge vers un opérateur de Carathéodory $a_{homrei}(x, y_1, \dots, y_n, \xi)$.*

Remarque 4 *Le Théorème 5 contient en particulier le cas où $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$, Ω_1 et Ω_2 étant des ouverts disjoints de frontière lipschitzienne et $a_\epsilon(x, \xi)$ est tel que $a_\epsilon(x, \xi) = a^1(x, \xi)$ si $x \in \Omega_1$, $a_\epsilon(x, \xi) = a^2(x, \frac{x}{\epsilon}, \frac{\{\frac{x}{\epsilon}\}_Y}{\delta(\epsilon)}, \xi)$ si $x \in \Omega_2$, où a^1 et a^2 sont continues par rapport à chacune des variables et satisfont (3) et (4). Cela recouvre des situations plus générales que ce qui a été envisagé dans [6].*

Remarque 5 *Comme dans le cas linéaire, [4,5], le Théorème 5 peut être étendu à l'étude de domaines perforés.*

1. Introduction

The reiterated homogenization was first introduced in Bensoussan, Lions and Papanicolaou [2] and in Sanchez-Palencia [8] for linear and periodic operators. The nonlinear case was studied for elliptic and convex problems in [3]. The non-linear case for periodic monotone operators was obtained in [6], using a method of energy and multiscales convergence, see Allaire and Briane [1] for this method. Applications of these results for nonlinear materials can be found in [3]. The non periodic case in homogenization was first studied in the linear case by Tartar [9] and Murat and Tartar [7], using the H -convergence theory.

In this Note, we study reiterated homogenization for monotone operators by using the periodic unfolding method introduced in Cioranescu, Damlamian and Griso [4].

We consider equations of the form:

$$\begin{cases} -\operatorname{div}(a_\epsilon(x, Du_\epsilon)) = f & \text{sur } \Omega, \\ u_\epsilon \in W_0^{1,p}(\Omega), \end{cases} \quad (2)$$

where Ω is a Lipschitz open bounded set of \mathbf{R}^N , $1 < p < \infty$, $p^{-1} + q^{-1} = 1$ and $f \in W^{-1,q}(\Omega)$.

2. Unfolding reiterated operator

Let Y and Z be two reference cells (sets having the paving property with respect to basis, defining the periods, (b_1, \dots, b_N) and (c_1, \dots, c_N) , respectively) associated with the scales ϵ and $\epsilon\delta(\epsilon)$, with $\lim_{\epsilon \rightarrow 0} \delta(\epsilon) = 0$. For $z \in \mathbf{R}^N$, we denote $[z]_Y$ the unique integer combination $\sum_{j=1}^N k_j b_j$ of the periods such that $z - [z]_Y$ belongs to Y , and we set $\{z\}_Y = z - [z]_Y$. Then, for each $x \in \mathbf{R}^N$, we immediately see that $x = \epsilon([\frac{x}{\epsilon}]_Y + \{\frac{x}{\epsilon}\}_Y)$.

We define similarly for all $y \in \mathbf{R}^N$, $[y]_Z$ and $\{y\}_Z$.

For $u \in L^p(\Omega)$, $p \in [1, \infty]$, extended by zero outside of Ω , we define the unfolding operator $\mathcal{T}_\epsilon : L^p(\Omega) \rightarrow L^p(\Omega \times Y)$ by

$$\mathcal{T}_\epsilon(u)(x, y) = u\left(\epsilon\left[\frac{x}{\epsilon}\right]_Y + \epsilon y\right), \quad \text{for } x \in \Omega \text{ and } y \in Y.$$

Now we apply to $\tilde{u} \in L^p(\Omega \times Y)$, $p \in [1, \infty]$, extended by zero outside of $\Omega \times Y$, a similar unfolding operation for the variable y , x being seen merely as a parameter. Adding a new variable $z \in Z$, we obtain the unfolding operator $\mathcal{T}'_{\delta(\epsilon)} : L^p(\Omega \times Y) \rightarrow L^p(\Omega \times Y \times Z)$ given by

$$\mathcal{T}'_{\delta(\epsilon)}(\tilde{u})(x, y, z) = \tilde{u}\left(x, \delta(\epsilon)\left[\frac{y}{\delta(\epsilon)}\right]_Z + \delta(\epsilon)z\right), \quad \text{for } (x, y, z) \in (\Omega \times Y \times Z).$$

Therefore, for $u \in L^p(\Omega)$, we can define the reiterated unfolding operator by

$$\mathcal{T}'_{\delta(\epsilon)}(\mathcal{T}_\epsilon(u))(x, y, z) = u\left(\epsilon\left[\frac{x}{\epsilon}\right]_Y + \epsilon\delta(\epsilon)\left[\frac{y}{\delta(\epsilon)}\right]_Z + \epsilon\delta(\epsilon)z\right), \quad \text{for } (x, y, z) \in (\Omega \times Y \times Z).$$

We immediately see that for every $x \in \Omega$, we have $\mathcal{T}'_{\delta(\epsilon)}(\mathcal{T}_\epsilon(u))(x, \{\frac{x}{\epsilon}\}_Y, \{\frac{\frac{x}{\epsilon}}{\delta(\epsilon)}\}_Z) = u(x)$.

3. Homogenization results

Theorem 1 *Let $1 < p < \infty$, $p^{-1} + q^{-1} = 1$ and $a_\varepsilon : \Omega \times \mathbf{R}^N \rightarrow \mathbf{R}^N$, with $a_\varepsilon(\cdot, \xi)$ measurable for all $\xi \in \mathbf{R}^N$ and $a_\varepsilon(x, \cdot)$ continuous for almost all $x \in \Omega$, be such that*

- *there exists $c > 0$ such that for all $x \in \Omega$, $\xi_1, \xi_2 \in \mathbf{R}^N$,*

$$|\xi_1 - \xi_2|^p \leq c(a_\varepsilon(x, \xi_1) - a_\varepsilon(x, \xi_2))|\xi_1 - \xi_2|, \quad (3)$$

- *there exists $C > 0$ such that for all $x \in \Omega$ et $\xi \in \mathbf{R}^N$,*

$$|a_\varepsilon(x, \xi)| \leq C(1 + |\xi|^{p-1}). \quad (4)$$

Furthermore, we assume that for almost every $(x, y) \in \Omega \times Y$,

$$\mathcal{T}_\varepsilon(a_\varepsilon)(x, y, \xi) \rightarrow a_{hom}(x, y, \xi), \text{ as } \varepsilon \text{ goes to zero,} \quad (5)$$

where $a_{hom}(x, y, \xi)$ is Carathéodory. Let $f_\varepsilon \in W^{-1,q}(\Omega)$ be such that $f_\varepsilon \rightarrow f$ strongly in $W^{-1,q}(\Omega)$.

Let $u_\varepsilon \in W_0^{1,p}(\Omega)$ be the solution of the following problem

$$\begin{cases} \int_{\Omega} (a_\varepsilon(x, Du_\varepsilon)|D\varphi) dx = \int_{\Omega} f_\varepsilon \varphi dx, \\ \forall \varphi \in W^{1,p}(\Omega), \end{cases} \quad (6)$$

then we have,

$$u_\varepsilon \rightharpoonup u_0 \text{ weakly in } W_0^{1,p}(\Omega),$$

where u_0 is the first term of the unique solution (u_0, \hat{u}) of the following variational problem:

$$\begin{cases} u_0 \in W_0^{1,p}(\Omega), \hat{u} \in L^p(\Omega; W_{per}^{1,p}(Y)), \text{ with } \int_Y \hat{u}(x, y) dy = 0, \\ \forall \Psi \in W_0^{1,p}(\Omega), \forall \Phi \in L^p(\Omega; W_{per}^{1,p}(Y)), \\ \frac{1}{|Y|} \int_{\Omega \times Y} (a_{hom}(x, y, D_x u_0 + D_y \hat{u})|D_x \Psi(x) + D_y \Phi(x, y)) dx dy = \int_{\Omega} f \Psi dx. \end{cases} \quad (7)$$

Moreover, the following strong convergence holds

$$\mathcal{T}_\varepsilon(Du_\varepsilon) \rightarrow D_x u_0 + D_y \hat{u} \text{ in } L^p(\Omega \times Y \times Z; \mathbf{R}^N), \text{ when } \varepsilon \text{ goes to zero.}$$

The proof of Theorem 1 is very simple with the unfolding method. We proceed in four steps, we shall give it in details in a forthcoming publication. First, we prove the weak convergences of the unfolding sequences of $(\mathcal{T}_\varepsilon(u_\varepsilon))_\varepsilon$ and $(\mathcal{T}_\varepsilon(Du_\varepsilon))_\varepsilon$, then we establish an equation satisfied by the limit. In the third step, we show that the unfolding sequence of the gradient $(\mathcal{T}_\varepsilon(Du_\varepsilon))_\varepsilon$ converges strongly and at the end we proceed to the identification of the homogenized equation.

Moreover, if we introduce the averaging operator $\mathcal{U}_\varepsilon : L^p(\Omega \times Y) \rightarrow L^p(\Omega \times Y \times Z)$ defined by $\mathcal{U}_\varepsilon(v) = \frac{1}{|Y|} \int_Y v \left(\varepsilon \left[\frac{x}{\varepsilon} \right]_Y + \varepsilon y, \left\{ \frac{x}{\varepsilon} \right\}_Y \right) dy$, we obtain the following result for correctors:

Theorem 2 *We have the following strong convergences:*

$$D_x u_\varepsilon - D_x u_0 - \mathcal{U}_\varepsilon(D_y \hat{u}) \rightarrow 0 \text{ in } L^p(\Omega).$$

Remark 3 Condition (3) used in Theorem 1 implies the following condition, which was used in [6]: there exists $c > 0$ such that for all $x \in \Omega$, $\xi_1, \xi_2 \in \mathbf{R}^N$,

$$(a_\varepsilon(x, \xi_1) - a_\varepsilon(x, \xi_2))|\xi_1 - \xi_2| \geq c(1 + |\xi_1| + |\xi_2|)^{p-\alpha}|\xi_1 - \xi_2|^\alpha,$$

with $\max\{p, 2\} \leq \alpha < +\infty$.

Remark 4 In particular, Theorem 1 applies to the case where $a_\varepsilon(x, \xi) = a(x, \frac{x}{\varepsilon}, \xi)$, $a(x, y, \xi)$ being Y -periodic in the second argument and continuous with respect to every argument.

4. Reiterated homogenization

We can now give the reiterated result.

Theorem 5 Let us assume that $p, q, a_\varepsilon, f_\varepsilon, u_\varepsilon$ are as in Theorem 1 and that for almost every $(x, y, z) \in \Omega \times Y \times Z$,

$$\mathcal{T}'_{\delta(\varepsilon)}(\mathcal{T}_\varepsilon(a_\varepsilon))(x, y, z, \xi) \rightarrow a_{homrei}(x, y, z, \xi), \quad (8)$$

as ε goes to zero, with $\lim_{\varepsilon \rightarrow 0} \delta(\varepsilon) = 0$ and $a_{homrei}(x, y, z, \xi)$ is Carathéodory. We have

$$u_\varepsilon \rightharpoonup u_0 \text{ weakly in } W_0^{1,p}(\Omega), \text{ as } \varepsilon \rightarrow 0,$$

where u_0 is the first term of the unique solution $(u_0, \hat{u}, \tilde{u})$ of the variational problem

$$\left\{ \begin{array}{l} u_0 \in W_0^{1,p}(\Omega), \hat{u} \in L^p(\Omega; W_{per}^{1,p}(Y)), \text{ with } \int_Y \hat{u}(x, y) dy = 0, \\ \tilde{u} \in L^p(\Omega \times Y; W_{per}^{1,p}(Z)), \text{ with } \int_Z \tilde{u}(x, y, z) dz = 0, \\ \forall \Psi \in W_0^{1,p}(\Omega), \forall \Phi \in L^p(\Omega; W_{per}^{1,p}(Y)), \forall \Theta \in L^p(\Omega \times Y; W_{per}^{1,p}(Z)) \\ \frac{1}{|Y||Z|} \int_{\Omega \times Y \times Z} (b(x, y, z, D_x u_0 + D_y \hat{u} + D_z \tilde{u})) \\ D_x \Psi(x) + D_y \Phi(x, y) + D_z \Theta(x, y, z) dx dy dz = \int_{\Omega} f \Psi dx. \end{array} \right. \quad (9)$$

Moreover, the following strong convergence holds, when ε goes to zero,

$$\mathcal{T}'_{\delta(\varepsilon)}(\mathcal{T}_\varepsilon(Du_\varepsilon)) \rightarrow D_x u_0 + D_y \hat{u} + D_z \tilde{u} \text{ in } L^p(\Omega \times Y \times Z; \mathbf{R}^N).$$

Furthermore, if we define the averaging reiterated operator $\mathcal{U}'_{\varepsilon, \delta(\varepsilon)}$ for all $w \in L^p(\Omega \times Y \times Z)$:

$$\mathcal{U}'_{\varepsilon, \delta(\varepsilon)}(w) = \frac{1}{|Y||Z|} \int_{Y \times Z} w \left(\varepsilon \left[\frac{x}{\varepsilon} \right]_Y + \varepsilon y, \delta(\varepsilon) \left[\frac{\frac{x}{\varepsilon}}{\delta(\varepsilon)} \right]_Z + \delta(\varepsilon) z, \left\{ \frac{\frac{x}{\varepsilon}}{\delta(\varepsilon)} \right\}_Z \right) dy dz,$$

we deduce the following result for the correctors:

Theorem 6 We have the following strong convergences:

$$D_x u_\varepsilon - D_x u_0 - \mathcal{U}'_{\varepsilon, \delta(\varepsilon)}(D_y \hat{u}) - \mathcal{U}'_{\varepsilon, \delta(\varepsilon)}(D_z \tilde{u}) \rightarrow 0 \text{ in } L^p(\Omega).$$

Remark 7 In particular, Theorem 5 applies to the case where $a_\epsilon(x, \xi) = a(x, \frac{x}{\epsilon}, \{\frac{x}{\delta(\epsilon)}\}_Y, \xi)$, $a(x, y, z, \xi)$ being Y -periodic in the second argument, Z -periodic in the third and continuous with respect to every argument. Furthermore, if $Y = Z$ and $\frac{1}{\delta(\epsilon)}$ is an integer, then $a_\epsilon(x, \xi) = a(x, \frac{x}{\epsilon}, \{\frac{x}{\delta(\epsilon)}\}_Y, \xi)$ also equals $a(x, \frac{x}{\epsilon}, \frac{x}{\epsilon\delta(\epsilon)}, \xi)$. This generalizes the results of [6] in which it was necessary to assume a stronger condition on $a(x, \cdot, z, \xi)$.

Remark 8 Theorem 5 can easily be generalized to the case of n times reiteration if we assume that for almost all $(x, y_1, \dots, y_n) \in \Omega \times Y_1 \times \dots \times Y_n$, the sequence $\left(\left(\mathcal{T}'_{\delta_n(\epsilon)} \circ \dots \circ \mathcal{T}'_{\delta_1(\epsilon)} \circ \mathcal{T}_\epsilon \right) a_\epsilon(x, y_1, \dots, y_n, \xi) \right)_\epsilon$ converges to a Carathéodory operator $a_{\text{homrei}}(x, y_1, \dots, y_n, \xi)$.

Remark 9 As a particular case, Theorem 5 applies to the following situation: $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$, with Ω_1, Ω_2 disjoint Lipschitzian open sets and $a_\epsilon(x, \xi)$ is such that $a_\epsilon(x, \xi) = a^1(x, \xi)$ if $x \in \Omega_1$ and $a_\epsilon(x, \xi) = a^2(x, \{\frac{x}{\epsilon}\}_Y, \{\frac{x}{\delta(\epsilon)}\}_Z, \xi)$ if $x \in \Omega_2$, where a^1 and a^2 are continuous with respect to every argument and satisfy (3), (4). This is more general than what was treated in [6].

Remark 10 As in the linear case, see [4, 5], Theorem 5 can be generalized to perforated domains.

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