Partial Differential Equations

Existence and concentration for nonlinear Schrödinger equations with fast decaying potentials

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Abstract

The existence of solutions to

$$-\varepsilon^2 \Delta u + Vu = u^p \quad \text{in } \mathbb{R}^N,$$

is proved for small ε, where $N \geq 3$, V is a nonnegative continuous potential which has a positive local minimum, and $\frac{N}{N-2} < p < \frac{N+2}{N-2}$. No restriction is imposed on the rate of decay of V; this includes in particular the case where V is compactly supported. *To cite this article: V. Moroz, J. Van Schaftingen, C. R. Acad. Sci. Paris, Ser. I ??? (200?).*

Résumé

Existence et concentration pour l'équation de Schrödinger non linéaire avec des potentiels à décroissance rapide. Nous prouvons l'existence de solutions positives non triviales de

$$-\varepsilon^2 \Delta u + Vu = u^p \quad \text{dans } \mathbb{R}^N,$$

pour ε petit, où $N \geq 3$, V est potentiel continu positif qui a un minimum local strictement positif et $\frac{N}{N-2} < p < \frac{N+2}{N-2}$. Nous n'imposons aucune restriction sur le taux de décroissance de V. En particulier, nous couvrons le cas où le support de V est compact. *Pour citer cet article : V. Moroz, J. Van Schaftingen, C. R. Acad. Sci. Paris, Ser. I ? ? ? (200 ?).*

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Version française abrégée

On considère l'équation de Schrödinger non linéaire (1). On sait que quand V atteint un minimum global strictement positif non dégénéré en $x_0 \in \mathbb{R}^N$ cette équation admet des solutions qui se concentrent autour de x_0 [7]. Ce travail a été étendu dans différentes directions dans le cas où $\inf_{\mathbb{R}^n} V > 0$ [1].

Nous nous intéressons au cas où $V > 0$ mais $\inf_{\mathbb{R}^N} V = 0$ dont l'étude a été initiée par A. Ambrosetti, V. Felli et A. Malchiodi [2] et par A. Ambrosetti, A. Malchiodi and D. Ruiz [3] qui ont établi l'existence sous la condition de décroissance lente (3). D. Bonheure et J. Van Schaftingen ont dans le cas $\frac{N}{N-2} < p < \frac{N+2}{N-2}$, affaibli cette condition en (4).

Nous annonçons dans cette note l'existence et la concentration de solutions de (1) dans le cas de potentiels à décroissance rapide (5) qui incluent en particulier les potentiels à support compact. Rappelons que si le support de V est compact, (1) n'a pas de solution dans un voisinage de l'infini si $p \leq \frac{N}{N-2}$ [8]. Nous obtenons

Théorème 0.1 *Soit $N \geq 3$, $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ et $V \in C(\mathbb{R}^N, \mathbb{R}^+)$. Supposons qu'il existe un ouvert régulier $\Lambda \subset \mathbb{R}^N$ tel que $0 < \inf_\Lambda V < \inf_{\partial\Lambda} V$. Alors, pour tout $\varepsilon > 0$ suffisament petit, l'équation* (1) *a une solution positive non triviale u_ε.*

Les méthodes s'appliquent en fait au problème plus général (6) où $N \geq 2$, $p > 1$, $\varepsilon > 0$ et $V, K \in C(\mathbb{R}^N, \mathbb{R}^+)$. Les points de concentration sont des points critiques de la fonction de concentration $\mathcal{A}$ définie en (7) et l'existence sera obtenue dans l'espace de Sobolev à poids défini en (8).

Théorème 0.2 *Soit $N \geq 2$, $1 < p < 2^*$ et soit , $K \in C(\mathbb{R}^N, \mathbb{R}^+)$. Supposons qu'il existe $\sigma < (N-2)p-N$ et $M > 0$ tels que* (9) *ait lieu et qu'il existe un ouvert borné régulier $\Lambda \subset \mathbb{R}^N$ tel que $0 < \inf_\Lambda \mathcal{A} < \inf_{\partial\Lambda} \mathcal{A}$. Alors il existe $\varepsilon_0 > 0$ tel que pour tout $\varepsilon \in (0, \varepsilon_0)$,* (6) *ait au moins une solution positive non triviale $u_\varepsilon \in \mathcal{D}^1_V(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$. De plus, $\|u_\varepsilon\|_\varepsilon = O(\varepsilon^{N/2})$ quand $\varepsilon \to 0$, u_ε atteint son maximum en $x_\varepsilon \in \Lambda$, x_ε satisfait* (10) *et il existe $C, \lambda > 0$ tels que pour tout $\varepsilon \in (0, \varepsilon_0)$, u_ε a la dcroissance* (11).

Si le support de V est compact, la restriction $\sigma < (N-2)p - N$ dans le théorème est optimale, dans le sens que (6) n'a pas de solutions positives pour $\sigma \geq (N-2)p - N$ [8], et la borne supérieure (11) est optimale dans le sens que $\liminf_{|x|\to\infty} |x|^{N-2} u_\varepsilon(x) > 0$, pour tout $\varepsilon \in (0, \varepsilon_0]$.

La preuve suit la méthode de pénalisation de M. del Pino et P. Felmer [6] adaptée ensuite par D. Bonheure et J. Van Schaftingen [5]. Le Théorème 0.2 est valable pour $N \geq 2$, mais la preuve du cas $N = 2$ est diffère de cemme du cas $N \geq 3$, sur lequel nous nous concentrons dans la suite.

On définit un potentiel de pénalisation H défini par (12) où $\beta > 0$, $0 < \rho_0 < \rho$, $B(0, \rho) \subset \Lambda$ et $0 < \frac{\kappa}{\log \frac{\rho}{\rho_0}} < \frac{(N-2)^2}{4}$. On vérifie entre autres que $-\Delta - H$ satisfait un principe de comparaison entre sur- et sous-solutions et qu'il possède des sursolutions qui décroissent comme $|x|^{-(N-2)}$ à l'infini. On définit ensuite par (14) une non-linéarité pénalisée, et on montre que la fonctionnelle associée (15) a un point critique qui satisfait l'équation (16).

Il reste alors à montrer que u_ε satisfait l'équation non pénalisée (6). Pour ce faire on commence par étudier le comportement asymptotique local de u_ε. On montre que (17) a lieu et que si $\varepsilon_n \to 0$, $(x_n) \subset \Lambda$, $\liminf_{n\to\infty} u_{\varepsilon_n}(x_n) > 0$ et $x_n \to x_0 \in \Lambda$, alors $\mathcal{A}(x_0) = \inf_\Lambda \mathcal{A}$, et à une sous-suite près la suite $v_n(x) := u_{\varepsilon_n}(x_n + \varepsilon x)$ converge vers une solution de (18). On en déduit qu'il existe $R > 0$ tel que pour $\varepsilon > 0$ u_ε satisfait (19). On considère ensuite une sur-solution w_ε de (19) définie par (20) où $\mu = \frac{1}{2}\sqrt{\inf_\Lambda V}$ et $\tilde{w} \in C^2(\mathbb{R}^N)$ satisfait $-\Delta\tilde{w} - H\tilde{w} = 0$ en dehors de Λ et vaut 1 sur un voisinage de la famille (x_ε). Par le principe de comparaison et par (17), pour $\varepsilon > 0$ petit, on conclut que $u_\varepsilon \leq w_\varepsilon$ in $B(x_\varepsilon, r)^c$. On en déduit u_ε a la décroissance (11), d'où par l'hypothèse (9), u_ε satisfait (21). Par construction de g_ε, u_ε est une solution du problème initial (1).

1. Introduction

We are interested in a nonlinear Schrödinger equations of the type

$$-\varepsilon^2 \Delta u + V u = u^p \quad \text{in } \mathbb{R}^N, \tag{1}$$

where $\varepsilon > 0$ is a small parameter, $N \geq 3$, $V \in C(\mathbb{R}^N, \mathbb{R}^+)$, and $1 < p < \frac{N+2}{N-2}$.

In the case where V achieves a nondegenerate positive global minimum at $x_0 \in \mathbb{R}^N$ it is known that for all small $\varepsilon > 0$ equation (1) admits positive solutions $u_\varepsilon \in H^1(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$, such that $\|u_\varepsilon\|_{H^1(\mathbb{R}^N)} \to 0$ as $\varepsilon \to 0$, and the family (u_ε) concentrates around x_0: there exists $x_\varepsilon \in \mathbb{R}^N$

$$\liminf_{\varepsilon \to 0} u_\varepsilon(x_\varepsilon) > 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \max_K u_\varepsilon(x) = 0 \quad \text{for every compact } K \subset \mathbb{R}^N \setminus \{x_0\},$$

where $x_0 = \lim_{\varepsilon \to 0} x_\varepsilon$. Moreover, when $\varepsilon \to 0$ the rescaled functions $v_\varepsilon(x) := u_\varepsilon(x_0 + \varepsilon x)$ converge in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$ to the unique positive radial solution $v_0 > 0$ of the limiting equation

$$-\Delta v_0 + V(x_0) v_0 = v_0^p \quad \text{in } \mathbb{R}^N. \tag{2}$$

This result goes back to the pioneering work by A. Floer and A. Weinstein [7], and has been extended in different directions (see e.g. [1] for an extensive list of references). In most of these works the assumption $\inf_{\mathbb{R}^N} V > 0$ was made.

We are interested in the case where $V > 0$ but $\inf_{\mathbb{R}^N} V = 0$, whose study was initiated by A. Ambrosetti, V. Felli and A. Malchiodi in [2] and by A. Ambrosetti, A. Malchiodi and D. Ruiz [3]. In particular, in [3] the existence and concentration of solutions to (1) was proved when V satisfies the *slow decay* assumption

$$\liminf_{|x| \to \infty} V(x)|x|^2 > 0. \tag{3}$$

D. Bonheure and J. Van Schaftingen [4,5] established the existence and concentration of solution to (1) when

$$\liminf_{|x| \to \infty} V(x)|x|^{(N-2)(p-1)} > 0, \tag{4}$$

which thus provides an improvement to the results in [3] when $p > \frac{N}{N-2}$.

In this note, we announce the existence and concentration of solutions to (1) in the case of *fast decaying potentials*, i.e. potentials for which

$$\liminf_{|x| \to \infty} V(x)|x|^2 = 0. \tag{5}$$

This includes, in particular, potentials with compact support in $\mathbb{R}^N$.

One issue in the study of equations (1) with fast decaying potentials is that its solutions may have slow polynomial decay as $x \to \infty$. Thus, any concentration result should capture a transition between polynomial decay of the concentrating solutions u_ε and exponential decay of the limiting solution that is obtained by rescaling the problem. Another issue is that polynomial decay of solutions brings Liouville-type nonexistence phenomena, which restrict the admissible exponents. For example, if V is compactly supported, equation (1) has no positive solutions in the neighborhood of infinity for $p \leq \frac{N}{N-2}$, cf. [8].

A special case of our results is the following

Theorem 1.1 *Let $N \geq 3$, $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ and $V \in C(\mathbb{R}^N, \mathbb{R})$ be a nonnegative potential. Assume that there exists a smooth bounded open set $\Lambda \subset \mathbb{R}^N$ such that $0 < \inf_\Lambda V < \inf_{\partial\Lambda} V$. Then for small $\varepsilon > 0$ equation* (1) *has a positive solution u_ε.*

2. The main result

Theorem 1.1 follows from results for the slightly more general problem

$$-\varepsilon^2\Delta u + Vu = Ku^p \quad \text{ in } \mathbb{R}^N, \tag{6}$$

where $N \geq 2$, $p > 1$, $\varepsilon > 0$ and $V, K \in C(\mathbb{R}^N, \mathbb{R}^+)$. The existence of solutions will be related to the local minimizers of the concentration function

$$\mathcal{A}(x) := V(x)^{\frac{p+1}{p-1}-\frac{N}{2}} K(x)^{-\frac{2}{p-1}}, \tag{7}$$

and will be obtained in the weighted Sobolev space induced by the linear part of the equation,

$$\mathcal{D}^1_V(\mathbb{R}^N) := \Big\{ u \in \mathcal{D}^1_0(\mathbb{R}^N) \;\Big|\; \|u\|_\varepsilon = \int_{\mathbb{R}^N} \big(\varepsilon^2 |\nabla u|^2 + V|u|^2\big) < +\infty \Big\}. \tag{8}$$

Here $\mathcal{D}^1_0(\mathbb{R}^N)$ is the closure of $C^\infty_c(\mathbb{R}^N)$ with respect to the L^2-norm of the gradient.

Theorem 2.1 *Let $N \geq 2$, $1 < p < 2^*$ and let $V, K \in C(\mathbb{R}^N, \mathbb{R}^+)$. Assume that there exists $\sigma < (N-2)p - N$ and $M > 0$ such that*

$$0 \leq K(x) \leq M(1+|x|)^\sigma \quad \text{for all } x \in \mathbb{R}^N, \tag{9}$$

and that there exists a smooth bounded open set $\Lambda \subset \mathbb{R}^N$ such that $0 < \inf_\Lambda \mathcal{A} < \inf_{\partial\Lambda} \mathcal{A}$. Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, equation (6) has at least one positive solution $u_\varepsilon \in \mathcal{D}^1_V(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$. Moreover, $\|u_\varepsilon\|_\varepsilon = O(\varepsilon^{N/2})$ as $\varepsilon \to 0$, u_ε attains its maximum at $x_\varepsilon \in \Lambda$,

$$\liminf_{\varepsilon \to 0} u_\varepsilon(x_\varepsilon) > 0, \qquad \lim_{\varepsilon \to 0} \mathcal{A}(x_\varepsilon) = \inf_{x \in \Lambda} \mathcal{A}(x), \tag{10}$$

and there exists $C, \lambda > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ holds

$$u_\varepsilon(x) \leq C \exp\Big(-\frac{\lambda}{\varepsilon} \frac{|x - x_\varepsilon|}{1 + |x - x_\varepsilon|} \Big) \big(1 + |x - x_\varepsilon|^2\big)^{-\frac{N-2}{2}}. \tag{11}$$

If V has compact support then the restriction $\sigma < (N-2)p - N$ in the theorem is sharp, in the sense that (6) has no positive solutions for $\sigma \geq (N-2)p - N$ (see e.g. [8]), and the upper bound (11) is sharp in the sense that $\liminf_{|x|\to\infty} |x|^{N-2} u_\varepsilon(x) > 0$, for each fixed $\varepsilon \in (0, \varepsilon_0]$.

3. Sketch of the proofs

The proof of Theorem 2.1 and its extensions will appear in [9]. The approach is based on an adequate modification of the penalization scheme of M. del Pino and P. Felmer [6], subsequently adapted by D. Bonheure and J. Van Schaftingen [5]. Whereas Theorem 2.1 holds for $N \geq 2$, there are substantial differences in the proof between the case $N = 2$ and the case $N \geq 3$. We sketch only the case $N \geq 3$.

3.1. *The penalization potential*

Assuming without loss of generality that $0 \in \Lambda$ and that $\overline{B(0,\rho)} \subset \Lambda$, define $H : \mathbb{R}^N \to \mathbb{R}$ by

$$H(x) := \frac{\kappa(1 - \chi_\Lambda(x))}{|x|^2 \big(\log \frac{|x|}{\rho_0}\big)^{1+\beta}}, \tag{12}$$

where χ_Λ is the characteristic function of Λ, $\beta > 0$, $0 < \rho_0$, and $0 < \frac{\kappa}{\log \frac{\rho}{\rho_0}} < \frac{(N-2)^2}{4}$. This potential has the following properties

(i) by the classical Hardy inequality, the quadratic form associated to $-\Delta - H$ on $\mathbb{R}^N$ is positive;

(ii) $-\Delta - H$ satisfies a comparison principle between sub- and super-solutions;

(iii) the minimal positive solution of

$$-\Delta w - Hw = 0 \quad \text{in } \bar{\Lambda}^c, \qquad\qquad w = 1 \quad \text{on } \partial\Lambda, \tag{13}$$

satisfies $c|x|^{-(N-2)} \leq w(x) \leq C|x|^{-(N-2)}$.

3.2. *The penalized problem*

The truncated nonlinearity $g_\varepsilon : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ is defined by

$$g_\varepsilon(x,s) := \chi_\Lambda(x)K(x)s^p + \min\bigl(\varepsilon^2 H(x),\, K(x)s^{p-1}\bigr)s, \tag{14}$$

and $G_\varepsilon(x,s) := \int_0^s g_\varepsilon(x,t)\,dt$. The penalized functional is

$$\mathcal{J}_\varepsilon(u) := \frac{1}{2}\int_{\mathbb{R}^N} \bigl(\varepsilon^2|\nabla u(x)|^2 + V(x)|u(x)|^2\bigr)\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} G_\varepsilon(x,u(x))\,dx. \tag{15}$$

One can verify that $\mathcal{J}_\varepsilon \in C^1(\mathcal{D}^1_V(\mathbb{R}^N),\mathbb{R})$, satisfies the Palais–Smale condition (cf. [5, Lemmas 5,6]) and has the mountain-pass geometry at 0. Hence the mountain-pass lemma implies the existence of a family of positive solutions $(u_\varepsilon) \subset \mathcal{D}^1_V(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ to the penalized equation

$$-\varepsilon^2\Delta u + V(x)u = g_\varepsilon(x,u) \quad \text{in } \mathbb{R}^N. \tag{16}$$

It remains to establish that (u_ε) solve the original problem (1) and has the required asymptotic behavior.

3.3. *Asymptotics of the solutions.*

By the comparison principle, we first prove that $\liminf_{\varepsilon\to 0}\|u_\varepsilon\|_{L^\infty(\Lambda)} > 0$. One then establishes as in [5, Proposition 16], that if the family $(x_\varepsilon) \subset \Lambda$ satisfies $\liminf_{n\to\infty} u_\varepsilon(x_\varepsilon) > 0$ then $\liminf_{\varepsilon\to 0} d(x_\varepsilon, \partial\Omega) > 0$ and

$$\lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \|u_\varepsilon\|_{L^\infty(\Lambda\setminus B(x_\varepsilon,\varepsilon R))} = 0. \tag{17}$$

Moreover, if $\varepsilon_n \to 0$, and $(x_n) \subset \Lambda$ is a sequence such that $\liminf_{n\to\infty} u_{\varepsilon_n}(x_n) > 0$ and $x_n \to x_0 \in \Lambda$, then $\mathcal{A}(x_0) = \inf_\Lambda \mathcal{A}$, and, up to a subsequence a subsequence, $v_n(x) := u_{\varepsilon_n}(x_n + \varepsilon x)$ converges in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$ to a positive solution $v \in H^1(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ of the limiting equation

$$-\Delta v + V(x_0)v = K(x_0)v^p \quad \text{in } \mathbb{R}^N. \tag{18}$$

This local information will be crucial to obtain sharp global decay estimates on the solutions from which we will deduce that the solutions solve the original problem (1).

3.4. *Barrier functions and solutions of the original problem.*

By construction of the penalization, u_ε is a subsolution to $-\varepsilon^2\Delta - \varepsilon^2 H + V$ in Λ^c. By the comparison principle argument and by (17) there exists $R > 0$ so that for all small $\varepsilon > 0$ one has

$$-\varepsilon^2\Delta u_\varepsilon - \varepsilon^2 H u_\varepsilon + \frac{1}{2}V u_\varepsilon \leq 0 \quad \text{in } \mathbb{R}^N \setminus B(x_\varepsilon, \varepsilon R). \tag{19}$$

Let $w \in C^2(\Lambda^c)$ be the minimal positive solution of (13). Set $r = \frac{1}{3}\liminf_{\varepsilon\to 0} d(x_\varepsilon, \partial\Lambda)$, and let $\tilde{w} \in C^2(\mathbb{R}^N)$ be a positive extension of w such that $\tilde{w}(x) = 1$ if $d(x, \Lambda^c) \geq r$. Define a family of barrier functions w_ε by

$$w_\varepsilon(x) = \begin{cases} \cosh \dfrac{\mu(r - |y|)}{\varepsilon} & \text{if } x \in B(x_\varepsilon, r), \\ \tilde{w}(x) & \text{if } x \in B(x_\varepsilon, r)^c, \end{cases} \tag{20}$$

where $\mu = \frac{1}{2}\sqrt{\inf_\Lambda V}$. A direct computation shows that w_ε is a supersolution to (19). By the comparison principle and (17), for small $\varepsilon > 0$ one concludes that $u_\varepsilon \leq w_\varepsilon$ in $B(x_\varepsilon, r)^c$. This, in particular, implies the sharp concentration bound (11).

Finally, using assumption (9) and (11), for small $\varepsilon > 0$ and all $x \in \Lambda^c$ we obtain

$$K(x)\big(u_\varepsilon(x)\big)^{p-1} \leq M(1+|x|)^\sigma \big(Ce^{-\frac{\lambda}{\varepsilon}}|x|^{-(N-2)}\big)^{p-1} \leq CMe^{-\frac{\lambda}{\varepsilon}}(1+|x|)^{-(N-2)(p-1)+\sigma} \leq \varepsilon^2 H(x). \tag{21}$$

By the construction of g_ε, one has then $g_\varepsilon\big(x, u_\varepsilon(x)\big) = K(x)\big(u_\varepsilon(x)\big)^p$ for all $x \in \mathbb{R}^N$, and therefore u_ε solves the original problem (1).

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