

THE CONTINUITY OF FUNCTIONS WITH N -TH DERIVATIVE MEASURE

AUGUSTO C. PONCE AND JEAN VAN SCHAFTINGEN

ABSTRACT. We study the continuity of functions u whose mixed derivative $\partial_1 \cdots \partial_N u$ is a measure. If $u \in W^{1,1}(\mathbb{R}^N)$, then we prove that u is continuous. The same conclusion holds for $u \in W^{k,p}(Q)$, with $kp > N - 1$, where Q denotes a cube in \mathbb{R}^N . The key step in the proof consists in showing that the measure $\partial_1 \cdots \partial_N u$ does not charge hyperplanes orthogonal to the coordinate axes.

1. INTRODUCTION

The classical Sobolev embedding theorem states that

$$W^{k,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N), \quad \text{where} \quad \frac{1}{q} = \frac{1}{p} - \frac{k}{N},$$

for every $k \geq 1$ integer and $1 \leq p < \infty$ such that $kp < N$. In the borderline case, namely $kp = N$, then functions in $W^{k,p}(\mathbb{R}^N)$ need not be bounded (or even locally bounded), except when $k = N$ and $p = 1$. In fact, (see, e.g., [4])

Proposition 1.1. *If $u \in W^{N,1}(\mathbb{R}^N)$, then $u \in L^\infty(\mathbb{R}^N)$ and*

$$(1.1) \quad \|u\|_{L^\infty} \leq \int_{\mathbb{R}^N} |\partial_1 \cdots \partial_N u|.$$

Moreover, u is continuous and

$$(1.2) \quad |u(x) - u(y)| \leq \int_{Q_\varepsilon} |\partial_1 \cdots \partial_N u| + C \|\nabla u\|_{L^N(Q_\varepsilon)} \quad \forall x, y \in Q_\varepsilon,$$

where Q_ε is any parallel cube of side $2\varepsilon > 0$ and $C > 0$ does not depend on ε .

We denote by ∂_i the derivative with respect to the x_i -variable; a cube Q_ε is *parallel* if its sides are parallel to the hyperplanes $[x_i = 0]$ for every $i = 1, \dots, N$. In Remark 2 we explain how one proves (1.2); in (1.2) we make use of the Sobolev embedding $W^{N,1} \hookrightarrow W^{1,N}$. The continuity of u can be deduced either from (1.1) (via approximation of u by convolution) or directly from (1.2).

If $u \in W^{N-1,1}(\mathbb{R}^N)$ and $D^N u$ is merely a finite measure, then (1.1) still holds (easily checked via approximation); in particular, one deduces that $u \in$

2000 *Mathematics Subject Classification.* 46E35.

Key words and phrases. Functions whose derivatives are measures; Sobolev spaces; functions of bounded variation; embedding theorems; continuous functions.

The second author (J.V.S.) is a Research Fellow of the Fonds National de la Recherche Scientifique (Belgium) and is supported by the Fonds Spécial de Recherche (Université catholique de Louvain).

$L^\infty(\mathbb{R}^N)$. A natural question is whether u is continuous. Simple examples show that the answer is *no* if $N = 1$ (here, $W^{0,1} = L^1$). In dimension $N \geq 2$, it turns out that the answer is *yes*, but it is more delicate to prove. Using Lorentz spaces, L. Tartar established the following

Theorem 1.2. *Let $N \geq 2$. If $u \in W^{N-1,1}(\mathbb{R}^N)$ is such that $D^N u$ is a measure, then u is continuous.*

We refer the reader to [2] for the proof of L. Tartar (with a contribution by A. Cohen).

An alternative approach to prove Theorem 1.2 goes as follows. By approximation, (1.2) still holds for a.e. $x, y \in Q_\varepsilon$. Since $D^N u$ is the derivative of $D^{N-1}u$ in the sense of distributions, it follows from the theory of *BV*-functions (see, e.g., [3]) that

$$(1.3) \quad |D^N u|(\{z\}) = 0 \quad \forall z \in \mathbb{R}^N,$$

i.e., the measure $|D^N u|$ does not contain Dirac masses. Thus, by dominated convergence, one deduces that

$$(1.4) \quad \lim_{\varepsilon \rightarrow 0} \int_{\overline{Q_\varepsilon}} |D^N u| = 0$$

for every family of cubes $(Q_\varepsilon)_{\varepsilon > 0}$. The continuity of u then easily follows from (1.2) and (1.4).

This argument simplifies Tartar's proof, but it still relies on the theory of *BV*-functions. On the other hand, in both inequalities (1.1) and (1.2), only $\partial_1 \cdots \partial_N u$ comes into play. The goal of this paper is to clarify the role of $\partial_1 \cdots \partial_N u$ and improve Theorem 1.2.

We assume from now on that $N \geq 2$. One of our main results is

Theorem 1.3. *If $u \in W^{1,1}(\mathbb{R}^N)$ and $\partial_1 \cdots \partial_N u$ is a measure, then u is continuous and bounded.*

We recall that $\partial_1 \cdots \partial_N u$ is a measure if there exists $C > 0$ such that

$$\left| \int_{\mathbb{R}^N} u \partial_1 \cdots \partial_N \varphi \right| \leq C \|\varphi\|_{L^\infty} \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N).$$

As one sees by considering u equal to the characteristic function of a cube, the condition that $\partial_1 \cdots \partial_N u$ is a measure is not sufficient to ensure the continuity of u . In this example, u belongs to $BV(\mathbb{R}^N)$, but not to $W^{1,1}(\mathbb{R}^N)$.

In what follows, we denote by $Q = Q_1$ the cube $(-1, 1)^N$. We point out that the conclusion of Theorem 1.3 still holds for functions $u \in W_0^{1,1}(Q)$ such that $\partial_1 \cdots \partial_N u$ is a measure; see Corollary 2.1. Without any restrictions on the boundary, one has the

Theorem 1.4. *Let $u \in W^{N-1,1}(Q)$ be such that $\partial_1 \cdots \partial_N u$ is a measure. Then, u is continuous in \overline{Q} .*

Note however that in dimension $N \geq 3$ there are functions $u \in W^{N-2,1}(Q)$, with $\partial_1 \cdots \partial_N u$ measure, which are not continuous. In fact, take any

$$v \in W^{N-2,1}((-1, 1)^{N-1})$$

which is discontinuous and define

$$u(x', x_N) = v(x') \quad \forall (x', x_N) \in Q.$$

Then, $\partial_1 \cdots \partial_N u = 0$ but u is not continuous.

In the same spirit, we have the

Theorem 1.5. *Let $u \in W^{k,p}(Q)$, $k \geq 1$ integer and $1 < p < \infty$, be such that $\partial_1 \cdots \partial_N u$ is a measure. If $kp > N - 1$, then u is continuous in \overline{Q} .*

According to Theorem 1.4, the conclusion of Theorem 1.5 holds if $kp = N - 1$ and $p = 1$. Simple examples show that this is no longer true if $kp = N - 1$ and $p > 1$.

As a corollary of Theorems 1.4 and 1.5, one immediately deduces counterparts of these results for functions $u \in W^{k,p}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is an open set. In this case, one shows that u is continuous in Ω . Note, however, that u need not be continuous on $\overline{\Omega}$, even if Ω is a ball. In fact, for any $a \in (0, \frac{1}{2})$, let

$$u(x_1, x_2) = \frac{1}{(1 - x_1)^a} \quad \forall (x_1, x_2) \in B_1 \subset \mathbb{R}^2.$$

Then, $u \in W^{1,1}(B_1)$ and $\partial_1 \partial_2 u = 0$ in B_1 , but u is not uniformly bounded in B_1 .

In Theorems 1.3–1.5, the measure $\partial_1 \cdots \partial_N u$ need not be the derivative of a BV -function. It is then unlikely that one can rely on the BV -theory in order to deduce (1.3) as above. One of the main ingredients to establish Theorems 1.3–1.5 is the next

Proposition 1.6. *Let $u \in W^{1,1}(Q)$ be such that $\partial_1 \cdots \partial_N u$ is a measure. Then, $|\partial_1 \cdots \partial_N u|$ does not charge hyperplanes of the form $[x_i = t]$, $\forall i \in \{1, \dots, N\}$, $\forall t \in (-1, 1)$.*

Here,

$$[x_i = t] := \{x \in \mathbb{R}^N; x_i = t\}$$

is the hyperplane passing through te_i and orthogonal to the x_i -axis. We say that a positive measure μ does not charge a measurable set A if $\mu(A) = 0$. Note that the measure $|\partial_1 \cdots \partial_N u|$ can charge other hyperplanes. For instance, if

$$u(x) = \max \{1 - |x_1|, \dots, 1 - |x_N|, 0\} \quad \forall x \in \mathbb{R}^N,$$

then $u \in W^{1,1}(\mathbb{R}^N)$ and $\partial_1 \cdots \partial_N u$ is a measure supported on the diagonals of the cube Q .

The paper is organized as follows. In Sections 2–3 we establish Theorems 1.3–1.5. In Sections 4–5, we prove some results—including Proposition 1.6—which we used in Sections 2–3. Finally, in Section 6 we explain how to deduce counterparts of Theorems 1.3 and 1.5 for fractionary Sobolev spaces $W^{s,p}$, with $s > 0$.

2. PROOF OF THEOREM 1.3

We assume Proposition 1.6, which will be established in Section 4 below. Let (ρ_n) be a sequence of nonnegative radial mollifiers such that $\rho_n \in C_0^\infty(B_{1/n})$, $\forall n \geq 1$. Since $u \in L^1(\mathbb{R}^N)$, for every fixed $n \geq 1$ the function $\rho_n * u$ converges uniformly to 0 as $|x| \rightarrow +\infty$. It is then easy to see that for each $x \in \mathbb{R}^N$

$$(2.1) \quad \rho_n * u(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_N} \rho_n * (\partial_1 \cdots \partial_N u) = \int_{\mathbb{R}^N} (\rho_n * F) \partial_1 \cdots \partial_N u,$$

where

$$F(y) = \begin{cases} 1 & \text{if } y_i < x_i \text{ for every } i \in \{1, \dots, N\}, \\ 0 & \text{otherwise.} \end{cases}$$

Note in particular that

$$\rho_n * F(y) \rightarrow F(y) \quad \text{for every } y \in \mathbb{R}^N \text{ such that } y_i \neq x_i, \forall i \in \{1, \dots, N\}.$$

Applying Proposition 1.6 to an arbitrary cube in \mathbb{R}^N , we have

$$(2.2) \quad |\partial_1 \cdots \partial_N u|([y_i = t]) = 0 \quad \forall t \in \mathbb{R}, \quad \forall i = 1, \dots, N.$$

Hence,

$$\rho_n * F \rightarrow F \quad \partial_1 \cdots \partial_N u \text{-a.e.}$$

By dominated convergence, as $n \rightarrow \infty$ in (2.1), we then get

$$(2.3) \quad u(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_N} \partial_1 \cdots \partial_N u \quad \text{a.e.}$$

In view of (2.2), the right-hand side of (2.3) is well-defined for every $x \in \mathbb{R}^N$ and, by dominated convergence, is continuous. Hence, u coincides a.e. with a continuous function. This concludes the proof of Theorem 1.3.

Remark 1. An easy variant of (2.3) shows that $u(x) \rightarrow 0$ uniformly as $|x| \rightarrow +\infty$. We leave the details to the reader.

As a consequence of Theorem 1.3, we have

Corollary 2.1. *If $u \in W_0^{1,1}(Q)$ and $\partial_1 \cdots \partial_N u$ is a measure, then u is continuous on \overline{Q} .*

Proof. Let $\bar{u} \in W^{1,1}(\mathbb{R}^N)$ be the extension of u as zero outside Q . By Lemma 2.2 below, $\partial_1 \cdots \partial_N \bar{u}$ is a measure in \mathbb{R}^N . Thus, $u = \bar{u}$ is continuous on \overline{Q} in view of Theorem 1.3. \square

We are left to prove the

Lemma 2.2. *Let $u \in W_0^{1,1}(Q)$. If $\partial_1 \cdots \partial_N u$ is a measure, then*

$$(2.4) \quad \int_Q u \partial_1 \cdots \partial_N \varphi = (-1)^N \int_Q \varphi \partial_1 \cdots \partial_N u \quad \forall \varphi \in C^\infty(\overline{Q}).$$

Proof. By definition, equality in (2.4) holds for every $\varphi \in C_0^\infty(Q)$. We now show that (2.4) still holds for functions $\varphi \in C^\infty(\overline{Q})$ with support in $[-1, 1] \times (-1, 1)^{N-1}$. Consider a sequence $(\theta_n) \subset C_0^\infty((-1, 1))$ such that

$0 \leq \theta_n \leq 1$ and $\theta_n(t) \rightarrow 1$ as $n \rightarrow \infty$ for every $t \in (-1, 1)$. Let $\varphi \in C^\infty(\overline{Q})$ have its support in $[-1, 1] \times (-1, 1)^{N-1}$. Clearly,

$$I_n := \int_Q u(x) \partial_1 \cdots \partial_N (\theta_n(x_1) \varphi(x)) dx = (-1)^N \int_Q \theta_n(x_1) \varphi(x) \partial_1 \cdots \partial_N u.$$

Since $u \in W_0^{1,1}(Q)$, we also have

$$I_n = - \int_Q (\partial_1 u(x)) \theta_n(x_1) \partial_2 \cdots \partial_N \varphi(x) dx.$$

Thus, applying dominated convergence and using that $u \in W_0^{1,1}(Q)$,

$$I_n \rightarrow - \int_Q (\partial_1 u) \partial_2 \cdots \partial_N \varphi = \int_Q u \partial_1 \partial_2 \cdots \partial_N \varphi \quad \text{as } n \rightarrow \infty.$$

On the other hand, since $\partial_1 \cdots \partial_N u$ is a finite measure, it also follows by dominated convergence that

$$\int_Q \theta_n(x_1) \varphi(x) \partial_1 \cdots \partial_N u \rightarrow \int_Q \varphi \partial_1 \cdots \partial_N u.$$

We conclude that (2.4) holds with functions supported in $[-1, 1] \times (-1, 1)^{N-1}$. Repeating the process with respect to the remaining $N - 1$ variables, one obtains the conclusion. \square

3. PROOFS OF THEOREMS 1.4 AND 1.5

Theorems 1.4 and 1.5 are based on the following

Theorem 3.1. *Let $u \in W^{k,p}(Q)$, $k \geq 1$ integer and $1 \leq p < \infty$, be such that $\partial_1 \cdots \partial_N u$ is a measure. Then, there exist $v, w_1, \dots, w_N \in W^{k,p}(Q)$ and a constant $C > 0$ (depending on N) such that*

- (i) $u = v + \sum_j w_j$;
- (ii) $\|v\|_{W^{k,p}} + \sum_j \|w_j\|_{W^{k,p}} \leq C \|u\|_{W^{k,p}}$;
- (iii) v is continuous and

$$(3.1) \quad |v(x) - v(y)| \leq \int_Q |\partial_1 \cdots \partial_N u| \quad \forall x, y \in Q;$$

- (iv) $\partial_j w_j = 0$ in $\mathcal{D}'(Q)$, for every $j = 1, \dots, N$.

We postpone the proof of Theorem 3.1 to Section 5.

Proofs of Theorems 1.4 and 1.5. Applying Theorem 3.1, we can write u as

$$(3.2) \quad u = v + \sum_{j=1}^N w_j,$$

where v is continuous and each w_j is a function of $(N - 1)$ -variables in $W^{N-1,1}$ ($W^{k,p}$ with $kp > N - 1$, resp.). Therefore, each w_j is also continuous; hence, u is continuous as well. \square

Remark 2 (Modulus of continuity of u). Assume $p > N - 1$. From Theorem 3.1, one deduces the following estimate on the modulus of continuity of a function $u \in W^{1,p}(Q)$ such that $\partial_1 \cdots \partial_N u$ is a measure:

$$(3.3) \quad |u(x) - u(y)| \leq \int_{Q_\varepsilon} |\partial_1 \cdots \partial_N u| + \frac{C}{\varepsilon^p} \|\nabla u\|_{L^p(Q_\varepsilon)} |x - y|^{1 - \frac{N-1}{p}},$$

for every $x, y \in Q_\varepsilon$, where $Q_\varepsilon \subset Q$ is an arbitrary parallel cube of side $2\varepsilon > 0$.

In fact, by scaling it suffices to establish (3.3) when $Q_\varepsilon = Q$ is the cube $(-1, 1)^N$. We then apply Theorem 3.1 and write u as in (i). Note that the proof of Theorem 3.1 provides the following estimate which is slightly better than (ii):

$$\sum_{j=1}^N \|\nabla w_j\|_{L^p(Q)} \leq C \|\nabla u\|_{L^p(Q)}$$

By condition (iii), we have

$$|v(x) - v(y)| \leq \int_Q |\partial_1 \cdots \partial_N u| \quad \forall x, y \in Q.$$

Since w_j is independent of the variable x_j , and $w_j \in W^{1,p}(Q)$ with $p > N - 1$, it follows from Morrey's inequality in dimension $N - 1$ that

$$\begin{aligned} |w_j(x) - w_j(y)| &\leq C \|\nabla w_j\|_{L^p(Q)} |x - y|^{1 - \frac{N-1}{p}} \\ &\leq C \|\nabla u\|_{L^p(Q)} |x - y|^{1 - \frac{N-1}{p}} \quad \forall x, y \in Q. \end{aligned}$$

Combining the inequalities for v and w_j , we obtain (3.3) with $\varepsilon = 1$.

4. PROOF OF PROPOSITION 1.6

The proof of Proposition 1.6 relies on the following two lemmas:

Lemma 4.1. *Let $v \in L^1(Q_2)$ be such that $\partial_1 \cdots \partial_N v$ is a measure. If v is odd with respect to the variables x_1, \dots, x_{N-1} , then there exists $\varepsilon > 0$ such that*

$$(4.1) \quad |v(y, s) - v(y, -t)| \geq |(\partial_1 \cdots \partial_N v)(S)|$$

for a.e. $y \in (1, 1 + \varepsilon)^{N-1}$ and a.e. $s, t \in (0, \varepsilon)$.

Here, we use the following notation

$$Q_2 := (-2, 2)^N \quad \text{and} \quad S := [-1, 1]^{N-1} \times \{0\}.$$

Proof of Lemma 4.1. By outer regularity of the Radon measure $\mu := \partial_1 \cdots \partial_N v$, there exists $\varepsilon > 0$, such that

$$(4.2) \quad \left| \int_{Q_2} \varphi d\mu \right| \geq \frac{1}{2^{N-1}} |\mu(S)|,$$

for every $\varphi \in C_0^\infty(Q_2)$ such that $0 \leq \varphi \leq 1$ in Q_2 , $\varphi = 1$ on S , and

$$\text{supp } \varphi \subset (-1 - 2\varepsilon, 1 + 2\varepsilon)^{N-1} \times (-2\varepsilon, 2\varepsilon).$$

Let (ρ_n) be a sequence of nonnegative radial mollifiers such that $\rho_n \in C_0^\infty(B_{1/n})$, $\forall n \geq 1$. Since v is odd with respect to the variables x_1, \dots, x_{N-1} , one has

$$\rho_n * v(y, s) - \rho_n * v(y, -t) = 2^{N-1} \int_{-y_1}^{y_1} \cdots \int_{-y_{N-1}}^{y_{N-1}} \int_{-t}^s \rho_n * \mu.$$

Take $n \geq 1$ large so that $\text{supp } \rho_n \subset B_\varepsilon$. By (4.2), one has

$$|\rho_n * v(y, s) - \rho_n * v(y, -t)| \geq |\mu(S)|$$

for every $y \in (1, 1 + \varepsilon)^{N-1}$ and $s, t \in (0, \varepsilon)$. As $n \rightarrow \infty$, we obtain (4.1). This establishes Lemma 4.1. \square

Lemma 4.2. *Assume $v \in W^{1,1}(Q_2)$ satisfies the assumptions of Lemma 4.1. Then,*

$$(\partial_1 \cdots \partial_N v)([-1, 1]^{N-1} \times \{0\}) = 0.$$

Proof. Let $\varepsilon > 0$ as in Lemma 4.1. Given $\delta > 0$, we denote by

$$A_\delta = (1, 1 + \varepsilon)^{N-1} \times (-\delta, \delta).$$

Since

$$\int_{(1, 1 + \varepsilon)^{N-1}} \int_0^\delta |v(y, s) - v(y, s - \delta)| dy ds \leq \delta \int_{A_\delta} |\nabla v|,$$

it follows from (4.1) that

$$\varepsilon^{N-1} \delta |(\partial_1 \cdots \partial_N v)(S)| \leq \delta \int_{A_\delta} |\nabla v|.$$

Dividing both sides by δ and letting $\delta \rightarrow 0$, we conclude that

$$|(\partial_1 \cdots \partial_N v)(S)| = 0.$$

\square

We can now establish Proposition 1.6.

Proof of Proposition 1.6. Without loss of generality, we may assume that $t = 0$ and $i = N$. We split the proof in two steps.

Step 1. We show that

$$(\partial_1 \cdots \partial_N u)(S_r) = 0,$$

where $S_r := [-r, r]^{N-1} \times \{0\}$, with $0 < r < \frac{1}{2}$.

Define $v : Q_1 \rightarrow \mathbb{R}$ as

$$v(x_1, \dots, x_N) = \sum_{\substack{\alpha_i \in \{0, 1\} \\ 1 \leq i \leq N-1}} (-1)^{|\alpha|} u((-1)^{\alpha_1} x_1, \dots, (-1)^{\alpha_{N-1}} x_{N-1}, x_N),$$

where $|\alpha| = \sum_{i=1}^{N-1} \alpha_i$. One immediately checks that v is odd with respect to the variables x_1, \dots, x_{N-1} and that

$$(\partial_1 \cdots \partial_N v)(S_r) = 2^{N-1} (\partial_1 \cdots \partial_N u)(S_r).$$

Moreover, $v \in W^{1,1}(\Omega)$; thus, the previous lemma gives the conclusion.

Step 2. Proof of the proposition completed.

Let \tilde{S} be any square of the form $\prod_{i=1}^{N-1} [a_i, b_i]^{N-1} \times \{0\}$. By a variant of Step 1, we have

$$(\partial_1 \cdots \partial_N u)(\tilde{S}) = 0.$$

Thus, by inner regularity of $\partial_1 \cdots \partial_N u$,

$$(\partial_1 \cdots \partial_N u)(\omega \times \{0\}) = 0,$$

for every open set $\omega \subset (-1, 1)^{N-1}$. This implies that

$$|\partial_1 \cdots \partial_N u|(S) = 0.$$

The proof of the proposition is complete. \square

5. PROOF OF THEOREM 3.1

We first introduce some notation. Let $v : Q \rightarrow \mathbb{R}$ and let $P \subset Q$ be an (oriented) parallelepiped. If $N = 1$ and $P = [a, b]$, then define

$$\Delta_P v = v(b) - v(a).$$

If $N > 1$ and $P = P' \times [a, b]$, then let

$$\Delta_P v = \Delta_{P'} v(\cdot, b) - \Delta_{P'} v(\cdot, a).$$

In particular, if $v \in C^\infty(Q)$, then one has the identity

$$(5.1) \quad \Delta_P v = \int_P \partial_1 \cdots \partial_N v.$$

The proof of Theorem 3.1 is based on the next

Lemma 5.1. *Let $u \in W^{k,p}(Q)$, with $k \geq 1$ integer and $1 \leq p < \infty$. Then, for a.e. $a \in Q$ there exist $v, w_1, \dots, w_N \in W^{k,p}(Q)$ such that*

- (i) $u = v + \sum_j w_j$;
- (iii') for every $x \in Q$,

$$v(x) = \Delta_{[a_1, x_1] \times \cdots \times [a_N, x_N]} u;$$

- (iv) $\partial_j w_j = 0$ in $\mathcal{D}'(Q)$, for every $j = 1, \dots, N$.

Moreover, $a \in Q$ can be chosen so that

$$(5.2) \quad \|v\|_{W^{k,p}} + \sum_{j=1}^N \|w_j\|_{W^{k,p}} \leq C \|u\|_{W^{k,p}},$$

for some $C > 0$ independent of u .

The proof of Lemma 5.1 is based on a standard Fubini-type argument. We present here a sketch of the proof.

Proof of Lemma 5.1. For every $a \in Q$, let

$$v_a(x) = \Delta_{[a_1, x_1] \times \cdots \times [a_N, x_N]} u \quad \text{and} \quad g_a(x) = u(x) - v_a(x).$$

Observe that g_a can be written as a sum of N functions $w_{1,a}, \dots, w_{N,a}$ depending on at most $(N-1)$ components of x , and each $w_{j,a}$ is obtained by (sums of) restrictions of u with respect to parallel affine subspaces passing through a .

Let (u_n) be a sequence in $C^\infty(\overline{Q})$ such that

$$u_n \rightarrow u \quad \text{in } W^{k,p}(Q).$$

For each $a \in Q$, define $v_{n,a}$ and $g_{n,a}$ accordingly. By Fubini, for a.e. $a \in Q$ we have

$$(5.3) \quad g_{n,a} \rightarrow g_a \quad \text{in } W^{k,p}(Q).$$

Therefore,

$$v_{n,a} \rightarrow v_a \quad \text{in } W^{k,p}(Q).$$

In order to obtain (5.2), note that by Fubini there exists a set of positive measure $A \subset Q$ such that

$$\|w_{j,a}\|_{W^{k,p}} \leq C\|u\|_{W^{k,p}} \quad \forall a \in A,$$

where the constant $C > 0$ depends on the (Lebesgue) measure of A . To obtain estimate (5.2), it suffices to take $a \in A$ for which (5.3) holds. \square

Proof of Theorem 3.1. Applying Lemma 5.1, we can find $v, w_1, \dots, w_N \in W^{k,p}(Q)$ such that (i), (ii) and (iv) hold. We now show that

$$(5.4) \quad v(x) = \int_{a_1}^{x_1} \cdots \int_{a_N}^{x_N} \partial_1 \cdots \partial_N u.$$

Recall that, by Proposition 1.6,

$$(5.5) \quad |\partial_1 \cdots \partial_N u|([y_i = t] \cap Q) = 0 \quad \forall t \in (-1, 1) \quad \forall i = 1, \dots, N.$$

In particular, the right-hand side of (5.4) is well-defined. Let (ρ_n) be a sequence of mollifiers such that $\text{supp } \rho_n \subset B_{1/n}$. By (iii') and (5.1), we have

$$\rho_n * v(x) = \int_{a_1}^{x_1} \cdots \int_{a_N}^{x_N} \partial_1 \cdots \partial_N (\rho_n * u)$$

for every $x \in Q$ such that $d(x, \partial Q) > \frac{1}{n}$. Proceeding as in the proof of Theorem 1.3, we deduce (5.4). It then follows from (5.5) that v is continuous on \overline{Q} ; moreover, (3.1) holds. This establishes Theorem 3.1. \square

6. FURTHER RESULTS

Most of our results can be extended to fractionary Sobolev spaces $W^{s,p}$ with $s > 0$ and $1 \leq p < \infty$.

We first recall the definition of $W^{s,p}(\Omega)$ for a domain $\Omega \subset \mathbb{R}^N$. Given $0 < s < 1$, we say that $u \in W^{s,p}(\Omega)$ if

$$u \in L^p(\Omega) \quad \text{and} \quad \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy < \infty.$$

If $s > 1$, $s \notin \mathbb{N}$, then we say that $u \in W^{s,p}(\Omega)$ if

$$u \in W^{k,p}(\Omega) \quad \text{and} \quad D^k u \in W^{s-k,p}(\Omega),$$

where $k \in \mathbb{N}$ is such that $k < s < k + 1$.

The counterparts of Theorems 1.3 and 1.5 for $W^{s,p}$ are

Theorem 6.1. *If $u \in W^{1/p,p}(\mathbb{R}^N)$, $1 < p < \infty$, and $\partial_1 \cdots \partial_N u$ is a measure, then u is continuous and bounded.*

Theorem 6.2. *Let $u \in W^{s,p}(Q)$, $s > 0$ and $1 < p < \infty$, be such that $\partial_1 \cdots \partial_N u$ is a measure. If $sp > N - 1$, then u is continuous on \overline{Q} .*

Simple examples show that the conclusion of Theorem 6.1 is no longer true if $sp < 1$ (take for instance $u = \chi_Q$, the characteristic function of Q). The proofs of Theorems 6.1 and 6.2 are based on counterparts of Lemma 5.1 and Proposition 1.6 for $W^{s,p}$. The analogue of Lemma 5.1 can be still established via a Fubini-type argument using the equivalent form of the Gagliardo seminorm in \mathbb{R}^N (see [1]):

$$(6.1) \quad |u|_{W^{s,p}}^p := \sum_{i=1}^N \int_0^\infty d\tau \int_{\mathbb{R}^N} \frac{|u(x + \tau e_i) - u(x)|^p}{\tau^{1+sp}} dx.$$

We shall focus on the counterpart of Proposition 1.6:

Proposition 6.3. *Let $u \in W^{s,p}(Q)$ be such that $\partial_1 \cdots \partial_N u$ is a measure. If $sp \geq 1$, then*

$$(6.2) \quad |\partial_1 \cdots \partial_N u|([x_i = t] \cap Q) = 0 \quad \forall t \in (-1, 1), \quad \forall i = 1, \dots, N.$$

We need the analogue of Lemma 4.2 in $W^{s,p}$ for $sp \geq 1$:

Lemma 6.4. *Assume $s > 0$, $p > 1$ and $sp \geq 1$. Let $v \in W^{s,p}(Q)$ satisfy the assumptions of Lemma 4.1. Then*

$$(\partial_1 \cdots \partial_N v)(S) = 0,$$

where $S = [-1, 1]^{N-1} \times \{0\}$.

Proof. If $s \geq 1$, then this follows from Lemma 4.2 and the embedding $W^{s,p}(Q) \subset W^{1,1}(Q)$. We now assume $0 < s < 1$. If the conclusion did not hold, then, by Lemma 4.1, there would exist $\varepsilon, \delta > 0$ such that

$$|v(y, s) - v(y, -t)| \geq \delta > 0$$

for a.e. $y \in (1, 1 + \varepsilon)^{N-1}$ and a.e. $s, t \in (0, \varepsilon)$. Together with $sp \geq 1$, this would yield

$$(6.3) \quad \int_{(1,1+\varepsilon)^{N-1}} \int_0^\varepsilon \int_0^\varepsilon \frac{|v(y, s) - v(y, -t)|^p}{(s+t)^{1+sp}} ds dt dy \geq \\ \geq \varepsilon^{N-1} \int_0^\varepsilon \int_0^\varepsilon \frac{\delta^p}{(s+t)^{1+sp}} ds dt = +\infty.$$

In view of the equivalent seminorm (6.2), this would be in contradiction with the hypothesis $u \in W^{s,p}(\Omega)$. \square

Proof of Proposition 6.3. One proceeds exactly as in the proof of Proposition 1.6, replacing Lemma 4.2 by Lemma 6.4. We leave the details to the reader. \square

REFERENCES

- [1] R.A. Adams, Sobolev spaces, Pure and Applied Mathematics Vol. 65, Academic Press, New York-London, 1975.
- [2] M. Campos Pinto, Développement et analyse de schémas adaptatifs pour les équations de transport, Ph.D. dissertation, Université Paris 6, 2005.
- [3] L.C. Evans and R.F. Gariepy, Measure theory and fine properties of functions, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.
- [4] A. Kufner, O. John and S. Fučík, Function spaces, Monographs and Textbooks on Mechanics of Solids and Fluids, Noordhoff International Publishing, Leyden, 1977.

- [5] J. Souček, Spaces of functions on domain Ω , whose k -th derivatives are measures defined on $\bar{\Omega}$, Časopis Pěst. Mat. 97 (1972), 10–46, 94.

(A.C. Ponce) LABORATOIRE DE MATHÉMATIQUES ET PHYSIQUE THÉORIQUE (UMR CNRS 6083), FÉDÉRATION DENIS POISSON, UNIVERSITÉ FRANÇOIS RABELAIS, 37200 TOURS, FRANCE

E-mail address: `ponce@lmpt.univ-tours.fr`

(J. Van Schaftingen) DÉPARTEMENT DE MATHÉMATIQUE, UNIVERSITÉ CATHOLIQUE DE LOUVAIN, CHEMIN DU CYCLOTRON 2, 1348 LOUVAIN-LA-NEUVE, BELGIUM

E-mail address: `vanschaftingen@math.ucl.ac.be`