A simple proof of an inequality of Bourgain, Brezis and Mironescu

Une preuve simple d'une inégalité de Bourgain, Brezis et Mironescu

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Abstract

A simpler proof of a recent inequality of Bourgain, Brezis and Mironescu is given. *To cite this article: J. Van Schaftingen, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

Résumé

Nous donnons une preuve plus simple d'une inégalité récente de Bourgain, Brezis et Mironescu. *Pour citer cet article : J. Van Schaftingen, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

Version française abrégée

Bourgain, Brezis et Mironescu ont établi dans [1] l'inégalité suivante.

Proposition 0.1 *Soit Γ une courbe fermée, orientée et rectifiable de $\mathbb{R}^3$, et soit $\vec{t}$ le vecteur unité tangent à Γ. Si $\vec{k} \in W^{1,3}(\mathbb{R}^3;\mathbb{R}^3)$, alors*

$$\left| \int_\Gamma \vec{k} \cdot \vec{t} \right| \le C \left\| k \right\|_{W^{1,3}} |\Gamma| \,.$$

La preuve de la proposition 0.1 dans [1] est assez complexe. Nous en donnons une preuve élémentaire qui se généralise à des surfaces de dimension quelconque dans un espace de dimension quelconque.

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Proposition 0.1 *Soit Γ une courbe fermée, orientée et lipschitzienne dans $\mathbb{R}^N$, $N \geq 2$; soit $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$. Si $\nabla u \in L^N(\mathbb{R}^N)$, alors*

$$\left| \int_\Gamma u \, d\gamma \right| \leq C_N \, \|\nabla u\|_{L^N(\mathbb{R}^N)} \, |\Gamma| \, , \tag{1}$$

où $|\Gamma|$ désigne la longueur de la courbe Γ.

La preuve de 0.1 commence par établir, avec la stratégie de [1], la formule (3). Ensuite, alors que Bourgain, Brezis et Mironescu utilisent une décomposition de Littlewood-Paley, nous utilisons les inégalités de Morrey et de Hölder pour conclure.

La preuve se généralise sans difficulté à des surfaces k-dimensionnelles (proposition 3.2).

1. Introduction

Bourgain, Brezis and Mironescu proved in [1] the following inequality.

Proposition 1.1 *Let Γ be a closed, oriented, rectifiable curve of $\mathbb{R}^3$, and denote by $\vec{t}$ the unit tangent vector along Γ; let $\vec{k} \in W^{1,3}(\mathbb{R}^3; \mathbb{R}^3)$. Then*

$$\left| \int_\Gamma \vec{k} \cdot \vec{t} \right| \leq C \, \|k\|_{W^{1,3}} \, |\Gamma| \, .$$

The proof of proposition 1.1 in [1] is technically involved. We provide an elementary proof and a generalization to k dimensional surfaces and N-dimensional space. For simplicity, we begin with the case of a curve in $\mathbb{R}^N$.

Proposition 1.2 *Let Γ be an oriented, compact and closed Lipschitz curve of $\mathbb{R}^N$, $N \geq 2$; let $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$. If $\nabla u \in L^N(\mathbb{R}^N)$, then*

$$\left| \int_\Gamma u \, d\gamma \right| \leq C_N \, \|\nabla u\|_{L^N(\mathbb{R}^N)} \, |\Gamma| \, , \tag{2}$$

where $|\Gamma|$ denotes the length of curve Γ.

Remark 1 *When $N = 1$, the left-hand side of (2) is 0; when $N = 2$ and γ is a Jordan curve, proposition 1.2 is a simple consequence of Green theorem and the isoperimetric inequality; and when $N = 3$ it is equivalent to proposition 1.1.*

2. Proof of proposition 1.2

PROOF. Without loss of generality, the curve Γ is connected and is the image of S^1 by a Lipschitz map γ. We assume first that u and $\gamma : S^1 \to \mathbb{R}^N$ are of class C^1. We start with the same strategy as [1], bounding

$$e \cdot \int_{S^1} u(\gamma(x)) \dot{\gamma}(x) \, dx,$$

for an arbitrary unit-norm vector $e \in \mathbb{R}^N$.

Let

$$\Gamma_t = \left\{ x \in S^1 \mid e \cdot \gamma(x) = t \right\} .$$

Since $e \cdot \gamma(s)$ is of class C^1, Sard's lemma implies that for almost every $t \in \mathbb{R}$, Γ_t is finite and $e \cdot \dot{\gamma}(s) \neq 0$ if $\gamma(s) \in \Gamma_t$. We have then

$$e \cdot \int_{S^1} u(\gamma(x)) \dot{\gamma}(x) \, dx = \int_{S^1} u(\gamma(x)) \, e \cdot \dot{\gamma}(x) \, dx = \int_{\mathbb{R}} \sum_{x \in \Gamma_t} \sigma(x) u(x) \, dt \, , \tag{3}$$

where $\sigma(x) = \operatorname{sign}(e \cdot \dot{\gamma}(x))$. Since Γ is closed, for almost every $t \in \mathbb{R}$ we can write $\Gamma_t = \left\{ P_1, \ldots, P_{r(t)} \right\} \cup \left\{ N_1, \ldots, N_{r(t)} \right\}$ so that $\sigma(P_i) = 1$, $\sigma(N_i) = -1$ and $\sum_{i=1}^{r} |\gamma(P_i) - \gamma(N_i)|$ is minimal (in particular, it is bounded by $|\Gamma|$).

In order to estimate $\sum_{x \in \Gamma_t} \sigma(x) u(x)$, we proceed differently from [1]. They used a Littlewood-Paley decomposition. Instead, we apply Morrey's inequality (see e.g. [2, theorem IX.12]) in $\mathbb{R}^{N-1}$ for $u_t = u|_{\{y \in \mathbb{R}^N \mid e \cdot y = t\}}$ before applying the discrete Hölder inequality to the sum

$$\sum_{x \in \Gamma_t} \sigma(x) u(x) \leq C_N \left\| \nabla u_t \right\|_N \sum_{i=1}^{r(t)} |\gamma(P_i) - \gamma(N_i)|^{\frac{1}{N}} \leq C_N \left\| \nabla u_t \right\|_p |\Gamma|^{\frac{1}{N}} \, r(t)^{\frac{N-1}{N}} \, .$$

We are now ready to estimate the integral of (3):

$$\begin{aligned} \int_{\mathbb{R}} \sum_{x \in \Gamma_t} \sigma(x) u(x) \, dt &\leq C_N |\Gamma|^{\frac{1}{N}} \int_{\mathbb{R}} \left\| \nabla u_t \right\|_N r(t)^{\frac{N-1}{N}} \, dt \\ &\leq C_N |\Gamma|^{\frac{1}{N}} \left(\int_{\mathbb{R}} \left\| \nabla u_t \right\|_N^N \, dt \right)^{\frac{1}{N}} \left(\int_{\mathbb{R}} r(t) \, dt \right)^{\frac{N-1}{N}} \leq C'_{p,N} \, |\Gamma| \left\| \nabla u \right\|_N \end{aligned}$$

since $2 \int_{\mathbb{R}} r(t) \, dt = \int_{S^1} |e \cdot \gamma'(x)| \, dx \leq \int_{S^1} |\gamma'(x)| \, dx = |\Gamma|$.

The result is extended to general Γ and u by standard smoothing arguments.

3. Generalization to surfaces

Proposition 1.2 generalizes straightforwardly to k-dimensional surfaces defined as follows.

Definition 3.1 *A pair $\Gamma = (M, \gamma)$ is a C^r k-dimensional Lipschitz surface of $\mathbb{R}^N$ if*

(i) M is a compact oriented k-dimensional C^r manifold without boundary,

(ii) $\gamma : M \to \mathbb{R}^N$ is a Lipschitz function.

When $\Gamma = (M, \gamma)$ is a k-dimensional Lipschitz surface of $\mathbb{R}^N$, it is possible, since M is oriented, to define the integral of Borel function $u : \mathbb{R}^N \to \mathbb{R}$ as the k-vector $\int_\Gamma u \, d\gamma(x)$, where $d\gamma(x)[a_1, \ldots, a_k] = \gamma'(x) a_1 \wedge \ldots \wedge \gamma'(x) a_k$, and the mass of Γ as $|\Gamma| = \int_M |d\gamma|$, where $|\cdot|$ denotes the euclidean norm of a k-vector.

Proposition 3.2 *Let Γ be a C^k k-dimensional surface. Then*

$$\left| \int_M u \, d\gamma \right| \leq C_N \left\| \nabla u \right\|_N |\Gamma| \, , \tag{4}$$

where the norm on the left is the comass-norm (see [3]).

PROOF. Since the proof is similar to the proof of proposition 1.2, we only give a sketch. For an arbitrary simple unit covector $e = e_1 \wedge \cdots \wedge e_k$, we write

$$e \cdot \int_\Gamma u \, d\gamma = \int_{\mathbb{R}^k} \sum_{x \in \Gamma_y} \sigma(x) u(x) \, dy,$$

where $\Gamma_y = \{x \in M \mid e_i \cdot \gamma(x) = y_i, 1 \leq i \leq k\}$ and $\sigma(x) = \operatorname{sign}(e_1 \gamma'(x) \wedge \ldots \wedge e_k \gamma'(x))$. This formula is valid by Sard's lemma because M is a C^k manifold. Then, using Morrey's and Hölder's inequalities, with the notations of the proof of proposition 1.2,

$$\int_{\mathbb{R}^k} \sum_{x \in \Gamma_y} \sigma(x) u(x) \, dy \leq \int_{\mathbb{R}} \|\nabla u_{y_1}\|_N \left(\int_{\mathbb{R}^{k-1}} \sum_{i=1}^{r(y)} |\gamma(P_i) - \gamma(N_i)| \, dy'' \right)^{\frac{1}{N-1}} \left(\int_{\mathbb{R}^{k-1}} r(y) \, dy'' \right)^{\frac{N}{N-1}} dy_1,$$

where $y'' = (y_2, \ldots, y_k)$ and one concludes using $\int_{\mathbb{R}^{k-1}} \sum_{i=1}^{r(y)} |\gamma(P_i) - \gamma(N_i)| \, dy'' \leq |\Gamma|$, Hölder's inequality and $2 \int_{\mathbb{R}^k} r(y) \, dy \leq |\Gamma|$.

Remark 2 *Proposition 3.2 can also be proved by induction on k. The case $k = 1$ is proposition 1.2 and for $k > 1$, Γ is cut into slices of dimension $k - 1$, for which the estimate of proposition 3.2 holds. The integration of this estimate, with Hölder's inequality gives the conclusion.*
Remark 3 *The inequality (4) is the limit case of*

$$\left| \int_\Gamma u \, d\gamma \right| \leq C_{p,N} \delta(\Gamma)^{1 - \frac{N}{p}} |\Gamma| \, \|\nabla u\|_p \,,$$

where $p > N$, $\nabla u \in L^p(\mathbb{R}^N)$ and $\delta(\Gamma)$ denotes the diameter of Γ. It is a simple consequence of Morrey's inequality.

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