FUNCTION SPACES BETWEEN BMO AND CRITICAL SOBOLEV SPACES

JEAN VAN SCHAFTINGEN

Abstract. The function spaces $\mathrm{D}_k(\mathbf{R}^n)$ are introduced and studied. The definition of these spaces is based on a regularity property for the critical Sobolev spaces $\mathrm{W}^{s,p}(\mathbf{R}^n)$, where $sp = n$, obtained by Bourgain and Brezis, C. R. Math. **338** (2004) (see also Van Schaftingen, C. R. Math. **338** (2004)). The spaces $\mathrm{D}_k(\mathbf{R}^n)$ contain all the critical Sobolev spaces. They are embedded in $\mathrm{BMO}(\mathbf{R}^n)$, but not in $\mathrm{VMO}(\mathbf{R}^n)$. Moreover they have some extension and trace properties that $\mathrm{BMO}(\mathbf{R}^n)$ does not have.

1. Introduction

1.1. **Integrals with divergence-free vector-fields.** When $p < n$, the Sobolev space $\mathrm{W}^{1,p}(\mathbf{R}^n)$ of functions whose distributional derivative is in $\mathrm{L}^p(\mathbf{R}^n)$ is continuously embedded in the space $\mathrm{L}^{p^*}(\mathbf{R}^n)$, with $p^* = np/(n-p)$, while when $p > n$ it is embedded in the space of Hőlder continuous function of exponent α, $C^{0,\alpha}(\mathbf{R}^n)$, with $\alpha = 1 - n/p$ [1, 5, 19].

The case $p = n$ is more delicate. When $n > 1$, functions in $\mathrm{W}^{1,n}(\mathbf{R}^n)$ do not need to be continuous or bounded, but have many properties in common with such functions. This is expressed for example by the embedding of $\mathrm{W}^{1,n}(\mathbf{R}^n)$ in the spaces $\mathrm{BMO}(\mathbf{R}^n)$ and $\mathrm{VMO}(\mathbf{R}^n)$ of functions of bounded and vanishing mean oscillation [6]. These considerations are also valid for fractional Sobolev spaces $\mathrm{W}^{s,p}(\mathbf{R}^n)$, with $sp = n$.

Another property of critical Sobolev space was recently obtained by Bourgain and Brezis [3, 22]: For every vector field $\varphi \in (\mathrm{L}^1 \cap C)(\mathbf{R}^n; \mathbf{R}^n)$ and $u \in \mathrm{W}^{s,p}(\mathbf{R}^n)$, if $\operatorname{div} \varphi = 0$ in the sense of distributions, then

$$\left| \int_{\mathbf{R}^n} u \varphi \, dx \right| \le C_{s,p} \|\varphi\|_{\mathrm{L}^1(\mathbf{R}^n)} \, \|u\|_{\mathrm{W}^{s,p}(\mathbf{R}^n)}. \tag{1.1}$$

There is no such property for $\mathrm{BMO}(\mathbf{R}^n)$ or for $\mathrm{VMO}(\mathbf{R}^n)$ (see [2] and Remark 5.2).

A natural question is the relationship between (1.1) and the embedding of $\mathrm{W}^{s,p}(\mathbf{R}^n)$ in the spaces $\mathrm{BMO}(\mathbf{R}^n)$ and $\mathrm{VMO}(\mathbf{R}^n)$. In order to answer it, we define, for $n \ge 1$, the seminorm

$$\|u\|_{\mathrm{D}_{n-1}(\mathbf{R}^n)} = \sup_{\substack{\varphi \in \mathcal{D}(\mathbf{R}^n; \mathbf{R}^n) \\ \operatorname{div} \varphi = 0 \\ \|\varphi\|_{\mathrm{L}^1(\mathbf{R}^n)} \le 1}} \left| \int_{\mathbf{R}^n} u \varphi \, dx \right|. \tag{1.2}$$

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and the vector space

$$\mathrm{D}_{n-1}(\mathbf{R}^n) = \bigl\{u \in \mathcal{D}'(\mathbf{R}^n) \ : \ \|u\|_{\mathrm{D}_{n-1}(\mathbf{R}^n)} < \infty\bigr\}.$$

Here $\mathcal{D}(\mathbf{R}^N;\mathbf{R}^N)$ is the space of compactly supported smooth vector fields and $\mathcal{D}'(\mathbf{R}^n)$ is the space of distributions [16]. The subscript $n-1$ will be justified by further extensions. By the inequality (1.1), $\mathrm{W}^{s,p}(\mathbf{R}^n)$ is embedded in $\mathrm{D}_{n-1}(\mathbf{R}^n)$.

The question of the previous paragraph is answered as follows: $\mathrm{VMO}(\mathbf{R}^n)$ is not embedded in $\mathrm{D}_{n-1}(\mathbf{R}^n)$ (Proposition 5.1), and $\mathrm{D}_{n-1}(\mathbf{R}^n)$ is embedded in $\mathrm{BMO}(\mathbf{R}^n)$ (Theorem 5.3). Moreover, if $u \in \mathrm{D}_{n-1}(\mathbf{R}^n)$ is continuous, and $k \geq 2$, then $\|u|_{\mathbf{R}^k}\|_{\mathrm{BMO}(\mathbf{R}^k)} \leq C\|u\|_{\mathrm{D}_{n-1}(\mathbf{R}^n)}$ (Theorems 3.4 and 5.3). This inequality remains open when $k = 1$.

The proof of the embedding of $\mathrm{D}_{n-1}(\mathbf{R}^n)$ in $\mathrm{BMO}(\mathbf{R}^n)$ is based on the duality between $\mathrm{BMO}(\mathbf{R}^n)$ and the Hardy space $\mathrm{H}^1(\mathbf{R}^n)$, and on a decomposition of every function in $\mathrm{H}^1(\mathbf{R}^n)$ as a sum of some components of divergence-free vector-fields, with a suitable control on the norms.

The inequality (1.1) was preceded by a geometric counterpart [4]: For every closed rectifiable curve $\gamma \in C^1(S^1;\mathbf{R}^n)$ and $u \in (C \cap \mathrm{W}^{1,n})(\mathbf{R}^n)$,

$$\left|\int_{\mathbf{R}^n} u(\gamma(t))\dot{\gamma}(t)\,dt\right| \leq C_{s,p}\|\dot{\gamma}\|_{\mathrm{L}^1(S^1)}\,\|u\|_{\mathrm{W}^{s,p}(\mathbf{R}^n)}. \tag{1.3}$$

(See [23] for an elementary proof.) The right-hand side of (1.3) could also be used to define a seminorm on continuous functions. By the arguments of [3], based on a decomposition of divergence-free vector-fields in solenoids of Smirnov [18], one has in fact

$$\|u\|_{\mathrm{D}_{n-1}(\mathbf{R}^n)} = \sup_{\gamma \in C^1(S^1;\mathbf{R}^n)} \frac{1}{\|\dot{\gamma}\|_{\mathrm{L}^1(S^1)}}\left|\int_{S^1} u(\gamma(t))\dot{\gamma}(t)\,dt\right|.$$

An open problem is whether restricting the curves on the right-hand side to be contained in k–dimensional planes, to triangles or to circles would yield an equivalent norm. The restriction to curves contained in k-dimensional planes is equivalent to requiring φ in (1.2) to have a range whose dimensions is at most k, see section 6.4 below.

If $s \leq 1$, $sp = n$, and $u \in \mathrm{W}^{s,p}(\mathbf{R}^n)$, then $u_+ \in \mathrm{W}^{s,p}(\mathbf{R}^n)$. This property also holds in $\mathrm{BMO}(\mathbf{R}^n)$. We do not know whether it holds for $\mathrm{D}_{n-1}(\mathbf{R}^n)$. The question whether, for a given $\varphi : \mathbf{R} \to \mathbf{R}$ one has $\varphi(u) \in \mathrm{D}_{n-1}(\mathbf{R}^n)$ whenever $u \in \mathrm{D}_{n-1}(\mathbf{R}^n)$ remains open when φ is not affine.

1.2. **Integrals with curl-free vector-fields.** When $s = 1$ and $p = n = 2$, the inequality (1.1) is in fact a dual statement of the Sobolev–Nirenberg embedding

$$\|g\|_{\mathrm{L}^{n/(n-1)}(\mathbf{R}^n)} \leq C\|Dg\|_{\mathrm{L}^1(\mathbf{R}^n)}. \tag{1.4}$$

When $s = 1$ and $p = n > 2$, the estimate (1.1) is stronger than the embedding (1.4). If $n = 3$, (1.4) yields by duality that, for every $\varphi \in \mathcal{D}(\mathbf{R}^3;\mathbf{R}^3)$ and $u \in \mathrm{W}^{1,3}(\mathbf{R}^3)$, if $\operatorname{curl}\varphi = 0$ in the sense of distributions,

$$\left|\int_{\mathbf{R}^3} u\varphi\,dx\right| \leq C\|\varphi\|_{\mathrm{L}^1(\mathbf{R}^3)}\,\|u\|_{\mathrm{W}^n(\mathbf{R}^3)}. \tag{1.5}$$

For $u \in \mathrm{W}^{s,p}(\mathbf{R}^3)$ with $sp = 3$, this inequality can be deduced from (1.1) recalling that, for every $e \in \mathbf{R}^3$

$$\operatorname{div}(\varphi \times e) = (\operatorname{curl} \varphi) \cdot e. \tag{1.6}$$

In $\mathbf{R}^3$, one can now investigate the relationship between (1.1), (1.5), and the embedding of of $\mathrm{W}^{s,p}(\mathbf{R}^n)$ in the spaces $\mathrm{BMO}(\mathbf{R}^n)$ and $\mathrm{VMO}(\mathbf{R}^n)$. We define therefore the seminorm

$$\|u\|_{\mathrm{D}_1(\mathbf{R}^3)} = \sup_{\substack{\varphi \in \mathcal{D}(\mathbf{R}^3;\mathbf{R}^3)\\ \operatorname{curl}\varphi = 0\\ \|\varphi\|_{\mathrm{L}^1(\mathbf{R}^3)} \le 1}} \left| \int_{\mathbf{R}^3} u\varphi \, dx \right|$$

and the vector space

$$\mathrm{D}_1(\mathbf{R}^3) = \left\{ u \in \mathcal{D}'(\mathbf{R}^3) \;:\; \|u\|_{\mathrm{D}_1(\mathbf{R}^3)} < \infty \right\}.$$

While $\mathrm{VMO}(\mathbf{R}^3) \not\subset \mathrm{D}_1(\mathbf{R}^3)$, one has the following continuous embeddings:

$$\mathrm{D}_2(\mathbf{R}^3) \subset \mathrm{D}_1(\mathbf{R}^3) \subset \mathrm{BMO}(\mathbf{R}^3).$$

The first embedding is a consequence of (1.6), and the second of the duality between $\mathrm{BMO}(\mathbf{R}^3)$ and the Hardy space $\mathrm{H}^1(\mathbf{R}^3)$, and of a decomposition of every function in $\mathrm{H}^1(\mathbf{R}^3)$ as a sum of some components of curl-free vector-fields.

If $u \in \mathrm{D}_1(\mathbf{R}^2)$, its extension $U(x, y) = u(x)$ to $\mathbf{R}^3$ is in $\mathrm{D}_1(\mathbf{R}^3)$. It would be in $\mathrm{D}_1(\mathbf{R}^3)$ if and only if U was bounded. On the other hand, if $u \in \mathrm{D}_2(\mathbf{R}^3)$ is continuous, one has the trace inequality $\|u|_{\mathbf{R}^2}\|_{\mathrm{D}_1(\mathbf{R}^2)} \le C\|u\|_{\mathrm{D}_2(\mathbf{R}^3)}$. The problem whether the trace inequalities $\|u|_{\mathbf{R}^2}\|_{\mathrm{BMO}(\mathbf{R}^2)} \le C\|u\|_{\mathrm{D}_1(\mathbf{R}^3)}$ and $\|u|_{\mathbf{R}}\|_{\mathrm{BMO}(\mathbf{R})} \le C\|u\|_{\mathrm{D}_2(\mathbf{R}^3)}$ hold is open.

The seminorm $\|\cdot\|_{\mathrm{D}_1(\mathbf{R}^3)}$ can also be characterized geometrically: By the co-area formula, for every $u \in C(\mathbf{R}^3)$,

$$\|u\|_{\mathrm{D}_1(\mathbf{R}^3)} = \sup_{\Omega} \frac{1}{\mathcal{H}^2(\partial\Omega)} \left| \int_{\partial\Omega} u(y)\nu(y) \, d\mathcal{H}^2(y) \right|,$$

where the supremum is taken over bounded domains $\Omega \subset \mathbf{R}^3$ with a smooth connected boundary, $\nu(y)$ is the unit exterior normal vector to the boundary at $y \in \partial\Omega$, and $\mathcal{H}^2$ is the two-dimensional Hausdorff measure.

1.3. **Integrals along differential forms.** In higher dimensions, the generalization of (1.1) corresponding to (1.5) in $\mathbf{R}^3$ is expressed with differential forms: If $1 \le k \le n-1$, then, for every compactly supported smooth k–differential form $\varphi \in \mathcal{D}(\mathbf{R}^n; \Lambda^k \mathbf{R}^n)$ and for every $u \in \mathrm{W}^{s,p}(\mathbf{R}^n)$ with $p \ge 1$ and $sp = n$, if $d\varphi = 0$, then

$$\left| \int_{\mathbf{R}^n} u\varphi \, dx \right| \le C_{s,p} \|\varphi\|_{\mathrm{L}^1(\mathbf{R}^n)} \, \|u\|_{\mathrm{W}^{s,p}(\mathbf{R}^n)}. \tag{1.7}$$

The previous definitions of $\mathrm{D}_k(\mathbf{R}^n)$ are generalized as follows: For $1 \le k \le n-1$, we define the seminorm

$$\|u\|_{\mathrm{D}_k(\mathbf{R}^n)} = \sup_{\substack{\varphi \in \mathcal{D}(\mathbf{R}^n;\Lambda^k\mathbf{R}^n)\\ d\varphi = 0\\ \|\varphi\|_{\mathrm{L}^1(\mathbf{R}^n)} \le 1}} \left| \int_{\mathbf{R}^n} u\varphi \, dx \right|,$$

and the vector space

$$\mathrm{D}_k(\mathbf{R}^n) = \left\{ u \in \mathcal{D}'(\mathbf{R}^n) \ : \ \|u\|_{\mathrm{D}_k(\mathbf{R}^n)} < \infty \right\}.$$

By (1.7), $\mathrm{W}^{s,p}(\mathbf{R}^n) \subset \mathrm{D}_k(\mathbf{R}^n)$. These spaces $\mathrm{D}_k(\mathbf{R}^n)$ also contain other functions, such as $\log(\sum_{i=1}^{k+1} x_i^2)$.

The spaces $\mathrm{D}_k(\mathbf{R}^n)$ contain neither $\mathrm{BMO}(\mathbf{R}^n)$ nor $\mathrm{VMO}(\mathbf{R}^n)$. Our main result is that $\mathrm{D}_k(\mathbf{R}^n)$ is embedded in $\mathrm{BMO}(\mathbf{R}^n)$. We first show that $\mathrm{D}_k(\mathbf{R}^n)$ is embedded in $\mathrm{D}_1(\mathbf{R}^n)$, then we prove that $\mathrm{D}_1(\mathbf{R}^n)$ is embedded in $\mathrm{BMO}(\mathbf{R}^n)$, in the same fashion as the embedding of $\mathrm{D}_1(\mathbf{R}^3)$ in $\mathrm{BMO}(\mathbf{R}^3)$ outlined above.

The other known properties of the spaces $\mathrm{D}_k(\mathbf{R}^n)$ can be summarized as follows. The seminorm $\|\cdot\|_{\mathrm{D}_k(\mathbf{R}^n)}$ is a norm modulo constants and the space $\mathrm{D}_k(\mathbf{R}^n)$ modulo constants is a Banach space. The spaces $\mathrm{D}_k(\mathbf{R}^n)$ are all different and are decreasing with respect to k. The spaces $\mathrm{D}_k(\mathbf{R}^n)$ also have a trace property: If $u \in \mathrm{D}_K(\mathbf{R}^N)$ is a limit of continuous functions, then it has a well-defined trace in $\mathrm{D}_k(\mathbf{R}^n)$ if $N - K = n - k$. On the other hand the function spaces $\mathrm{D}_k(\mathbf{R}^n)$ do not have better integrability properties then the exponential integrability of functions in $\mathrm{BMO}(\mathbf{R}^n)$.

1.4. **Organization of the paper.** We define in section 2 the spaces $\mathrm{D}_k(\mathbf{R}^n)$ for $1 \le k \le n$ by duality on closed smooth forms, and characterize them by duality on exact forms (Proposition 2.6). The space $\mathrm{D}_n(\mathbf{R}^n)$ is in fact $\mathrm{L}^\infty(\mathbf{R}^n)/\mathbf{R}$ (Proposition 2.9). We characterize geometrically the seminorm for continuous functions in the cases $k = 1$ (Proposition 2.10) and $k = n - 1$ (Proposition 2.11). For $1 < k < n - 1$, there is an equivalent seminorm defined by integration on real polyhedral chains without boundary (Proposition 2.14).

Section 3 reviews the basic properties of the spaces $\mathrm{D}_k(\mathbf{R}^n)$: mutual injections (Theorem 3.1), extension theory (Theorem 3.2) and trace theory (Theorem 3.4).

Section 4 gives examples of functions in $\mathrm{D}_k(\mathbf{R}^n)$: critical Sobolev spaces (Theorem 4.1) and other functions (Propositions 4.3, 4.6 and 4.10).

The embedding of the spaces $\mathrm{D}_k(\mathbf{R}^n)$ in $\mathrm{BMO}(\mathbf{R}^n)$ is proved in section 5 (Theorem 5.3). This provides an easy proof of the completeness of $\mathrm{D}_k(\mathbf{R}^n)$ (Theorem 5.5). We also address the question of improved integrability of functions in $\mathrm{D}_k(\mathbf{R}^n)$ (Proposition 5.6).

The paper ends with considerations about further problems in the study of the spaces $\mathrm{D}_k(\mathbf{R}^n)$ (section 6).

An appendix is devoted to density properties of closed and exact smooth forms.

2. Definitions and characterizations

2.1. **Preliminaries.** The space of k–forms on $\mathbf{R}^n$ is denoted by $\Lambda^k\mathbf{R}^n$. The exterior product of $\alpha \in \Lambda^k\mathbf{R}^n$ and $\beta \in \Lambda^\ell\mathbf{R}^n$ is $\alpha \wedge \beta \in \Lambda^{k+\ell}\mathbf{R}^n$. The space $\Lambda^1\mathbf{R}^n$ is the dual of $\mathbf{R}^n$ and has a canonical basis $\omega_1, \dots, \omega_n$ biorthogonal to the canonical basis $e_1, \cdots, e_n$ of $\mathbf{R}^n$. Moreover, $\Lambda^k\mathbf{R}^n$ has a canonical Euclidean norm denoted by $|\cdot|$.

A differential form is a function $\varphi : \mathbf{R}^n \to \Lambda^k \mathbf{R}^n$. The exterior differential d is defined by

$$d\varphi = \sum_{i=1}^{n} \omega_i \wedge \frac{\partial \varphi}{\partial x_i}.$$

(This makes sense if e.g. φ is a C^1 function or a distribution.)

If V is finite-dimensional, the space of V–valued compactly supported smooth (C^∞) functions (test functions) is denoted by $\mathcal{D}(\mathbf{R}^n; V)$ and is endowed with its usual topology [16]. The space of distributions $\mathcal{D}'(\mathbf{R}^n)$ is the dual of $\mathcal{D}(\mathbf{R}^n) = \mathcal{D}(\mathbf{R}^n; \mathbf{R})$.

Lebesgue's measure on $\mathbf{R}^n$ is denoted by $\mathcal{L}^n$ and the r–dimensional Hausdorff measure by $\mathcal{H}^r$.

2.2. **Definition.** The spaces $\mathrm{D}_k(\mathbf{R}^n)$ are defined in terms of appropriate test function spaces.

Definition 2.1. For $1 \le k \le n$, define

$$\mathcal{D}_\#(\mathbf{R}^n; \Lambda^k \mathbf{R}^n) = \Big\{ \varphi \in \mathcal{D}(\mathbf{R}^n; \Lambda^k \mathbf{R}^n) \ : \ d\varphi = 0 \text{ and } \int_{\mathbf{R}^n} \varphi \, dx = 0 \Big\},$$

$$\mathrm{L}^1_\#(\mathbf{R}^n; \Lambda^k \mathbf{R}^n) = \Big\{ \varphi \in \mathrm{L}^1(\mathbf{R}^n; \Lambda^k \mathbf{R}^n) \ : \ d\varphi = 0 \text{ and } \int_{\mathbf{R}^n} \varphi \, dx = 0 \Big\}.$$

Remark 2.2. The restriction $1 \le k \le n$, is justified by the fact that

$$\mathrm{L}^1_\#(\mathbf{R}^n; \Lambda^k \mathbf{R}^n) = \mathcal{D}_\#(\mathbf{R}^n; \Lambda^k \mathbf{R}^n) = \{0\}$$

when $k = 0$ or $k > n$.

Remark 2.3. If $1 \le k \le n-1$, $\varphi \in \mathrm{L}^1(\mathbf{R}^n; \Lambda^k \mathbf{R}^n)$ and $d\varphi = 0$, then $\int_{\mathbf{R}^n} \varphi = 0$, while for every $\varphi \in \mathrm{L}^1(\mathbf{R}^n; \Lambda^n \mathbf{R}^n)$, $d\varphi = 0$. Therefore, for a given $1 \le k \le 1$, only one condition in the definition is essential.

Definition 2.4. For $1 \le k \le n$, and $u \in \mathcal{D}'(\mathbf{R}^n)$, let

$$\|u\|_{\mathrm{D}_k(\mathbf{R}^n)} = \sup_{\substack{\varphi \in \mathcal{D}_\#(\mathbf{R}^n; \Lambda^k \mathbf{R}^n) \\ \|\varphi\|_{\mathrm{L}^1(\mathbf{R}^n)} \le 1}} \left| \int_{\mathbf{R}^n} u\varphi \, dx \right|,$$

and define

$$\mathrm{D}_k(\mathbf{R}^n) = \left\{ u \in \mathcal{D}'(\mathbf{R}^n) \ : \ \|u\|_{\mathrm{D}_k(\mathbf{R}^n)} < \infty \right\}.$$

The integral appearing in the definition of $\|\cdot\|_{\mathrm{D}_k(\mathbf{R}^n)}$ should be understood as a duality product between $\mathcal{D}'(\mathbf{R}^n)$ and $\mathcal{D}(\mathbf{R}^n; \Lambda^k \mathbf{R})$, which takes its values in $\Lambda^k \mathbf{R}^n$, while $|\cdot|$ is the standard Euclidean norm of this integral. The set $\mathrm{D}_k(\mathbf{R}^n)$ is a vector space; the function $\|\cdot\|_{\mathrm{D}_k(\mathbf{R}^n)}$ is a seminorm on $\mathrm{D}_k(\mathbf{R}^n)$, and vanishes for constant distributions.

Remark 2.5. In fact, by Theorem 5.3, if $\|u\|_{\mathrm{D}_k(\mathbf{R}^n)} = 0$, then $\|u\|_{\mathrm{BMO}(\mathbf{R}^n)} = 0$, whence u is constant.

The seminorm $\|\cdot\|_{\mathrm{D}_k(\mathbf{R}^n)}$ can also be computed by considering exterior differentials of compactly supported smooth forms in place of closed compactly supported smooth forms.

Proposition 2.6. *Let $1 \le k \le n$. For every $u \in \mathcal{D}'(\mathbf{R}^n)$,*

$$\|u\|_{\mathrm{D}_k(\mathbf{R}^n)} = \sup_{\substack{\psi \in \mathcal{D}(\mathbf{R}^n;\Lambda^{k-1}\mathbf{R}^n)\\ \|d\psi\|_{\mathrm{L}^1(\mathbf{R}^n)} \le 1}} \left| \int_{\mathbf{R}^n} u \, d\psi \, dx \right|.$$

Proof. This follows from Theorems A.5 and A.8. □

Theorems A.5 and A.8 also allow to extend u by density to a linear operator from $\mathrm{L}^1_{\#}(\mathbf{R}^n, \Lambda^k\mathbf{R}^n)$ to $\Lambda^k\mathbf{R}^n$.

Proposition 2.7. *If $u \in \mathrm{D}_k(\mathbf{R}^n)$, then $\langle u, \varphi\rangle \in \Lambda^k\mathbf{R}^n$ is well-defined for every $\varphi \in \mathrm{L}^1_{\#}(\mathbf{R}^n, \Lambda^k\mathbf{R}^n)$.*

We shall also consider the subspace generated by continuous functions.

Definition 2.8. The space $\mathrm{V}_k(\mathbf{R}^n)$ is the closure of the set of bounded continuous functions in $\mathrm{D}_k(\mathbf{R}^n)$.

2.3. **Characterization of $\mathrm{D}_n(\mathbf{R}^n)$.** The space $\mathrm{D}_n(\mathbf{R}^n)$ is well-known; it is $\mathrm{L}^\infty(\mathbf{R}^n)/\mathbf{R}$.

Proposition 2.9. *The spaces $\mathrm{D}_n(\mathbf{R}^n)$ and $\mathrm{L}^\infty(\mathbf{R}^n)/\mathbf{R}$ are isometrically isomorphic.*

Proof. If $u \in \mathrm{L}^\infty(\mathbf{R}^n)$, then for every $\varphi \in \mathcal{D}_{\#}(\mathbf{R}^n; \Lambda^n\mathbf{R}^n)$ and $\lambda \in \mathbf{R}$,

$$\left| \int_{\mathbf{R}} u\varphi \, dx \right| = \left| \int_{\mathbf{R}^n} (u - \lambda)\varphi \, dx \right| \le \|u - \lambda\|_{\mathrm{L}^\infty(\mathbf{R}^n)} \|\varphi\|_{L^1(\mathbf{R}^n)}.$$

Conversely, if $u \in \mathrm{D}_n(\mathbf{R}^n)$, then

$$\varphi \mapsto \ell(\varphi) = \int_{\mathbf{R}^n} u\varphi \, dx,$$

is a linear continuous mapping from $\mathrm{L}^1_{\#}(\mathbf{R}^n, \Lambda^n\mathbf{R}^n)$ to $\Lambda^n\mathbf{R}^n \cong \mathbf{R}$. By the Hahn–Banach Theorem there is an extension $\bar{\ell}$ to $\mathrm{L}^1(\mathbf{R}^n, \Lambda^n\mathbf{R}^n)$. This extension is represented as $\bar{\ell}(\varphi) = \int_{\mathbf{R}^n} \bar{u}\varphi \, dx$ with $\|\bar{u}\|_{\mathrm{L}^\infty(\mathbf{R}^n)} \le \|u\|_{\mathrm{D}_n(\mathbf{R}^n)}$. Since $\int_{\mathbf{R}^n} \bar{u}\varphi \, dx = \int_{\mathbf{R}^n} u\varphi \, dx$ for every $\varphi \in \mathcal{D}_{\#}(\mathbf{R}^n, \Lambda^n\mathbf{R}^n)$, there is $\lambda \in \mathbf{R}$ such that $u - \lambda = \bar{u}$. □

2.4. **Geometric characterization of $\mathrm{V}_1(\mathbf{R}^n)$.** The definition of $\mathrm{D}_k(\mathbf{R}^n)$, and hence that of $\mathrm{V}_k(\mathbf{R}^n)$, rely on compactly supported closed smooth forms (Definition 2.4), or equivalently on compactly supported exact forms (Proposition 2.6). Compactly supported smooth forms ensure that the definition makes sense for distributions. There is a more geometrical characterization for continuous functions, which extends by density to $\mathrm{V}_1(\mathbf{R}^n)$ and $\mathrm{V}_{n-1}(\mathbf{R}^n)$.

Proposition 2.10. *For every $u \in C(\mathbf{R}^n)$,*

$$\|u\|_{\mathrm{D}_1(\mathbf{R}^n)} = \sup_{\Omega} \frac{1}{\mathcal{H}^{n-1}(\partial\Omega)} \left| \int_{\partial\Omega} u(y)\nu(y) \, d\mathcal{H}^{n-1}(y) \right|, \tag{2.1}$$

where the supremum is taken over bounded domains Ω with a smooth connected boundary, and $\nu(y)$ is the unit exterior normal vector to the boundary at $y \in \partial\Omega$.

Proof. Let Ω be a bounded domains with a smooth connected boundary. Let $\rho \in \mathcal{D}(B(0,1))$ be such that $\rho \geq 0$ and $\int_{\mathbf{R}^n} \rho\, dx = 1$, let $\rho_\varepsilon(x) = \rho(x/\varepsilon)/\varepsilon^n$, and define

$$\varphi_\varepsilon(x) = \frac{1}{\mathcal{H}^{n-1}(\partial\Omega)} \int_{\partial\Omega} \rho_\varepsilon(y-x)\nu(y)\, d\mathcal{H}^{n-1}(y).$$

Since $\|\varphi_\varepsilon\|_1 \leq 1$ and $d\varphi_\varepsilon = 0$,

$$\left| \int_{\mathbf{R}^n} u\varphi_\varepsilon\, dx \right| \leq \|u\|_{\mathrm{D}_1(\mathbf{R}^n)}.$$

Since u is continuous, $\rho_\varepsilon * u \to u$ as $\varepsilon \to 0$ uniformly on every compact subset of $\mathbf{R}^n$, and

$$\begin{aligned} \left| \int_{\mathbf{R}^n} u\varphi_\varepsilon\, dx \right| &= \frac{1}{\mathcal{H}^{n-1}(\partial\Omega)} \left| \int_{\partial\Omega} (\rho_\varepsilon * u)(y)\nu(y)\, d\mathcal{H}^{n-1}(y) \right| \\ &\to \frac{1}{\mathcal{H}^{n-1}(\partial\Omega)} \left| \int_{\partial\Omega} u(y)\nu(y)\, d\mathcal{H}^{n-1}(y) \right|, \end{aligned}$$

as $\varepsilon \to 0$. Therefore

$$\frac{1}{\mathcal{H}^{n-1}(\partial\Omega)} \left| \int_{\partial\Omega} u(y)\nu(y)\, d\mathcal{H}^{n-1}(y) \right| \leq \|u\|_{\mathrm{D}_1(\mathbf{R}^n)}.$$

Conversely, let A denote the right-hand side of (2.1). First note that for every bounded open set Ω with a smooth boundary that is not necessarily connected,

$$\left| \int_{\partial\Omega} u(y)\nu(y)\, d\mathcal{H}^{n-1}(y) \right| \leq A\mathcal{H}^{n-1}(\partial\Omega).$$

By Proposition 2.6, we need to evaluate, for $\varphi \in \mathcal{D}(\mathbf{R}^n, \Lambda^0\mathbf{R})$,

$$\left| \int_{\mathbf{R}^n} u\nabla\varphi\, dx \right| = \left| \int_{\mathbf{R}^n} \varphi\nabla u\, dx \right|.$$

One has

$$\int_{\mathbf{R}^n} \varphi\nabla u\, dx = \int_0^\infty \int_{\{x\in\mathbf{R}^n \,:\, \varphi(x)>s\}} \nabla u(x)\, dx\, ds - \int_{-\infty}^0 \int_{\{x\in\mathbf{R}^n \,:\, \varphi(x)<s\}} \nabla u(x)\, dx\, ds.$$

For every $s > 0$, the set $\{x \in \mathbf{R}^n \,:\, \varphi(x) > s\}$ is open and bounded. Moreover, by Sard's Lemma, for almost every $s > 0$, for every $y \in \varphi^{-1}(\{s\})$, $\nabla\varphi(y) \neq 0$. Hence $\partial\{x \in \mathbf{R}^n \,:\, \varphi > s\}$ is smooth and

$$\begin{aligned} \left| \int_{\{x\in\mathbf{R}^n \,:\, \varphi>s\}} \nabla u(x)\, dx \right| &= \left| \int_{\partial\{x\in\mathbf{R}^n \,:\, \varphi(x)>s\}} u(y)\nu(y)\, d\mathcal{H}^{n-1}(y) \right| \\ &\leq A\mathcal{H}^{n-1}(\partial\{x \in \mathbf{R}^n \,:\, \varphi(x) > s\}). \end{aligned}$$

A similar reasoning for $s < 0$, and the integration with respect to s allow to conclude with the co-areaa formula [9, 10]:

$$\begin{aligned}\left|\int_{\mathbf{R}^n} u \nabla \varphi \, dx\right| \\ \leq A \int_{\mathbf{R}^+} \mathcal{H}^{n-1}(\partial \{x \in \mathbf{R}^n \;:\; \varphi(x) > s\}) + \mathcal{H}^{n-1}(\partial \{x \in \mathbf{R}^n \;:\; \varphi(x) < -s\}) \, ds \\ = A \int_{\mathbf{R}^n} |\nabla \varphi| \, dx.\end{aligned}$$

□

2.5. Geometric characterization of $\mathrm{V}_{n-1}(\mathbf{R}^n)$.

Proposition 2.11. *If $n \geq 2$, for every $u \in C(\mathbf{R}^n)$,*

$$\|u\|_{\mathrm{D}_{n-1}(\mathbf{R}^n)} = \sup_{\gamma \in C^1(S^1;\mathbf{R}^n)} \frac{1}{\|\dot\gamma\|_{\mathrm{L}^1(S^1)}} \left| \int_{S^1} u(\gamma(t))\dot\gamma(t)\, dt \right|.$$

The proof repeats the argument of Bourgain and Brezis for the equivalence between the inequality (1.7) and

$$\int_{S^1} u(\gamma(t))\dot\gamma(t)\, dt \leq C_{s,p} \|\dot\gamma\|_{\mathrm{L}^1(S^1)} \|u\|_{\mathrm{W}^{s,p}(\mathbf{R}^n)},$$

for every $u \in \mathrm{W}^{s,p}(\mathbf{R}^n)$ with $sp = n$ [3].

Proof. First note that

$$\|u\|_{\mathrm{D}_{n-1}(\mathbf{R}^n)} = \sup_{\substack{f \in \mathcal{D}(\mathbf{R}^n;\mathbf{R}^n) \\ \operatorname{div} f = 0 \\ \|f\|_{\mathrm{L}^1(\mathbf{R}^n;\mathbf{R}^n)} \leq 1}} \left| \int_{\mathbf{R}^n} u f \, dx \right|.$$

Let $\rho \in \mathcal{D}(B(0,1))$ be such that $\rho \geq 0$ and $\int_{\mathbf{R}^n} \rho \, dx = 1$, and let $\rho_\varepsilon(x) = \rho(x/\varepsilon)/\varepsilon^n$. Define

$$f_\varepsilon(x) = \frac{1}{\|\dot\gamma\|_{\mathrm{L}^1(S^1)}} \int_{S^1} \rho(\gamma(t) - x)\dot\gamma(t)\, dt.$$

One has $\|f_\varepsilon\|_{\mathrm{L}^1(\mathbf{R}^n)} \leq 1$ and $\operatorname{div} f_\varepsilon = 0$, therefore

$$\left| \int_{\mathbf{R}^n} u f_\varepsilon \, dx \right| \leq \|u\|_{\mathrm{D}_{n-1}(\mathbf{R}^n)}.$$

The proof continues as for the first part of Proposition 2.10.

The converse inequality comes from on a result of Smirnov [18], which states that for every $R > 0$ and for every $f \in \mathcal{D}(B(0,R);\mathbf{R}^n)$ there exists $(\gamma^\ell_m)_{1 \leq m,\ell}$ in $C^1(S^1; B(0,R))$ and $(\lambda^\ell_m)_{1 \leq m,\ell}$ in $\mathbf{R}$ such that for every $m \geq 1$,

$$\sum_{\ell \geq 1} |\lambda^\ell_m| \, \|\dot\gamma^\ell_m\|_{\mathrm{L}^1(S^1)} \leq \|f\|_{\mathrm{L}^1(\mathbf{R}^n)},$$

and for every $u \in C(B(0,R))$

$$\sum_{\ell=1}^\infty \lambda^\ell_m \int_{S^1} u(\gamma^\ell_m(t))\dot\gamma^\ell_m \, dt \to \int_{\mathbf{R}^n} u f \, dx,$$

as $m \to \infty$. □

2.6. **Geometric characterization of $\mathrm{V}_k(\mathbf{R}^n)$.** The characterization of the seminorm $\|\cdot\|_{\mathrm{D}_k(\mathbf{R}^n)}$ of Proposition 2.10, relies essentially on the fact that the seminorm could be evaluated by considering differential of scalar functions, while in Proposition 2.11 it relied on the decomposition result of Smirnov. Those facts do not hold anymore for $1 < k < n-1$, but there is an equivalent geometrical seminorm expressed in terms of real polyhedral chains. Let us first recall some basic facts and notations about currents and polyhedral chains [10, 17].

Definition 2.12. The space of *k–dimensional currents* $\mathcal{D}_k(\mathbf{R}^n)$ is the topological dual of $\mathcal{D}(\mathbf{R}^n; \Lambda^k\mathbf{R}^n)$.

The boundary of a current $T \in \mathcal{D}_k(\mathbf{R}^n)$, is $\partial T \in \mathcal{D}_{k-1}(\mathbf{R}^n)$ defined by

$$\langle \partial T, \varphi\rangle = \langle T, d\varphi\rangle.$$

The *mass* of a current is

$$\mathbf{M}(T) = \sup\Big\{\langle T, \varphi\rangle \ : \ \varphi \in \mathcal{D}(\mathbf{R}^n; \Lambda^k\mathbf{R}^n) \text{ and } \forall x \in \mathbf{R}^n, |\varphi(x)| \le 1\Big\}.$$

The *support* $\operatorname{supp} T$ of a current $T \in \mathcal{D}_k(\mathbf{R}^n)$ is the complement of the largest open set U such that $\langle T, \varphi\rangle = 0$ when $\operatorname{supp}\varphi \subset U$. For every integer k, let

$$S_k = \Big\{\lambda \in \mathbf{R}^k \ : \ \sum_{i=1}^k \lambda_k \le 1 \text{ and } \forall 1 \le i \le k,\ \lambda_i \ge 0\Big\}.$$

For every finite collection $(x_i)_{1\le i\le k}$ in $\mathbf{R}^n$, the current $[\![x_0, \dots, x_k]\!] \in \mathcal{D}_k(\mathbf{R}^n)$ is defined by

$$\langle [\![x_0, \dots, x_k]\!], \varphi\rangle = \int_{S_k} \langle \varphi(x_0 + \sum_{i=1}^k \lambda_i x_i), x_1 - x_0 \wedge \dots \wedge x_k - x_0\rangle \, d\lambda.$$

Definition 2.13. A current $T \in \mathcal{D}_k(\mathbf{R}^n)$ is a real polyhedral chain if there is $(x^i_j)_{1\le i\le m, 1\le j\le k}$ in $\mathbf{R}^n$ and $(\mu_i)_{1\le i\le m}$ in $\mathbf{R}$ such that

$$T = \sum_{i=1}^m \mu_i [\![x^i_0, \dots, x^i_k]\!].$$

The set of k–dimensional real polyhedral chains is denoted by $\mathbf{P}_k(\mathbf{R}^n)$. Every real polyhedral chain has a compact support and a finite mass. Hence, $\langle T, u\rangle$ is well-defined when $u : \mathbf{R}^n \to \Lambda^k\mathbf{R}^n$ is continuous. Moreover, if $u : \mathbf{R}^n \to \mathbf{R}$ is continuous then $\langle T, u\rangle \in (\Lambda^k\mathbf{R}^n)' \cong \Lambda^k\mathbf{R}^n$ is naturally defined.

Definition 2.14. If $u : \mathbf{R}^n \to \mathbf{R}$ is continuous, let

$$\|u\|_{\tilde{\mathrm{V}}_k(\mathbf{R}^n)} = \sup_{\substack{P \in \mathbf{P}_{n-k}(\mathbf{R}^n)\\ \partial P = 0\\ \mathbf{M}(P) \le 1}} \langle P, u\rangle.$$

The seminorm $\|u\|_{\tilde{\mathrm{V}}_k(\mathbf{R}^n)}$ measures the oscillation of the function u though its integral on k–dimensional real polyhedral chains without boundary.

Theorem 2.15. *For every $n \ge 1$ and $1 \le k \le n-1$, there exist $c > 0$ such that for every $u \in C(\mathbf{R}^n)$,*

$$\|u\|_{\tilde{\mathrm{V}}_k(\mathbf{R}^n)} \le \|u\|_{\mathrm{D}_k(\mathbf{R}^n)} \le c\|u\|_{\tilde{\mathrm{V}}_k(\mathbf{R}^n)}.$$

Proof. Given P in $\mathbf{P}_{n-k}(\mathbf{R}^n)$, let
$$\varphi_\varepsilon(x) = \langle P, \rho_\varepsilon(\cdot - x)\rangle,$$
where $\rho_\varepsilon = \rho(\cdot/\varepsilon)/\varepsilon^n$ with $\rho \in \mathcal{D}(\mathbf{R}^n)$, $\rho \geq 0$ and $\int_{\mathbf{R}^n} \rho \, dx = 1$ and where $*$ denotes the Hodge duality between $\Lambda^k \mathbf{R}^n$ and $\Lambda^{n-k}\mathbf{R}^n$. One checks that $d\varphi_\varepsilon = 0$, $\|\varphi_\varepsilon\|_{\mathrm{L}^1(\mathbf{R}^n)} \leq \mathbf{M}(P)$ and
$$\int_{\mathbf{R}^n} u\varphi_\varepsilon \, dx = *\langle P, \rho_\varepsilon * u\rangle.$$
Since $\rho_\varepsilon * u \to u$ uniformly as $\varepsilon \to 0$,
$$\|u\|_{\mathrm{V}_k(\mathbf{R}^n)} \leq \|u\|_{\mathrm{D}_k(\mathbf{R}^n)}.$$

The converse inequality is based on the deformation Theorem for currents [10, 17]. It states that given $T \in \mathcal{D}_k(\mathbf{R}^n)$ with $\mathbf{M}(T) + \mathbf{M}(\partial T) < \infty$, for every $\varepsilon > 0$, there exists $P \in \mathbf{P}_k(\mathbf{R}^n)$, $S \in \mathcal{D}_k(\mathbf{R}^n)$ and $R \in \mathcal{D}_{k+1}(\mathbf{R}^n)$ such that
$$T - P = \partial R + S,$$
with
$$\begin{aligned} \mathbf{M}(P) &\leq c\mathbf{M}(T), & \mathbf{M}(\partial P) &\leq c\mathbf{M}(\partial T), \\ \mathbf{M}(R) &\leq c\varepsilon\mathbf{M}(T), & \mathbf{M}(S) &\leq c\varepsilon\mathbf{M}(\partial T), \end{aligned}$$
and
$$\begin{aligned} \operatorname{supp} P \cup \operatorname{supp} R \ &\subset \ \bigl\{x \in \mathbf{R}^n \ : \ \operatorname{dist}(x, \operatorname{supp} T) < 2\sqrt{n+k}\bigr\}, \\ \operatorname{supp} \partial P \cup \operatorname{supp} S \ &\subset \ \bigl\{x \in \mathbf{R}^n \ : \ \operatorname{dist}(x, \operatorname{supp} \partial T) < 2\sqrt{n+k}\bigr\}. \end{aligned}$$
This implies that if $T \in \mathbf{P}_k(\mathbf{R}^n)$ and $U \subset \mathbf{R}^n$ is open and bounded, and $\operatorname{supp} T \subset U$ and $\partial T = 0$, there exists a sequence $(P_\varepsilon)_{\varepsilon > 0}$ in $\mathbf{P}_k(\mathbf{R}^n)$ with $\partial P_\varepsilon = 0$ and $\operatorname{supp} P_\varepsilon \subset U$ such that for every $u \in C(\mathbf{R}^n; \mathbf{R}^n)$,
$$\mathbf{M}(P_\varepsilon) \leq c\mathbf{M}(T), \tag{2.2}$$
and
$$\langle P_\varepsilon, u\rangle \to \langle T, u\rangle \tag{2.3}$$
as $\varepsilon \to 0$.

Now given $\varphi \in \mathcal{D}_\#(\mathbf{R}^n; \Lambda^k \mathbf{R}^n)$, consider $T \in \mathcal{D}_{n-k}(\mathbf{R}^n)$ defined by
$$\langle T, v\rangle = \int_{\mathbf{R}^n} \varphi \wedge v \, dx.$$
Since T has compact support, $\partial T = 0$ and $\mathbf{M}(T) \leq \|\varphi\|_{\mathrm{L}^1(\mathbf{R}^n)}$, there is a sequence $(P_\varepsilon)_{\varepsilon>0}$ in $\mathbf{P}_{n-k}(\mathbf{R}^n)$ such that $\partial P_\varepsilon = 0$, $\operatorname{supp} P_\varepsilon \subset U$, (2.2) and (2.3), where U is a fixed open bounded set such that $\operatorname{supp} T \subset U$. □

3. Basic properties of $\mathrm{D}_k(\mathbf{R}^n)$

3.1. **Mutual injections.** The collection of spaces $\mathrm{D}_k(\mathbf{R}^n)$ is a decreasing sequence of spaces.

Theorem 3.1. *Let $k \leq \ell$. If $u \in \mathrm{D}_\ell(\mathbf{R}^n)$, then $u \in \mathrm{D}_k(\mathbf{R}^n)$, and*
$$\|u\|_{\mathrm{D}_k(\mathbf{R}^n)} \leq C\|u\|_{\mathrm{D}_\ell(\mathbf{R}^n)},$$
where C does not depend on u.

Proof. Let $\varphi \in \mathcal{D}_\#(\mathbf{R}^n; \Lambda^k \mathbf{R}^n)$. If $\alpha \in \Lambda^{\ell-k}\mathbf{R}^n$, then $\alpha \wedge \varphi \in \mathcal{D}_\#(\mathbf{R}^n; \Lambda^\ell \mathbf{R}^n)$. Therefore

$$\left| \int_{\mathbf{R}^n} u\, \alpha \wedge \varphi \, dx \right| \le \|u\|_{\mathrm{D}_\ell(\mathbf{R}^n)} \|\alpha \wedge \varphi\|_{\mathrm{L}^1(\mathbf{R}^n)} \le \|u\|_{\mathrm{D}_\ell(\mathbf{R}^n)} \, |\alpha| \, \|\varphi\|_{\mathrm{L}^1(\mathbf{R}^n)}.$$

Taking the supremum over $\alpha \in \Lambda^{\ell-k}\mathbf{R}^n$ with $|\alpha| \le 1$ leads to the conclusion. □

3.2. **Extension theory.** If $n < N$, functions in $\mathrm{D}_k(\mathbf{R}^n)$ can be extended to functions in $\mathrm{D}_k(\mathbf{R}^N)$. This extension operator is an isomorphism on its image.

Theorem 3.2. *Let $u \in \mathcal{D}'(\mathbf{R}^n)$. Define $U \in \mathcal{D}'(\mathbf{R}^N)$ by*

$$\int_{\mathbf{R}^N} U(z)\varphi(z)\, dz = \int_{\mathbf{R}^n} u(x) \int_{\mathbf{R}^{N-n}} \varphi(x,y)\, dy\, dx.$$

For every $1 \le k \le n$,

$$c\|U\|_{\mathrm{D}_k(\mathbf{R}^N)} \le \|u\|_{\mathrm{D}_k(\mathbf{R}^n)} \le C\|U\|_{\mathrm{D}_k(\mathbf{R}^N)},$$

where $c, C > 0$ are independent of u and U.

Proof. By induction, it is sufficient to consider the case $N = n+1$.

First let us estimate $\|U\|_{\mathrm{D}_k(\mathbf{R}^N)}$. Consider $\Phi \in \mathcal{D}_\#(\mathbf{R}^N; \Lambda^k \mathbf{R}^N)$. It can be written as

$$\Phi = \Phi_0 + \Phi_1 \wedge \omega_N,$$

where $\Phi_0 \in \mathcal{D}_\#(\mathbf{R}^N; \Lambda^k \mathbf{R}^n)$ and $\Phi_1 \in \mathcal{D}_\#(\mathbf{R}^N; \Lambda^{k-1} \mathbf{R}^n)$. Define

$$\varphi(x) = \int_{\mathbf{R}} \Phi(x,t)\, dt, \quad \varphi_0(x) = \int_{\mathbf{R}} \Phi_0(x,t)\, dt, \quad \varphi_1(x) = \int_{\mathbf{R}} \Phi_1(x,t)\, dt.$$

For $m = 1, 2$,

$$d\varphi_m(x) = \sum_{i=1}^n \omega_i \wedge \frac{\partial \varphi_m}{\partial x_i} = \int_{\mathbf{R}} \sum_{i=1}^N \omega_i \wedge \frac{\partial \Phi_m}{\partial x_i}\, dt = 0,$$

and

$$\int_{\mathbf{R}^n} \varphi_m \, dx = \int_{\mathbf{R}^n} \int_{\mathbf{R}} \Phi_m \, dt\, dx = 0.$$

Since $\varphi = \varphi_0 + \varphi_1 \wedge \omega_N$, one has

$$\int_{\mathbf{R}^N} U\Phi\, dz = \int_{\mathbf{R}^n} u\varphi\, dx = \int_{\mathbf{R}^n} u\varphi_0\, dx + \int_{\mathbf{R}^n} u\varphi_1\, dx \wedge \omega_N.$$

and therefore,

$$\left| \int_{\mathbf{R}^N} U\Phi\, dz \right| \le \|u\|_{\mathrm{D}_k(\mathbf{R}^n)} \|\varphi_0\|_{\mathrm{L}^1(\mathbf{R}^n)} + \|u\|_{\mathrm{D}_{k-1}(\mathbf{R}^n)} \|\varphi_1\|_{\mathrm{L}^1(\mathbf{R}^n)} \, |\omega_N| \, .$$

When $k > 1$, the conclusion comes from Theorem 3.1 and from the inequality

$$\begin{aligned} \|\varphi_0\|_{\mathrm{L}^1(\mathbf{R}^n)} + \|\varphi_1\|_{\mathrm{L}^1(\mathbf{R}^n)} &\le \|\Phi_0\|_{\mathrm{L}^1(\mathbf{R}^n)} + \|\Phi_1 \wedge \omega_N\|_{\mathrm{L}^1(\mathbf{R}^n)} \\ &\le \|\Phi_0 + \Phi_1 \wedge \omega_N\|_{\mathrm{L}^1(\mathbf{R}^n)}, \end{aligned}$$

The last inequality comes from the fact that $\Phi_0(x)$ and $\Phi_1(x) \wedge \omega_N$ are orthogonal for every $x \in \mathbf{R}^N$. When $k = 1$, one has $\varphi_1 = 0$, and the conclusion comes similarly.

Conversely, let us now estimate $\|u\|_{\mathrm{D}_k(\mathbf{R}^n)}$ by Proposition 2.6. Let $\psi \in \mathcal{D}(\mathbf{R}^n; \Lambda^{k-1}\mathbf{R}^n)$. Consider a family $(\eta_\lambda)_{\lambda>0}$ in $\mathcal{D}(\mathbf{R})$ such that $\eta_\lambda \geq 0$, $\int_{\mathbf{R}} \eta_\lambda \, dt = 1$ and $\int_{\mathbf{R}} |\eta_\lambda'| \, dt \to 0$ as $\lambda \to \infty$, and let $\Psi_\lambda(x,t) = \eta_\lambda(t)\psi(x)$. For every $\lambda > 0$,

$$\int_{\mathbf{R}^N} U d\Psi_\lambda \, dz = \int_{\mathbf{R}^N} U(d\eta_\lambda \wedge \psi + \eta_\lambda d\psi) \, dz = 0 + \int_{\mathbf{R}^n} u d\psi \, dx.$$

Therefore,

$$\begin{aligned}\Bigl|\int_{\mathbf{R}^n} u \, d\psi \, dx\Bigr| &= \Bigl|\int_{\mathbf{R}^N} U d\Psi_\lambda \, dz\Bigr| \\ &\leq \|U\|_{\mathrm{D}_k(\mathbf{R}^N)} \bigl(\|\psi\|_{\mathrm{L}^1(\mathbf{R}^n)} + \|d\psi\|_{\mathrm{L}^1(\mathbf{R}^n)} \|\eta_\lambda'\|_{\mathrm{L}^1(\mathbf{R})}\bigr).\end{aligned}$$

Letting $\lambda \to \infty$ yields the conclusion. □

Remark 3.3. When $k < n$, the result of Theorem 3.2 is optimal: There are functions in $\mathrm{D}_k(\mathbf{R}^n)$ whose extension to $\mathbf{R}^N$ does not belong to $\mathrm{D}_K(\mathbf{R}^N)$ for $K > k$ (see Proposition 4.6). On the other hand, the extension of a function in $\mathrm{D}_n(\mathbf{R}^n)$ lies in $\mathrm{D}_N(\mathbf{R}^N)$ by Proposition 2.9. In view of the trace theory of the next section, one could wonder whether when $1 \leq k < n$, there is *another* extension in $\mathrm{D}_K(\mathbf{R}^N)$ with $K > k$.

3.3. **Trace theory.** The restriction of continuous functions from $\mathbf{R}^N$ to $\mathbf{R}^n$ can be extended to a continuous operator from $\mathrm{V}_K(\mathbf{R}^n)$ to $\mathrm{V}_k(\mathbf{R}^n)$ when $N - K = n - k$.

Theorem 3.4. *Let $n \leq N$, $1 \leq K \leq N$ and $k = K - (N - n)$. Let $U \in \mathrm{V}_k(\mathbf{R}^N)$ be continuous. Define for $x \in \mathbf{R}^n$,*

$$u(x) = U(x, 0).$$

Then $u \in \mathrm{V}_k(\mathbf{R}^n)$, and

$$\|u\|_{\mathrm{D}_k(\mathbf{R}^n)} \leq \|U\|_{\mathrm{D}_K(\mathbf{R}^N)}.$$

Proof. By induction, we can assume $N = n + 1$. Let $\varphi \in \mathcal{D}_\#(\mathbf{R}^n; \Lambda^k\mathbf{R}^n)$, let $\rho \in \mathcal{D}(\mathbf{R})$ such that $\rho \geq 0$ and $\int_{\mathbf{R}} \rho \, dt = 1$ and let $\rho_\varepsilon(t) = \rho(t/\varepsilon)/\varepsilon$. Let $\Phi_\varepsilon(x,t) = \rho_\varepsilon(t)\psi(x) \wedge \omega_N$. Since $\Phi_\varepsilon \in \mathcal{D}_\#(\mathbf{R}^N; \Lambda^K\mathbf{R}^N)$ and u is continuous,

$$\begin{aligned}\Bigl|\int_{\mathbf{R}^n} u\varphi \, dx\Bigr| &= \lim_{\varepsilon \to 0} \Bigl|\int_{\mathbf{R}^n} \int_{\mathbf{R}} u\Phi_\varepsilon \, dt \, dx\Bigr| \\ &\leq \|U\|_{\mathrm{D}_K(\mathbf{R}^N)} \lim_{\varepsilon \to 0} \|\Phi_\varepsilon\|_{\mathrm{L}^1(\mathbf{R}^N)} \leq \|U\|_{\mathrm{D}_K(\mathbf{R}^N)} \|\varphi\|_{\mathrm{L}^1(\mathbf{R}^n)}.\end{aligned}$$

□

4. Examples of functions in $\mathrm{D}_k(\mathbf{R}^n)$

4.1. **Sobolev spaces.** The first class of functions in the space $\mathrm{D}_k(\mathbf{R}^n)$ are functions in critical Sobolev spaces, which motivated the definition.

Theorem 4.1 (Bourgain and Brezis [3])**.** *If $u \in \mathrm{W}^{s,p}(\mathbf{R}^n)$, $p > 1$ and $sp = n$, then for every $1 \leq k \leq n-1$, $u \in \mathrm{D}_k(\mathbf{R}^n)$, and*

$$\|u\|_{\mathrm{D}_k(\mathbf{R}^n)} \leq C_{k,s,p} \|u\|_{\mathrm{W}^{s,p}(\mathbf{R}^n)}$$

The seminorm on the right is the Sobolev semi-norm. For $0 < s < 1$ it is defined as

$$\|u\|^p_{\mathrm{W}^{s,p}(\mathbf{R}^n)} = \int_{\mathbf{R}^n\times\mathbf{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy.$$

Proof. This follows from the inequality

$$\left|\int_{\mathbf{R}^n} u\varphi\,dx\right| \le C_{s,p}\|u\|_{\mathrm{W}^{s,p}}\|\varphi\|_{\mathrm{L}^1(\mathbf{R}^n)}$$

for every $\varphi \in \mathcal{D}_\#(\mathbf{R}^n;\Lambda^k\mathbf{R}^n)$ and $u \in \mathrm{W}^{s,p}(\mathbf{R}^n)$ [3, 22]. □

Remark 4.2. The Besov spaces $\mathrm{B}^{s,p}_q(\mathbf{R}^n)$ and the Triebel–Lizorkin spaces $\mathrm{F}^{s,p}_q(\mathbf{R}^n)$, with $sp = n$ and $1 < q < \infty$, are embedded in $\mathrm{D}_k(\mathbf{R}^n)$ for $1 \le k \le n$. This follows from the standard embeddings of these spaces in

$$\mathrm{W}^{s',p'}(\mathbf{R}^n) = \mathrm{B}^{s',p'}_{p'}(\mathbf{R}^n) = \mathrm{F}^{s',p'}_{p'}(\mathbf{R}^n),$$

when $0 < s' < \min(s,1)$, $p' \ge q$ and $s'p' = n$ [14, 21].

4.2. **Locally Lipschitz functions in $\mathbf{R}^n \setminus \{0\}$.** The space $\mathrm{D}_{n-1}(\mathbf{R}^n)$ is larger than critical Sobolev spaces. There is a simple condition for locally Lipschitz functions in $\mathbf{R}^n \setminus \{0\}$ to be in $\mathrm{D}_{n-1}(\mathbf{R}^n)$, which is satisfied e.g. by the function $\log|x|$.

Proposition 4.3. *Let $n \ge 2$ and $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbf{R}^n\setminus\{0\})$. If $|x|\,\nabla u \in \mathrm{L}^\infty(\mathbf{R}^n)$, then $u \in \mathrm{D}_{n-1}(\mathbf{R}^n)$ and*

$$\|u\|_{\mathrm{D}_{n-1}(\mathbf{R}^n)} \le \|\,|x|\,\nabla u\|_{\mathrm{L}^\infty(\mathbf{R}^n)}.$$

Remark 4.4. In general, $u \notin \mathrm{D}_n(\mathbf{R}^n)$ as shows the function $u(x) = \log|x|$.

Proof. Let $f \in \mathcal{D}(\mathbf{R}^n\setminus\{0\}\,;\mathbf{R}^n)$ be such that $\operatorname{div} f = 0$. Lemma 4.5 yields

$$\left|\int_{\mathbf{R}^n} uf\,dx\right| = \left|\int_{\mathbf{R}^n} x\;(\nabla u\cdot f)\,dx\right| \le \|\nabla u\,|x|\,\|_{\mathrm{L}^\infty(\mathbf{R}^n)}\|f\|_{\mathrm{L}^1(\mathbf{R}^n)}.$$

Therefore, for every $\varphi \in \mathcal{D}_\#(\mathbf{R}^n\setminus\{0\}\,;\mathbf{R}^n)$,

$$\left|\int_{\mathbf{R}^n} u\varphi\,dx\right| \le \|\nabla u\,|x|\,\|_{\mathrm{L}^\infty(\mathbf{R}^n)}\|\varphi\|_{\mathrm{L}^1(\mathbf{R}^n)}.$$

Since $\{0\}$ has vanishing n–capacity (Lemma A.2), $\mathcal{D}_\#(\mathbf{R}^n\setminus\{0\}\,;\Lambda^{n-1}\mathbf{R})$ is dense in $\mathcal{D}_\#(\mathbf{R}^n;\Lambda^{n-1}\mathbf{R})$ (Theorem A.5). This concludes the proof. □

Lemma 4.5. *Let $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbf{R}^n\setminus\{0\})$ and $f \in \mathcal{D}(\mathbf{R}^n\setminus\{0\}\,;\mathbf{R}^n)$. If $\operatorname{div} f = 0$, then*

$$\int_{\mathbf{R}^n} uf\,dx = -\int_{\mathbf{R}^n} x\;(f\cdot\nabla u)\,dx.$$

Proof. By integration by parts,

$$\int_{\mathbf{R}^n} x\;(f\cdot\nabla u)\,dx = -\int_{\mathbf{R}^n} x\;(\operatorname{div} f)u\,dx - \int_{\mathbf{R}^n} uf\,dx.$$

The conclusion comes from the assumption $\operatorname{div} f = 0$. □

4.3. **Examples of functions in $\mathrm{D}_k(\mathbf{R}^n)$.** Proposition 4.3 and Theorem 3.2 yield examples of functions in the spaces $\mathrm{D}_k(\mathbf{R}^n)$ showing that these spaces are distinct.

Proposition 4.6. *If $1 \le \ell \le n$, then*

$$\log\Bigl(\sum_{i=1}^{\ell} |x_i|^2\Bigr) \in \mathrm{D}_k(\mathbf{R}^n)$$

if and only if $1 \le k < \ell$.

Remark 4.7. An immediate consequence of Proposition 4.6 is that the spaces $\mathrm{D}_k(\mathbf{R}^n)$ are all different.

Remark 4.8. Proposition 4.6 is consistent with the isomorphism $\mathrm{D}_n(\mathbf{R}^n) \cong \mathrm{L}^\infty(\mathbf{R}^n)/\mathbf{R}$ (Proposition 2.9).

Proof of Proposition 4.6. For every $x \in \mathbf{R}^n \setminus \{0\}$, $|x|\,|\nabla(\log|x|)| = 1$. Therefore, for $\ell \ge 2$, by Proposition 4.3,

$$\log\Bigl(\sum_{i=1}^{\ell} |x_i|^2\Bigr) \in \mathrm{D}_{\ell-1}(\mathbf{R}^\ell).$$

By Theorem 3.2, this remains valid for the extension to $\mathbf{R}^n$

$$\log\Bigl(\sum_{i=1}^{\ell} |x_i|^2\Bigr) \in \mathrm{D}_{\ell-1}(\mathbf{R}^n).$$

Hence, by Theorem 3.1, if $1 \le k < \ell$,

$$\log\Bigl(\sum_{i=1}^{\ell} |x_i|^2\Bigr) \in \mathrm{D}_k(\mathbf{R}^n).$$

On the other hand, suppose for contradiction that, for some $k \ge \ell$,

$$\log\Bigl(\sum_{i=1}^{\ell} |x_i|^2\Bigr) \in \mathrm{D}_k(\mathbf{R}^n).$$

By Theorem 3.1, this would be true for $k = \ell$. By Theorem 3.2,

$$\log\Bigl(\sum_{i=1}^{\ell} |x_i|^2\Bigr) \in \mathrm{D}_\ell(\mathbf{R}^\ell) = \mathrm{L}^\infty(\mathbf{R}^\ell)/\mathbf{R},$$

which is absurd. □

Proposition 4.6 includes as a special case estimates obtained by Bourgain and Brezis:

Corollary 4.9 (Bourgain and Brezis [3])**.** *Let $f \in \mathrm{L}^1(\mathbf{R}^2;\mathbf{R}^2)$. If $\operatorname{div} f = 0$ in the sense of distributions, then*

$$\log\frac{1}{|x|} * f \in \mathrm{L}^\infty(\mathbf{R}^2;\mathbf{R}^2).$$

Other interesting examples can be obtained in a similar way:

Proposition 4.10. *If* $1 \le \ell \le n$ *and* $0 < \alpha \le 1$, *then*

$$\Big(\big|\log(\sum_{i=1}^{\ell} |x_i|^2)\big| + 1\Big)^{\alpha} \in \mathrm{D}_k(\mathbf{R}^n)$$

if and only if $1 \le k < \ell$.

Note that

$$\Big(\big|\log(\sum_{i=1}^{\ell} |x_i|^2)\big| + 1\Big)^{\alpha} \notin \mathrm{W}^{s,p}_{\mathrm{loc}}(\mathbf{R}^n)$$

when $\ell < n$ (otherwise it would be continuous on almost every hyperplane). Hence the examples provided here do not belong to critical Sobolev spaces.

5. Relation with BMO($\mathbf{R}^n$)

5.1. **Preliminaries.** Let us first recall some facts about the space of functions with bounded mean oscillation BMO($\mathbf{R}^n$) [20].

The space BMO($\mathbf{R}^n$) is defined as the set of functions $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n)$ such that

$$\|u\|_{\mathrm{BMO}(\mathbf{R}^n)} = \sup_B \frac{1}{\mathcal{L}^n(B)^2} \int_B \int_B |u(x) - u(y)| \, dx \, dy < \infty,$$

where the supremum is taken on balls $B \subset \mathbf{R}^n$.

The space BMO($\mathbf{R}^n$) is the dual space of the real Hardy space $\mathrm{H}^1(\mathbf{R}^n)$, which can be characterized as the space of functions $f \in \mathrm{L}^1(\mathbf{R}^n)$ such that $\mathcal{R}_i f \in \mathrm{L}^1(\mathbf{R}^n)$, where $\mathcal{R}_i$ denotes the Riesz transform (defined by $\mathcal{R}_i f = x_i / |x|^{n+1} * f$ when $f \in \mathcal{D}(\mathbf{R}^n)$). One can take

$$\|f\|_{\mathrm{H}^1(\mathbf{R}^n)} = \|f\|_{\mathrm{L}^1(\mathbf{R}^n)} + \sum_{i=1}^{n} \|\mathcal{R}_i f\|_{\mathrm{L}^1(\mathbf{R}^n)}.$$

If K_n denotes the fundamental solution of the Laplacian $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$ (i.e. K_n is Newton's kernel), and $f \in \mathrm{H}^1(\mathbf{R}^n)$, one has:

$$\frac{\partial^2}{\partial x_i \partial x_j}\Big(K_n * f\Big) \in \mathrm{L}^1(\mathbf{R}^n),$$

and

$$\Big\|\frac{\partial^2}{\partial x_i \partial x_j}\Big(K_n * f\Big)\Big\|_{\mathrm{L}^1(\mathbf{R}^n)} \le C\|f\|_{\mathrm{H}^1(\mathbf{R}^n)}.$$

Finally, $\mathcal{D}(\mathbf{R}^n) \cap \mathrm{H}^1(\mathbf{R}^n)$ is dense in $\mathrm{H}^1(\mathbf{R}^n)$.

The space of functions with vanishing mean oscillations VMO($\mathbf{R}^n; \mathbf{R}^n$) is the closed subspace of VMO($\mathbf{R}^n$) that is characterized by

$$\lim_{\varepsilon \to 0} \sup_{\mathcal{L}^n(B) \le \varepsilon} \frac{1}{\mathcal{L}^n(B)^2} \int_B \int_B |u(x) - u(y)| \, dx \, dy = 0,$$

where the supremum is taken over balls $B \subset \mathbf{R}^n$. The critical Sobolev spaces $\mathrm{W}^{s,p}(\mathbf{R}^n)$ with $sp = n$ are embedded in VMO($\mathbf{R}^n$).

Going back to the examples of the preceding section, for every $\ell \ge 1$,

$$\log\Big(\sum_{i=1}^{\ell} |x_i|^2\Big) \in \mathrm{BMO}(\mathbf{R}^n),$$

but it does not belong to $\mathrm{VMO}(\mathbf{R}^n)$, while for $0 < \alpha < 1$ and $\ell \ge 1$,

$$\Bigl(\bigl|\log\bigl(\sum_{i=1}^{\ell} |x_i|^2\bigr)\bigr| + 1\Bigr)^{\alpha} \in \mathrm{VMO}(\mathbf{R}^n).$$

(It is still in $\mathrm{BMO}(\mathbf{R}^n)$ when $\alpha = 1$.) Comparing with Proposition 4.10, this gives a first insight on the relationship between $\mathrm{D}_k(\mathbf{R}^n)$, $\mathrm{VMO}(\mathbf{R}^n)$ and $\mathrm{BMO}(\mathbf{R}^n)$:

Proposition 5.1. *For every $1 \le k \le n$, the space $\mathrm{VMO}(\mathbf{R}^n)$ is not embedded in $\mathrm{D}_k(\mathbf{R}^n)$.*

Remark 5.2. There is thus no inequality

$$\Bigl|\int_{\mathbf{R}^n} u\,\varphi\,dx\Bigr| \le C\|u\|_{\mathrm{BMO}(\mathbf{R}^n)}\|\varphi\|_{\mathrm{L}^1(\mathbf{R}^n)},$$

for $u \in \mathrm{VMO}(\mathbf{R}^n)$. This was remarked indirectly by Bethuel, Orlandi and Smets [2].

5.2. **Embedding in** BMO. The spaces $\mathrm{D}_k(\mathbf{R}^n)$ do not contain $\mathrm{BMO}(\mathbf{R}^n)$ nor $\mathrm{VMO}(\mathbf{R}^n)$. Since $\mathrm{L}^\infty(\mathbf{R}^n)/\mathbf{R} \subset \mathrm{D}_k(\mathbf{R}^n)$ for $1 \le k \le n$, they are not contained in $\mathrm{VMO}(\mathbf{R}^n)$, but there is an embedding of $\mathrm{D}_k(\mathbf{R}^n)$ in $\mathrm{BMO}(\mathbf{R}^n)$:

Theorem 5.3. *Let $1 \le k \le n$. If $u \in \mathrm{D}_k(\mathbf{R}^n)$, then $u \in \mathrm{BMO}(\mathbf{R}^n)$, and*

$$\|u\|_{\mathrm{BMO}(\mathbf{R}^n)} \le C\|u\|_{\mathrm{D}_k(\mathbf{R}^n)},$$

where C is independent of u.

Proof. By Theorem 3.1, we can assume $k = 1$. The seminorm of u in $\mathrm{BMO}(\mathbf{R}^n)$ will be estimated by duality with the Hardy space $\mathrm{H}^1(\mathbf{R}^n)$.

Let $f \in \mathcal{D}(\mathbf{R}^n) \cap \mathrm{H}^1(\mathbf{R}^n)$. Let

$$\varphi_i = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\Bigl(K_N * f\Bigr)\omega_j.$$

Note $\varphi_j \in \mathrm{L}^1_{\#}(\mathbf{R}^n; \Lambda^1\mathbf{R}^n)$. Moreover

$$\begin{aligned}(5.1)\qquad \Bigl|\int_{\mathbf{R}^n} f u\,dx\Bigr| &\le \sum_{i=1}^{n}\Bigl|\int_{\mathbf{R}^n} \partial^2 f/\partial x_i^2 u\,dx\Bigr| \le \sum_{i=1}^{n}\Bigl|\int_{\mathbf{R}^n} \varphi_i\,u\,dx\Bigr| \\ &\le \|u\|_{\mathrm{D}_1(\mathbf{R}^n)}\sum_{i=1}^{n}\|\varphi_i\|_{\mathrm{L}^1(\mathbf{R}^n)} \le C\|u\|_{\mathrm{D}_1(\mathbf{R}^n)}\|f\|_{\mathrm{H}^1(\mathbf{R}^n)}.\end{aligned}$$

(Note that Proposition 2.7 about the welldefinitenesss of the duality product between $\mathrm{D}_1(\mathbf{R}^n)$ on and $\mathrm{L}^1_{\#}(\mathbf{R}^n; \Lambda^1\mathbf{R}^n)$ was used.) □

Remark 5.4. A similar argument shows that for $1 < p < \infty$

$$\|u\|_{\mathrm{L}^p(\mathbf{R}^n)} \le \sup_{\substack{\varphi \in \mathcal{D}_{\#}(\mathbf{R}^n;\mathbf{R}^n)\\ \|\varphi\|_{\mathrm{L}^p(\mathbf{R}^n)} \le 1}} \Bigl|\int_{\mathbf{R}^n} u\,\varphi\,dx\Bigr|.$$

The extension of the spaces $\mathrm{D}_k(\mathbf{R}^n)$ to the case $1 < p < \infty$ would thus not be interesting.

5.3. **Completeness of $\mathrm{D}_k(\mathbf{R}^n)$.** The injection of $\mathrm{D}_k(\mathbf{R}^n)$ provides an easy proof of the completeness of $\mathrm{D}_k(\mathbf{R}^n)$.

Theorem 5.5. *For $1 \le k \le n$, the space $\mathrm{D}_k(\mathbf{R}^n)$ is a complete Banach space modulo constants.*

Proof. Let $(u_m)_{m\ge 1}$ be a Cauchy sequence in $\mathrm{D}_k(\mathbf{R}^n)$: $\lim_{m,l\to\infty} \|u_m - u_\ell\|_{\mathrm{D}_k(\mathbf{R}^n)} = 0$. By Theorem 5.3, $(u_m)_{m\ge 1}$ is a Cauchy sequence in the Banach space $\mathrm{BMO}(\mathbf{R}^n)$ and has thus a limit $u \in \mathrm{BMO}(\mathbf{R}^n)$. For every $\varphi \in \mathcal{D}_\#(\mathbf{R}^n; \Lambda^k \mathbf{R}^n)$ and $m \ge 1$

$$\begin{aligned}\left|\int_{\mathbf{R}^n} (u_m - u)\, \varphi \, dx\right| &= \lim_{l\to\infty} \left|\int_{\mathbf{R}^n} (u_m - u_l)\, \varphi \, dx\right| \\ &\le \lim_{l\to\infty} \|u_m - u_l\|_{\mathrm{D}_k(\mathbf{R}^n)} \|\varphi\|_{\mathrm{L}^1(\mathbf{R}^n)}.\end{aligned}$$

Taking the supremum over $\varphi \in \mathcal{D}_\#(\mathbf{R}^n; \Lambda^k \mathbf{R}^n)$ with $\|\varphi\|_{\mathrm{L}^1(\mathbf{R}^n)} \le 1$, one obtains $u \in \mathrm{D}_k(\mathbf{R}^n)$ and $u_m \to u$ in $\mathrm{D}_k(\mathbf{R}^n)$ as $m \to \infty$.

Finally, if $u \in \mathrm{D}_k(\mathbf{R}^n)$ then, by Theorem 5.3, $\|u\|_{\mathrm{BMO}(\mathbf{R}^n)} = 0$, and therefore u is constant. □

5.4. **Integrability of functions in $\mathrm{D}_k(\mathbf{R}^n)$.** If $u \in \mathrm{BMO}(\mathbf{R}^n)$, $\|u\|_{\mathrm{BMO}(\mathbf{R}^n)} \le 1$ and $\int_{B(0,1)} u \, dx = 0$, the John-Nirenberg Theorem states that

$$\int_{B(0,1)} \exp(\mu\, |u|)\, dx \le c, \tag{5.2}$$

where $\mu > 0$ and $c > 0$ can bechosenn independently of u. Since the spaces $\mathrm{D}_k(\mathbf{R}^n)$ are embedded in $\mathrm{BMO}(\mathbf{R}^n)$, this might be improved on $\mathrm{D}_k(\mathbf{R}^n)$.

On the other hand if $sp = n$, $0 < s < 1$, $u \in \mathrm{W}^{s,p}(\mathbf{R}^n)$, $\|u\|_{\mathrm{W}^{s,p}(\mathbf{R}^n)} \le 1$ and $\int_{B(0,1)} f \, dx = 0$, then

$$\int_{B(0,1)} \exp\bigl(\mu\, |u|^{p/(p-1)}\bigr)\, dx \le c \tag{5.3}$$

where the constants c and μ are independent of u [12]. The exponent $p/(p-1)$ can not be improved, since $\log(1/\,|x|)_+^\alpha \in \mathrm{W}^{s,p}(\mathbf{R}^n)$ when $\alpha < p/(p-1)$ [14, p. 47]. There is not much room between (5.2) and (5.3), and it is therefore not surprising that (5.2) is optimal also in $\mathrm{D}_k(\mathbf{R}^n)$.

Proposition 5.6. *Suppose that for everycompactlyy supported $u \in C(\mathbf{R}^n)$, if $\|u\|_{\mathrm{D}_k(\mathbf{R}^n)} \le 1$ and $\int_{B(0,1)} u = 0$,*

$$\int_{B(0,1)} F(u)\, dx < \infty,$$

then there exists $\lambda > 0$ such that

$$\int_0^\infty \Phi(s) e^{-\lambda s}\, ds < \infty.$$

Proof. Choose $y \in \mathbf{R}^n$ with $|y| = 1/2$. Consider the functions $(u_m)_{m\ge 1}$ defined by

$$u_m(x) = \min\bigl(m, \bigl(\log(2\,|x-y|)\bigr)_-\bigr) - \min\bigl(m, \bigl(\log(2\,|x+y|)\bigr)_-\bigr).$$

By Proposition 4.3, $\|u_m\|_{\mathrm{D}_k(\mathbf{R}^n)}$ is bounded uniformly in m. Choose $\alpha > 0$ such that $\|\alpha u_m\|_{\mathrm{D}_k(\mathbf{R}^n)} \leq 1$. On the other hand,

$$\int_{B(0,1)} \Phi(\alpha u_m)\,dx = 2^{1-n} \int_{B(0,1)} \Phi\big(\alpha \max\big(\log(1/\,|x|), m\big)\big)\,dx$$
$$= K \int_1^{e^{-m}} \Phi(-\alpha \log r) r^{n-1}\,dr = \frac{K}{\alpha} \int_0^{\alpha m} \Phi(s) e^{-ns/\alpha}\,ds \leq c.$$

Letting $m \to \infty$ yields the conclusion with $\lambda = n/\alpha$. □

The improvement of $\mathrm{D}_k(\mathbf{R}^n)$ on $\mathrm{BMO}(\mathbf{R}^n)$ should thus not be seen as an improvement of the integrand, but as an improvement on the domains of integration: By the trace Theorem 3.4, the embedding Theorem 5.3, and the John–Nirenberg inequality, functions in $\mathrm{V}_k(\mathbf{R}^n)$ areexponentiallyy integrable on $n-k+1$–dimensional subspaces.

6. Further problems

6.1. **Traces of $\mathrm{V}_1(\mathbf{R}^n)$ on $\mathrm{VMO}(\mathbf{R}^{n-1})$.** By Theorem 3.4 and Theorem 5.3, functions in $\mathrm{V}_k(\mathbf{R}^n)$ have VMO traces on $n-k+1$–dimensional spaces. The dimension $n-k$ seems more natural: Functions in $\mathrm{V}_n(\mathbf{R}^n)$ are continuous, and hence have traces on 0–dimensional spaces, i.e. points. If there was such a trace inequality, one could define $\mathrm{D}_0(\mathbf{R}^n) = \mathrm{BMO}(\mathbf{R}^n)$. This notation would be consistent with the mutual injection Theorem 3.1, the extension Theorem 3.2 and the examples of Proposition 4.6. It would then be nice to have a definition of $\mathrm{D}_k(\mathbf{R}^n)$ whichencompassess the case $k = 0$. The two-dimensional case would already solve the problem of traces of $\mathrm{V}_{n-1}(\mathbf{R}^n)$ on lines.

6.2. **Geometric characterizations.** By Propositions 2.10 and 2.11, the spaces $\mathrm{V}_1(\mathbf{R}^n)$ and $\mathrm{V}_{n-1}(\mathbf{R}^n)$ can be defined by oscillations respectively along boundaries of bounded domains and along closed curves. Further refinements would restrict the set of domains and of curves. The most striking result would be if the oscillation could be simply evaluatedrespectivelyy on spheres and on circles.

The spaces $\mathrm{V}_k(\mathbf{R}^n)$ for $1 < k < n-1$ do not have such a simple characterization. Proposition 2.14 gives an equivalent seminorm, obtained by integration on closed real polyhedral chains without boundary. This result needs to be improved by restricting the class of sets on which the oscillations are computed for example to integer polyhedral chains without boundary, or to embedded oriented $n-k$–dimensional manifolds without boundary.

6.3. **Closure of the space of continuous functions.** The two equivalent definitions of $\mathrm{VMO}(\mathbf{R}^n)$ [15] suggest the definition of the closure of the bounded uniformly continuous functions $\mathrm{U}_k(\mathbf{R}^n)$ and of

$$\mathrm{W}_k(\mathbf{R}^n) = \{u \in \mathrm{D}_k(\mathbf{R}^n) \;:\; N_k(u) = 0\}$$

where

$$N_k(u) = \lim_{\varepsilon \to 0} \sup_{\substack{\varphi \in \mathcal{D}_{\#}(B(x,r);\Lambda^k \mathbf{R}^n) \\ \|\varphi\|_{\mathrm{L}^1(\mathbf{R}^n)} \le 1 \\ x \in \mathbf{R}^n \\ r \le \varepsilon}} \left| \int_{B(x,r)} u\varphi \, dx \right|.$$

The space $U_k(\mathbf{R}^n)$ is contained in none of the spaces $\mathrm{V}_k(\mathbf{R}^n)$, $\mathrm{W}_k(\mathbf{R}^n)$ and $\mathrm{VMO}(\mathbf{R}^n)$, while $\mathrm{V}_k(\mathbf{R}^n)$ is not contained in $\mathrm{VMO}(\mathbf{R}^n)$. The remaining problems are thus the understanding of the mutual relationship between $\mathrm{W}_k(\mathbf{R}^n)$ and $\mathrm{VMO}(\mathbf{R}^n) \cap \mathrm{D}_k(\mathbf{R}^n)$ and of their possible embeddings in $\mathrm{V}_k(\mathbf{R}^n)$ and in $\mathrm{U}_k(\mathbf{R}^n)$.

6.4. **Similar spaces $\mathrm{E}_k(\mathbf{R}^n)$.** It would have been possible to define another family of spaces with properties similar to $\mathrm{D}_k(\mathbf{R}^n)$. For $1 \le k \le n$, let

$$\begin{aligned}\mathcal{D}_{\#,k}(\mathbf{R}^n;\mathbf{R}^n) = \bigl\{\varphi \in \mathcal{D}(\mathbf{R}^n;\mathbf{R}^n) \ : \ &\operatorname{div}\varphi = 0 \\ &\text{and the dimension of the range of } \varphi \text{ is at most } k\bigr\}.\end{aligned}$$

(The set $\mathcal{D}_{\#,k}(\mathbf{R}^n;\mathbf{R}^n)$ is not a vector space when $k < n$.) Define, for $1 \le k \le n-1$,

$$\|u\|_{\mathrm{E}_k(\mathbf{R}^n)} = \sup_{\substack{\varphi \in \mathcal{D}_{\#,k+1}(\mathbf{R}^n;\mathbf{R}^n) \\ \|\varphi\|_{\mathrm{L}^1(\mathbf{R}^n)} \le 1}} \left| \int_{\mathbf{R}^n} u\varphi \, dx \right|.$$

For $1 \le k \le n-1$, the space $\mathrm{E}_k(\mathbf{R}^n)$ is continuously embedded in $\mathrm{D}_k(\mathbf{R}^n)$, and $\mathrm{E}_{n-1}(\mathbf{R}^n)$ is isomorphic to $\mathrm{D}_{n-1}(\mathbf{R}^n)$. Therefore, the critical Sobolev spaces are embedded in $\mathrm{E}_k(\mathbf{R}^n)$ which in turn are embedded in $\mathrm{BMO}(\mathbf{R}^n)$. Moreover, the trace property of Theorem 3.4 holds for the closure of continuous function in $\mathrm{E}_k(\mathbf{R}^n)$. Continuous functions in $\mathrm{E}_k(\mathbf{R}^n)$ can be characterized by Proposition 2.11 provided one restricts in the supremum $\gamma(S^1)$ to be contained in a $k+1$-dimensinal affine plane. The question is about where $\mathrm{E}_k(\mathbf{R}^n)$ lies between the spaces $\mathrm{D}_k(\mathbf{R}^n)$ or to $\mathrm{D}_{n-1}(\mathbf{R}^n)$. In particular, is it isomorphic to one of those?

6.5. **Composition of functions in $\mathrm{D}_k(\mathbf{R}^n)$.** Let $F : \mathbf{R} \to \mathbf{R}$ be a Lipchitz function. If $u \in \mathrm{W}^{s,p}(\mathbf{R}^n)$ with $0 < s \le 1$ and $sp = n$, then $F \circ u \in \mathrm{W}^{s,p}(\mathbf{R}^n)$ [24]. If $u \in \mathrm{BMO}(\mathbf{R}^n)$, then $F \circ u \in \mathrm{BMO}(\mathbf{R}^n)$ [20]. Is $F \circ u \in \mathrm{D}_k(\mathbf{R}^n)$ when F is a Lipschitz smooth function? A simpler problem is whether $|u| \in \mathrm{D}_k(\mathbf{R}^n)$ whenever $u \in \mathrm{D}_k(\mathbf{R}^n)$?

If $F : \mathbf{R}^n \to \mathbf{R}^n$ is smooth, the mapping $u \mapsto u \circ F$ is a continuous operator on $\mathrm{W}^{1,n}(\mathbf{R}^n)$ and $\mathrm{BMO}(\mathbf{R}^n)$ if and only if F is quasiconformal [13], i.e.

$$\sup_{x \in \mathbf{R}^n} \frac{|DF(x)|^n}{|\det DF(x)|} < \infty.$$

Is it true that $u \mapsto u \circ F$ is continuous from $\mathrm{D}_k(\mathbf{R}^n)$ to $\mathrm{D}_k(\mathbf{R}^n)$ if F is quasiconformal, or even for F smooth and bilipschitzian?

6.6. **Localization of $\mathrm{D}_k(\mathbf{R}^n)$.** There should be localized versions of the spaces $\mathrm{D}_k(\mathbf{R}^n)$. There are two different definitions, depending on whether the supremum

$$\sup_{\|\varphi\|_{\mathrm{L}^1(\mathbf{R}^n)}\le 1} \left| \int_\Omega u\varphi\,dx \right|$$

is taken over smooth closed forms on Ω or over smooth closed forms on $\mathbf{R}^n$ with support in Ω. They contain respectively $\mathrm{W}^{1,n}_0(\Omega)$ and $\mathrm{W}^{1,n}(\Omega)$ [7]. It would be natural for these spaces to be embedded respectively in $\mathrm{bmo}_z(\bar\Omega)$ (functions whose extension by 0 to $\mathbf{R}^n$ is in $\mathrm{BMO}(\mathbf{R}^n)$) and the second in the larger space $\mathrm{bmo}_r(\bar\Omega)$ (restrictions of functions in $\mathrm{BMO}(\mathbf{R}^n)$ to Ω) (see [8] for the definitions).

Appendix A. Density of compactly supported forms

A.1. **The closure of closed k–forms.** This appendix is devoted to the study of dense sets in the space $\mathrm{L}^1(\mathbf{R}^n;\Lambda^k\mathbf{R}^n)$ of summable closed forms.

Definition A.1. Let $1 \le p < \infty$ and $\Omega \subset \mathbf{R}^n$ be open and bounded. The p–capacity of a compact set $\Sigma \subset \Omega$ is defined as

$$\mathrm{capa}_p(\Sigma,\Omega) = \inf\Bigl\{ \int_\Omega |\nabla\eta|^p \;:\; \eta \in \mathcal{D}(\Omega),\, \eta = 1 \text{ in a neighborhood of } \Sigma \Bigr\}.$$

In general $\mathrm{capa}_p(\Sigma,\Omega)$ depends on Ω, but if $\mathrm{capa}_p(\Sigma,\Omega) = 0$ for some bounded open set Ω, then $\mathrm{capa}_p(\Sigma,\Omega') = 0$ for every bounded open set $\Omega' \supset \Sigma$. Therefore it makes sense to speak of sets with vanishing p–capacity without specifying Ω.

Lemma A.2. *If $n \ge 2$, for every $a \in \mathbf{R}^n$, the set $\{a\}$ has vanishing n–capacity.*

Proof. Consider the sequence

$$\eta_m(x) = \theta\bigl(\log(1/|x-a|)/m\bigr),$$

with $\theta \in C^\infty(\mathbf{R})$ and $\operatorname{supp}\theta' \subset (0,1)$, $\theta(0) = 0$ and $\theta(1) = 1$. □

Remark A.3. More generally, if the 0–dimensional Hausdorff measure of Σ vanishes, then $\mathcal{H}^0(\Sigma) = 0$. Conversely, if $\mathrm{capa}_n(\Sigma) = 0$, then for every $s > 0$, the s–dimensional Hausdorff measure of Σ vanishes [9].

In a similar way, one can prove

Lemma A.4. *There exists a sequence $(\zeta_m)_m$ in $\mathcal{D}(\mathbf{R}^n)$ such that $0 \le \zeta_m \le 1$, $\zeta_m \to 1$ almost everywhere and*

$$\int_{\mathbf{R}^n} |\nabla\zeta_m|^n\,dx \to 0$$

as $m \to \infty$.

Theorem A.5. *If $\Sigma \subset \mathbf{R}^n$ is compact and has vanishing n–capacity, then $d\bigl(\mathcal{D}(\mathbf{R}^n\setminus\Sigma;\Lambda^{k-1}\mathbf{R}^n)\bigr)$ is dense in $\mathrm{L}^1_\#(\mathbf{R}^n;\Lambda^k\mathbf{R}^n)$.*

The proof makes use of a result of Bourgain and Brezis.

Theorem A.6 (Bourgain and Brezis [3])**.** *Let $1 \le k \le n-1$. For every $\varphi \in \mathrm{L}^1_{\#}(\mathbf{R}^n; \Lambda^k \mathbf{R}^n)$, there exists $\psi \in L^{n/(n-1)}(\mathbf{R}^n; \Lambda^{k-1}\mathbf{R}^n)$ such that*

$$\begin{cases} d\psi = \varphi, \\ \delta\psi = 0. \end{cases}$$

Here δ denotes the codifferential, i.e. the adjoint of d with respect to Hodge star. This result is based on inequality (1.7). When $k = 1$, the meaningless condition $\delta\psi = 0$ is dropped and this is equivalent with the Nirenberg–Sobolev embedding.

Proof of Theorem A.5. Since the exterior differential d commutes with translations, by classical smoothing arguments, $(C^\infty \cap \mathrm{L}^1_{\#})(\mathbf{R}^n; \Lambda^k \mathbf{R}^n)$ is dense in $\mathrm{L}^1_{\#}(\mathbf{R}^n; \Lambda^k \mathbf{R}^n)$.

Let $\varphi \in (C^\infty \cap \mathrm{L}^1_{\#})(\mathbf{R}^n; \Lambda^k \mathbf{R}^n)$ and let $\Sigma \subset \Omega \subset \mathbf{R}^n$ be open and bounded. Since Σ has vanishing capacity, there is a sequence $(\eta_m)_{m \ge 1}$ in $\mathcal{D}(\Omega)$ such that $0 \le \eta_m \le 1$, $\eta_m = 1$ on a neighborhood of Σ and $\|\nabla \eta_m\|_{\mathrm{L}^n(\mathbf{R}^n)} \to 0$ as $n \to \infty$. Moreover, by Poincaré's inequality, up to a subsequence, $\eta_m \to 0$ almost everywhere.

Consider now the sequence

$$\psi_m = (1 - \eta_m)\zeta_m \psi,$$

where ζ_m is given by Lemma A.4. By definition, $\psi_m \in \mathcal{D}(\mathbf{R}^n; \Lambda^{k-1}\mathbf{R}^n)$. We claim that $d\psi_m \to \varphi$ in $\mathrm{L}^1(\mathbf{R}^n; \Lambda^k \mathbf{R}^n)$.

In fact,

$$d\psi_m = -\zeta_m \, d\eta_m \wedge \psi + (1 - \eta_m) \, d\zeta_m \wedge \psi + (1 - \eta_m) \, \zeta_m \, \varphi. \tag{A.1}$$

By Hőlder's inequality,

$$\| - \zeta_m \, d\eta_m \wedge \psi\|_{\mathrm{L}^1(\mathbf{R}^n)} \le \|\zeta_m\|_{\mathrm{L}^\infty(\mathbf{R}^n)} \|d\eta_m\|_{\mathrm{L}^n(\mathbf{R}^n)} \|\psi\|_{\mathrm{L}^{n/(n-1)}(\mathbf{R}^n)}.$$

Since $\|\nabla \eta_m\|_{\mathrm{L}^n(\mathbf{R}^n)} \to 0$ and $\|\psi\|_{\mathrm{L}^{n/(n-1)}(\mathbf{R}^n)} < \infty$, the first term in (A.1) tends to zero. A similar reasoning holds for the second term, and the last term converges to φ as $m \to \infty$ by Lebesgue's dominated convergence Theorem. □

Corollary A.7. *The set $\mathcal{D}_{\#}(\mathbf{R}^n; \Lambda^k \mathbf{R}^n)$ is dense in $\mathrm{L}^1_{\#}(\mathbf{R}^n; \Lambda^k \mathbf{R}^n)$.*

A.2. **The closure of exact n–forms.** Theorem A.6 fails when $k = n$, and therefore the proof Theorem A.5 fails in this case, but there is in fact a stronger result.

Theorem A.8. *The set $d\big(\mathcal{D}(\mathbf{R}^n; \Lambda^{n-1}\mathbf{R}^n)\big)$ is dense in $\mathcal{D}_{\#}(\mathbf{R}^n; \Lambda^n \mathbf{R}^n)$.*

Remark A.9. The density is with respect to the usual topology on the space of test functions [16].

Proof. Let $\varphi \in \mathcal{D}_{\#}(\mathbf{R}^n; \Lambda^n \mathbf{R}^n)$. Therefore $\varphi = f \; \omega_1 \wedge \dots \wedge \omega_n$, with $f \in \mathcal{D}(\mathbf{R}^n)$ and $\int_{\mathbf{R}^n} f \, dx = 0$. Let $(\rho_\varepsilon)_{\varepsilon > 0}$ be a sequence of mollifiers. Define $g_\varepsilon \in \mathcal{D}(\mathbf{R}^n; \mathbf{R}^n)$ by

$$g_\varepsilon(z) = \frac{2}{\|f\|_{\mathrm{L}^1(\mathbf{R}^n)}} \int_{\mathbf{R}^n \times \mathbf{R}^n} (x-y) \int_0^1 \rho_\varepsilon\big(z - tx - (1-t)y\big) \, f_+(x) f_-(y) \, dt \, dx \, dy.$$

Next, note

(A.2)

$$\begin{aligned}\operatorname{div} g_\varepsilon(z) &= \frac{2}{\|f\|_{\mathrm{L}^1(\mathbf{R}^n)}} \int_{\mathbf{R}^n\times\mathbf{R}^n} \int_0^1 (x-y)\cdot\nabla\rho_\varepsilon\big(z-tx-(1-t)y\big)\, f_+(x) f_-(y)\, dt\, dx\, dy\\ &= \frac{2}{\|f\|_{\mathrm{L}^1(\mathbf{R}^n)}} \int_{\mathbf{R}^n\times\mathbf{R}^n} \big(\rho_\varepsilon(z-x) - \rho_\varepsilon(z-y)\big) f_+(x) f_-(y)\, dx\, dy\\ &= (\rho_\varepsilon * f)(z).\end{aligned}$$

Therefore $\operatorname{div} g_\varepsilon \to f$ in $\mathcal{D}(\mathbf{R}^n)$ as $\varepsilon \to 0$. Letting

$$\psi_\varepsilon = \sum_{i=1}^n g_\varepsilon^i (-1)^{i+1} \omega_1 \wedge \cdots \wedge \widehat{\omega}_i \wedge \cdots \wedge \omega_n,$$

one concludes

$$d\psi_\varepsilon = \operatorname{div} g_\varepsilon\, \omega_1 \wedge \cdots \wedge \omega_n \to f\, \omega_1 \wedge \cdots \wedge \omega_n = \varphi,$$

as $\varepsilon \to 0$. □

Remark A.10. The construction (A.2) is inspired from the construction of a non-optimal mass displacement plan in the Monge–Kantorovich mass displacement problem [11].

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Département de Mathématique, Université catholique de Louvain, 2 chemin du Cyclotron, 1348 Louvain-la-Neuve, Belgium

E-mail address: vanschaftingen@math.ucl.ac.be