# Estimates for $\mathrm{L}^1$ vector fields with a second order condition

Jean Van Schaftingen *

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**Abstract**

An estimate on the integral of the product of a vector field $f \in \mathrm{L}^1(\mathbf{R}^N; \mathbf{R}^{2N-1})$ and a function $u \in \mathrm{W}^{1,N}(\mathbf{R}^N)$ when $f$ satisfies a condition involving the sum of some second-order derivatives. This generalizes a previous result concerning vector fields whose divergence is a summable function [1, 3]. A relationship between this inequality and a Korn–Sobolev inequality of Strauss [2] is established.

## 1 Introduction

This note originates in the inequality proved by the author in [3].

**Theorem 1.** *There exists a constant $C_N$ such that for each $f \in \mathrm{L}^1(\mathbf{R}^N; \mathbf{R}^N)$ such that $\operatorname{div} f \in \mathrm{L}^1$ and $u \in (\mathrm{L}^\infty \cap \mathrm{W}^{1,N})(\mathbf{R}^N; \mathbf{R}^N)$,*

$$\left| \int_{\mathbf{R}^N} f \cdot u \, dx \right| \leq C_N (\|f\|_1 \|\nabla u\|_N + \|\operatorname{div} f\|_1 \|u\|_N).$$

Theorem 1 was proved when $\operatorname{div} f = 0$ by Bourgain and Brezis [1]. In this note, a variant of Theorem 1 is proved where the divergence is replaced by a second order operator.

**Theorem 2.** *Let $u \in (\mathrm{L}^\infty \cap \mathrm{W}^{1,N})(\mathbf{R}^N)$ and $f_{ij} \in \mathrm{L}^1(\mathbf{R}^N)$, $g_i \in \mathrm{L}^1(\mathbf{R}^N)$ for $N-1 \leq i \leq N$ and $1 \leq j \leq i$. If*

$$\sum_{\substack{N-1 \leq i \leq N \\ 1 \leq j \leq i}} \frac{\partial^2 f_{ij}}{\partial x_i \partial x_j} = \sum_{N-1 \leq i \leq N} \frac{\partial g_i}{\partial x_i}, \tag{1}$$

*in the sense of distributions, then for each $N-1 \leq i \leq N$ and $1 \leq j \leq i$*

$$\left| \int_{\mathbf{R}^N} f_{ij} u \right| \leq C_N (\|f\|_1 \|\nabla u\|_N + \|g\|_1 \|u\|_N),$$

*where*

$$\|f\|_1 = \sum_{\substack{N-1 \leq i \leq N \\ 1 \leq j \leq i}} \|f_{ij}\|_1$$

*and*

$$\|g\|_1 = \sum_{N-1 \leq i \leq N} \|g_i\|_1 .$$

*Remark* 1. Theorem 2 implies Theorem 1. Indeed, suppose $f$ satisfies the hypotheses of Theorem 1. If $f_{Nj} = f_j$, $f_{N-1j} = 0$ for each $j$, $g_N = \operatorname{div} f$ and $g_{N-1} = 0$, then $f$ and $g$ satisfy the hypotheses of Theorem 2. The conclusion of Theorem 2 implies the conclusion of Theorem 1.

*Département de Mathématique, Université catholique de Louvain, 2 chemin du Cyclotron, 1348 Louvain-la-Neuve, Belgium. vanschaftingen@math.ucl.ac.be

The restriction $N-1 \leq i \leq N$ does not seem natural when $N \geq 3$. In particular, Theorem 2 does not answer the question whether

$$\left|\int_{\mathbf{R}^3} f \cdot u \, dx\right| \leq C_N \|f\|_1 \|\nabla u\|_3 .$$

for each $u \in (\mathrm{L}^\infty \cap \mathrm{W}^{1,3})(\mathbf{R}^3; \mathbf{R}^3)$ and $f \in \mathrm{L}^1(\mathbf{R}^3; \mathbf{R}^3)$ such that $\sum_{i=1}^3 \partial_i^2 f_i = 0$ excepted when one of the components $f_i$ vanishes. More generally one can ask whether Theorem 2 is true under more natural assumptions:

**Open Problem 1.** Let $u \in (\mathrm{L}^\infty \cap \mathrm{W}^{1,N})(\mathbf{R}^N)$, $f_{ij} \in \mathrm{L}^1(\mathbf{R}^N)$ and $g_i \in \mathrm{L}^1(\mathbf{R}^N)$ for $1 \leq i \leq N$ and $1 \leq j \leq i$. If

$$\sum_{\substack{1 \leq i \leq N \\ 1 \leq j \leq i}} \frac{\partial^2 f_{ij}}{\partial x_i \partial x_j} = \sum_{1 \leq i \leq N} \frac{\partial g_i}{\partial x_i},$$

in the sense of distributions, then is it true that for each $1 \leq i \leq N$ and $1 \leq j \leq i$,

$$\left|\int_{\mathbf{R}^N} f_{ij} u\right| \leq C_N (\|f\|_1 \|\nabla u\|_N + \|g\|_1 \|u\|_N),$$

where

$$\|f\|_1 = \sum_{\substack{1 \leq i \leq N \\ 1 \leq j \leq i}} \|f_{ij}\|_1$$

and

$$\|g\|_1 = \sum_{1 \leq i \leq N} \|g_i\|_1 ?$$

The problem is open even in the simple case where $g_i = 0$ for all $i$ and $f_{ij} = 0$ for $i \neq j$.

**Open Problem 2.** Let $u \in (\mathrm{L}^\infty \cap \mathrm{W}^{1,N})(\mathbf{R}^N; \mathbf{R}^N)$ and $f \in \mathrm{L}^1(\mathbf{R}^N; \mathbf{R}^N)$. If

$$\sum_{i=1}^N \frac{\partial^2 f_i}{\partial {x_i}^2} = 0,$$

in the sense of distributions, then is it true that

$$\left|\int_{\mathbf{R}^N} f \cdot u\right| \leq C_N \|f\|_1 \|\nabla u\|_N ?$$

# 2 Proof of Theorem 2

The key estimate is in the following

**Lemma 3.** *Let $u \in C^1(\mathbf{R}^{N-1})$. Let $f_{ij} \in \mathrm{L}^1(\mathbf{R}^N)$ and $g_i \in \mathrm{L}^1(\mathbf{R}^N)$ for $N-1 \leq i \leq N$ and $1 \leq j \leq i$. If* (1) *holds in the sense of distributions, then for each $t \in \mathbf{R}$,*

$$\left|\int_{\mathbf{R}^{N-1}} f_{NN}(x,t) u(x) \, dx\right| \leq \frac{1}{2}\Big(\|f_{NN}\|_1 \|\partial_{N-1} u\|_\infty + \sum_{\substack{N-1 \leq i \leq N \\ 1 \leq j \leq N-1}} \|f_{ij}\|_1 \|\partial_j u\|_\infty + \sum_{N-1 \leq i \leq N} \|g_i\|_1 \|u\|_\infty\Big).$$

*Proof.* Let $y \in \mathbf{R}^{N-2}$ and $z \in \mathbf{R}$. Write the integrand as

$$\begin{aligned} f_{NN}(y,z,t) = \frac{1}{2} \int_{-\infty}^0 \Big(\frac{\partial}{\partial x_{N-1}} + \frac{\partial}{\partial x_N}\Big) & f_{NN}(y, z+s, t+s) \\ & + \Big(\frac{\partial}{\partial x_{N-1}} - \frac{\partial}{\partial x_N}\Big) f_{NN}(y, z+s, t-s) \, ds. \end{aligned}$$

This gives

$$
\begin{aligned}
2\int_{\mathbf{R}^{N-1}} & f_{NN}(y,z,t)u(y,z)\,dz\,dy \\
&= \int_{\mathbf{R}^{N-1}}\int_{-\infty}^{0} u(y,z)\Big(\frac{\partial f_{NN}}{\partial x_{N-1}}(y,z+s,t+s)+\frac{\partial f_{NN}}{\partial x_{N-1}}(y,z+s,t-s)\Big)\,ds\,dz\,dy \\
&\quad + \int_{\mathbf{R}^{N-1}}\int_{-\infty}^{0} u(y,z)\Big(\frac{\partial f_{NN}}{\partial x_{N}}(y,z+s,t+s)-\frac{\partial f_{NN}}{\partial x_{N}}(y,z+s,t-s)\Big)\,ds\,dz\,dy. \qquad (2)
\end{aligned}
$$

The first term is estimated by integration by parts

$$
\begin{aligned}
\int_{\mathbf{R}^{N-2}}\int_{-\infty}^{0}\int_{\mathbf{R}} & u(y,z)\left(\frac{\partial f_{NN}}{\partial x_{N-1}}(y,z+s,t+s)+\frac{\partial f_{NN}}{\partial x_{N-1}}(y,z+s,t-s)\right)\,dz\,ds\,dy \\
&= -\int_{\mathbf{R}^{N-2}}\int_{-\infty}^{0}\int_{\mathbf{R}} \frac{\partial u}{\partial x_{N-1}}(y,z)\Big(f_{NN}(y,z+s,t+s)+f_{NN}(y,z+s,t-s)\Big)\,dz\,ds\,dy \\
&= -\int_{\mathbf{R}^{N-2}}\int_{-\infty}^{0}\int_{\mathbf{R}} \frac{\partial u}{\partial x_{N-1}}(y,z'-s)\Big(f_{NN}(y,z',t+s)+f_{NN}(y,z',t-s)\Big)\,dz'\,ds\,dy \\
&\le \left\|\frac{\partial u}{\partial x_{N-1}}\right\|_{\infty}\int_{\mathbf{R}^N}|f_{NN}|. \qquad (3)
\end{aligned}
$$

For any $y,z,t$ and $s$, the integrand of the second term of (2) can be written as

$$
\frac{\partial f_{NN}}{\partial x_N}(y,z+s,t+s)-\frac{\partial f_{NN}}{\partial x_N}(y,z+s,t-s)=\int_{-s}^{s}\frac{\partial^2 f_{NN}}{\partial {x_N}^2}(y,z+s,t+\tau)\,d\tau. \qquad (4)
$$

Bringing (4) and (1) together yields

$$
\begin{aligned}
\int_{\mathbf{R}^{N-2}}\int_{\mathbf{R}}\int_{-\infty}^{0} & u(y,z)\Big(\frac{\partial f_{NN}}{\partial x_N}(y,z+s,t+s)-\frac{\partial f_{NN}}{\partial x_N}(y,z+s,t-s)\Big)\,ds\,dz\,dy \\
&= \int_{\mathbf{R}^{N-2}}\int_{\mathbf{R}}\int_{-\infty}^{0} u(y,z)\int_{-s}^{s}\frac{\partial^2 f_{NN}}{\partial {x_N}^2}(y,z+s,t+\tau)\,d\tau\,ds\,dz\,dy \\
&= -\sum_{\substack{N-1\le i\le N\\ 1\le j\le N-1}}\int_{\mathbf{R}^{N-2}}\int_{\mathbf{R}}\int_{-\infty}^{0} u(y,z)\int_{-s}^{s}\frac{\partial^2 f_{ij}}{\partial x_i\partial x_j}(y,z+s,t+\tau)\,d\tau\,ds\,dz\,dy \\
&\quad + \sum_{N-1\le i\le N}\int_{\mathbf{R}^{N-2}}\int_{\mathbf{R}}\int_{-\infty}^{0} u(y,z)\int_{-s}^{s}\frac{\partial g_i}{\partial x_i}(y,z+s,t+\tau)\,d\tau\,ds\,dz\,dy.
\end{aligned}
$$

Each term of the sum will now be bounded separately. For $i=N$ and $1\le j\le N-1$, one has

$$
\begin{aligned}
\int_{\mathbf{R}^{N-2}}\int_{\mathbf{R}}\int_{-\infty}^{0} & u(y,z)\int_{-s}^{s}\frac{\partial^2 f_{Nj}}{\partial x_N\partial x_j}(y,z+s,t+\tau)\,d\tau\,ds\,dz\,dy \\
&= \int_{-\infty}^{0}\int_{\mathbf{R}^{N-2}}\int_{\mathbf{R}} u(y,z)\Big(\frac{\partial f_{Nj}}{\partial x_j}(y,z+s,t+s)-\frac{\partial f_{Nj}}{\partial x_j}(y,z+s,t-s)\Big)\,dz\,dy\,ds \\
&= -\int_{-\infty}^{0}\int_{\mathbf{R}^{N-2}}\int_{\mathbf{R}} \frac{\partial u}{\partial x_j}(y,z)\Big(f_{Nj}(y,z+s,t+s)-f_{Nj}(y,z+s,t-s)\Big)\,dz\,dy\,ds \\
&\le \left\|\frac{\partial u}{\partial x_j}\right\|_{\infty}\int_{\mathbf{R}^N}|f_{Nj}|. \qquad (5)
\end{aligned}
$$

If $i = N-1$ and $1 \leq j \leq N-1$, then

$$\begin{aligned}
&\int_{\mathbf{R}^{N-2}} \int_{\mathbf{R}} \int_{-\infty}^{0} \int_{-s}^{s} u(y,z) \frac{\partial^2 f_{N-1j}}{\partial x_{N-1} \partial x_j}(y, z+s, t+\tau)\, d\tau\, ds\, dz\, dy \\
&\qquad = \int_{\mathbf{R}^{N-2}} \int_{\mathbf{R}} \int_{\mathbf{R}} \int_{-\infty}^{-|\tau|} u(y,z) \frac{\partial^2 f_{N-1j}}{\partial x_{N-1} \partial x_j}(y, z+s, t+\tau)\, ds\, d\tau\, dz\, dy \\
&\qquad = \int_{\mathbf{R}^{N-2}} \int_{\mathbf{R}} \int_{\mathbf{R}} u(y,z) \frac{\partial f_{N-1j}}{\partial x_j}(y, z-|\tau|, t+\tau)\, d\tau\, dz\, dy \\
&\qquad = \int_{\mathbf{R}} \int_{\mathbf{R}^{N-2}} \int_{\mathbf{R}} u(y,z) \frac{\partial f_{N-1j}}{\partial x_j}(y, z-|\tau|, t+\tau)\, dz\, dy\, d\tau \\
&\qquad = \int_{\mathbf{R}} \int_{\mathbf{R}^{N-2}} \int_{\mathbf{R}} \frac{\partial u}{\partial x_j}(y,z) f_{N-1j}(y, z-|\tau|, t+\tau)\, dz\, dy\, d\tau \\
&\qquad \leq \left\| \frac{\partial u}{\partial x_j} \right\|_\infty \int_{\mathbf{R}^N} |f_{N-1j}|\,. \qquad (6)
\end{aligned}$$

The reasoning is similar for the terms with $g_i$. The sum of all these inequalities yields the result. $\square$

Next lemma proves an estimate for $u \in \mathrm{W}^{1,N}(\mathbf{R}^N)$ when there is an estimate of the type of Lemma 3 for $v \in \mathrm{C}^1(\mathbf{R}^{N-1})$.

**Lemma 4.** *Let $f \in \mathrm{L}^1(\mathbf{R}^N)$ and $a, b \in \mathbf{R}^+$ such that, for any function $v \in \mathrm{C}^1(\mathbf{R}^{N-1})$ and for any $t \in \mathbf{R}$,*

$$\left| \int_{\mathbf{R}^{N-1}} f(x,t) v(x)\, dx \right| \leq a \|\nabla v\|_\infty + b \|v\|_\infty\,,$$

*then, for any $u \in (\mathrm{L}^\infty \cap \mathrm{W}^{1,N})(\mathbf{R}^N)$,*

$$\left| \int_{\mathbf{R}^N} fu \right| \leq C_N (\|f\| \, \|\nabla u\|_N)^{1-(1/N)} (a \|\nabla u\|_N + b \|u\|_N)^{1/N}.$$

*Proof.* Let $\rho : \mathbf{R}^{N-1} \to \mathbf{R}$ be a measurable bounded function with compact support such that $\int_{\mathbf{R}^{N-1}} \rho = 1$ and let $\rho_\varepsilon(\cdot) = \varepsilon^{1-N} \rho(\frac{\cdot}{\varepsilon})$. If $u^t(y) = u(t,y)$ and $f^t(y) = f(t,y)$, then

$$\int_{\mathbf{R}^{N-1}} f^t u^t\, dy = \int_{\mathbf{R}^{N-1}} f^t (u^t - \rho_\varepsilon * u^t)\, dy + \int_{\mathbf{R}^{N-1}} f^t (\rho_\varepsilon * u^t)\, dy.$$

The Sobolev-Morrey embedding in $\mathbf{R}^{N-1}$ gives

$$\left| \int_{\mathbf{R}^{N-1}} f^t (u^t - \rho_\varepsilon * u^t)\, dy \right| \leq C'_N \varepsilon^{1/N} \left\| f^t \right\|_1 \left\| \nabla u^t \right\|_N.$$

On the other hand

$$\left| \int_{\mathbf{R}^{N-1}} f^t (\rho_\varepsilon * u^t)\, dy \right| \leq C''_N \varepsilon^{(1/N)-1} (a \left\| \nabla u^t \right\|_N + b \left\| u^t \right\|_N).$$

The constants $C'_N$ and $C''_N$ depend only on the dimension $N$ (and of $\rho$). For each $t \in \mathbf{R}$, if $\|f^t\|_1 \|\nabla u^t\|_N \neq 0$, the choice $\varepsilon = (a \|\nabla u^t\|_N + b \|u^t\|_N)/(\|f^t\|_1 \|\nabla u^t\|_N)$ yields

$$\left| \int_{\mathbf{R}^{N-1}} f^t u^t\, dy \right| \leq C'''_N (\left\| f^t \right\|_1 \left\| \nabla u^t \right\|_N)^{1-(1/N)} (a \left\| \nabla u^t \right\|_N + b \left\| u^t \right\|_N)^{1/N}.$$

If $\|f^t\|_1 \|\nabla u^t\|_N \neq 0$, let $\varepsilon \to \infty$ to obtain the same inequality. The inequality is thus valid for any $t \in \mathbf{R}$.

Finally, by Hölder's inequality

$$\begin{aligned}\left|\int_{\mathbf{R}^N} fu\,dx\right| &\le \int_{\mathbf{R}} C_N'''(\|f^t\|_1 \|\nabla u^t\|_N)^{1-(1/N)}(a\|\nabla u^t\|_N + b\|u^t\|_N)^{1/N}\,dt\\ &\le C_N'''\left(\int_{\mathbf{R}} \|f^t\|_1\,dt\right)^{(N-1)/N}\left(\int_{\mathbf{R}} \|\nabla u^t\|_N^N\,dt\right)^{(N-1)/N^2}\\ &\qquad\left(\int_{\mathbf{R}} (a\|\nabla u_1^t\|_N + b\|u_1^t\|_N)^N\,dt\right)^{1/N^2}\\ &\le C_N(\|f\|_1\|\nabla u\|_N)^{(N-1)/N}(a\|\nabla u\|_N + b\|u\|_N)^{1/N}. \quad \square\end{aligned}$$

The combination of Lemmas 3 and 4 yields a special case of Theorem 2.

**Lemma 5.** *Under the hypotheses of Theorem 2,*

$$\left|\int_{\mathbf{R}^N} f_{NN}u\right| \le C_N(\|f\|_1\|\nabla u\|_N + \|g\|_1\|u\|_N),$$

*where $C_N$ is a constant independent of $f$, $g$ and $u$.*

*Proof.* This is a direct consequence of Lemmas 3 and 4. □

By appropriate changes of variable, Theorem 2 can be deduced from Lemma 5.

*Proof of Theorem 2.* By Lemma 5, the result is true for $i = j = N$. It is also true for $i = j = N-1$, by interverting $N$ and $N-1$ in the hypotheses.

If $j < N-1$ and $i = N$, define new variables by $x'_j = x_j - x_N$ and $x'_k = x_k$ if $k \ne j$ and a new vector field $f'$, defined by $f'_{NN} = f_{NN} + f_{Nj}$ and $f'_{NN-1} = f_{NN-1} + f_{N-1j}$ (for the other components, let $f'_{ij} = f_{ij}$). Let $g' = g$. One checks that $f'$ verifies the same hypotheses as $f$ and that $\|f'\|_1 \le 2\|f\|_1$. Since the inequality is true for $f'_{NN}$ and for $f_{NN}$, it is true for $f'_{NN} - f_{NN} = f_{Nj}$.

The situation is somewhat more tedious when $j = N-1$. Define new variables by $x'_{N-1} = x_{N-1} - x_N$ and $x'_k = x_k$ for $k \ne N-1$. Let

$$f'_{Nk} = \begin{cases} f_{Nk} + f_{N-1k} & \text{if } k < N-1,\\ f_{NN-1} + 2f_{N-1N-1} & \text{if } k = N-1,\\ f_{NN} + f_{NN-1} - f_{N-1N-1} & \text{if } k = N,\end{cases}$$

and $f'_{N-1k} = f_{N-1k}$. Let $g'_{N-1} = g_{N-1}$ and $g'_N = g_N + g_{N-1}$. The condition (1) is checked by $f'$ and $g'$. Since the inequality holds for $f_{NN}$, $f_{N-1N-1}$ and $f'_{NN}$, it holds for $f_{NN-1}$. □

# 3 Relationship with a Korn-Sobolev inequality

The Sobolev-Gagliardo-Nirenberg inequality

$$\|u\|_{N/(N-1)} \le C\|\nabla u\|_1$$

can be obtained by a combination of Theorem 1 and the classical Calderón–Zygmund estimates.

In a similar way a Korn–Sobolev inequality of Strauss results from Theorem 2 and the classical Calderón–Zygmund estimates.

**Theorem 6 (Strauss [2]).** *For any $u \in \mathcal{D}(\mathbf{R}^N;\mathbf{R}^N)$,*

$$\|u\|_{\frac{N}{N-1}} \le K_N \sum_{1\le i\le j\le N} \|\partial_i u_j + \partial_j u_i\|_1.$$

*Sketch of the proof of Theorem 6 using Theorem 2.* Let $H \in \mathcal{D}(\mathbf{R}^N; \mathbf{R}^N)$. Let $A$ be the differential operator defined for $\mathbf{R}^N$-valued functions by

$$(Au)_{ij} = (\partial_i u_j + \partial_j u_i).$$

Its formal adjoint is defined for $\mathbf{R}^{N^2}$-valued functions by

$$(A^* v)_i = -\sum_{j=1}^{N} (\partial_j v_{ij} + \partial_j v_{ji}).$$

Consider the system $A^* A p = H$. It is equivalent to

$$\Delta p_i + \partial_i \sum_{j=1}^{N} \partial_j p_j = -\frac{H_i}{2}. \tag{7}$$

This system is elliptic and has a solution in $p \in (\mathrm{W}^{1,\infty} \cap \mathrm{W}^{2,1}_{\mathrm{loc}})(\mathbf{R}^N; \mathbf{R}^{N^2})$. Furthermore, there exists a constant $B_N$ independent of $H$ such that

$$\left\| D^2 p \right\|_N \leq B_N \left\| H \right\|_N .$$

Since $p$ solves (7),

$$\int_{\mathbf{R}^N} uH = \int_{\mathbf{R}^N} u\, A^* A p = \int_{\mathbf{R}^N} Au\, Ap.$$

Recalling $\partial_i^2 (Au)_{jj} + \partial_j^2 (Au)_{ii} = 2 \partial_i \partial_j (Au)_{ij}$, the application of Theorem 2 to each $2 \times 2$ submatrix of $Au$ gives

$$\left| \int_{\mathbf{R}^N} uH \right| = \left| \int_{\mathbf{R}^N} Au\, Ap \right| \leq C_N \left\| Au \right\|_1 \left\| \nabla A p \right\|_N \leq B_N C_N \left\| Au \right\|_1 \left\| H \right\|_N .$$

Since $H$ is arbitrary, the result follows. □

*Remark* 2. The proof of Theorem 6 needs only a weak version of Theorem 2 where $f_{ij} = 0$ and $g_i = 0$ for $j < N - 1$ and $N - 1 \leq i \leq N$.


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