APPROXIMATION OF SYMMETRIZATIONS AND SYMMETRY OF CRITICAL POINTS

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Abstract. We give a sufficient condition in order that a sequence of cap or Steiner symmetrizations or of polarizations approximates some fixed cap or Steiner symmetrization. This condition is used to obtain the almost sure convergence for random sequences of symmetrization taken in an appropriate set. The results are applicable to the symmetrization of sets. An application is given to the study of the symmetry of critical points obtained by minimax methods based on the Krasnoselskii genus.

1. Introduction

A symmetrization by rearrangement transforms a set or a function into a more symmetric one, while some quantities remain under control. For example, for each $u \in W_0^{1,p}(B(0,R))$ with $1 \leqslant p < \infty$ and $u \geqslant 0$, one can construct a radial and radially decreasing function u^* such that for every Borel-measurable function $f : \mathbb{R} \to \mathbb{R}^+$,

$$\int_{B(0,R)} f(u^*)\,dx = \int_{B(0,R)} f(u)\,dx\,.$$

In particular, $u^* \in L^p(B(0,R))$ and $\|u^*\|_p = \|u\|_p$. While the map $u \mapsto u^*$ is nonlinear, it is still non-expansive in $L^p(B(0,R))$. Furthermore, $u^* \in W_0^{1,p}(B(0,R))$ and one has the Pólya-Szegő inequality:

$$\int_{B(0,R)} |\nabla u^*|^p\,dx \leqslant \int_{B(0,R)} |\nabla u|^p\,dx\,.$$

Other useful inequalities, such as the Riesz-Sobolev rearrangement inequality hold. For symmetrization inequalities, we refer to [12,16]. Symmetrizations were defined for sets in the nineteenth century by Steiner and Schwarz. Symmetrizations of functions go back to Hardy, Littlewood and Pólya [11] and to Pólya and Szegő [19].

Applications of symmetrization by rearrangement are multiple. Symmetrizations were used by Talenti and Aubin to compute the optimal constants for the Sobolev inequality [2,27]. They can be used to obtain estimates on the first eigenvalue of the Laplacian with Dirichlet boundary conditions (Faber-Krahn inequality [19,28,33]). By symmetrization techniques, it is also possible to prove that solutions of problems in the calculus of variations are symmetric functions [23]. In some cases they provide also an alternative to concentration-compactness [8].

Since symmetrizations and symmetrization inequalities are useful, it would be nice to have general, simple and elegant methods to construct symmetrizations and prove the associated inequalities. The main difficulty is that symmetrizations are nonlinear and nonlocal transformations. One way to manage these problems is the level-sets method. The functional for which an inequality is needed is decomposed in integrals on level sets. For example, if $u : \Omega \to \mathbb{R}^+$ is nonnegative and measurable

Supported by a Research Fellow Grant of the Fonds National de la Recherche Scientifique.

and $f \in C^1(\mathbb{R}^+, \mathbb{R}^+)$, one has

$$\int_\Omega f(u)\,dx = \int_{\mathbb{R}^+} \mathcal{L}^N(\{x \in \Omega \,:\, f(x) \leqslant t\}) f'(t)\,dt.$$

This can be thought as localizing the functional with respect to the u variable. As long as the functionals in consideration do not involve gradients or convolution products, the inequalities are proved trivially. — For example, the proof of the Hardy-Littlewood inequality becomes very elegant [10, 33]. — When it is not the case any more, the set inequalities become nontrivial geometric inequalities. For example, the Pólya–Szegö inequality follows from the classical isoperimetric inequality [18], and the Riesz-Sobolev rearrangement inequality is a consequence of the same inequality for characteristic functions of sets [16]. In those cases the level set method does not essentially simplify the proof. The method of level-sets is used extensively by Mossino [18].

Another method to study symmetrization is to approximate a symmetrization by a sequence of simpler symmmetrizations — which are more localized than more elaborated symmetrizations. This goes back to the original definition of the Steiner symmetrization as a tool to prove the classical isoperimetric Theorem. Later, inequalities for capacitors were proved by approximation of Steiner and cap symmetrizations by lower-order Steiner and cap symmetrizations [21]; the Riesz-Sobolev inequality was proved by approximation of a Steiner symmetrization by lower-order Steiner symmetrizations [5]; Recently, a still simpler transformation, the polarization, was used to approximate many symmetrizations in order to obtain simple proofs of the isoperimetric inequality, the Pólya-Szegő inequality and a weak form of the Riesz-Sobolev rearrangement inequality [3, 6, 23, 31].

In a recent work [30], we used approximation of symmetrization in order to investigate the symmetry properties of critical points obtained by minimax methods. The key point was the use of polarizations to obtain a continuous approximation of a Steiner or cap symmetrization which is not continuous in general in Sobolev spaces [1].

In this paper, we investigate further the approximation of symmetrizations by simpler symmetrizations. We study which sequences of symmetrizations approximate a given symmetrization, and we give a simple sufficient condition. Since almost every sequence of symmetrizations in a well-chosen set satisfies this condition, we solve by the way a conjecture of Mani-Levitska concerning random sequences of Steiner symmetrizations [17]. This sufficient condition allows us to obtain some information about the symmetry of critical points of symmetric functionals obtained by minimax methods using the Krasnoselskii genus.

The paper begins by reviewing in section 2 the main facts about symmetrizations used in the sequel. We define in section 2.1 the Steiner with respect to an affine subspace and cap symmetrizations with respect to a closed affine half subspace. The set of affine subspaces and closed affine half subspaces is denoted by $\mathcal{S}$, and the symmetrization of u with respect to $S \in \mathcal{S}$ is denoted by u^S. The simplest cap symmetrizations are the polarizations; they are symmetrizations with respect to $H \in \mathcal{H}$, where $H \subset \mathcal{H}$ is the set of closed affine halfspaces. Many of their properties are easy to prove (section 2.2). We introduce a partial order $\prec$, such that $S \prec T$ if the symmetrization with respect to T can be used to approximate the symmetrization with respect to S (Definition 2.19 and Proposition 2.20). For $S \in \mathcal{S}$, the set of $T \in \mathcal{S}$ (resp. $\in \mathcal{H}$) such that $S \prec T$ is denoted by $\mathcal{S}_S$ (resp. $\mathcal{H}_S$). With these notations, we restate in a common framework all the approximation results of [31]:

Theorem 2.28. *Let $S \in \mathcal{S}$ and $\mathcal{T} \subset \mathcal{S}_S$. If for every $H \in \mathcal{H}_S$, there exists $T \in \mathcal{T}$ such that $T \prec H$, then there exists a sequence $(T_n)_{n \geqslant 1} \subset \mathcal{T}$ such that if $\Omega \subset \mathbb{R}^N$ is open, $u \in \mathcal{K}(\Omega)$ and (u, S) is admissible, then*

$$\|u^{T_1 \dots T_n} - u^S\|_\infty \to 0.$$

The condition "(u, S) is admissible" simply means that the symmetrization u^S is defined. In order to state a sufficient condition for a sequence of symmetrizations to approximate a symmetrization, we define a metric d on $\mathcal{S}$ for which the mapping $(u, S) \mapsto u^S$ is continuous (Definition 2.35, Proposition 2.38 and Corollary 2.39).

With all the machinery of section 2, we can state and prove the main result of Section 3,

Theorem 3.2. *Let $S \in \mathcal{S}$, $\mathcal{T} \subset \mathcal{S}_S$ and $(T_n)_{n \geqslant 1} \subset \mathcal{S}_S$ be such that*

a) for every $H \in \mathcal{H}_S$, there exists $T \in \mathcal{T}$ such that $T \prec H$,
b) for each $m \geqslant 1$ and $S_1, \dots, S_m \in \mathcal{T}$, there exists $k \geqslant 0$ such that for every $1 \leqslant i \leqslant m$, $\mathrm{d}(S_i, T_{k+i}) \leqslant \delta$,

Then for each open set $\Omega \subset \mathbb{R}^N$ and $u \in \mathcal{K}(\Omega)$ such that (u, S) is admissible,

$$\|u^{T_1 \dots T_n} - u^S\|_\infty \to 0, \qquad \text{as } n \to \infty.$$

The proof relies on the fact that for every $m \geqslant 1$ and $\delta > 0$, the m first terms of the sequence of Theorem 2.28 are contained up to an error δ in the sequence $(T_n)_{n \geqslant 1}$.

Given $\mathcal{T}$, it is easy to construct sequences satisfying the hypotheses of Theorem 3.2. In fact, if the approximating symmetrizations are symmetrization with respect to random variables that are distributed throughout the whole of $\mathcal{T}$, then the convergence occurs almost surely (Theorem 3.4).

All the preceding results can be extended to the approximation of the symmetrization of compact sets in Hausdorff distance d_H (Proposition 3.10). For example, if $\mathfrak{K}(\mathbb{R}^N)$ denotes the set of compact sets of $\mathbb{R}^N$, one has:

Theorem 3.13. *Let $S \in \mathcal{S}$ with $\partial S = \phi$ and let (E, Σ, P) be a probability space. Let $\ell > \dim S$ and*

$$\mathcal{T}_S^\ell = \{T \in \mathcal{S}_S \,:\, \partial T = \phi \text{ and } \dim T = \ell\}.$$

If $(T_n)_{n \geqslant 1}$ are independent random variables with values in $\mathcal{T}_S^\ell$ whose distribution functions are invariant under isometries that preserve S, then

$$P\left(sete \in E \,:\, \forall K \in \mathfrak{K}(\mathbb{R}^N),\ \lim_{n \to \infty} \mathrm{d}_H(K^{T_1(e) \dots T_n(e)}, K^S) = 0\right) = 1.$$

Finally, in section 4, Theorem 3.2 is applied to the proof of symmetry properties of critical points obtained by minimax methods using the Krasnoselskii genus. If A is a symmetric (i.e. $A = -A$) set in a Banach space V, its Krasnoselkii genus $\gamma(A)$ is the least integer k such that there is an odd mapping in $C(A, S^{k-1})$. The properties of γ are developed in section 4.1. For $\varphi : M \subset V \to \mathbb{R}$, let

$$\beta_\ell = \inf_{\substack{A \subset M \\ A \text{ is closed} \\ \gamma(A) \geqslant \ell}} \sup_{u \in A} \varphi(u).$$

Theorem 3.2 allows us to construct, given a set of small Krasnoselskii genus, a set of more symmetric functions that has not a smaller Krasnoselskii genus (Propositions 4.7).

The main result is that when the functional φ satisfies some symmetry assumptions, then there are symmetric critical points on the levels β_ℓ for small ℓ:

Theorem 4.8. *Let $\Omega = \Omega' \times \Omega'' \subset \mathbb{R}^N$ be open, with $\Omega' \subset \mathbb{R}^k$ invariant under $O(k)$. Let $M \subset W^{1,p}(\Omega) \setminus \{0\}$ be a complete symmetric $C^{1,1}$-manifold. Suppose $\varphi \in C^1(M)$ is an even functional that satisfies the Palais-Smale condition, and is bounded from below on M. Also suppose that if $H \in \mathcal{H}$, $\{0\} \times \mathbb{R}^{N-k} \subset \partial H$ and $u \in M$, then $u^H \in M$ and $\varphi(u^H) \leqslant \varphi(u)$. If $\ell \leqslant k$, then there is a critical point $u \in M$ and $x \in S^{k-1}$ such that $\varphi(u) = \beta_\ell$ and $u^{S_x} = u$.*

Here S_x denotes the cap symmetrization with respect to $\mathbb{R}x \times \mathbb{R}^{N-k}$. We end with simple applications of this result. The method applies to Dirichlet and Neumann problems (Theorems 4.9 and 4.10).

2. Symmetrizations

2.1. Definitions.

In the following, $\mathcal{H}^k$ denotes the k-dimensional outer Hausdorff measure, while for $x \in \mathbb{R}^N$ and $0 \leqslant r \leqslant \infty$, $B(x, r) = \{y \in \mathbb{R}^N : |x - y| < r\}$. The extended set of real numbers is denoted by $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. The set of compactly supported continuous functions on the open set Ω is denoted by $\mathcal{K}(\Omega)$ and the modulus of continuity of a function $u \in \mathcal{K}(\Omega)$ is the function $\omega_u : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\omega_u(\delta) = \sup \{|u(x) - u(y)| \; : \; x, y \in \Omega \text{ and } |x - y| \leqslant \delta\}.$$

We define the Steiner and spherical cap symmetrizations according to Sarvas [21]. In contrast with Sarvas, our definition does not make difference between compact and open sets, but is valid for any set, possibly non-measurable. This ensures a good pointwize definition of the symmetrization of measurable sets and functions.

Definition 2.1 (Steiner symmetrization)**.** Let S be a k-dimensional affine subspace of $\mathbb{R}^N$, $0 \leqslant k \leqslant N-1$. The symmetrization of a set $A \subset \mathbb{R}^N$ with respect to S is the unique set A^S such that for any $x \in S$, if L is the $(N-k)$-dimensional hyperplane orthogonal to S that contains x,

$$A^S \cap L = B(x, r) \cap L,$$

where $0 \leqslant r \leqslant \infty$ is defined by $\mathcal{H}^{N-k}(B(x, r) \cap L) = \mathcal{H}^{N-k}(A \cap L)$.

Remark 2.2. The symmetrization with respect to a 0-dimensional plane is called point symmetrization or Schwarz symmetrization. (Some authors call Schwarz symmetrization a symmetrization with respect to a 1-dimensional plane and Steiner symmetrization a symmetrization with respect to a $(N-1)$-dimensional plane [16].)

Definition 2.3 (Cap symmetrization)**.** Let S be a k-dimensional closed affine half subspace of $\mathbb{R}^N$, $1 \leqslant k \leqslant N$ and let ∂S be the boundary of S inside the affine plane generated by S. The symmetrization of a set $A \subset \mathbb{R}^N$ with respect to S is the unique set A^S such that $A^S \cap \partial S = A \cap \partial S$ and for each $x \in \partial S$, if L is the $(N-k+1)$-dimensional hyperplane orthogonal to ∂S that contains x and y is the unique point of the intersection $\partial B(x, \varrho) \cap S$, then for every $\varrho > 0$

$$A^S \cap \partial B(x, \varrho) \cap L = B(y, r) \cap \partial B(x, \varrho) \cap L,$$

where $r \geqslant 0$ is defined by $\mathcal{H}^{N-k}(B(y, r) \cap \partial B(x, \varrho) \cap L) = \mathcal{H}^{N-k}(A \cap \partial B(x, \varrho) \cap L)$.

Remark 2.4. The symmetrization with respect to a one dimensional closed affine subspace is also called foliated Schwarz symmetrization [23].

Definition 2.5. The set of all the k-dimensional affine subspaces of $\mathbb{R}^N$ for $0 \leqslant k \leqslant N-1$, and of all the k-dimensional closed affine half subspaces of $\mathbb{R}^N$ for $1 \leqslant k \leqslant N$ is denoted by $\mathcal{S}$.

Symmetrizations have the following basic properties:

Proposition 2.6. *Let A, $B \subset \mathbb{R}^N$ and $S \in \mathcal{S}$. If $A \subset B$, then $A^S \subset B^S$. If A is measurable, then A^S is measurable and $\mathcal{L}^N(A^S) = \mathcal{L}^N(A)$. If A is open, then A^S is open.*

We need some condition to ensure that the symmetrization of a function is meaningful.

Definition 2.7. Let $\Omega \subset \mathbb{R}^N$, $u : \Omega \to \bar{\mathbb{R}}$ and $S \in \mathcal{S}$. The pair (u, S) is *admissible* if $\Omega^S = \Omega$, and, for every $c > 0$,

$$\mathcal{L}^N(\{x \in \Omega \,:\, |u(x)| > c\}) < \infty$$

and either $u \geqslant 0$, or $\partial S \neq \phi$ and $(\mathbb{R}^N \setminus \Omega)^S = \mathbb{R}^N \setminus \Omega$.

Definition 2.8. Let $\Omega \subset \mathbb{R}^N$, $u : \Omega \to \bar{\mathbb{R}}$ and $S \in \mathcal{S}$. Suppose that (u, S) is admissible. The *symmetrization* of u with respect to S is the unique function u^S such that for each $c \in \bar{\mathbb{R}}$,

$$\left\{x \in \Omega \,:\, u^S(x) > c\right\} = \{x \in \Omega \,:\, u(x) > c\}^S .$$

Remark 2.9. The function u^S can be defined as

$$u^S(x) = \sup\left\{c \in \mathbb{R} \,:\, x \in \{y \in \Omega \,:\, u(y) > c\}^S\right\}.$$

The definitions with open balls of symmetrization of sets are of crucial importance in order to obtain the existence of u^S satisfying Definition 2.8 (see [29]).

The symmetrization of a function does not essentially depend on the domain:

Proposition 2.10. *Let $u : \Omega \to \bar{\mathbb{R}}$, $\tilde{u} : \mathbb{R}^N \to \bar{\mathbb{R}}$ be defined by $\tilde{u}|_\Omega = u$ and $\tilde{u}|_{\mathbb{R}^N \setminus \Omega} = 0$ and $S \in \mathcal{S}$. If (u, S) is admissible, then $(\tilde{u}, S)$ is admissible and $\tilde{u}^S|_\Omega = u^S$.*

The symmetrization of functions in L^p is a non-expansive nonlinear mapping that preserves the norm:

Proposition 2.11 (L^p properties of symmetrizations)**.** *Let $1 \leqslant p \leqslant \infty$, $\Omega \subset \mathbb{R}^N$ be measurable and $u, v \in L^p(\Omega)$. If (u, S) and (v, S) are admissible, then $u^S, v^S \in L^p(\Omega)$, $\|u^S\|_p = \|u\|_p$, $\|v^S\|_p = \|v\|_p$ and $\|u^S - v^S\|_p \leqslant \|u - v\|_p$.*

Proof. See e.g. [10, 32]. □

Remark 2.12. If $u \in W^{1,p}(\Omega)$ then $u^S \in W^{1,p}(\Omega)$ and $\|\nabla u^S\|_p \leqslant \|\nabla u\|_p$, but if $\partial S = \phi$, the mapping $u \mapsto u^S$ is continuous in $W^{1,p}(\Omega)$ if and only if $\dim S = N - 1$ [1, 7, 9]. If $\partial S \neq \phi$, $u \mapsto u^S$ is continuous if $\dim S = N$ (see [30] and Corollary 2.40 below). If $\dim S < N - 1$, then a reasoning in the spirit of Lemma 2.33 and the results of Almgren and Lieb [1] shows that $u \mapsto u^S$ is not continuous. The case $\dim S = N - 1$ remains open, but it is likely that the method of Burchard would show that the cap symmetrization is then continuous.

We introduce the complementary of a affine half subspace.

Definition 2.13. Let $u \in \mathcal{S}$ and $S \in \mathcal{S}$ with $\partial S \neq \phi$. The *complementary* of S is the reflexion of S with respect to ∂S. It is denoted by S^*.

As a straightforward consequence of the definitions, one has

Proposition 2.14. *Let $S \in \mathcal{S}$ and $u : \Omega \to \bar{\mathbb{R}}$. If (u, S) and $(-u, S^*)$ are admissible, then*

$$(-u)^{S^*} = -(u^S).$$

2.2. **Polarizations.** We recall briefly some facts about the simplest symmetrizations, the polarizations.

Definition 2.15. The symmetrization with respect to $H \in \mathcal{S}$ is a *polarization* if ∂H is a hyperplane, or, equivalently, $\dim H = N$. The reflexion of $x \in \mathbb{R}^N$ with respect to ∂H is denoted by x_H. The set of $H \in \mathcal{S}$ such that $\dim H = N$ is denoted by $\mathcal{H}$.

Proposition 2.16. *Let $H \in \mathcal{H}$, $\Omega \subset \mathbb{R}^N$ and $u : \Omega \to \bar{\mathbb{R}}$. If (u, H) is admissible, then*

$$u^H(x) = \begin{cases} \max(u(x), u(x_H)) & \text{if } x \in H\,, \\ \min(u(x), u(x_H)) & \text{if } x \notin H\,. \end{cases}$$

Remark 2.17. The characterization of Proposition 2.16 is the classical definition of the polarization of a function [6].

Proposition 2.18. *Let $H \in \mathcal{H}$, $\Omega \subset \mathbb{R}^N$ be open and $u : \Omega \to \bar{\mathbb{R}}$ be measurable. If (u, H) is admissible, $f : \Omega \times \bar{\mathbb{R}} \to \mathbb{R}^+$ is a Borel measurable function, and for every $t \in \bar{\mathbb{R}}$ and $x \in \Omega$ such that $x_H \in \Omega$, $f(x_H, t) = f(x, t)$, then*

$$\int_\Omega f(x, u^H(x))\, dx = \int_\Omega f(x, u(x))\, dx\,.$$

Furthermore, if $1 \leqslant p < \infty$, $u \in W^{1,p}_0(\Omega)$ (resp. $(-u, H)$ is admissible and $u \in W^{1,p}(\Omega)$) then $u^H \in W^{1,p}_0(\Omega)$ (resp. $u^H \in W^{1,p}(\Omega)$) and

$$\int_\Omega |\nabla u^H|^p\, dx = \int_\Omega |\nabla u|^p\, dx\,.$$

If $u \in \mathcal{K}(\Omega)$, then $u^H \in \mathcal{K}(\Omega)$ and for any $\delta > 0$,

$$\omega_{u^H}(\delta) \leqslant \omega_u(\delta)\,.$$

Proof. See [6, 30]. □

2.3. **Approximating symmetrization.** In order to study the approximations of a symmetrization by other symmetrizations we introduce a partial order $\prec$ on the symmetrizations such that $S \prec T$ if the symmetrization with respect to T can be used to approximate the symmetrization with respect to S.

Definition 2.19. Let $S, T \in \mathcal{S}$. We write $S \prec T$ if $S \subseteq T$ and $\partial S \subseteq \partial T$. For $S \in \mathcal{S}$, let

$$\mathcal{S}_S = \{T \in \mathcal{S} \;:\; S \prec T\}$$

and

$$\mathcal{H}_S = \{H \in \mathcal{H} \;:\; S \prec H\}\,.$$

This definition is justified by the next proposition.

Proposition 2.20. *Let $S, T \in \mathcal{S}$ and suppose $S \prec T$. If A is Borel measurable, then $A^{ST} = A^{TS} = A^S$.*

If $\Omega \subset \mathbb{R}^N$ and $u : \Omega \to \bar{\mathbb{R}}$ are Borel measurable, and (u, S) is admissible, then (u, T), (u^T, S) and (u^S, T) are admissible and $u^{ST} = u^{TS} = u^S$.

Proof. The definitions yields $A^{TS} = A^{ST} = A^S$ for any Borel measurable set $A \subset \mathbb{R}^N$. The conclusion follows from the definitions of the admissibility and of the symmetrization of a function. □

Remark 2.21. By Proposition 2.11, if $S \prec T$, then

$$\|u^T - u^S\|_p \leqslant \|u - u^S\|_p,$$

i.e. T does not increase the distance between u and u^S and T 0can be used to approximate S.

Remark 2.22. If A is merely measurable, its intersection with some affine subspace could be $\mathcal{H}^k$-non-measurable, resulting in $A^{TS} \supsetneq A^S = A^{ST}$. However, one can still conclude that $A^S \subset A^{TS}$ and that $\mathcal{L}^N(A^{TS} \setminus A^S) = 0$.

Many properties of the symmetrizations can be deduced from the next

Theorem 2.23. *Let $S \in \mathcal{S}$. There exists a sequence $(H_n)_{n\geqslant 1} \subset \mathcal{H}_S$ such that if $\Omega \subset \mathbb{R}^N$ is open, $u \in \mathcal{K}(\Omega)$ and (u, S) is admissible, then*

$$\|u^{H_1\dots H_n} - u^S\|_\infty \to 0.$$

Proof. See [31]. □

Remark 2.24. Weaker forms of Theorem 2.23, where the sequence could depend on the function to symmetrize were proved by Brock and Solynin [6] and by Smets and Willem [23].

Corollary 2.25. *Let $S \in \mathcal{S}$ and $u \in \mathcal{K}(\Omega)$. If (u, S) is admissible, then $u^S \in \mathcal{K}(\Omega)$ and for any $\delta > 0$,*

$$\omega_{u^S}(\delta) \leqslant \omega_u(\delta).$$

Proof. This follows from Proposition 2.18 and Theorem 2.23. □

Among the consequences, there is the compactness of the set of functions obtained by symmetrizations compatible with a given symmetrization:

Proposition 2.26. *Let $S \in \mathcal{S}$, $\Omega \subset \mathbb{R}^N$ and $u \in \mathcal{K}(\Omega)$. If (u, S) is admissible, then*

$$\mathcal{U} = \left\{u^{T_1\dots T_n} \;:\; n \geqslant 1,\, T_i \in \mathcal{S}_S \text{ for each } 1 \leqslant i \leqslant n\right\}$$

is totally bounded in $L^\infty(\Omega)$.

Proof. By Proposition 2.11, if $v \in \mathcal{U}$, then $\|v\|_\infty = \|u\|_\infty$. Since u is compactly supported, there exists $x \in \partial S$ ($x \in S$ if $u \geqslant 0$) and $r \geqslant 0$ such that $\operatorname{supp} u \subset B(x, r)$. Since $S \prec T$, $B(x, r)^T = B(x, r)^{ST} = B(x, r)^S = B(x, r)$. By Proposition 2.6, for each $v \in \mathcal{U}$, one has $\operatorname{supp} v \subset B(x, r)$. Finally, by Corollary 2.25, for every $v \in \mathcal{U}$, we have $v \in \mathcal{K}(\Omega)$ and

$$\omega_v(\delta) \leqslant \omega_u(\delta).$$

The conclusion comes from the Ascoli-Arzela Theorem. □

Remark 2.27. In fact, $\mathcal{U}$ is totally bounded in $L^p(\mathbb{R}^N)$ for every $1 \leqslant p \leqslant \infty$.

Proposition 2.26 is one of the ingredients of

Theorem 2.28. *Let $S \in \mathcal{S}$ and $\mathcal{T} \subset \mathcal{S}_S$. If for every $H \in \mathcal{H}_S$, there exists $T \in \mathcal{T}$ such that $T \prec H$, then there exists a sequence $(T_n)_{n\geqslant 1} \subset \mathcal{T}$ such that if $\Omega \subset \mathbb{R}^N$ is open, $u \in \mathcal{K}(\Omega)$ and (u, S) is admissible, then*

$$\|u^{T_1\dots T_n} - u^S\|_\infty \to 0.$$

Proof. See [31]. □

Remark 2.29. For every $1 \leqslant p < \infty$, the convergence happens for any $u \in L^p(\Omega)$ such that (u, S) is admissible.

2.4. **The metric structure of $\mathcal{S}$.** In order to construct other sequences of symmetrizations approximating a symmetrization by some kind of perturbation, we give a metric structure to the set $\mathcal{S}$. Since the definition of the metric on $\mathcal{S}$ relies on isometries of $\mathbb{R}^N$, we briefly investigate the relationship between symmetrizations and isometries. We call $i : \mathbb{R}^N \to \mathbb{R}^N$ an isometry provided that for every $x, y \in \mathbb{R}^N$, one has $|i(x) - i(y)| = |x - y|$.

Proposition 2.30. *Let $i : \mathbb{R}^N \to \mathbb{R}^N$ be an isometry and $S \in \mathcal{S}$. If $A \subset \mathbb{R}^N$, then $i(A^S) = i(A)^{i(S)}$. If $(u, i(S))$ is admissible, then $(u \circ i, S)$ is admissible, and $u^{i(S)} \circ i = (u \circ i)^S$.*

Proof. Since the definitions of the symmetrizations are invariant by isometry, this is straightforward. □

Remark 2.31. The isometries is the largest class of transformations of $\mathbb{R}^N$ for which Proposition 2.30 holds for every $S \in \mathcal{S}$.

We need also some information about elements of $\mathcal{S}$ which are identical in a ball.

Proposition 2.32. *There exist constants $K_1 > 1$ and $K_2 > 0$ that depend only on the dimension of the space N such that the following holds: Let $r \geqslant 0$, $R \geqslant K_1 r$, $S, T \in \mathcal{S}$, $x \in S$, and $u \in \mathcal{K}_+(\Omega)$. If (u, S) and (u, T) are admissible, $\operatorname{supp} u \subset B(x, r)$ and $B(x, R) \cap S = B(x, R) \cap T$, then*

$$\|u^S - u^T\|_\infty \leqslant \omega_u(K_2 r^2 / R).$$

Proof. This follows from the next Lemma applied to $u|_{B(x,r)}$ and from Proposition 2.10, since u^S and u^T are the extensions by 0 outside of $B(x, r)$ of $(u|_{B(x,r)})^S$ and $(u|_{B(x,r)})^T$. □

Lemma 2.33. *There exist constants $K_1 > 1$ and $K_2 > 0$ that depend only on the dimension of the space N such that the following holds: Let $r \geqslant 0$, $R \geqslant K_1 r$, $S, T \in \mathcal{S}$, and $x \in S$. If $B(x, R) \cap S = B(x, R) \cap T$ then there exists an injective map $g : B(0, r) \to \mathbb{R}^N$ such that for each $x \in B(x, r)$, $|g(x) - x| \leqslant K_2 r^2 / R$. Furthermore, for any $A \subset B(x, r)$, $g(A^S) = g(A)^T$ and if $\Omega \subset B(x, r)$, $u : \Omega \to \mathbb{R}$ and (u, T) is admissible, then $(u \circ g, S)$ is admissible and $u^T \circ g = (u \circ g)^S$.*

Remark 2.34. This was proved by Sarvas when $\dim S = N - 1$ [21].

Proof. If $\partial S \cap B(x, R) = \partial T \cap B(x, R) \neq \phi$ the proposition is trivial. The result is also trivial when $\dim S = \dim T = N$. Assume thus $\partial S \cap B(x, R) = \partial T \cap B(x, R) = \phi$ and $\dim S < N$. For any y, let C_{Sy} denote the circle that contains y, whose center is in ∂S and that is contained in an affine (two-dimensional) plane perpendicular to ∂S. If $\partial S = \phi$, define C_{Sy} to be the straight line perpendicular to S that contains y. Define C_{Ty} analogously.

The mapping g is the unique mapping such that if $y \in S \cap B(x, r)$, $g(C_{Sy} \cap B(x, r)) \subset C_{Tx}$, and if $A \subset C_{Sy} \cap B(x, r)$ is Borel measurable, then $\mathcal{H}^{N-k}(A) = \mathcal{H}^{N-k}(g(A))$, where k is the dimension of S and of T. A direct computation shows that for sufficiently large K_1 and K_2, the map g has the required properties. □

Now we define a distance on $\mathcal{S}$.

Definition 2.35. Let $S, T \in \mathcal{S}$ and

$$\varrho(S, T) = \inf\Big\{ \ln\Big(1 + \sup_{x \in \mathbb{R}^N} \frac{|x - i(x)|}{1 + |x|} + \sup_{x \in i(S) \,\Delta\, T} \frac{1}{1 + |x|}\Big) : i : \mathbb{R}^N \to \mathbb{R}^N \text{ is an isometry} \Big\}.$$

The *distance* between S, T is

$$\mathrm{d}(S,T) = \varrho(S,T) + \varrho(T,S).$$

Proposition 2.36. *The pair* $(\mathcal{S}, \mathrm{d})$ *is a separable metric space.*

Remark 2.37. The metric space $(\mathcal{S}, \mathrm{d})$ is not complete, but it is locally compact.

The symmetrization is continuous with respect to this distance. More precisely,

Proposition 2.38. *Let* $\Omega \subset \mathbb{R}^N$ *be open. The mapping*

$$\{(u,S) \in (\mathcal{K}(\Omega), \|\cdot\|_\infty) \times (\mathcal{S}, \mathrm{d}) \;:\; (u,S) \textit{ is admissible}\} \\ \to (\mathcal{K}(\Omega), \|\cdot\|_\infty) : (u,S) \mapsto u^S$$

is continuous.

Proof. Let $(u,S) \in (\mathcal{K}(\Omega), \|\cdot\|_\infty) \times (\mathcal{S}, \mathrm{d})$ be admissible, and let $\varepsilon > 0$. By Proposition 2.10, we can assume $\Omega = \mathbb{R}^N$.

First suppose $u \geqslant 0$. Let $(u,S) \in \mathcal{K}_+(\mathbb{R}^N) \times \mathcal{S}$ be admissible. Let K_1 and K_2 be given by Proposition 2.32. Fix $x \in S$ and $r \geqslant \varepsilon K_1/K_2$ such that $\operatorname{supp} u \in B(x,r)$. There exists $\delta > 0$, depending only on ε, x and r, such that if $T \in \mathcal{S}$ and $\mathrm{d}(S,T) \leqslant \delta$, then there is an isometry $i : \mathbb{R}^N \to \mathbb{R}^N$ with $|y - i(y)| \leqslant \varepsilon$ for each $y \in B(x,r)$ and $i(T) \cap B(x, K_2 r^2/\varepsilon) = S \cap B(x, K_2 r^2/\varepsilon)$. By Proposition 2.32, since $K_2 r^2/\varepsilon \geqslant K_1 r$, $\|u^S - u^{i(T)}\|_\infty \leqslant \omega_u(\varepsilon)$. Moreover, since by Proposition 2.30, $u^{i(T)} \circ i = (u \circ i)^T$,

$$\|u^{i(T)} - u^T\|_\infty = \|u^{i(T)} \circ i - u^T \circ i\|_\infty = \|(u \circ i)^T - u^T \circ i\|_\infty \\ \leqslant \|(u \circ i)^T - u^T\|_\infty + \|u^T - u^T \circ i\|_\infty.$$

Since by Proposition 2.11 the symmetrization is non-expansive in $L^\infty(\mathbb{R}^N)$,

$$\|(u \circ i)^T - u^T\|_\infty \leqslant \|u \circ i - u\|_\infty \leqslant \omega_u(\varepsilon).$$

By Corollary 2.25, the modulus of continuity does not increase by symmetrization:

$$\|u^T - u^T \circ i\|_\infty \leqslant \omega_{Tu}(\varepsilon) \leqslant \omega_u(\varepsilon).$$

For any $(v,T) \in \mathcal{K}_+(\mathbb{R}^N) \times \mathcal{S}$, if $\mathrm{d}(T,S) \leqslant \delta$ and $\|u - v\|_\infty \leqslant \varepsilon$, then, by the non-expansiveness of the symmetrizations,

$$\|u^S - v^T\|_\infty \leqslant \|u^S - u^T\|_\infty + \|u^T - v^T\|_\infty \leqslant 3\omega_u(\varepsilon) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, our claim is proved.

If $u \ngeqslant 0$, then by definition of admissibility, $\partial S \neq \phi$. Let $x \in \partial S$ and choose $r > 0$ such that $\operatorname{supp} u \subset B(x,r)$. By definition of d, there is $\delta > 0$ such that if $d(S,T) \leqslant \delta$, there exists an isometry $i : \mathbb{R}^N \to \mathbb{R}^N$ such that $|y - i(y)| \leqslant \varepsilon$ for $y \in B(x,r)$ and $i(T) \cap B(x,r) = S \cap B(x,r)$. Since $x \in \partial S$, S and T are closed affine half subspaces, and i is an isometry, $i(T) = S$. By Proposition 2.30,

$$\|u^S - u^T\|_\infty = \|u^{i(T)} \circ i - u^T \circ i\|_\infty = \|(u \circ i)^T - u^T \circ i\|_\infty.$$

The end of the proof is similar to the case when $u \geqslant 0$. $\square$

Corollary 2.39. *Let* $\Omega \subset \mathbb{R}^N$ *be open and* $1 \leqslant p < \infty$. *The mapping*

$$\Big\{(u,S) \in (L^p(\Omega), \|\cdot\|_p) \times (\mathcal{S}, \mathrm{d}) \;:\; (u,S) \textit{ is admissible}\Big\} \\ \to (L^p(\Omega), \|\cdot\|_p) : (u,S) \mapsto u^S$$

is continuous.

This remains true if $p = \infty$, *provided* $L^p(\Omega)$ *is replaced by* $C_0(\Omega)$.

As in [30], we can obtain the

Corollary 2.40. *Let $\Omega \subset \mathbb{R}^N$ be open and $1 < p < \infty$. The mapping*

$$\Big\{(u,H) \in W^{1,p}(\Omega) \times (\mathcal{H}, \mathrm{d}) \;:\; (u,H) \text{ and } (-u,H) \text{ are admissible}\Big\} \to W^{1,p}(\Omega) : (u,H) \mapsto u^H$$

is continuous.

Proof. This is a consequence of Proposition 2.18, of Corollary 2.39 and of the uniform convexity of the norm $\|\nabla u\|_p$. □

3. Constructing approximating sequences

3.1. **A sufficient condition.** Since the result of a symmetrization is stable under small perturbations on the symmetrization (Proposition 2.38), we can prove that some perturbations of an approximating sequence are approximating sequences.

Proposition 3.1. *Let $S \in \mathcal{S}$, $(S_n)_{n\geqslant 1} \subset \mathcal{S}_S$ and $(T_n)_{n\geqslant 1} \subset \mathcal{S}_S$. If for each open set $\Omega \subset \mathbb{R}^N$ and $u \in \mathcal{K}(\Omega)$ such that (u,S) is admissible,*

$$\|u^{S_1\dots S_n} - u^S\| \to 0, \qquad \text{as } n \to \infty,$$

and if for every $\delta > 0$ and $m \geqslant 1$, there exists $k \geqslant 0$ such that for each $1 \leqslant i \leqslant m$,

$$\mathrm{d}(S_i, T_{k+i}) \leqslant \delta,$$

then for each open set $\Omega \subset \mathbb{R}^N$ and $u \in \mathcal{K}(\Omega)$ such that (u,S) is admissible,

$$\|u^{T_1\dots T_n} - u^S\| \to 0, \qquad \text{as } n \to \infty.$$

Proof. Let $u \in \mathcal{K}(\Omega)$ and $\varepsilon > 0$. Since by Proposition 2.26, the sequence $(u^{T_1\dots T_n})_{n\geqslant 1}$ is totally bounded in $L^\infty(\Omega)$ and since by hypothesis

$$u^{T_1\dots T_n S_1\dots S_m} \to u^S, \qquad \text{as } m \to \infty,$$

there exists $m \geqslant 1$ such that for every $n \geqslant 0$,

$$\|u^{T_1\dots T_n S_1\dots S_m} - u^S\|_\infty \leqslant \varepsilon.$$

By the continuity of symmetrization (Proposition 2.38) and the fact that $(u^{T_1\dots T_n})_{n\geqslant 1}$ is totally bounded, there exists $\delta > 0$ such that for each $1 \leqslant i \leqslant m$, for each $n \geqslant 0$ and for each $T \in \mathcal{S}_S$, if $\mathrm{d}(S_i, T) \leqslant \delta$, then

$$\|u^{T_1\dots T_n S_i} - u^{T_1\dots T_n T}\|_\infty \leqslant \varepsilon/m.$$

By hypothesis, there is $k \geqslant 0$ such that for each $1 \leqslant i \leqslant m$, $\mathrm{d}(S_i, T_{k+i}) \leqslant \delta$. We can then use the non-expansiveness of symmetrizations (Proposition 2.11) and the preceding estimates to obtain, for every $\ell \geqslant m + k$,

$$\begin{aligned}
\|u^S - u^{T_1\dots T_\ell}\|_\infty &\leqslant \|u^S - u^{T_1\dots T_{m+k}}\|_\infty \\
&\leqslant \|u^S - u^{T_1\dots T_k S_1\dots S_m}\|_\infty + \sum_{i=1}^m \|u^{T_1\dots T_{k+i-1} S_i\dots S_m} - u^{T_1\dots T_{k+i} S_{i+1}\dots S_m}\| \\
&\leqslant \|u^S - u^{T_1\dots T_k S_1\dots S_m}\|_\infty + \sum_{i=1}^m \|u^{T_1\dots T_{k+i-1} S_i} - u^{T_1\dots T_{k+i}}\| \leqslant 2\varepsilon. \qquad \square
\end{aligned}$$

Theorem 3.2. *Let $S \in \mathcal{S}$, $\mathcal{T} \subset \mathcal{S}_S$ and $(T_n)_{n\geqslant 1} \subset \mathcal{S}_S$ be such that*

a) for every $H \in \mathcal{H}_S$, there exists $T \in \mathcal{T}$ such that $T \prec H$,

b) for each $m \geqslant 1$ and $S_1, \dots, S_m \in \mathcal{T}$, there exists $k \geqslant 0$ such that for every $1 \leqslant i \leqslant m$, $\mathrm{d}(S_i, T_{k+i}) \leqslant \delta$,

Then for each open set $\Omega \subset \mathbb{R}^N$ and $u \in \mathcal{K}(\Omega)$ such that (u,S) is admissible,

$$\|u^{T_1\dots T_n} - u^S\|_\infty \to 0, \qquad \text{as } n \to \infty.$$

Remark 3.3. Since $(\mathcal{S}, \mathrm{d})$ is separable, $(\mathcal{T}, \mathrm{d})$ is also separable so that given a countable dense set of $\mathcal{T}$ it is possible to construct explicitly a sequence $(T_n)_{n \geqslant 1}$ satisfying the hypotheses of Theorem 3.2.

Proof. This follows from Theorem 2.28 and Proposition 3.1. □

3.2. **Random sequences of symmetrizations.** As a first application of Theorem 3.2, we prove that symmetrizations can be approximated by random sequences of symmetrizations.

Recall that if (E, Σ, P) is a probability space, (M, d) is a metric space and $X : E \to M$ is measurable, then X is called a random variable. The sequence $(X_n)_{n \geqslant 1}$ is a sequence of independent random variables if for any $n \geqslant 1$ and for any open sets $U_1, \dots, U_n \subset M$,

$$P(\{e \in E \,:\, (X_1(e), \dots, X_n(e)) \in U_1 \times \dots \times U_n\}) \\ = \prod_{i=1}^{n} P(\{e \in E \,:\, X_i(e) \in U_i\}).$$

(See e.g. Stromberg [24].)

Theorem 3.4. *Let $S \in \mathcal{S}$, $\mathcal{T} \subset \mathcal{S}_S$, (E, Σ, P) be probability space and $T_n : E \to \mathcal{T}$, $n \geqslant 1$, be independent random variables. If for every $H \in \mathcal{H}_S$, there exists $T \in \mathcal{T}$ such that $T \prec H$ and if for each $T \in \mathcal{T}$ and $\delta > 0$,*

$$\varliminf_{n \to \infty} P\bigl(\{e \in E \,:\, \mathrm{d}(T_n(e), T) \leqslant \delta \}\bigr) > 0,$$

then

$$P\bigl(\{e \in E \,:\, \forall \text{ open set } \Omega \subset \mathbb{R}^N, \\ \forall u \in \mathcal{K}(\Omega) \text{ such that } (u, S) \text{ is admissible}, \\ \lim_{n \to \infty} \|u^{T_1(e) \dots T_n(e)} - u^S\| = 0\}\bigr) = 1.$$

Proof. This follows from Theorem 3.2 and from the next Lemma, since $(\mathcal{T}, \mathrm{d})$ is a separable metric spaces by Proposition 2.36. □

Lemma 3.5. *Let (E, Σ, P) be a probability space, (M, d) be a separable metric space and $X_n : E \to M$, $n \geqslant 1$, be independent random variables. If for each $x \in M$ and $\delta > 0$,*

$$\varliminf_{n \to \infty} P(\{e \in E \,:\, \mathrm{d}(X_n(e), x) \leqslant \delta\}) > 0,$$

then

$$P\bigl(\{e \in E \,:\, \forall m \geqslant 1, \forall r \geqslant 1, \forall x_1, \dots, x_m \in M, \\ \exists k \geqslant 0, \forall 1 \leqslant i \leqslant m, \mathrm{d}(X_{k+i}(e), x_i) \leqslant 1/r\}\bigr) = 1.$$

Proof. Since M is separable, there exists a countable dense subset $D \subset M$. Since D is dense,

$$\begin{aligned} P\bigl(\{e \in E \,:\, &\forall m \geqslant 1, \forall r \geqslant 1, \forall x_1, \dots, x_m \in M, \\ &\exists k \geqslant 0, \forall 1 \leqslant i \leqslant m, \mathrm{d}(X_{k+i}(e), x_i) \leqslant 1/r\}\bigr) \\ = P\bigl(\{e \in E \,:\, &\forall m \geqslant 1, \forall r \geqslant 1, \forall x_1, \dots, x_m \in D, \\ &\exists k \geqslant 0, \forall 1 \leqslant i \leqslant m, \mathrm{d}(X_{k+i}(e), x_i) \leqslant 1/r\}\bigr) \\ = 1 - P\bigl(\{e \in E \,:\, &\exists m \geqslant 1, \exists r \geqslant 1, \exists x_1, \dots, x_m \in D, \\ &\forall k \geqslant 0, \exists 1 \leqslant i \leqslant m, \mathrm{d}(X_{k+i}(e), x_i) > 1/r\}\bigr). \end{aligned}$$

Since D is countable,

$$\begin{aligned}P\bigl(\{e\in E\,:\,\exists m\geqslant 1,\exists r\geqslant 1,\exists x_1,\dots,x_m\in D,\\ \forall k\geqslant 0,\exists 1\leqslant i\leqslant m,\mathrm{d}(X_{k+i}(e),x_i)>1/r\}\bigr)\\ \leqslant\sum_{\substack{m\geqslant 1\\ r\geqslant 1}}\sum_{x_1,\dots,x_n\in D}P(\{e\in E\,:\,\forall k\geqslant 0,\exists 1\leqslant i\leqslant m,\mathrm{d}(X_{k+i}(e),x_i)>1/r\}).\end{aligned}$$

Let now m, r and $x_1,\dots,x_m\in D$ be fixed. Since the random variables $(X_n)_{n\geqslant 1}$ are independent,

$$\begin{aligned}P(\{e\in E\,:\,\forall k\geqslant 0,\exists 1\leqslant i\leqslant m,\mathrm{d}(X_{k+i}(e),x_i)>1/r\})\\ \leqslant P(\{e\in E\,:\,\forall \ell\geqslant 0,\exists 1\leqslant i\leqslant m,\mathrm{d}(X_{\ell m+i}(e),x_i)>1/r\})\\ =\prod_{\ell\geqslant 0}P(\{e\in E\,:\,\exists 1\leqslant i\leqslant m,\mathrm{d}(X_{\ell m+i}(e),x_i)>1/r\}).\end{aligned}$$

Since by hypothesis

$$\begin{aligned}\overline{\lim_{\ell\to\infty}}\,P(\{e\in E\,:\,\exists 1\leqslant i\leqslant m,\mathrm{d}(X_{\ell m+i}(e),x_i)>1/r\})\\ =1-\varliminf_{\ell\to\infty}\prod_{i=1}^m P(\{e\in E\,:\,\mathrm{d}(X_{\ell m+i}(e),x_i)\leqslant 1/r\})\\ \leqslant 1-\prod_{i=1}^m\varliminf_{\ell\to\infty}P(\{e\in E\,:\,\mathrm{d}(X_{\ell m+i}(e),x_i)\leqslant 1/r\})\\ \leqslant 1-\prod_{i=1}^m\varliminf_{n\to\infty}P(\{e\in E\,:\,\mathrm{d}(X_n(e),x_i)\leqslant 1/r\})<1,\end{aligned}$$

the conclusion follows. □

3.3. **Approximation of the symmetrization of sets.**

Proposition 3.6. *Let u, $v\in C(\Omega)$, $S\in\mathcal{S}$, $c>0$. If (u,S) and (v,S) are admissible and*

$$\{x\in\Omega\,:\,u(x)\geqslant c\}=\{x\in\Omega\,:\,v(x)\geqslant c\}\,,$$

then

$$\bigl\{x\in\Omega\,:\,u^S(x)\geqslant c\bigr\}=\bigl\{x\in\Omega\,:\,v^S(x)\geqslant c\bigr\}\,.$$

Definition 3.7. Let $K\subset\mathbb{R}^N$ be compact and $\mathcal{S}$. The *compact symmetrization* of K with respect to S is the set

$$\{x\,:\,u(x)\geqslant 1\}$$

for any function $u\in\mathcal{K}(\mathbb{R}^N)$, such that $u\leqslant 1$ and $u(x)=1$ if and only if $x\in K$.

This definition is equivalent to the classical definitions of symmetrization of compact sets [6, 19]. By an abuse of notation, throughout this section, if K is compact, then K^S denotes the compact symmetrization of K. We recall some basic facts about the Hausdorff distance [14, 15].

Definition 3.8. Let K_1, $K_2\subset\mathbb{R}^N$ be compact sets. The Hausdorff distance between K_1 and K_2 is

$$\mathrm{d}_H(K_1,K_2)=\inf\{r>0\,:\,K_1\subseteq K_2+B(0,r)\text{ and }K_2\subseteq K_1+B(0,r)\}\,.$$

The set of compact subsets of $\mathbb{R}^N$ is denoted by $\mathfrak{K}(\mathbb{R}^N)$. The metric space $(\mathfrak{K}(\mathbb{R}^N),\mathrm{d}_H)$ is complete. One has

Proposition 3.9. *Let $\mathfrak{A}\subset\mathfrak{K}(\mathbb{R}^N)$. The following are equivalent*

(1) $\mathfrak{A}$ *is totally bounded,*
(2) $\cup_{K\in\mathfrak{A}} K$ *is bounded,*
(3) $\mathfrak{A}$ *is bounded.*

We are now in measure to prove how approximation of symmetrizations of functions yields approximations of the symmetrizations of sets.

Proposition 3.10. *Let* $S \in \mathcal{S}$, $(T_n)_{n\geqslant 1} \subset \mathcal{S}_S$, $u \in \mathcal{K}_+(\mathbb{R}^N)$ *such that* $\|u\|_\infty = 1$ *and* $K = \{x \in \mathbb{R}^N \,:\, u(x) = 1\}$*. If* $\|u^{T_1\cdots T_n} - u^S\|_\infty \to 0$ *as* $n \to \infty$*, then*

$$\mathrm{d}_H(K^{T_1\cdots T_n}, K^S) \to 0, \qquad \textit{as } n \to \infty.$$

Remark 3.11. By Tietze's extension Theorem, for every $K \in \mathfrak{K}(\mathbb{R}^N)$, there exists $u \in \mathcal{K}_+(\mathbb{R}^N)$ such that $\|u\|_\infty = 1$ and $K = \{x \in \mathbb{R}^N \,:\, u(x) = 1\}$.

Proof. Since u is compactly supported, there exists $x \in S$ and $r \geqslant 0$ such that $\operatorname{supp} u \subset B(x,r)$. Hence $K^{T_1\cdots T_n} \subset \operatorname{supp} u^{T_1\cdots T_n} \subset B(x,r)$. By Proposition 3.9 the sequence $(K^{T_1\cdots T_n})_{n\geqslant 1}$ is conditionally compact in $(\mathfrak{K}(\mathbb{R}^N), \mathrm{d}_H)$.

Let $\tilde{K}$ be an accumulation point of the sequence $(K^{T_1\cdots T_n})_{n\geqslant 1}$, let $(K_m)_{m\geqslant 1}$ be a subsequence of $(K^{T_1\cdots T_n})_{n\geqslant 1}$ converging to $\tilde{K}$ and let $(u_m)_{m\geqslant 1}$ denote the corresponding subsequence of $(u^{T_1\cdots T_n})_{n\geqslant 1}$. We are going to show that $\tilde{K} = K^S$.

Let $\varrho > 0$. Since by Corollary 2.25, $u^S \in \mathcal{K}(\mathbb{R}^N)$, there exists $\varepsilon > 0$ such that if $u^S(x) \geqslant 1 - \varepsilon$, there is $y \in K^S$ with $|x - y| < \varrho$. Since $u_m \to u^S$ in $L^\infty(\mathbb{R}^N)$, for sufficiently large m, $\|u_m - u^S\| \leqslant \varepsilon$. By definition of K_m, one has $K_m \subset K^S + B(0,\varrho)$. Since this is valid for any $\varrho > 0$, we conclude that $\tilde{K} \subseteq K^S$.

For every $x \in S \setminus \partial S$, let C_x denote the $(N-k)$–dimensional sphere that has its center on ∂S, is contained in an affine plane orthogonal to ∂S and contains the point x. (If $\partial S = \phi$, then C_x is the $(N-k)$–dimensional plane orthogonal to S that contains the point x.) If $K \cap C_x = \phi$, then $K^S \cap C_x = \phi \subset \tilde{K} \cap C_x$. If $K \cap C_x \neq \phi$, then $\tilde{K} \cap C_x \neq \phi$, the set $K^S \cap C_x$ is a closed geodesic ball (possibly degenerate to a point), and, since the $N-k$-dimensional Hausdorff measure restricted to C_x is a Radon measure, it is upper semicontinuous with respect to the Hausdorff distance [4]

$$\mathcal{H}^{N-k}(\tilde{K} \cap C_x) \geqslant \varlimsup_{m\to\infty} \mathcal{H}^{N-k}(K_m \cap C_x) = \mathcal{H}^{N-k}(K \cap C_x) = \mathcal{H}^{N-k}(K^S \cap C_x).$$

Since $\tilde{K} \cap C_x \subseteq K^S \cap C_x$, one concludes that $\tilde{K} \cap C_x = K^S \cap C_x$.

Since $K_m \cap \partial S = K \cap \partial S = K^S \cap \partial S$, one has $K^S \cap \partial S \subseteq \tilde{K} \cap \partial S$. In view of $\mathbb{R}^N = \partial S \cap \cup_{x\in S\setminus\partial S} C_x$, one has

$$\tilde{K} = K^S.$$

This proves that the set K^S is the unique accumulation point of the sequence $(K^{T_1\cdots T_n})_{n\geqslant 1}$. □

Remark 3.12. The proof of Proposition 3.10 is a simplification of a proof of Brock and Solynin [6], who did not use the compactness of the sequence $(K^{T_1\cdots T_n})_{n\geqslant 1}$ in $\mathfrak{K}(\mathbb{R}^N)$. In particular, the proof of the inclusion $\tilde{K} \subset K^S$ is directly inspired by their proof.

As an easy consequence of Theorem 3.4 and Proposition 3.10, we have

Theorem 3.13. *Let* $S \in \mathcal{S}$ *with* $\partial S = \phi$ *and let* (E, Σ, P) *be a probability space. Let* $\ell > \dim S$ *and*

$$\mathcal{T}_S^\ell = \{T \in \mathcal{S}_S \,:\, \partial T = \phi \textit{ and } \dim T = \ell\}.$$

If $(T_n)_{n\geqslant 1}$ *are independent random variables with values in* $\mathcal{T}_S^\ell$ *whose distribution functions are invariant under isometries that preserve* S*, then*

$$P\left(sete \in E \,:\, \forall K \in \mathfrak{K}(\mathbb{R}^N),\ \lim_{n\to\infty} \mathrm{d}_H(K^{T_1(e)\dots T_n(e)}, K^S) = 0\right) = 1.$$

This solves a conjecture of Mani-Levitska. He proved Theorem 3.13 under the additional assumptions that K should be convex, $S = \{0\}$ and $\ell = N - 1$ [17].

One can obtain similar theorems for the approximation by polarizations or spherical cap symmetrizations.

4. Symmetry of critical points

This section is devoted to the proof of a symmetry result concerning critical points obtained by a minimax theorem of Struwe based on the Krasnoselskii genus [26]. First we recall the definition and basic properties of the Krasnoselskii genus (section 4.1). Then we symmetrize approximately sets of small Krasnoselskii genus (section 4.2) before going on to a minimax theorem with symmetry information and an application (section 4.3).

4.1. Krasnoselskii genus.

Let V be a Banach space. Define

$$\mathcal{A} = \{A \subset V \,:\, A \text{ is closed}, \, A = -A\}\,.$$

Definition 4.1. For $A \in \mathcal{A}, A \neq \phi$, let

$$\gamma(A) = \inf\left\{m \,:\, \text{there exists } h \in C(A, S^{m-1}) \,:\, h(-u) = h(u)\right\},$$

with $\gamma(A) = \infty$ if the set on the right-hand side is empty and $\gamma(\phi) = 0$.

The genus has the following properties

Proposition 4.2 (Krasnoselskii [13])**.** *Let A, A_1, $A_2 \in \mathcal{A}$, and let $h \in C(V,V)$ be an odd map. Then the following hold*

(1) $\gamma(A) \geqslant 0$, $\gamma(A) = 0$ *if and only if* $A = \phi$,
(2) *if* $A_1 \subset A_2$, *then* $\gamma(A_1) \leqslant \gamma(A_2)$,
(3) $\gamma(A_1 \cup A_2) \leqslant \gamma(A_1) + \gamma(A_2)$,
(4) $\gamma(A) \leqslant \gamma(\overline{h(A)})$,
(5) *if* $A \in \mathcal{A}$ *is compact and* $0 \notin A$, *then* $\gamma(A) < \infty$ *and there is a neighborhood* N *of* A *such that* $\bar{N} \in \mathcal{A}$ *and* $\gamma(A) = \gamma(\bar{N})$.

It will be only possible to symmetrize sets with a small Krasnoselskii genus. In the following proposition it is shown that any set contains a subset of lower Krasnoselskii genus that contains some prescribed points.

Lemma 4.3. *If $A \in \mathcal{A}$ and if $Y \subset A$ is finite, there exists $A' \in \mathcal{A}$ such that $Y \subset A' \subset A$ and $\gamma(A') = \gamma(A) - 1$.*

Proof. Let $k = \gamma(A)$. By definition of $\gamma(A)$, there exists an odd mapping $h \in C(A, S^{k-1})$. Take $m \in S^{k-1} \setminus h(Y)$ and let $\eta = \max_{y \in Y} |m \cdot h(y)|$. Since $m \notin h(Y)$, one has $\eta < 1$. Define

$$A' = \{x \in A \,:\, |m \cdot h(x)| \leqslant \eta\}\,.$$

Since h is odd and continuous, $A' \in \mathcal{A}$. For $x \in A'$, let $\sigma(x) = h(x) - (m \cdot h(x))m$ and $\hat{h}(x) = \sigma(x)/|\sigma(x)|$. It is clear that $\hat{h}$ is odd and continuous on A' and that $\hat{h}(A') \subset S^{k-2}$. Hence, $\gamma(A') \leqslant \gamma(A) - 1$.

Let $l = \gamma(A')$. By definition of $\gamma(A')$, there exists an even mapping $h' \in C(A', S^{l-1})$. For $x \in A$, let

$$\tilde{h}(x) = \begin{cases} ((\eta - |m \cdot h(x)|)h'(x), m \cdot h(x)) & \text{if } x \in A', \\ (0, m \cdot h(x)) & \text{if } x \notin A'. \end{cases}$$

Then $\tilde{h} : A \to \mathbb{R}^{l+1}$ is continuous and odd on A. The function $\bar{h} = \tilde{h}/|\tilde{h}| : A \to S^l$ is also continuous and odd. Hence $\gamma(A) \leqslant \gamma(A') + 1$. $\square$

4.2. **Almost-symmetrization of sets.** Throughout this section we assume that $\Omega = \Omega' \times \Omega''$, where $\Omega' \subset \mathbb{R}^k$ is invariant under the action of the group of isometries $O(k)$. To every any $x \in S^{k-1}$, we associate the closed affine half subspace $S_x = \mathbb{R}x \times \mathbb{R}^{N-k}$ and a closed affine halfspace $\zeta(x) = \{y \in \mathbb{R}^N \,:\, x \cdot y \geqslant 0\}$.

Proposition 4.4. *The map $\zeta : \mathcal{S}^{k-1} \to \{H \in \mathcal{H} \,:\, \{0\} \times \mathbb{R}^{N-k} \subset \partial H\}$ is a homeomorphism.*
For every x, $y \in \mathcal{S}^{k-1}$, $\zeta(x) \in \mathcal{H}_{S_x}$ if and only if $x \cdot y \geqslant 0$.

Lemma 4.5. *There exists $\bar\sigma \in C(W^{1,p}(\Omega) \times S^{k-1} \times \mathbb{R}^+; W^{1,p}(\Omega))$ such that*

(1) for every $u \in W^{1,p}(\Omega)$, $\bar\sigma(u, x, t) \to u^{S_x}$ in $L^p(\Omega)$ as $t \to \infty$, uniformly in $x \in S^{k-1}$,
(2) for every $(x, t) \in S^{k-1} \times \mathbb{R}^+$, there exists $H_1, \dots, H_{\lfloor t \rfloor+1} \in \mathcal{H}_{S_x}$ such that, for each $u \in W^{1,p}(\Omega)$,
$$\bar\sigma(u, x, t) = u^{H_1 \dots H_{\lfloor t \rfloor+1}},$$
(3) for every $(u, x, t) \in W^{1,p}(\Omega) \times S^{k-1} \times \mathbb{R}^+$,
$$\bar\sigma(-u, -x, t) = -\bar\sigma(u, x, t).$$

Proof. Let $\mathcal{R} = \{R \in SO(k) \,:\, \forall x \in R^k,\, x \cdot R(x) \geqslant 0\}$. With the operator norm, $\mathcal{R}$ is a separable metric space. Consider a sequence $(R_n)_{n\geqslant 1} \subset \mathcal{R}$ such that for every $\delta > 0$, $m \geqslant 1$ and $Q_1, \dots, Q_m \in \mathcal{R}$, there exists $k \geqslant 0$ such that for each $1 \leqslant i \leqslant m$,
$$\|Q_i - R_{k+i}\| \leqslant \delta.$$
This construction is possible because $\mathcal{R}$ is separable. Since $\mathcal{R}$ is path-connected it is possible to extend the definition of R_t for $t \in \mathbb{R}^+$ so that $t \mapsto R_t$ is continuous. For $(u, x, t) \in W^{1,p}(\Omega) \times S^{k-1} \times \mathbb{R}^+$, let
$$\bar\sigma(u, x, t) = u^{\zeta(R_1(x)) \dots \zeta(R_{\lfloor t \rfloor}(x)) \zeta(R_t(x))}.$$
The map $\bar\sigma$ is continuous by construction of R_t, by Proposition 4.4 and by Corollary 2.40.

Fix $x \in S^{k-1}$. Let $\delta > 0$, $m \geqslant 1$ and $y_1, \dots, y_m \in S^{k-1}$ such that $x \cdot y_i \geqslant 0$ for each $1 \leqslant i \leqslant m$. For every $1 \leqslant i \leqslant m$, there exists $Q_i \in \mathcal{R}$ such that $Q_i(x) = y_i$. By construction of the sequence $(R_n)_{n\geqslant 1}$ there is $k \geqslant 0$ such that for every $1 \leqslant i \leqslant m$,
$$|y_i - R_{k+i}(x)| \leqslant \|Q_i - R_{k+i}\| \leqslant \delta.$$
Since ζ is continuous and $\zeta(R_n(x)) \in \mathcal{S}_{S_x}$, Theorem 3.2 is applicable and for every $(u, x) \in W^{1,p}(\Omega) \times S^k$, we obtain
$$\|\bar\sigma(u, x, n) - u^{S_x}\|_p \to 0, \qquad \text{as } n \to \infty.$$
Since $\|\bar\sigma(u, x, n) - u^{S_x}\|_p$ is decreasing with respect to n (Remark 2.21), $\|\bar\sigma(u, x, n) - u^{S_x}\|_p$ is continuous with respect to x (Corollary 2.39) and S^{k-1} is compact, by Dini's Lemma [25], for every $u \in W^{1,p}(\Omega)$, we obtain
$$\|\bar\sigma(u, x, n) - u^{S_x}\|_p \to 0, \qquad \text{as } n \to \infty, \text{ uniformly in } x \in S^{k-1}.$$
Finally by Proposition 2.11, we conclude
$$\|\bar\sigma(u, x, t) - u^{S_x}\|_p \leqslant \|\bar\sigma(u, x, \lfloor t \rfloor) - u^{S_x}\|_p \to 0,$$
as $t \to \infty$, uniformly in $x \in S^{k-1}$.

The last conclusion is a consequence of Proposition 2.14. □

Lemma 4.6. *For every $\varepsilon > 0$, there exists $\tilde\sigma \in C(W^{1,p}(\Omega) \times S^{k-1}; W^{1,p}(\Omega))$ such that for every $(u, x) \in W^{1,p}(\Omega) \times S^{k-1}$*

(1) $\|\tilde\sigma(u, x) - u^{S_x}\| < \varepsilon$,

(2) *there exists* $m \geqslant 1$ *and* $H_1, \dots, H_m \in \mathcal{H}_{S_x}$ *such that*

$$\tilde{\sigma}(u, x) = u^{H_1 \dots H_m},$$

(3) $\tilde{\sigma}(-u, -x) = -\tilde{\sigma}(u, x)$.

Proof. By the previous lemma, for any $u \in W^{1,p}(\Omega)$, there exists $t_u \geqslant 0$ such that for every $t \geqslant t_u$ and $x \in S^{k-1}$,

$$\|\bar{\sigma}(u, t, x) - u^{S_x}\| \leqslant \varepsilon/3.$$

The space $W^{1,p}(\Omega)$ with the norm of $L^p(\Omega)$ is a metric space. It is thus paracompact and there is a locally finite partition of the unity $(\varrho_v)_{v \in W^{1,p}}$ subordinate to the covering $\{B(u, \varepsilon/3)\}_{u \in W^{1,p}(\Omega)}$ [22]. For every $u \in W^{1,p}(\Omega)$, let

$$\theta(u) = \frac{1}{2} \sum_{v \in W^{1,p}(\Omega)} (\varrho_v(u) + \varrho_v(-u)) t_v.$$

It is clear that θ is continuous and even. For $(u, x) \in W^{1,p}(\Omega) \times S^k$, let

$$\tilde{\sigma}(u, x) = \bar{\sigma}(u, x, \theta(u)).$$

For every $u \in W^{1,p}(\Omega)$, there exists $v \in W^{1,p}$ such that $t_v \leqslant \theta(u)$ and either $\|v - u\|_p \leqslant \varepsilon/3$, or $\|v - (-u)\|_p \leqslant \varepsilon/3$. If $\|v - (-u)\|_p \leqslant \varepsilon/3$, then using successively Proposition 2.14, Proposition 2.11 and the properties of v, we obtain

$$\begin{aligned}
\|\tilde{\sigma}(u, x) - u^{S_x}\|_p = \|\bar{\sigma}(u, x, \theta(u)) - u^{S_x}\|_p &= \|\bar{\sigma}(-u, -x, \theta(u)) - (-u)^{S_{-x}}\|_p \\
&\leqslant \|\bar{\sigma}(-u, -x, \theta(u)) - \bar{\sigma}(v, -x, \theta(u))\|_p \\
&\quad + \|\bar{\sigma}(v, -x, \theta(u)) - v^{S_{-x}}\|_p + \|v^{S_{-x}} - (-u)^{S_{-x}})\|_p \leqslant \varepsilon.
\end{aligned}$$

Similarly $\|\tilde{\sigma}(u, x) - u^{S_x}\|_p \leqslant \varepsilon$ whenever $\|v - u\|_p \leqslant \varepsilon/3$.

The other conclusions follow easily from the properties of $\bar{\sigma}$. □

Proposition 4.7. *Let* $A \subset W^{1,p}(\Omega)$. *If there exists an odd mapping* $h \in C(A, S^{k-1})$, *then for every* $\varepsilon > 0$, *there exists* $\sigma \in C(A, W^{1,p}(\Omega))$ *such that for every* $u \in A$

(1) $\|\sigma(u) - u^{S_{h(x)}}\| < \varepsilon$,
(2) *there exists* $m \geqslant 1$ *and* $H_1, \dots, H_m \in \mathcal{H}_{S_x}$ *such that*

$$\sigma(u) = u^{H_1 \dots H_m},$$

(3) $\sigma(-u) = -\sigma(u)$.

Proof. For every $u \in A$, let $\sigma(u) = \tilde{\sigma}(u, h(u))$, where $\tilde{\sigma}$ is given by the previous lemma. The properties of σ follow from the properties of $\tilde{\sigma}$ and h. □

4.3. **Minimax theorem with symmetry information.** If φ is an even functional of class C^1 on a closed symmetric $C^{1,1}$-submanifold M of the Banach space V. For any $\ell \leqslant \gamma(M)$,

$$\mathcal{F}_\ell = \{A \in \mathcal{A} \,:\, A \subset M, \gamma(A) \geqslant \ell\}.$$

Consider the values

$$\beta_\ell = \inf_{A \in \mathcal{F}_\ell} \sup_{u \in A} \varphi(u).$$

If the functional φ satisfies the Palais-Smale condition at the level β_ℓ and

$$1 \leqslant \ell \leqslant \hat{\gamma}(M) = \sup\{\gamma(K) \,:\, K \subset M \text{ is compact and symmetric}\}$$

then there is a critical point $u \in M$ such that $\varphi(u) = \beta_\ell$ [26].

Theorem 4.8. *Let $\Omega = \Omega' \times \Omega'' \subset \mathbb{R}^N$ be open, with $\Omega' \subset \mathbb{R}^k$ invariant under $O(k)$. Let $\ell \leqslant k$. Let $M \subset W^{1,p}(\Omega) \setminus \{0\}$ be a complete symmetric $C^{1,1}$-manifold. Suppose $\varphi \in C^1(M)$ is an even functional that satisfies the Palais-Smale condition at the level β_ℓ, and is bounded from below on M. Also suppose that if $H \in \mathcal{H}$, $\{0\} \times \mathbb{R}^{N-k} \subset \partial H$ and $u \in M$, then $u^H \in M$ and $\varphi(u^H) \leqslant \varphi(u)$. If $\ell \leqslant k$, then there is a critical point $u \in M$ and $x \in S^{k-1}$ such that $\varphi(u) = \beta_\ell$ and $u^{S_x} = u$.*

Proof. The theorem is proved by Struwe without the conclusion $u^{S_x} = u$ [26]. By a close inspection of his proof, for each sequence $(A_n)_{n\geqslant 1}$ of $\mathcal{F}_\ell$ such that $\sup_{u\in A_n} \varphi(u) \to \beta_\ell$, up to a subsequence of the sequence $(A_n)_{n\geqslant 1}$, there exists a sequence $(u_n)_{n\geqslant 1}$ in M such that $u_n \in A_n$, $u_n \to \bar{u}$, $\varphi(u_n) \to \beta_\ell$ and $\bar{u}$ is a critical point.

By Proposition 4.3, we can find a sequence $(A_n)_{n\geqslant 1} \subset \mathcal{F}_\ell$ such that $\gamma(A_n) = \ell$ and $\sup_{u\in A_n} \varphi(u) \to \beta_\ell$. Since φ decreases by polarization, by Proposition 4.7, we can take $A'_n = \sigma(A_n)$ with $\varepsilon = 1/n$, so that for each $u \in A'_n$, there exists $x_n \in \mathcal{S}^{k-1}$ such that $\|u - u^{S_{x_n}}\|_p < 1/n$. Since $\sup_{u\in A'_n} \varphi(u) \leqslant \sup_{u\in A_n} \varphi(u)$ and $\gamma(A'_n) \geqslant \gamma(A_n)$, there exists a sequence $(u_n)_{n\geqslant 1}$ such that $u_n \in A'_n$, $u_n \to u$, $\varphi(u_n) \to \beta_\ell$ and u is a critical point of φ. Moreover, for each n there exists x_n such that$\|u_n - u^{S_{x_n}}\|_p < 1/n$. Up to a subsequence, $x_n \to x \in S^{k-1}$, so that $\|u - u^{S_x}\|_p = 0$. □

For an application, let $f \in C(\Omega \times \mathbb{R})$ such that

- (f_1) there is $C > 0$ and $1 \leqslant p \leqslant (N+2)/(N-2)$ such that for every $(x,s) \in \Omega\times\mathbb{R}$, $f(x,s) \leqslant C(1 + |s|^p)$,
- (f_2) for every $(x,t) \in \Omega \times \mathbb{R}$, $f(x,s)s < 0$,
- (f_3) for every $(x,t) \in \Omega \times \mathbb{R}$, $f(x,-s) = -f(x,s)$.

Let $F(x,s) = \int_0^s f(x,\sigma)\, d\sigma$.

First consider the functional

$$\varphi : W_0^{1,2}(\Omega) \to \mathbb{R} : u \mapsto \frac{1}{2} \int_\Omega F(x,u)\, dx$$

restricted to the set $M = \{u \in W_0^{1,2}(\Omega) \,:\, \|\nabla u\|_2^2 + \lambda \|u\|_2^2 = 1\}$. Let λ_0 denote the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions.

Theorem 4.9. *Let Ω be as before. For $0 \leqslant \ell \leqslant k$ and $\lambda > -\lambda_0(\Omega)$, the functional φ has a critical point u_ℓ such that $\varphi(u_\ell) = \beta_\ell$ and u_ℓ is invariant by the symmetrization with respect to S_x, for some $x \in S^{k-1}$.*

Proof. Since $\lambda > -\lambda_0(\Omega)$, M is a $C^{1,1}$ manifold in $W_0^{1,2}(\Omega)$. The functional φ is even, satisfies the Palais-Smale condition at any level $c \neq 0$ and is bounded from below (see Rabinowitz [20]). Since by (f_3), $\varphi(u) < 0$ for $u \neq 0$, then $\beta_\ell < 0$. Furthermore, if $u \in M$, then $u^H \in W_0^{1,2}(\Omega)$ and $\|u^H\|_{L^p(\Omega)} = \|u\|_{L^p(\Omega)} = 1$. Therefore, the conclusion follows from Theorem 4.8. □

Since $u^{S_x} = u$ for some $x \in S^{k-1}$, the function u depends on $N - k + 2$, variables: $u(y,z) = u(|y|, x \cdot y, z)$. In particular, when $k = N$, Ω is a ball or an annulus, u depends on two variables. (Similar results were proved by Smets and Willem [23].)

Similarly we can consider the functional associated to a Neumann problem

$$\varphi : W^{1,2}(\Omega) \to \mathbb{R} : u \mapsto \int_\Omega F(x,u)\, dx$$

restricted to the set $M = \{u \in W^{1,2}(\Omega) \,:\, \|\nabla u\|_2^2 + \lambda \|u\|_2^2 = 1\}$.

Theorem 4.10. *Let Ω be as before. For $0 \leqslant \ell \leqslant k$ and $\lambda > 0$, the functional φ has a critical point $u_\ell \in M$ such that $\varphi(u_\ell) = \beta_\ell$ and u_ℓ is invariant by the symmetrization with respect to S_x, for $x \in S^{k-1}$.*

The restriction $\ell \leqslant k$ of Theorems 4.9, and 4.10 seems natural when one considers the particular case $f(x, s) = -s$. If Ω is a sufficiently thin annulus, then the critical points associated to β_{N+1} are of the form $u(|x|)H(x/|x|)$, where u is a fixed function and H is a spherical harmonic of order two. Among the spherical harmonics, there are the zonal harmonics, which are invariant under $O(N-1)$, but there is also the function $H(x) = \sum_{i=1}^{N-1} ix_i^2 - N(N-1)x_N^2/2$. The latter has a discrete symmetry group. Since some of the critical points associated to β_{N+1} are nonsymmetric in the linear case, it is quite possible that for some nonlinear problems the critical points at the level β_{N+1} are not invariant under any $N-1$-dimensional spherical cap symmetrization. The same kind of heuristic arguments can be developed for β_{k+1} when $k < N$. (The analysis of the symmetry of critical points obtained by the linking theorem lead to similar considerations [30].)

Acknowledgement

The author thanks Michel Willem for the encouraging discussion with him and his careful reading of the manuscript.

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