

UNIVERSAL APPROXIMATION OF SYMMETRIZATIONS BY POLARIZATIONS

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ABSTRACT. Any symmetrization (Schwarz, Steiner, cap or increasing rearrangement) can be approximated by a universal sequence of polarizations which converges in L^p norm for any admissible function in L^p , $1 \leq p < +\infty$, and uniformly with respect to admissible continuous functions. A new Pólya-Szegő inequality is proved for the increasing rearrangement.

1. INTRODUCTION

A symmetrization $*$ (or rearrangement) maps any function $u : \Omega \rightarrow \mathbb{R}$ to a more symmetrical function $u^* : \Omega^* \rightarrow \mathbb{R}$. Under some technical assumptions, it has the following properties:

$$(C) \quad \int_{\Omega} f(u) \, dx = \int_{\Omega^*} f(u^*) \, dx,$$

$$(HL) \quad \int_{\Omega^*} |u^* - v^*|^p \, dx \leq \int_{\Omega} |u - v|^p \, dx,$$

$$(PS) \quad \int_{\Omega} |\nabla u|^p \, dx \leq \int_{\Omega^*} |\nabla u^*|^p \, dx.$$

Rearrangements are used to prove the symmetry and the existence of solutions of some variational problems.

The symmetrization is defined for sets before being extended to functions. The inequalities (C) and (HL) are straightforward consequences of the preservation of both the inclusions and the measure after rearrangement of sets. Pólya-Szegő's inequality (PS) involves the gradient, and a proof that uses directly the definition of the rearrangement relies on an isoperimetric inequality for sets and on the coarea formula. The inequality (PS) can also be proved using approximation by polarizations, as Brock and Solynin did [5] and as we do in Corollary 6.3. Lieb and Loss [9] and Baernstein [3] deduced it from Riesz-like inequalities that they obtained using approximations.

The first approximation of symmetrizations by simpler symmetrizations appeared in the proof of the classical isoperimetric Theorem. A well-chosen sequence of Steiner symmetrization of a convex body converges with respect to the Hausdorff distance to a ball of the same volume. Mani-Levitska proved that random sequences of Steiner symmetrizations converge [10]. Brascamp, Lieb and Luttinger

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approximated in measure the Schwarz symmetrization of sets by lower order symmetrizations in order to prove a generalized Riesz rearrangement inequality [4, 9]. Sarvas approximated the symmetrization of sets by spherical cap and Steiner symmetrizations [12], while Baernstein [3], Brock and Solynin used polarizations [5]. This result was extended to the cap symmetrization by Smets and Willem [13].

For all the methods of approximations of symmetrizations by polarizations quoted above, the sequence of polarizations depends on the function that has to be symmetrized. Our main result (Theorem 4.4) is that there exists a sequence that depends neither on the function nor on the function space, and that the increasing rearrangement can also be approximated by polarizations. This symmetrization coincides in the one-dimensional case, with the rearrangement introduced by Carbou [6] and studied by Alberti [1]. The increasing rearrangement inequalities allow to prove the existence of solutions of variational problems that increase in some direction. Badiale obtained with the moving plane method similar results concerning the monotonicity of solutions of some elliptic systems [2].

By the same method, we prove that cap and Steiner symmetrizations approximate higher order Steiner and cap symmetrization. The approximating symmetrizations can be of any order, but they must be compatible with the symmetrization that they approximate.

The definitions and basic properties of symmetrizations (Section 2) and of polarizations (Section 3) are recalled. Section 4 is devoted to the proof of the main result. Finally the method is briefly extended to approximation by symmetrization (Section 5) and a Pólya-Szegő inequality is proven (Section 6).

2. SYMMETRIZATIONS

The Lebesgue outer measure on \mathbb{R}^N is denoted \mathcal{L}^N , the Hausdorff k -dimensional outer measure on \mathbb{R}^N is denoted by \mathcal{H}^k , the scalar product between x and y , $x \cdot y$, and the Euclidean norm $|x| = \sqrt{x \cdot x}$. If $x \in \mathbb{R}^N$, $0 \leq r \leq +\infty$, then

$$B(p, r) = \{x \in \mathbb{R}^N \mid |x - p| < r\}.$$

The characteristic function of a set A is denoted χ_A , and the symmetric difference of the sets A and B is denoted $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

Definition 2.1. If T is a proper affine subspace of \mathbb{R}^N , the *Steiner symmetrization* of a set $A \subset \mathbb{R}^N$ with respect to T is the unique set A^T for which the following holds: for any $p \in T$, if L is the maximal affine subspace orthogonal to T that contains p , then

$$A^T \cap L = B(p, r) \cap L,$$

where r is defined such that $\mathcal{H}^{N-\dim T}(B(p, r) \cap L) = \mathcal{H}^{N-\dim T}(A \cap L)$.

Remark 2.2. The Schwarz symmetrization with respect to $p \in \mathbb{R}^N$ is the Steiner symmetrization with respect to $\{p\}$, it is also sometimes the Steiner symmetrization with respect to a straight line [9, 11].

Definition 2.3. A set $S \subset \mathbb{R}^N$ is a *closed half affine subspace* of \mathbb{R}^N if it is a closed affine halfspace with respect to its affine span. The boundary of S with respect to its affine span is denoted ∂S and its dimension is $\dim S = \dim \partial S + 1$.

Definition 2.4. If S is a closed half affine subspace \mathbb{R}^N and $0 < \dim S < N$, the *cap symmetrization* of a set A with respect to S is the unique set A^S for which the

following holds: $A^S \cap \partial S = A \cap \partial S$ and, for any $q \in \partial S$ and any $s > 0$, if L is the maximal affine subspace orthogonal to ∂S that contains q , and p is the only point in the intersection $S \cap (L \cap \partial B(q, s))$, then

$$A^S \cap (L \cap \partial B(q, s)) = B(p, r) \cap (L \cap \partial B(q, s)),$$

where $0 \leq r \leq +\infty$ is defined by $\mathcal{H}^{N-\dim S}(B(p, r) \cap (L \cap \partial B(q, s))) = \mathcal{H}^{N-\dim S}(A \cap (L \cap \partial B(q, s)))$.

Definition 2.5. Let $A \subseteq \mathbb{R}^N$ and $v \in \mathbb{R}^N \cap \partial B(0, 1)$,

$$c_v(A) = \mathcal{H}^1(\{x \in A \mid v \cdot x > 0\}) - \mathcal{H}^1(\{x \in \mathbb{R}^N \setminus A \mid v \cdot x \leq 0\})$$

if the formula makes sense and $c_v(A) = -\infty$ otherwise.

Definition 2.6. The increasing rearrangement of $A \subset \mathbb{R}^N$ with respect to $v \in \mathbb{R}^N \cap \partial B(0, 1)$ is the unique set $A^{(v, \infty)}$ such that for any $x \in \mathbb{R}^N$,

$$(x + v\mathbb{R}) \cap A^{(v, \infty)} = \{y \in (x + v\mathbb{R}) \mid v \cdot y > c_v(A \cap (x + v\mathbb{R}))\}.$$

In the sequel, $*$ denotes indifferently a Steiner or cap symmetrization, or an increasing rearrangement.

For any sets $A, B \subseteq \mathbb{R}^N$,

$$(2.1) \quad A \subseteq B \implies A^* \subseteq B^*.$$

Proposition 2.7. *If $A \subset \mathbb{R}^N$ is measurable, then A^* is measurable.*

Proof. If $*$ is the increasing rearrangement with respect to $v \in \mathbb{R}^N \cap \partial B(0, 1)$, A^* can be written by definition as

$$A^* = \{x \in \mathbb{R}^N \mid v \cdot x > c(A \cap (x + v\mathbb{R}))\}.$$

Fubini's theorem implies that the function $x \mapsto c_v(A \cap (x + v\mathbb{R}))$ is measurable. Hence A^* is measurable. The proof is similar for the Steiner and cap symmetrizations. \square

Definition 2.8. A set A is admissible for a Steiner or cap symmetrization $*$ if A is measurable, and $\mathcal{L}^N(A) < +\infty$. If $*$ is the increasing rearrangement with respect to v , A is admissible if and only if

$$\mathcal{L}^N(A \Delta \{x \in \mathbb{R}^N \mid v \cdot x > 0\}) < \infty.$$

If $A \subset B \subset \mathbb{R}^N$ are admissible sets, then

$$(2.2) \quad \mathcal{L}^N(B^* \setminus A^*) = \mathcal{L}^N(B \setminus A).$$

When the sets A and B may have infinite measure, which is the case for the increasing rearrangement, the second condition is more restrictive than the preservation of the measure of sets ($\mathcal{L}^N(A) = \mathcal{L}^N(A^*)$). If $A, B \subset \mathbb{R}^N$ are admissible sets, then

$$(2.3) \quad \mathcal{L}^N(B^* \setminus A^*) \leq \mathcal{L}^N(B \setminus A).$$

Notation 2.9. For any function $u : \Omega \rightarrow \mathbb{R}$ and $c \in \mathbb{R}$, we write

$$\{u > c\} = \{x \in \Omega \mid u(x) > c\}.$$

Definition 2.10. The symmetrization of a function $u : \mathbb{R}^N \rightarrow \mathbb{R}$ is, for $y \in \mathbb{R}^N$,

$$u^*(y) = \sup \{c \in \mathbb{R} \mid y \in \{u > c\}^*\}.$$

Proposition 2.11. *If $*$ is a rearrangement and $u : \mathbb{R}^N \rightarrow \mathbb{R}$ is measurable, then u^* is measurable.*

Proof. Since $*$ is monotone on sets, $\{u^* > c\} = \cup_{n \geq 1} \{u > c + 1/n\}^*$, and the conclusion follows from Proposition 2.7. \square

Definition 2.12. A function $u : \Omega \rightarrow \mathbb{R}$ is *admissible* if it is measurable and if for any c with $\text{ess inf } u < c < \text{ess sup } u$ the set $\{u > c\}$ is admissible.

Definition 2.13. If $*$ is a Steiner or cap symmetrization and $1 \leq p < +\infty$, we define $L_*^p(\mathbb{R}^N) = L_+^p(\mathbb{R}^N)$ to be the set of nonnegative functions of $L^p(\mathbb{R}^N)$, $C_*(\mathbb{R}^N) = C_{0,+}(\mathbb{R}^N)$ to be the set of nonnegative continuous functions whose limit at the infinity is 0 and $\mathcal{K}_*(\mathbb{R}^N) = \mathcal{K}_+(\mathbb{R}^N)$ to be the set of nonnegative continuous functions with compact support. If $*$ is the increasing rearrangement with respect to $v \in \mathbb{R}^N \cap \partial B(0, 1)$, we write

$$L_*^p(\mathbb{R}^N) = \{u : \mathbb{R}^N \rightarrow [0, 1] \mid \exists h : \mathbb{R} \rightarrow \mathbb{R} \text{ such that } h \text{ is increasing,} \\ (h - \chi_{\mathbb{R}^+}) \in L^p(\mathbb{R}), \text{ and } (h(v \cdot) - u) \in L^p(\mathbb{R}^N)\},$$

$$C_*(\mathbb{R}^N) = \{u : \mathbb{R}^N \rightarrow [0, 1] \mid u \text{ is continuous,} \\ \lim_{v \cdot x \rightarrow -\infty} u(x) = 0, \text{ and } \lim_{v \cdot x \rightarrow +\infty} u(x) = 1\}$$

and

$$\mathcal{K}_*(\mathbb{R}^N) = \{u \in C_*(\mathbb{R}^N) \mid \exists h : \mathbb{R} \rightarrow \mathbb{R} \text{ such that } h \text{ is increasing,} \\ (h - \chi_{\mathbb{R}^+}) \text{ has compact support, and } (h(v \cdot) - u) \in \mathcal{K}(\mathbb{R}^N)\}.$$

The functions of the sets $\mathcal{K}_*(\mathbb{R}^N)$, $C_*(\mathbb{R}^N)$ and $L_*^p(\mathbb{R}^N)$ are all admissible. If $u \in \mathcal{K}_*(\mathbb{R}^N)$, then $u^* \in \mathcal{K}_*(\mathbb{R}^N)$ and, for any $c \in \mathbb{R}$,

$$(2.4) \quad \{u > c\}^* = \{u^* > c\}.$$

The preservation of inclusions (2.1) and measure (2.2) imply that the symmetrization of functions is nonexpansive for any L^p norm, $1 \leq p \leq \infty$. The ideas of Crowe, Zweibel and Rosenbloom [7], and of Alberti [1] can be generalized to encompass the case of the increasing rearrangement.

Proposition 2.14. For any $1 \leq p \leq \infty$ and for any admissible functions u and v , we have

$$\|u^* - v^*\|_p \leq \|u - v\|_p.$$

Proof. If $1 \leq p < \infty$, for any admissible functions u and v , we have

$$\int_{\Omega} |u - v|^p dx = \int_{\sigma \leq \tau} (\mathcal{L}^N(\{v > \tau\} \setminus \{u > \sigma\}) \\ + \mathcal{L}^N(\{u > \tau\} \setminus \{v > \sigma\})) p(p-1) |\sigma - \tau|^{p-2} d\sigma d\tau.$$

The properties (2.4) and (2.3) yield the conclusion. If $p = \infty$, the conclusion follows from preservation of inclusions. \square

3. POLARIZATIONS

Definition 3.1. A *polarizer* is a closed affine halfspace of \mathbb{R}^N .

Remark 3.2. A set H is a polarizer if and only if there exists $a \in \mathbb{R}^N$, $|a| = 1$ and $b \in \mathbb{R}^N$ such that $H = \{x \in \mathbb{R}^N \mid a \cdot x \geq b\}$.

Notation 3.3. If $x \in \mathbb{R}^N$ and $H \subseteq \mathbb{R}^N$ is a polarizer, x_H denotes the reflection of x with respect to ∂H . Using the description of Remark 3.2, $x_H = x - 2(a \cdot x - b)a$.

Definition 3.4. The *polarization* of a function $u : \mathbb{R}^N \rightarrow \mathbb{R}$ by a polarizer H is the function $u^H : \mathbb{R}^N \rightarrow \mathbb{R}$ defined by

$$u^H(x) = \begin{cases} \max \{u(x), u(x_H)\} & \text{if } x \in H, \\ \min \{u(x), u(x_H)\} & \text{if } x \notin H. \end{cases}$$

Remark 3.5. The polarization is also called two-point rearrangement. The polarization by H is the natural extension of the cap symmetrization with respect to $S = H$ when $\dim S = N$ (compare with Definition 2.4).

Notation 3.6. If T is an affine subspace, let

$$\mathcal{H}_T = \{H \subset \mathbb{R}^N \mid H \text{ is a polarizer and } T \subset H\},$$

if S is a closed half affine subspace, let

$$\mathcal{H}_S = \{H \subset \mathbb{R}^N \mid H \text{ is a polarizer, } S \subset H \text{ and } \partial S \subset \partial H\}$$

and, if $v \in \mathbb{R}^N \cap \partial B(0, 1)$, let

$$\mathcal{H}_{(v, \infty)} = \{H \subset \mathbb{R}^N \mid H \text{ is a polarizer and } a = v \text{ in the description of Remark 3.2}\}.$$

For any symmetrization $*$ and for any function $u : \mathbb{R}^N \rightarrow \mathbb{R}$,

$$u = u^* \iff \forall H \in \mathcal{H}_*, u = u^H.$$

Polarizations satisfy the properties (2.1) and (2.2). Thus they are nonexpansive. For $u \in L^p_*(\mathbb{R}^N)$ and $H \in \mathcal{H}_*$, the inequality

$$\|u^H - u^*\|_p = \|u^H - (u^*)^H\|_p \leq \|u - u^*\|_p$$

suggests that well chosen polarization can approximate the symmetrization $*$ for a given function. The proof goes in two steps: first the relative compactness of any sequence of iterated polarizations is established (Lemma 3.7), then a convergence condition ensures the convergence to the symmetrized function (Lemma 3.9).

Lemma 3.7. *Let $u \in \mathcal{K}_*(\mathbb{R}^N)$ and $(H_m)_{m \geq 0} \subset \mathcal{H}_*$ be a sequence of polarizers and $u_m = u^{H_1 \cdots H_m}$. Then, there is $v \in \mathcal{K}_*(\mathbb{R}^N)$ and an increasing sequence $(m_k)_{k \in \mathbb{N}}$ in \mathbb{N} such that, for any $1 \leq p \leq \infty$,*

$$\lim_{k \rightarrow \infty} \|v - u_{m_k}\|_p = 0.$$

Remark 3.8. This lemma is essentially due to Brock and Solynin [5, lemmas 6.1 and 6.2], and the main part of the arguments was given by Baernstein [3]. Smets and Willem proved it for the cap symmetrization [13].

Proof. The compactness of the sequence $(u_m)_{m \geq 1}$ is proven by Arzelá-Ascoli's Theorem. The sequence is equibounded: for any polarization H , $\|u^H\|_\infty = \|u\|_\infty$ and thus, by induction, $\|u_m\|_\infty = \|u\|_\infty < +\infty$.

Secondly, the sequence is equicontinuous. Let

$$\omega_v(\delta) = \sup \{v(x) - v(y) \mid |x - y| \leq \delta\}$$

be the modulus of continuity of a function v . By a tedious analysis of the possible different cases, it can be proved that for any polarization H , $\omega_{u^H} \leq \omega_u$, and thus,

by induction, $\omega_{u_m} \leq \omega_u$. Since $u \in \mathcal{K}_*(\mathbb{R}^N)$ is uniformly continuous, the sequence is equicontinuous.

It remains to prove that the supports are uniformly bounded. For the Steiner or the cap symmetrizations, $u \in \mathcal{K}_*(\mathbb{R}^N)$ implies that, for some p in T or in ∂S , and for some $R > 0$, $\{u > 0\} \subseteq B(p, R)$. Thus, because polarizations are monotone, $\{u^H > 0\} \subseteq B(p, R)^H = B(p, R)$ and, by induction, $\{u_m > 0\} \subseteq B(p, R)$.

For the increasing rearrangement with respect to v , we have, for some $c \in \mathbb{R}$,

$$\{u > 0\} \subseteq \{x \in \mathbb{R}^N \mid v \cdot x > c\}$$

and

$$\{u^H > 0\} \subseteq \{x \in \mathbb{R}^N \mid v \cdot x > c\}^H = \{x \in \mathbb{R}^N \mid v \cdot x > c\}.$$

Therefore we have

$$\{u_m > 0\} \subseteq \{x \in \mathbb{R}^N \mid v \cdot x > c\}$$

and similarly there exists $d \in \mathbb{R}$ such that

$$\{x \in \mathbb{R}^N \mid v \cdot x > d\} \subseteq \{u_m < 1\}.$$

There exists $R > 0$ such that $u_m(x) \neq h(x)$ implies $\text{dist}(x, x+v\mathbb{R}) \leq R$. Therefore, there is a bounded set $B \subset \mathbb{R}^N$ such that $\text{supp}(u_m - h) \subset B$ for $m \in \mathbb{N}$. We conclude, by Arzela-Ascoli's Theorem that any subsequence has a subsequence converging uniformly to some $v \in \mathcal{K}_*(\mathbb{R}^N)$.

The convergence for $1 \leq p < +\infty$ follows from the convergence for $p = +\infty$ and from the fact that all the supports of the functions of the sequence $(u_m - v)$ lie in the same compact set. \square

A second lemma states that for any nonsymmetrical function, there exists a polarizer $H \in \mathcal{H}_*$ that makes it closer to its symmetrization.

Lemma 3.9. *Let $u \in \mathcal{K}_*(\mathbb{R}^N)$. If $u \neq u^*$, then there is a polarizer $H \in \mathcal{H}_*$ such that, for any $1 \leq p < +\infty$,*

$$\|u^H - u^*\|_p < \|u - u^*\|_p.$$

Remark 3.10. This lemma is due to Brock and Solynin [5] for the Steiner symmetrization and to Smets and Willem [13] for the cap symmetrization.

Proof. Since $u \neq u^*$, there exists $c > 0$ such that the set $\{u > c\} \Delta \{u^* > c\}$ is not empty. Choose a point $y \in \{u^* > c\} \setminus \{u > c\}$. There is a polarizer $H \in \mathcal{H}_*$ such that $y_H \in \{u > c\} \setminus \{u^* > c\}$. In a sufficiently small neighborhood $N \subset H$ of y , we have then

$$u^H(x) = u(x_H) > c \geq u^*(x_H) \quad \text{and} \quad u^*(x) > c \geq u(x) = u^H(x_H),$$

whence, for $p \geq 1$,

$$|u(x) - u^*(x)|^p + |u(x_H) - u^*(x_H)|^p > |u^H(x) - u^*(x)|^p + |u^H(x) - u^*(x_H)|^p.$$

If $x \in N$, the corresponding nonstrict inequality holds. The integral inequality is obtained by integration of the preceding inequality over N and of the nonstrict inequality on $H \setminus N$. \square

4. APPROXIMATION BY POLARIZATIONS

We first establish the convergence of a sequence of polarizations for a single function.

Lemma 4.1. *Let $u \in \mathcal{K}_*(\mathbb{R}^N)$, $0 < \kappa < 1$, let $(m_k)_{k \geq 1} \subset \mathbb{N}$ be an increasing sequence of indices, and $(H_m)_{m \geq 1} \subset \mathcal{H}_*$ a sequence of polarizers such that for all $k \in \mathbb{N}$,*

$$(4.1) \quad \|u_{m_k} - u^*\|_1 - \|u_{m_k}^{H_{m_k}} - u^*\|_1 \geq \kappa \sup_{H \in \mathcal{H}_*} (\|u_{m_k} - u^*\|_1 - \|u_{m_k}^H - u^*\|_1).$$

Then the sequence $u_m = u^{H_1 \dots H_m}$ converges to u^ for any $1 \leq p \leq +\infty$.*

Remark 4.2. For any function $u \in \mathcal{K}_*(\mathbb{R}^N)$, a sequence of polarizers verifying condition (4.1) can be constructed.

Remark 4.3. We use the same strategy of proof that Smets and Willem [13], except that the inequality (4.1) is weaker than imposing to (H_m) to be optimal as they do.

Proof. By Lemma 3.7, there exists a subsequence $u_{m'_k}$ of $(u_{m_k})_{k \geq 1}$ that converges to $v \in \mathcal{K}_*$ for any L^p norm. Since the rearrangement $*$ is nonexpansive,

$$\|u^* - v^*\|_p = \lim_{k \rightarrow \infty} \|u_{m'_k}^* - v^*\|_p \leq \lim_{k \rightarrow \infty} \|u_{m'_k} - v\|_p = 0,$$

and $v^* = u^*$. For any polarizer $H \in \mathcal{H}_*$, using the nonexpansiveness of polarizations and equation (4.1), we have then

$$\begin{aligned} \|u_{m'_{k+1}} - u^*\|_1 &\leq \|u_{m'_k+1} - u^*\|_1 \\ &\leq \|u_{m'_k} - u^*\|_1 + \kappa (\|u_{m'_k}^H - u^*\|_1 - \|u_{m'_k} - u^*\|_1) \\ &= (1 - \kappa) \|u_{m'_k} - u^*\|_1 + \kappa \|u_{m'_k}^H - u^*\|_1 \leq \|u_{m'_k} - u^*\|_1. \end{aligned}$$

Passing to the limit, we obtain

$$\|v - u^*\|_1 \leq (1 - \kappa) \|v - u^*\|_1 + \kappa \|v^H - u^*\|_1 \leq \|v - u^*\|_1.$$

Hence and since $u^* = v^*$, we obtain $\|v - v^*\|_1 = \|v^H - v^*\|_1$. By Lemma 3.9 this is absurd if $v \neq u^*$. Therefore the subsequence $(u_{m'_k})_{k \in \mathbb{N}}$ converges to u^* for any L^p norm. The nonexpansiveness of polarizations allows to conclude

$$\lim_{k \rightarrow \infty} \|u_k - u^*\|_p \leq \lim_{k \rightarrow \infty} \|u_{m'_k} - u^*\|_p = 0. \quad \square$$

Theorem 4.4. *For any symmetrization $*$, there exists a sequence of polarizers $(H)_{m \geq 0} \subset \mathcal{H}_*$ such that, for any $1 \leq p < \infty$, if $u \in L_*^p(\mathbb{R}^N)$, the sequence*

$$u_m = u^{H_1 \dots H_m}$$

converges to u^ :*

$$\lim_{m \rightarrow \infty} \|u_m - u^*\|_p = 0.$$

If $u \in C_(\mathbb{R}^N)$, the sequence converges for $p = \infty$.*

Proof of Theorem 4.4. If $*$ is a Steiner or spherical cap symmetrization, first note that there is a countable set $N \subset \mathcal{K}_*(\mathbb{R}^N)$ dense in $L_*^p(\mathbb{R}^N)$ and in $C_*(\mathbb{R}^N)$ (see [14]). Choose a sequence (H_m) for which (4.1) holds for all $u \in N$. The sequence of iterated polarizations approaches the symmetrization for any $u \in N$.

Let $u \in L_*^p(\mathbb{R}^N)$ and $\epsilon > 0$. By density, there is $\tilde{u} \in N$ such that $\|u - \tilde{u}\|_p \leq \epsilon/3$. By contraction, for m sufficiently large and if $\tilde{u}_m = \tilde{u}^{H_1 \cdots H_n}$ we obtain

$$\|u_m - u^*\|_p \leq \|u_m - \tilde{u}_m\|_p + \|\tilde{u}_m - \tilde{u}^*\|_p + \|\tilde{u}^* - u^*\|_p \leq 2\|u - \tilde{u}\|_p + \|\tilde{u}_m - \tilde{u}^*\|_p \leq \epsilon.$$

If $*$ is the increasing rearrangement with respect to v and $h : \mathbb{R} \rightarrow [0, 1]$ is a continuous function such that $\text{supp}(h - \chi_{\mathbb{R}^+})$ is compact, then the same reasoning shows the convergence for any $w \in L_*^p(\mathbb{R}^N) \cap (h(v \cdot) + L^p(\mathbb{R}^N))$. Let $u \in L_*^p(\mathbb{R}^N)$ and

$$C_R = \{x \in \mathbb{R}^N \mid |(v \cdot x)v - x| \leq R\}.$$

Consider the function u_R which is equal to u on C_R and equal to h outside of it. Then $u_R \in L_*^p(\mathbb{R}^N) \cap (h(v \cdot) + L^p(\mathbb{R}^N))$ and thus $\int_{C_R} |u_m - u^*|^p dx \rightarrow 0$. Since

$$\int_{\mathbb{R}^N \setminus C_R} |u_m - u^*|^p dx \leq 2 \int_{\mathbb{R}^N \setminus C_R} |u - h|^p dx,$$

$u_m \rightarrow u^*$ follows.

The proof is similar for $u \in C_*(\mathbb{R}^N)$. \square

Remark 4.5. Theorem 4.4 implies that the symmetrization of any set can be approximated in measure and in Hausdorff distance [5, Lemma 7.2]. Conversely, if the symmetrization of any set can be approximated in measure by some fixed sequence of polarizations, then Theorem 4.4 follows by the approximation of functions in $L^p(\mathbb{R}^N)$ by simple functions.

5. APPROXIMATION BY SYMMETRIZATIONS

The method of proof of Theorem 4.4 can be extended to approximations of Steiner or cap symmetrizations by lower order Steiner or cap symmetrizations.

Definition 5.1. Let T be an affine subspace. A set of affine subspaces \mathcal{T} approximates T if, for any $T' \in \mathcal{T}$, $T \subset T'$, and for any affine subspace $T'' \subset \mathbb{R}^N$ of codimension 1 such that $T \subset T''$, there exists $T' \in \mathcal{T}$ such that $T' \subset T''$.

Theorem 5.2. Let T be an affine subspace of \mathbb{R}^N and \mathcal{T} be a set of affine subspaces. If \mathcal{T} approximates T , there is a sequence $(T_m)_{m \geq 1}$ in \mathcal{T} such that $u^{T_1 \cdots T_m} \rightarrow u^T$ for $u \in L_+^p(\mathbb{R}^N)$ or $u \in C_0(\mathbb{R}^N)$.

Definition 5.3. Let T be an affine subspace. A set \mathcal{S} of closed half affine subspaces of \mathbb{R}^N approximates T if, for any $S' \in \mathcal{S}$, $T \subset S'$, and for any affine subspace $T'' \subset \mathbb{R}^N$ of codimension 1 which is parallel to T , there exists $S' \in \mathcal{S}$ such that $\partial S' \subset T''$.

Example 5.4. If $\mathcal{T} = \{T\}$, then \mathcal{T} trivially approximates T . If $T = \{0\}$, the set of polarizers \mathcal{H}_T and the set of closed halflines containing 0 both approximate T .

Definition 5.5. Let S be a closed half affine subspace. A set \mathcal{S} of closed half affine subspaces of \mathbb{R}^N approximates S if, for any $S' \in \mathcal{S}$, $S \subset S'$ and $\partial S \subset S'$, and for any affine subspace $T'' \subset \mathbb{R}^N$ of codimension 1 which is parallel to T , there exists $S' \in \mathcal{S}$ such that $\partial S' \subset T''$.

Theorem 5.6. Let T be an affine subspace of \mathbb{R}^N (resp. S be a half affine subspace of \mathbb{R}^N) and \mathcal{S} be a set of closed half affine subspaces of \mathbb{R}^N . If \mathcal{S} approximates T (resp. S), then there exists a sequence $(S_m)_{m \geq 1}$ in \mathcal{S} such that $u^{S_1 \cdots S_m} \rightarrow u^T$ (resp. $u^{S_1 \cdots S_m} \rightarrow u^S$) for $u \in L_+^p(\mathbb{R}^N)$ or $u \in C_0(\mathbb{R}^N)$.

Proof. The proofs of Theorems 5.2 and 5.6 are similar. The proof is essentially the same as the proof of Theorem 4.4. The modifications in the lemmas are sketched for a closed half affine subspace S in Theorem 5.6. Suppose $u \in \mathcal{K}_+(\mathbb{R}^N)$. It is clear that for any $S' \in \mathcal{S}$, $u^{SS'} = u^S$ and $\|u^{S'} - u^S\|_p \leq \|u - u^S\|_p$. Therefore the sequence $\|u^{S_1 \dots S_n} - u^S\|_p$ is nonincreasing. Theorem 4.4 implies that the modulus of continuity decreases along the sequence. This allows to prove an analogue to Lemma 3.7. An analogue of Lemma 3.9 is also needed. Suppose $u \neq u^S$. Then by Lemma 3.9 there exists $H \in \mathcal{H}_S$ such that $\|u^H - u^S\|_p < \|u - u^S\|_p$. Since ∂H is parallel to ∂S and \mathcal{S} approximates S , there exists S' such that $S' \subset H$ and $\partial S' \subset \partial H'$. Hence $u^{HS'} = u^{S'}$ and

$$\|u^{S'} - u^S\|_p \leq \|u^H - u^S\|_p < \|u - u^S\|_p.$$

The remaining part of the proof is the same as the proof of Lemma 4.1 and of Theorem 4.4. \square

6. PÓLYA-SZEGÖ'S INEQUALITY

Definition 6.1. A set Ω is totally invariant with respect to a symmetrization $*$ if for any $H \in \mathcal{H}_*$, Ω is invariant under the reflection with respect to ∂H .

Definition 6.2. If $*$ is a symmetrization, Ω is a totally invariant set and $u : \Omega \rightarrow \mathbb{R}$ is a function, then the symmetrization of u is $u^* = \tilde{u}^*|_{\Omega}$, where \tilde{u} is any extension of u to \mathbb{R}^N .

The definition of the symmetrization of $u : \Omega \rightarrow \mathbb{R}$ does not depend on the extension \tilde{u} because Ω is totally invariant.

Corollary 6.3. If Ω is a totally invariant open set, $*$ is a Steiner or cap symmetrization, if $u \in W_{\text{loc}}^{1,1}(\Omega)$ is admissible, $1 < p < +\infty$ and $\nabla u \in L^p(\Omega)$, then

$$(6.1) \quad \|\nabla u^*\|_p \leq \|\nabla u\|_p.$$

Proof. Suppose first $u \in L^p_*(\Omega)$. Let u_m be the restrictions to Ω of the sequence of iterated polarizations of Theorem 4.4 applied to an extension $\tilde{u} \in L^p_*(\mathbb{R}^N)$ of u to \mathbb{R}^N . For any compactly supported smooth function $h \in \mathcal{D}(\Omega)^N$,

$$\begin{aligned} - \int_{\Omega} u^* \operatorname{div} h \, dx &= - \lim_{m \rightarrow \infty} \int_{\Omega} u_m \operatorname{div} h \, dx = \lim_{m \rightarrow \infty} \int_{\Omega} \nabla u_m h \, dx \\ &\leq \liminf_{m \rightarrow \infty} \|\nabla u_m\|_p \|h\|_{p'} = \|\nabla u\|_p \|h\|_{p'}, \end{aligned}$$

since $\|\nabla u^H\|_{\Omega,p} = \|\nabla u\|_{\Omega,p}$ [5] for any $u \in W_{\text{loc}}^{1,1}(\Omega)$ such that $\nabla u \in L^p(\mathbb{R}^N)$, and for any polarizer H . There exist thus $v \in L^p(\Omega)^N$ that is the weak limit of ∇u_m and the weak gradient of u^* .

In general, if $*$ is an increasing rearrangement, for $m \geq 3$, let

$$u_m(x) = \frac{m}{m-2} \min(\max(0, u(x) - 1/m), 1 - 2/m).$$

Since u is admissible, $u_m \in L^1_*(\Omega)$. From the first part of the proof, $\|\nabla u_m^*\|_p \leq \|\nabla u_m\|_p$. Since $\frac{m-2}{m} |\nabla u_m| \nearrow |\nabla u|$ and $\frac{m-2}{m} |\nabla u_m^*| \nearrow |\nabla u^*|$ almost everywhere, the conclusion comes from the monotone convergence Theorem. The end of the proof is similar for the Steiner and cap symmetrizations. \square

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