Merging Multiple Formations: A Meta-Formation Prospective

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***Abstract*— This paper considers the problem of merging of more than two (minimally) rigid formations which do not have any common agent to obtain a single (minimally) rigid formation in $\Re^2$ and $\Re^3$. Following previously developed strategies for sequential merging of two rigid formations, a new set of enhanced merging operations is developed. They can be performed in a formalized *meta-formation* framework, where the individual rigid formations are considered as *meta-vertices* and they can be merged into a *meta-formation*. These operations for growing meta-formations offer a level of control to the merging quality and optimality, in the sense of minimizing the number of *meta-edges* (that is, edges between different meta-vertices) required. It is also proved that all minimally rigid meta-formations in $\Re^2$ can be obtained by successively merging two or more *meta-vertices* using the proposed set of *meta-operations*.**

I. Introduction

Recently there has been growing interest in cooperative control of autonomous agents in formation [1]–[4], [8], [9], [13]. Agents are often modelled as vertices of graphs in order to investigate the information structures of formations [4], [12]. Communications and/or information flow and formation control architecture are usually modelled using either undirected or directed graphs, representing respectively, the symmetric [4] and asymmetric control and/or sensing strategies [5], [19].

One way to keep the autonomous agents in formation, i.e. as an assembled multi-agent system in which agents' relative positions are fixed, is to maintain enough of the distances between certain pairs of agents, such that all the inter-agent distances are preserved as a consequence. The formation, modelled using underlying undirected graphs, can be therefore studied using graph rigidity theory, traced back to [10]. Consequently, and for convenience in this paper, we may (with some abuse of nomenclature) use the two terms, formation and its (underlying undirected) graph, interchangeably. A framework has been established to study the information structure for control of formations with symmetric control/sensing [4], and extended to the asymmetric (directed) case [5], [19]. And for the first case, the ideas have been taken further to treat operations on rigid formations, rather than on individual agents forming the formation.

An algebra that consists of performing some basic operations on (minimally rigid) formations is introduced in [12], including examples of rejoining/splitting maneuvers. In [4], operations on formations for the problems of closing ranks, splitting and merging are studied. In particular, some results about merging formations were presented in [4], [12] and more complete results, with guiding principles to control formation merging, are found in [18]. Merging (of rigid formations) literally means combining two or more rigid formations into a single rigid formation, of course through the introduction of new distance constraints that will involve agent pairs with agents drawn from different formations among those merging. All these works focus on systematic ways of merging two formations at a time. For larger multiagent formations, adoption of a hierarchical structure is needed to be considered as well as merging of multiple (more than two) formations to form a formation of formations [17], or a meta-formation [1].

Whiteley [16] gives a detailed explanation of the merging problem using notions from rigidity theory. In fact, in the course of development of rigidity theory, merging problems are studied implicitly under the so-called "body-and-bar" framework [14], where each rigid body is equivalent to a rigid formation and they are linked (or merged) by bars. Later in [11], the testing of rigidity of a formation in $\Re^2$ is made more efficient by "gluing (or merging)" smaller rigid formations or single agents. The rigid graph theoretical results for the "body-and-bar" framework are very promising for showing the possibility of merging a collection of multiple formations and agents, and preserving the rigidity properties of the merged formation. However, as previously mentioned, existing works focus on operational merging procedures that can only cope with *two* formations at a time, and hence can only achieve a certain class of merging tasks involving multiple formations. There is clearly a gap between the existing operational procedures and the theoretical results.

The main contribution of this paper is to tackle the problem of merging multiple (more than two) disjoint autonomous formations, viewed as construction of meta-formations; and to propose a set of operations that can be used to obtain all minimally rigid meta-formations in $\Re^2$. More specifically, the work in [18] on merging two formations is extended by defining a new set of operations

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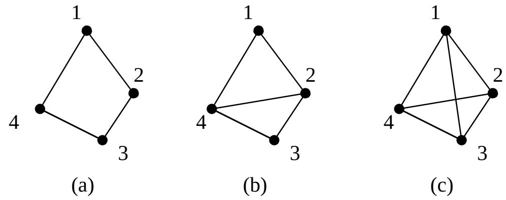

Fig. 1. Illustration of (a) non-rigid formation, (b) (minimally) rigid formation, and (c) non-minimally but rigid formation

that one can perform on multiple formations, while still maintaining rigidity of the merged formation. The operations are built upon a well-structured methodology by Henneberg [7], and the results and principles are proven to be consistent with theories developed in [14]. The paper will illustrate theories using examples in $\Re^2$, although some results are also fully or partially generalized to $\Re^3$.

The paper is organized as follows. Section II reviews the notions and some properties of graph rigidity theory, for its application to control of autonomous multiagent formations. In section III, results for merging of two minimally rigid formations are summarized and results from the bar-and-body framework are linked to the merging problem. The main results of this paper are presented in Section IV, including a set of operations that permit one to merge multiple formations, and associated theorems. Conclusions with a description of proposed future work appear in Section V. Discussion on computational aspects, and proofs of most results are omitted due to space limitations and will appear in a full length version later.

II. A Brief Review of Rigidity

In this section, we review the notions and some properties of (minimal) rigidity. Our description will be largely based on graphs. The graph modelling a formation is what is obtained when vertex position information and edge length information is thrown away. Different formations can have the same graph. A rigid graph is one for which for almost all choices of edge lengths and vertex positions for which a corresponding formation exists, the corresponding formation is rigid [19]. Intuitively, if enough of the distances between certain pairs of agents are maintained, such that all the inter-agent distances are preserved as a consequence, then the formation is said to be rigid.

Figure 1 shows several examples of two dimensional graphs, two of which are rigid and one of which is not rigid. In a non-rigid graph part of the graph can flex or move, while the rest of the graph stays still. The notion of rigidity conforms to one's normal intuition.

It proves possible in two dimensions to characterize rigidity in purely combinatorial terms, i.e. counting-type conditions related to the graph (discarding therefore the agent coordinates) can be used to conclude the rigidity or otherwise of a generic formation corresponding to the graph. This is the celebrated Laman's Theorem [19], for which no three-dimensional equivalent exists. In three dimensions, differing necessity and sufficiency conditions are known for a graph to correspond to a formation which will be rigid for generic values of the constrained inter-agent distances [15].

In some scenarios of multi-agent formation control, an information structure with a minimum number of communication links (or distance constraints) is to be exploited while preserving the rigidity of the formation. This leads to a widely used notion of *minimal rigidity*. A graph is called *minimally rigid* if it is rigid and if there exists no rigid graph with the same number of vertices and a smaller number of edges, i.e., a graph is *minimally rigid* if it is rigid and if no single edge can be removed without losing rigidity. Figure 1(b) gives an example of a minimally rigid graph.

Note here that there exists another graph theoretical notion found useful in analyzing formations, *global rigidity*, which eliminates the flip and/or flex ambiguity of the rigid graphs due to discontinuous motion. In this paper, however, we do not consider global rigidity.

III. Formation Merging Problem

In this section, we revisit the formation merging problem [4], [18]. We review the existing results for merging of two formations and the results dealing with rigidity of multiple bodies linked by bars. Where possible, we present the theorems in a meta-formation framework.

A. Merging Two Rigid Formations

A rigid formation merging problem is one of constructing a single post-merged rigid formation by adding new edges (i.e., new sensing and communication links corresponding to agent pairs between which the distances must be maintained) between two or more pre-merging rigid formations.

A recent work [18] provides a complete description of possible scenarios of merging two (but only two) minimally rigid formations to obtain a single minimally rigid formation, respectively, both in $\Re^2$ and $\Re^3$. A strategy is developed based on simplification of the merging problem to a problem of growing a minimally rigid graph. In summary, the strategy presented in [18] is built upon a Henneberg Construction (HC) procedure (explained below) to grow one of the two formations to include the vertices of the other, such that when the two formations share a sufficient number of common vertices, the rigidity of the (merged) formation is guaranteed by Lemma 1 below. Based on this strategy, three principles are provided to control the merging efficiently and optimally, in the sense of minimizing the number of added edges and the number of vertices incident to these edges. Any possible scenario of merging two given minimally rigid formations can be handled using a combination of these three principles. This paper adopts the framework and strategy in [18] as summarized below:

Lemma 1: [18] If two minimally rigid formations $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ in $\Re^d$ $d \in \{2, 3\}$ satisfy $|V_c| \geq d$ and $|E_c| = d|V_c| - d(d+1)/2$, where $V_c = V_1 \cap V_2$ and $E_c = E_1 \cap E_2$, then the (meta-) formation $G_1 \cup G_2$ is minimally rigid.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be underlying rigid graphs of two rigid formations, possibly with common vertices and/or edges. The *merging problem* is to find a set of new edges E_{new} such that the resulting graph $G' = (V', E')$, where $V' = V_1 \cup V_2$ and $E' = E_1 \cup E_2 \cup E_{new}$, is rigid. Note that E_{new} can be the empty set. We further note that the above definition can be easily generalized to encompass minimal rigidity. An *optimal procedure* that can solve the merging problem is one which minimizes both $|E_{new}|$ *and* the number of vertices in V' incident to the edges in E_{new}. Consider two minimally rigid graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ in $\Re^d$ $d \in \{2, 3\}$. If $V_c = V_1 \cap V_2$ satisfies $|V_c| \geq d$, Lemma 1 indicates that $G_1 \cup G_2$ is already minimally rigid and we do not need to add any more edges. Therefore, we only need to consider the case where $|V_c| < d$ and hence $n_{new} = d + 1 - |V_c| > 0$. In this case, we look for a *systematic* way for *optimal* selection of a vertex set $V_{new} \subseteq V_2 \backslash V_1$ with $|V_{new}| = n_{new}$ and an edge set E_{new} such that $G_1' = (V_1', E_1') = (V_1 \cup V_{new}, E_1 \cup E_{new})$ is minimally rigid.

In the strategy of [18], G_1 is grown to G_1' in n_{new} steps, where in each step we add a single vertex and a certain set of edges incident on this vertex. Let us denote the resultant graph in step m ($m \in \{1, ..., n_{new}\}$) by $\bar{G}_1(m)$, e.g., $\bar{G}_1(n_{new}) = G_1'$ and let $\bar{G}_1(0) = G_1$. In order to produce the graphs $\bar{G}_1(m)$ ($m \in \{1, ..., n_{new}\}$) and eventually G_1', [18] follows a well-known procedure in the literature which is called the Henneberg construction (HC) [4] and its extensions which deals with merging globally rigid formations [18].

The HC is a systematic way of constructing, from any given minimally rigid graph $G = (V, E)$ in $\Re^d$, ($d \in \{2, 3\}$), a larger minimally rigid graph $G' = (V', E')$ $V \subset V'$ in $m' = |V'| - |V|$ steps. The HC produces a sequence of graphs $\bar{G}(m) = (\bar{V}(m), \bar{E}(m))$ ($m \in \{0, ..., m'\}$), where $\bar{G}(0) = G$ and $\bar{G}(m') = G'$, which is called a Henneberg sequence (HS). Each $\bar{G}(m)$ ($m \in \{1, ..., n'\}$) is obtained in step m of the HC and is proven to be minimally rigid once one of the following two normal HC operations is used at each step m [4]:

- Vertex addition: Adding a new vertex i and d edges between i and d other vertices in $\bar{V}(m-1)$.
- Edge splitting: Removing an edge $(j, k) \in \bar{E}(m-1)$ and then adding a new vertex i together with $d+1$ edges incident on i, two of which are (i, j) and (i, k).

B. Formation Merging in a Meta-Formation Prospective

Merging of formations naturally resembles some graphical equivalences, such as gluing of rigid subgraphs (clusters) [11], which treats $\Re^2$ problems; or linking rigid bodies [14], which treats problems in $\Re^3$. The formulation in [14] uses the so-called *body-bar-joint* framework, where joints, bodies and bars correspond to vertices, rigid formations and edges connecting rigid formations, respectively. In this framework, each *body* can be considered as a rigid object that in $\Re^3$ has six degrees of freedom (DOF), three translational and three rotational. The bars used to link rigid bodies are called external bars. Intuitively, the merging process can be simply thought of as adding (external) bars to eliminate extra DOFs until the linked bodies only have six DOFs in total, making a single rigid body. Since each (independent) bar removes one DOF while connecting two bodies, one needs six external bars to link two bodies without losing rigidity. This result is consistent with the propositions in [18], stating that one needs at least six independent edges to merge two disjoint (minimally) rigid formations. The result in [11] can also be formulated in the *body-bar-joint* framework.

In this paper, considering the formation control applications aspect, we prefer a *meta-formation framework* view of the pure graph theoretical results. A meta-formation $F_m = (V_m, E_m)$ is a formation of meta-vertices $V_m = \{G_1, \ldots, G_{|V_m|}\}$ that are connected by a set of meta-edges $E_m = \{e_1, \ldots, e_{|E_m|}\}$. Each *meta-vertex* G_i is a rigid formation. If G_i contains only a single agent (or a pair of connected agents in $\Re^3$), it is called a *trivial meta-vertex*. Otherwise, G_i is called a *non-trivial meta-vertex*. The main difference between a non-trivial meta-vertex and a vertex representing a point agent is that the non-trivial meta-vertex has a two state descriptor (position and orientation), while the vertex (point agent) only has position but no orientation.

Before we proceed with further discussions using meta-formation framework, we to make two global remarks applying to the remaining parts of the paper.

Remark 1: In this paper, we present the results in a meta-formation prospective, which is a high-level abstraction of "formation of formations". We assume, unless otherwise stated, that: (a) all the formations (meta-vertices) are rigid; (b) no agent belonging to a *non-trivial* meta-vertex is incident on more than 2 meta-edges in $\Re^d$ ($d \in \{2, 3\}$). Note that the merging strategy presented in previous subsection implicitly guarantees this assumption in $\Re^2$.

Remark 2: Given that the problem of merging trivial meta-vertices can be simply treated as addition of agents to formation using HC vertex additions, we do not consider the trivial meta-vertices in the sequel. Therefore unless otherwise stated, we assume that all the meta-vertices are non-trivial disjoint rigid formations (i.e. each meta-vertex has at least d vertices in $\Re^d$ ($d \in \{2, 3\}$) and there is no common vertex between any pair of meta-vertices).

We restate in our terminology two fundamental theorems which are first established in *body-bar-joint* frameworks [11], [14], which are a development of Laman's theorem. Note both theorems below are obtained under the assumption stated in Remark 1. Analogous but stronger results exist in [6] which removes this assumption.

Theorem 1: Consider a meta-formation $F_m = (V_m, E_m)$ in $\Re^2$ satisfying the assumptions stated in Remark 1, where V_m is a set of disjoint non-trivial meta-vertices representing rigid formations and E_m is the set of meta-edges, F_m is rigid if and only if there exists subset $E_m' \subseteq E_m$ of F_m such that:

- $|E_m'| = 3|V_m| - 3$.
- There is no non-empty subset $E_m'' \subset E_m'$ such that $|E_m''| > 3|V_m''| - 3$, where V_m'' is the set of meta-vertices that are incident on meta-edges of E_m''.

Theorem 2: [11] A meta-formation $F_m = (V_m \cup V_t, E)$ containing at least 2 meta-vertices in $\Re^2$ satisfying the assumption of Remark 1, where V_m is the set of disjoint non-trivial meta-vertices, V_t is the set of trivial meta-vertices (single agents) and E_m is the set of meta-edges, is rigid if and only if there exists subset $E'_m \subseteq E_m$ of F_m such that:

- $E'_m \subseteq E_m$ of F_m such that $|E'_m| = 3|V_m| + 2|V_t| - 3$.
- There is no non-empty subset $E'' \subset E'_m$ of F_m such that $|E''_m| > 3|V''_m| + 2|V''_t| - 3$, where V''_n is the set of meta-vertices that are incident on meta-edges of E''_m and V''_t is the set of trivial meta-vertices that are incident on meta-edges of E''_m.

We shall end this section with a motivating example to show the linkage of the HC-based operational procedure and the results generated from the meta-formation framework. Consider the simple merging of three minimally rigid formations in $\Re^2$; the HC-based procedures can be performed by first merging two of the three formations with three meta-edges, and then merging the post-merged meta-formation with the other formation to form a single minimally rigid meta-formation after inserting another three meta-edges. Theorem 1 states that if and only if there are six meta-edges between the three meta-vertices, and between each two meta-vertices there is at most three meta-edges, the resulting framework is minimally rigid. It is thus easy to see that the formation obtained using the HC-based procedure satisfies the necessary and sufficient condition of Theorem 1. However, one may ask that if the HC-based procedure for merging two formations is able to progressively (i.e., through a sequence of steps) obtain all possible merged formations. In the next section, we answer this question formally and illustrate the answer with some specific examples.

IV. Merging formations as growing Meta-formations

In this section, we investigate the problem of merging multiple (more than two) formations. For illustration of the essential concepts, we first consider the problem in its simplest form of merging three minimally rigid formations in $\Re^2$. We then formalize the results in a systematic way in a meta-formation framework, introducing meta-operations. We also generalize the operations to $\Re^3$, to provide a flavor of this analysis framework.

A. Merging Three Minimally Rigid Formations in the Plane

For the specific problem raised at the end of previous section, considering the pre-merging formations as "meta-vertices", one can identify three types of possible structures from the problem of merging three rigid formations as shown in Fig. 2. Note that the figures reflect only the connectivity and the merging patterns and the actual positions of the agents are assumed to be algebraically independent. In each case, the resultant post-merged formation can be considered as a meta-formation with the three pre-merging formations as its meta-vertices and the edges used in merging these meta-vertices as "meta-edges". The three types of merging structures and the corresponding three types of post-merged

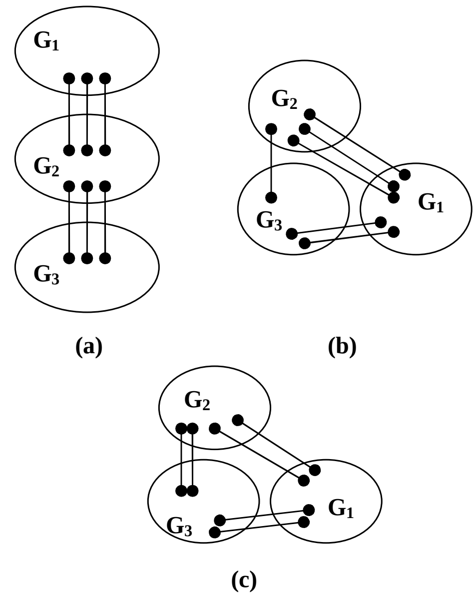

Fig. 2. Illustration of merging of three formations in $\Re^2$: (a) Serial, (b) Unbalanced, (c) Balanced

meta-formations are differentiated by their meta-degrees, which is the number of meta-edges incident on the formation. The practical implication about implementing a meta-edge is that it may require long-distance communication, or communication and control of a hybrid type with a more complicated protocol than for the "normal" case represented by edges between pairs of agents belonging to a single formation. In the case of merging, certain agents are nominated to maintain meta-edges and coordinate. This hierarchical structure is similar to that is proposed in [17].

Lemma 2: Each meta-formation $F_m = (V_m, E_m)$ in Fig. 2, where $V_m = (G_i,\ i = 1, 2, 3)$ is a set of minimally rigid formations (meta-vertices) and E_m is the set of (six) meta-edges, is minimally rigid.

For the convenience of explaining Fig. 2, we refer to the merging structure found in Fig. 2 (a) as *serial-merging*, for the formation G_2 is incident to 6 meta-edges and each of the formations G_1 and G_3 has 3 meta-edges, while in the case of Fig. 2(c), there is a better balance: each formation (meta-vertex) maintains 4 meta-edges. We refer to the merging patterns found in Fig. 2(b) and (c) as *unbalanced-merging* and *balanced-merging*, respectively.

We attempt to use the existing results for merging formations [4], [12], [18] to construct the merging of three formations in Fig. 2. Essentially, the operations presented in [4], [18] for merging two disjoint minimally rigid formations can be reformulated as a single operation, called *meta-addition*, in the meta-formation framework as described in the following lemma:

Lemma 3: Let $F'_m = (V'_m, E'_m)$ in $\Re^d (d = 2, 3)$ be a rigid meta-formation satisfying the assumption of Remark 1 and let G_i be a rigid formation and let F_m be the new meta-formation obtained by a *meta-addition* operation, that is, a set S of $d(d+1)/2$ meta-edges incident on G_i and F'_m.

Then the new meta-formation $F_m = F'_m \cup G_i \cup S$ is rigid. Moreover, F_m is minimally rigid if and only if both the meta-formation F'_m and the meta-vertex G_i are minimally rigid.

We find that: *serial-merging* can be easily constructed by merging the pairs (G_1, G_2) and (G_2, G_3) sequentially, using Lemma 3. This can be viewed in the meta-formation framework by considering first G_1 as a meta-formation by itself and a meta-vertex G_2 is added using meta-addition, and subsequently a meta-vertex G_3 is added. Similarly, *unbalanced-merging* can be obtained by two consecutive/progressive meta-addition operations (refer to Lemma 3) to add G_2 to G_1 which is treated as a meta-formation by itself, and then to add G_3 to the meta-formation consisting of G_1 and G_2. Note that the second meta-addition involves all three formations. However, the strategies given in [18] and summarized as meta-addition fails to construct a *balanced-merging* structure. Therefore, to obtain the balanced-merging structure by HC operations, the set of operations has to be expanded to a fuller set of operations that can be performed for merging that involves any number of formations; indeed, they need to be able to create any configuration type whose rigidity can be validated using Theorem 1.

B. Extending Henneberg Constructions

In this subsection, we seek to solve the problem of using a single merging operation to create the merging structure which could not otherwise be created using *meta-addition*. Our approach is to combine multiple steps of normal HC operations performed on formations into a single meta-operation that can be operated between multiple meta-vertices within a meta-formation.

Observe that the main difference between the sample balanced-merging structure (c), and the other two (a, b) as shown in Fig. 2, is that one edge ends up re-distributed from between G_1 and G_2 to between G_2 and G_3. This suggests that one needs to look at procedures such that, when G_3 is merged, the operation would allow removing some of the existing meta-edges and allow G_3 to be connected with four edges. Obviously this would be an edge-splitting operation as opposed to vertex addition. Hence, a new operation is proposed when merging the third formation to two merged formations, as illustrated below for merging three minimally rigid formations G_1, G_2 and G_3, and summarized in Lemma 4. Let us explain this new operation using standard HC ideas.

Following Lemma 3, G_1 and G_2 are merged by connecting them with 3 meta-edges, e_1, e_2 and (k, l), see Fig. 3(a). Formation G_3 is to be merged next and it is assumed that G_3 consists of at least three vertices, i, j, s, and interconnected by implicit or explicit edges denoted by dotted lines.

Refer to Fig. 3(b) and (c):

- At the first step, edge splitting is used when adding vertex j of G_3 to G', with new edges $(j, m), (j, k)$ and (j, l) and existing meta-edge (k, l) removed.
- At the second step, edge splitting is applied again to the newly added meta-edge (j, k), and observing the existence of an implicit (or explicit) edge (i, j) due to the rigidity property of G_3, only two new edges (i, k) and (i, r) have to be added so that the new vertex i of G_3 is added to $G' \cup \{j\}$. From Lemma 1 we conclude that the formations G' and G_3 are merged into a single minimally rigid formation in $\Re^2$.

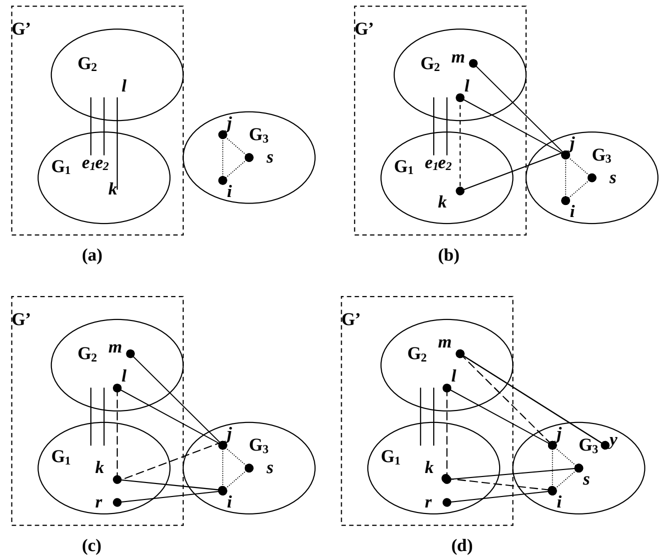

Fig. 3. Illustration of implementation of a meta-1-splitting operation in $\Re^2$: (a) before performing a meta-1-splitting (b) performing first HC operation to split an existing meta-edge (l,k) (c) performing second HC operation to split a newly added meta-edge (j,k), obtaining a rigid meta-formation (d) performing additional HC operations to obtain meta-edges which do not incident on common agents

Detailed in Fig. 3(d), we further extend the Third Principle[1] of [18] to redistribute the meta-edges among connecting vertices. Applying two successive edge splitting operations to meta-edges (m, j) and (k, i), one at each time, we obtain the desired formation *balanced-merging* in Fig. 2(c).

We combine the multiple HC operations (for example, two possible combinations are described above in Fig. 3(b) and (c)) into a single meta-operation as described below in Lemma 4, termed *meta-1-splitting*, to fit into the meta-formation framework. Note that every such operation, as defined below, can be implemented by a sequence of normal HC operations.

Lemma 4: Let $F'_m = (V'_m, E'_m)$ in $\Re^d (d = 2, 3)$ be a rigid meta-formation satisfying the assumption of Remark 1, and G_i a rigid formation and let F_m be the new meta-formation obtained by a *meta-1-splitting* operation, viz, removing a meta-edge e from F'_m and adding a set S of $d(d+1)/2+1$ meta-edges between G_i and F'_m, such that two are incident on the vertices (agents) connected by e. Then the new meta-formation $F_m = F'_m \cup G_i \cup S \backslash \{e\}$ is rigid. Moreover, F_m is minimally rigid if and only if both F'_m and G_i are minimally rigid.

[1]The third principle of formation merging asserts the genericity of the merging procedure is independent of the specific vertices concerned. That is, in a given merged formation, it is possible to re-distribute, as long as the hypothesis of Remark 1 is satisfied, the edges linking the two formations to other agents without violating the (minimal or global) rigidity property.

C. Meta-Operations for Growing Rigid Meta-Formations

In order to solve more general questions involving multiple formation merging, we propose expanding the concept of combining HC operations to meta-level, following the creation of *meta-addition* (Lemma 3) and *meta-1-splitting* (Lemma 4), to a full set of meta-operations that are sufficient to obtain all minimally rigid meta-formations in $\Re^2$. The generalization of this set of meta-operations to $\Re^3$ can be found to obtain not all, but a class of minimally rigid meta-formations.

It is noted that we have implicitly used the following principle in the operation described above, which we call the "principle of encapsulation": In a meta-formation framework, each meta-vertex G_i is encapsulated and no modification is made to its internal rigid structure, so the only edges that can be split are meta-edges. This principle protects the rigidity properties of meta-vertices that could otherwise be violated by any operations performed within these meta-vertices; and it also draws a clear distinction between normal HC operations and meta-operations.

As an extension of the meta-1-splitting operation, we note that one could perform edge splitting to further remove one more existing meta-edge. In this case, where a total of two existing meta-edges are removed after merging, the operation is referred to as *meta-2-splitting* and is detailed in Lemma 5. Note that though "meta-3-splitting" may be defined analogously in $\Re^2$, which removes 3 existing meta-edges, it cannot be implemented by normal HC operations, and it can also be proved that *meta-3-splitting* is in fact not necessary for generating all minimally rigid meta-formations in $\Re^2$.

Lemma 5: Let $F'_m = (V'_m, E'_m)$ in $\Re^d (d = 2, 3)$ be a rigid meta-formation satisfying the assumption of Remark 1, and G_i a rigid formation and let F_m be the new meta-formation obtained by a *meta-2-splitting* operation, viz, removing two meta-edges e_1 and e_2 from F'_m and adding a set S of $d(d+1)/2+2$ meta-edges between G_i and F'_m, such that two are incident on the vertices (agents) connected by e_1 and two are incident on the vertices (agents) connected by e_2. Then the new meta-formation $F_m = F'_m \cup G_i \cup S \backslash \{e_1, e_2\}$ is rigid. Moreover, F_m is minimally rigid if and only if both F'_m and G_i are minimally rigid.

Fig. 4 gives an abstracted illustration of the three meta-operations for merging multiple minimally rigid formations in $\Re^2$, highlighting the differences while hiding the complicated details concerning the undirected normal HC operations. The set of meta-operations proposed for merging of multiple formations in $\Re^2$ is summarized in the following theorem. Note that all the definitions of meta-operations are given in their general form, but it can be proved that their varieties can always be implemented using combinations of normal HC operations.

Theorem 3: If F'_m is a (minimally) rigid meta-formation in $\Re^2$, then F_m obtained by performing *meta-addition, meta-1-splitting or meta-2-splitting*, to add new meta-vertex G_i (with meta-degree k $(k = 3, 4, 5)$) is also (minimally) rigid.

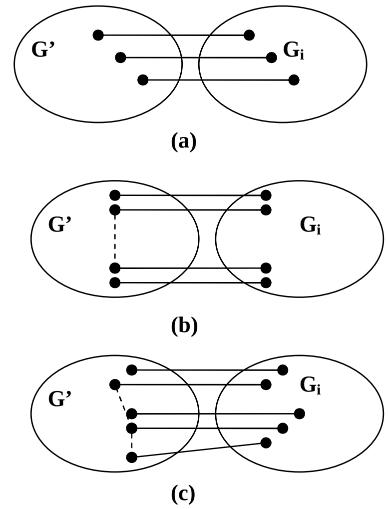

Fig. 4. Illustration of the three meta-operations for constructing minimally rigid meta-formations in $\Re^2$: (a) meta-addition, (b) meta-1-splitting, (c) meta-2-splitting

Moreover, those three meta-operations can be implemented by sequences of normal Henneberg Construction operations.

Note that in the case of merging multiple minimally rigid formations in $\Re^3$, one could have a fourth operation to remove in total 3 existing meta-edges, namely meta-3-splitting; and it is always possible to implement a meta-3-splitting in $\Re^3$ using corresponding Henneberg Construction operations.

In the sequel, we first define reverse meta-operations and then state in Theorem 4 that reverse meta-operations can always be performed for removing meta-vertices with a certain meta-degree in $\Re^2$. And it also can be proved, see Theorem 5, that all minimally rigid meta-formations in $\Re^2$ can be obtained using the set of meta-operations as stated in Theorem 3. However, whether these two results can be generalized to $\Re^3$ is still open problem.

The reverse meta-operations in $\Re^2$ can be defined in their general forms as follows: Let $F_m = (V_m, E_m)$ in $\Re^2$ be a rigid meta-formation, and G_i be a meta-vertex of set V_m, and E_m be the set of meta-edges.

- If G_i is incident to a set $E_i \subset E_m$ containing three meta-edges, a *reverse meta-addition* is removal of these three meta-edges and G_i from F_m.
- If G_i is incident to a set $E_i \subset E_m$ containing four meta-edges, a *reverse meta-1-splitting* is removal of these four meta-edges and G_i from F_m and inserting a new meta-edge e (and indeed there is at least one) between two meta-vertices that were adjacent to G_i such that the meta-formation $F'_m = (V_m \backslash G_i, E_m \backslash E_i \cup \{e\})$ is rigid.
- If G_i is incident to a set $E_i \subset E_m$ containing five meta-edges, a *reverse meta-2-splitting* is removal of these five meta-edges and G_i from F_m and inserting two new meta-edge e_1, e_2 (and indeed there are at least two) between two meta-vertices that were adjacent to G_i such that the meta-formation $F'_m = (V_m \backslash G_i, E_m \backslash E_i \cup \{e_1, e_2\})$ is rigid.

Since the meta-operations can always each be implemented using multiple steps of normal HC operations; and since normal HC operations can always be reversed [7], reverse meta-operations can always be implemented using a combination of reverse normal HC operations (a trivial example is that a reverse meta-operation can be performed in the reverse order of implementing the corresponding meta-operation). Furthermore, it also can be shown that it is always possible to perform one of the three reverse meta-operations on a minimally rigid meta-formation in $\Re^2$.

Theorem 4: Let F_m be a rigid meta-formation in $\Re^2$, and suppose it containsat least one non-trivial meta-vertex G_i with meta-degree k $(k = 3, 4, 5)$. It is always possible to perform one of the three reverse meta-operations, *reverse meta-addition, reverse meta-1-splitting or reverse meta-2-splitting* for the case of $k = 3, 4, 5$ respectively, to remove G_i and to obtain a smaller meta-formation F'_m that is rigid. Moreover, F'_m is minimally rigid if both F_m and G_i are minimally rigid.

Theorem 5: Under the assumption made in Remark 1 and 2, a meta-formation $F_m = (V_m, E_m)$ in $\Re^2$ consisting of $|V_m|$ non-trivial meta-vertices is minimally rigid if and only if it can be obtained by performing $|V_m| - 1$ meta-operations to merge $|V_m|$ disjoint meta-vertices (minimally rigid formation) G_i $(i = 1 \ldots |V_m|)$ successively.

Proof Sketch: The sufficiency holds due to Theorem 3. It follows from Theorem 1 that a minimally rigid F_m always contains a meta-vertex with a meta-degree smaller than 6, and since F_m is rigid, no meta-vertex can have a meta-degree smaller than 3. Therefore there is a meta-vertex with a degree 3, 4 or 5, which by Theorem 4 can be removed using one of the three reverse meta-operations to obtain a smaller minimally rigid meta-formation F'_m with $|V_m| - 1$ non-trivial meta-vertices. Doing this recursively one can reduce any minimally rigid meta-formation to a single meta-vertex using the reverse operations. This implies that it is possible to build any minimally rigid meta-formation starting from one meta-vertex, indeed, F_m can be obtained by applying a sequence of meta-operations (that is reversing the reverse meta-operations) on this last meta-vertex G_i.

V. Conclusion and Future work

We have provided a solution to the problem of merging multiple minimally rigid formations to obtain a single minimally rigid formation in $\Re^2$ and $\Re^3$. Following the strategies developed for sequential merging of two rigid formations [18], which constitute meta-addition in a meta-formation framework, we proposed a new set of enhanced merging operations, called meta-n-splittings ($n = 1, 2$ for $\Re^2$). The meta-addition and the meta-n-splittings can be considered as a set of meta-operations that can be performed in a formalized meta-formation framework, when the problem of merging meta-vertices arises. We engineered the operations to offer a level of control to the merging quality and optimality, by minimizing the number of meta-edges required and balancing the distribution of meta-degrees among formations. We also proved that all minimally rigid meta-formations in $\Re^2$ can be obtained by merging two or more formations using the proposed procedure and a set of three meta-operations.

This work has a few natural extensions. Paramount is the problem of merging of globally rigid formations, which is particularly relevant to certain problems in sensor network localization, including increasing the speed of current (sequential) localization algorithms. We have commenced work on this. Another direction of possible work would be solving the merging problems for multiple (minimally) persistent formations, in which directed graph representations are used. Partial results can be found in [6]. Other formation operations than merging constitute another future research topic under the meta-formation framework.

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