On the $2R$ conjecture for multi-agent systems

Vincent D. Blondel, Julien M. Hendrickx and John N. Tsitsiklis

***Abstract*— We consider a simple dynamical model of agents distributed on the real line. The agents have a limited vision range and they synchronously update their positions by moving to the average position of the agents that are within their vision range. This dynamical model was initially introduced in the social science literature as a model of opinion dynamics and is known there as the "Krause model". It gives rise to surprising and partly unexplained dynamics that we describe and discuss in this paper. One of the central observations is the $2R$-conjecture: when sufficiently many agents are distributed on the real line and have their position evolve according to the above dynamics, the agents eventually merge into clusters that have inter-cluster distances roughly equal to $2R$ (R is the vision range of the agents). This observation is supported by extensive numerical evidence and is robust under various modifications of the model. It is easy to see that clusters need to be separated by at least R. On the other hand, the unproved bound $2R$ that is observed in practice can probably only be obtained by taking into account the specifics of the model's dynamics. In this paper, we study these dynamics and consider a number of issues related to the $2R$ conjecture that explicitly uses the model's dynamics. In particular, we provide bounds for the vision range that lead all agents to merge into only one cluster, we analyze the relations between agents on finite and infinite intervals, and we introduce a notion of equilibrium stability for which clusters of equal weights need to be separated by at least $2R$ to be stable. These results, however, do not prove the conjecture. To understand the system behavior for a large agent density, we also consider a version of the model that involves a continuum of agents. We study properties of this continuous model and of its equilibria, and investigate the connections between the discrete and continuous versions.**

I. Introduction

We consider a simple multi-agent system. There are n agents and every agent has an opinion represented by a real number $x(i)$, $i \in 1, \ldots, n$. At every time step, the agents update their opinion by taking the average of all opinions distant from their own by no more than a pre-specified tolerance R:

$$x_{t+1}(i) = \frac{\sum_{j \in N_t(i)} x_t(j)}{|N_t(i)|}, \tag{1}$$

This research was supported by the National Science Foundation under grant ECS-0426453, by the Concerted Research Action (ARC) "Large Graphs and Networks" of the French Community of Belgium and by the Belgian Programme on Interuniversity Attraction Poles initiated by the Belgian Federal Science Policy Office. The scientific responsibility rests with its authors. Julien Hendrickx holds a FNRS fellowship (Belgian Fund for Scientific Research)

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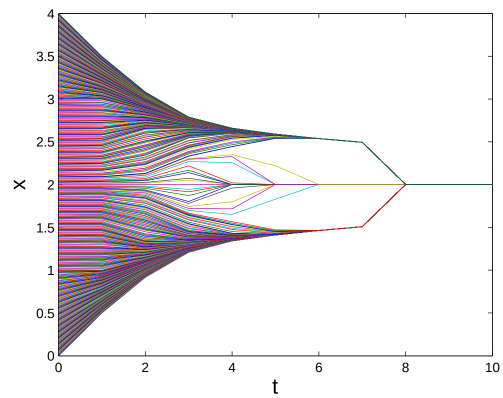
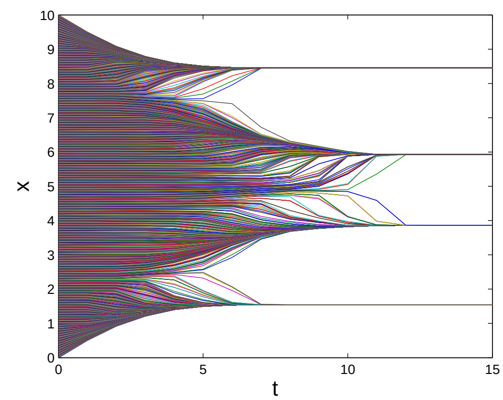


(a)



(b)

Fig. 1. Evolution with time $t \in [0, 15]$ of agent opinions initially equidistantly located on intervals of length $L = 4$ (a) and $L = 10$ (b), for $100L$ agents. Only one cluster is produced in (a), and inter-cluster distances in (b) are much larger than the vision range $R = 1$ of the agents.

where $N_t(i) = \{j\colon |x_t(j) - x_t(i)| \leq R\}$.

The model (1) was introduced by Krause [8] and has been studied in a number of contributions. It can be proved that opinions converge in finite time to clusters separated by a distance larger than R [9], [11] (For simplicity and without loss of generality we assume in the sequel that $R = 1$.) Thus, if the initial opinions lie in an interval of length L, we could, in principle, obtain up to L clusters. Fig. 1 shows, however, that for equidistant initial opinions, only one cluster appears when $L = 4$, and only four clusters appear when $L = 10$. Moreover, these clusters are separated by distances much larger than 1. Similar behaviors can be observed in the case of initial opinions that are randomly distributed [2]. This has already been observed in the literature [9], [11] and it has been conjectured that the number of clusters is asymptotically equal to $L/2$ [12]. But no explanation of this phenomenon nor any nontrivial lower bound on the inter-cluster distance have been provided so far. We analyze this issue by introducing a particular notion of equilibrium stability, under which a nontrivial lower bound on the distance between any two clusters is obtained. To better understand the system behavior for large numbers of agents, we also introduce and study a version allowing a continuous opinion distribution, and we link it to the discrete system. Note that similar phenomena are observed in a variation of this model, introduced by Weisbuch and is co-authors [4]. In Weisbuch's model, only two randomly selected agents update their opinion at each time step. If their previous opinions differ by more than a certain pre-defined distance they remain unaffected, otherwise each agent assumes a new opinion which is a weighted average of its previous opinion and of the other agent's opinion.

The opinion dynamics models considered here are similar to the linearized version of the Vicsek model [16]. In that model, agents are all moving in the plane at the same speed. Each agent updates its bearing by forming an average of its neighbors' bearings, where the neighbors are defined as those agents within a given radius. The main difference with the opinion dynamics model is thus the presence of two variables for each agent in [16], one which is involved in averaging and one that determines the neighborhood relation. A wide range of such systems have been studied in the literature, see [14] for a survey. But until now, with very few exceptions, as in [3], [7], the explicit dynamics of the neighborhood relation have not been taken into account, and the sequence of neighborhood graphs (where agents are connected if they are neighbors) is always considered as exogenous. In contrast, our analysis explicitly considers the dynamics of the model.

The linearized Vicsek model is a special case of more general multi-agent systems where every agent has a value which it updates by taking a linear (possibly convex) combination of the neighbors' values. Such systems are studied for example in [1], [5], [6], [13], [15].

Finally, opinion dynamics models also present similarities with certain rendezvous algorithms (see [10] for an example) in which the objective is to have all agents meet at one point. Agents are neighbors if their positions are within a radius R, and they update their position by taking an average of their neighbors' positions. There is an additional constraint, that once two agents become neighbors they must remain so forever and therefore need to make sure that they remain within a distance of at most R. This ensures that the connectivity of the neighborhood graph is preserved and that an initially connected set of agents is never split into smaller groups, so that all agents converge to a same point.

In Section II we explicitly use the neighborhood relation construction to analyze the equilibria to which the system (1) converges. We characterize their stability and attempt to explain the experimentally observed distances between opinion clusters at the equilibrium. We then consider in Section III a version of the model allowing a continuous opinion distribution, and prove results on the equilibria of such systems. We also explore the issue of convergence to an equilibrium. Finally, we analyze in Section IV the link between the continuous and discrete systems. Some proofs and numerical observations omitted here due to space constraints are available in an extended version of this paper [2].

II. Finite number of agents

Throughout the rest of the paper, we will assume, without loss of generality, that $R = 1$.

A. Convergence

If the number of agents is finite, it is a consequence of well known results in [1], [5], [9], [11], [13] that every x_i in the system (1) converges to a limiting value $\bar{x}_i$ and that the limiting values of two agents are either equal or separated by more than 1 [1]. We call *clusters* these limiting values to which opinions converge. Due to the particular dynamics of the system, the opinions converge to their clusters in finite time [9]: The distance between any two opinions converging to different clusters eventually becomes larger than 1, while the distance between two opinions converging to the same cluster decays to 0. So after a certain time, the interaction graph (where agents are connected if their distance is no more than 1) contains as many connected components as the number of clusters, and each of them is fully connected. In every connected component the updated opinion at the next time step is then the same for all agents so that their opinions stop changing.

Fact 1: For a finite number of agents and the dynamics (1), the opinions converge in finite time to clusters whose distance from each other is more than 1 [9].

The above results do not hold without the assumption that the number of agents is finite. Indeed, consider an infinite number of agents, all with an initial opinion which is a positive integer. Call $m(p)$ the number of agents having an initial opinion p. If $m(0) = 0$, $m(1) = 1$, and for all larger p, $m(p+1) = m(p) + 3m(p-1)$, then for each agent i and for each t, $x(t+1) = x(t) + 1/2$. All agent opinions are thus shifting indefinitely towards infinity. An infinite number of agents also allows equilibria where clusters are separated by less than 1, as in the case where there is one agent on every integer p and on every $p + \frac{1}{2}$.

Fact 2: For an infinite number of agents and the dynamics (1), the opinions may not converge or may converge to clusters whose distance from each other is less than 1.

B. Analysis of numerical experiments

Fig. 2 shows the evolution of 1000 initially random two-dimensional opinions according to (1). The distance

[1] Actually those results imply that the distance between the opinions of such agents is larger than 1 for all time steps t after a certain time T, and could thus converge to 1 when $t \to \infty$. But we explain in the sequel that the system converges in finite time, which forbids a convergence to 1.

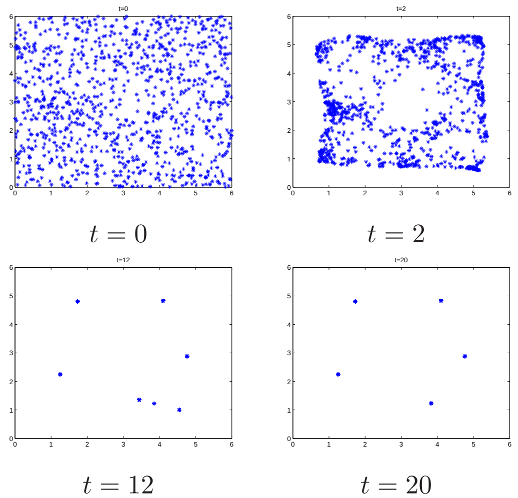


Fig. 2. Representation of the evolution of a set of 1000 two-dimensional initially random opinions according to (1). The opinions converge to clusters which are separated by more than 1. At $t = 12$, two clusters and some isolated agents in the lower right part of the figure are in a meta-stable situation. They eventually merge, as seen at $t = 20$.

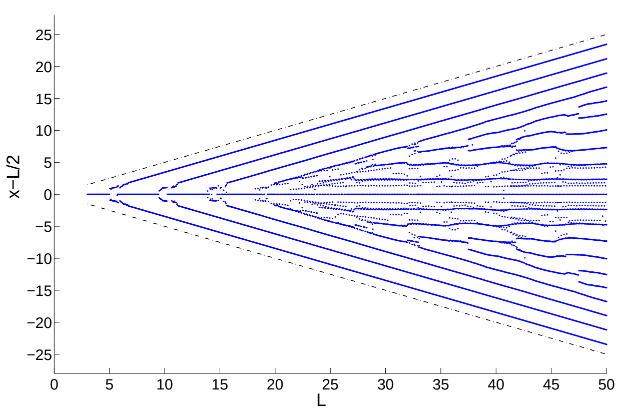


Fig. 3. The location of the different clusters at equilibrium, as a function of L, for $5000L$ agents with opinions initially equidistantly located on $[0, L]$. Clusters are represented in terms of their distance from $L/2$, and the dashed lines represent the endpoints 0 and L of the initial opinion distribution.

between agents is measured according to the Euclidean norm. All opinions converge to clusters, but these clusters are separated by distances significantly larger than 1 from their closest neighbors. The same can be observed in Fig. 1, which involves one-dimensional opinions, initially equidistantly located on $[0, 4]$ and $[0, 10]$, respectively. Although the results of Section II-A would allow up to 4 and 10 clusters, respectively, only 1 and 4 clusters are observed. To further investigate this phenomenon, we consider in Fig. 3, as a function of L, the number and positions of the final clusters, when the initial opinions are are located equidistantly in the interval $[0, L]$. The cluster positions tend to change with L in a piecewise continuous (or even linear) manner. The discontinuity points correspond to the emergence of new clusters, or to "splitting" of a cluster into two smaller ones. The number of clusters tends to increase linearly with L, as already observed in [12], and in [4] for the Weisbuch model. It is conjectured in [4] that the number of clusters is asymptotically equal to L/γ with $\gamma \simeq 2$. From our own numerical experiments, it seems however that γ is close to 2.2. In any case, the inter-cluster distance appears to be much larger than the minimal distance 1 between any two clusters, and no better asymptotic upper bound has yet been found. In the next three subsections we analyze three problems that relate to these observations, summarized below.

Observation 1: For initial opinions uniformly distributed over $[0, L]$, the typical equilibrium inter-cluster distance is $\gamma \simeq 2.2$, and the number of clusters is asymptotically L/γ. Similar observations are made for Weisbuch's model [].

C. Largest interval for one cluster

Consider n agent opinions equidistantly located on $[0, L]$. Assume that n is odd, so that one agent has an initial opinion of $L/2$. Explicit computations of the first two iterations show that at $t = 1$, all opinions belong to $[\frac{1}{2}, L - \frac{1}{2}]$, and at $t = 2$ to $[\frac{11}{12}, L - \frac{11}{12}]$. Moreover, since opinions are updated by making convex combinations, all further opinions also belong to these intervals. Observe now that our system cannot produce more than one cluster if all opinions are in $(L/2 - 1, L/2 + 1)$, as the agent opinion initially on $L/2$ always remains so by symmetry, and the agents' equilibrium opinions are either equal or at least 1 apart. So this system cannot produce more than one cluster if $\frac{11}{12} > L/2 - 1$, that is if $L < \frac{23}{6} \simeq 3.833$. This bound is smaller than the experimental bound which we observed to be between 5 and 5.1. The explicit bound of 23/6 provided here could of course be improved by explicitly computing further iterations but these calculations become tedious for $t > 2$. Let us also mention that, provided there are sufficiently many agents, a similar analysis is possible for the case where there is no agent opinion at the origin and for the case where opinions are randomly distributed on the interval.

Fact 3: Opinions initially equidistantly distributed on $[0, L]$ converge to one cluster if $L < \frac{23}{6}$.

Observation 2: Experimentally the maximal L leading to a single cluster is between 5 and 5.1.

D. Finite and semi-infinite intervals

When L is sufficiently large, Fig. 3 shows that distance of the first cluster from zero becomes independent of L. This can be explained by analyzing the "information" propagation: During an iteration, an agent is only influenced by those opinions within distance 1 of its own, and its opinion is modified by less than 1. So information is propagated by at most a distance 2 at every iteration. In the case of an initial uniform distribution on $[0, L]$ for a large L, during the first iterations the agents with initial opinions close to 0 behave as if opinions were initially distributed uniformly on $[0, +\infty)$. Moreover, once a group of opinions is separated from other opinions by more than 1, they are not influenced

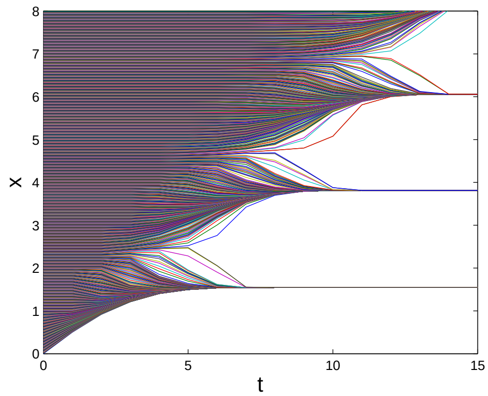


Fig. 4. Evolution with time of the opinions for an initial semi-infinite equidistant distribution of opinion (initially, there are 100 agents within each unit of the x axis).

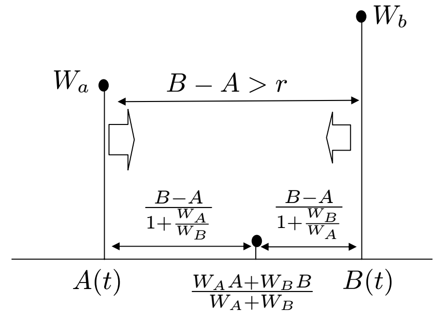


Fig. 5. Meta-stable situation: A and B are disconnected but interact both with isolated agents between them, so that they slowly get closer one from each other. The isolated agents remain approximately at the weighted average of the clusters. W_A and W_B represented by the heights are the number of agents in A and B and are supposed to be large.

by them at any subsequent iteration. Therefore, agents with initial opinions close to 0 and getting separated from the other opinions after some finite time follow exactly the same trajectories when the initial uniform distribution is on $[0, +\infty)$ or on $[0, L]$ for a sufficiently large L.

We performed simulations with an initial semi-infinite interval, i.e. opinions equidistantly distributed between 0 and $+\infty$. It appears that every agent eventually gets disconnected from the semi-infinite set but remains connected with some other agents. Each group behaves then independently of the rest of the system and converges to a single cluster. As shown in Fig. 4, the distance between two consecutive clusters converges to approximately 2.2. This asymptotic inter-cluster distance would partially explain the evolution of the number of clusters (as a function of L) shown in Fig. 3, but at the time of writing, there is no proof for the above observations.

Observation 3: For an initial semi-infinite opinion distribution, every agent opinion converges to a cluster and the distance between consecutive clusters converges to approximately 2.2.

E. Equilibrium stability

Another phenomenon preventing inter-cluster distances close to 1 often happens for values of L just smaller than those at which one cluster is separated in two, and has also been observed in [12]. Locally, the system first converges to a "meta-stable equilibrium" where two clusters are separated by a distance larger than 1 but smaller than 2, with some isolated agents between them. Since these isolated agents interact with both clusters, their opinions become approximately the weighted average of these clusters. At the same time, they attract the clusters that slowly move towards each other, as illustrated in Fig. 5. When the distance separating them finally becomes smaller than 1, they merge in one time-step and the system reaches an equilibrium. An example of such meta-stable situation can also be seen in Fig. 2 for $t = 12$, or in Fig. 1(a) between $t = 5$ and $t = 7$ with a small clusters instead of isolated agents. Based on these observations, we define in the sequel a notion of equilibrium stability characterizing the robustness of an equilibrium with respect to the addition of an "arbitrarily small" agent. The stability of an equilibrium (under this new notion) then implies that clusters are separated by distances larger than a bound that depends on the weights of the clusters. For clusters of identical weight this lower bound is equal to 2.

Observation 4: If the distance between two clusters is more than 1 (so that they do not interact), but within some threshold, the addition of a few isolated agents can cause structural changes.

To introduce the notion of stable equilibrium, we consider a weighted version of (1) where every agent i has a weight w_i and where the update rule is

$$x_{t+1}(i) = \frac{\sum_{j \in N_t(i)} w_j x_t(j)}{\sum_{j \in N_t(i)} w_j}. \quad (2)$$

The results described in Section II-A about convergence to clusters can be generalized to this weighted case, and we call *weight of a cluster* the sum of the weights of all agents at this cluster. Let $\bar{x}$ be a vector of agent opinions at equilibrium. Suppose that one adds a new agent of weight δ and opinion z, and let the system re-converge to a perturbed equilibrium opinion vector $\bar{x}'$ (which does not contain the perturbing agent opinion). We denote by $\Delta_{z,\delta} = w^T |\bar{x} - \bar{x}'|$ the distance between the initial and perturbed equilibria. We say that $\bar{x}$ is *stable* if $\max_z \Delta_{z,\delta}$, the largest distance between initial and perturbed equilibria for a perturbing agent of weight δ can be made arbitrarily small by choosing a sufficiently small δ. An equilibrium is thus unstable if some modification of fixed size can be achieved by adding an agent of arbitrarily small weight.

Theorem 1: An equilibrium is stable if and only if the distance between any two clusters A and B is larger than $1 + \min\left(\frac{W_A}{W_B}, \frac{W_B}{W_A}\right)$. In this expression, W_A and W_B are the weights of the clusters.

Proof: Consider an equilibrium $\bar{x}$ and an additional agent of opinion z and weight δ. If this agent is disconnected from all clusters, it has no influence and $\Delta_{z,\delta} = 0$. If it is connected to one cluster A of position x_A and weight W_A, the system reaches a new equilibrium after one time step, where both the additional agent and the cluster have an opinion $(z\delta + x_A W_A)/(\delta + W_A)$. So $\Delta_{z,\delta} \leq \delta\,|z - x_A|$. Finally, suppose that the perturbing agent is connected to two clusters A, B (it is never connected to more than two clusters). For a sufficiently small δ, its position after one time step is approximately

$$z' = x_A + \frac{x_B - x_A}{1 + \frac{W_A}{W_B}} = \frac{x_A - x_B}{1 + \frac{W_B}{W_A}} + x_B, \tag{3}$$

while the new positions of the clusters are $(z\delta + x_A W_A)/(\delta + W_A)$ and $(z\delta + x_B W_B)/(\delta + W_B)$. If $|x_A - x_B| > 1 + \min\left(\frac{W_A}{W_B}, \frac{W_B}{W_A}\right)$, it follows from (3) that for small δ the agent is then connected to only one cluster and that equilibrium is thus reached at the next time step, with a $\Delta_{z,\delta}$ proportional to δ. The condition of this theorem is thus sufficient for stability of the equilibrium as $\Delta_{z,\delta}$ is proportional to δ when it is satisfied.

If the condition is not satisfied, the agent is still connected to both clusters as represented in Fig. 5. An explicit recursive computation shows that in the sequel its opinion remains approximately at the weighted average of the two clusters (3), while these get steadily closer one to each other. Note that their weighted average moves at each iteration in the direction of the largest cluster by a distance bounded by $\delta/(W_A + W_B)$. Once the distance separating the clusters becomes smaller than or equal to 1, they merge in one central cluster of opinion z'. Thus, in this case, the addition of a perturbing agent of arbitrary small weight δ connected to both A and B results in the merging of the clusters independently of δ, so that $\Delta_{z,\delta}$ does not decay to 0 with δ. ■

We have observed in most cases that, provided that the number of agents is sufficiently large, the equilibrium to which the system converges is stable [2]. Intuitively this could be explained by the fact that when the system is converging to an unstable equilibrium, there are almost always a few agents remaining in the zone where they can produce large changes and redirect the system to a stable equilibrium. Of course, not every system converges to a stable equilibrium. For instance if an unstable equilibrium is taken as initial condition, the system remains at this unstable equilibrium forever. We make the following conjecture.

Conjecture 1: The probability of convergence to a stable equilibrium tends to 1 when the number of agents increases, for initial agent opinions that are randomly distributed according to a continuous probability function with connected support.

We give in the next sections partial results that support this conjecture. If the conjecture proves to be be correct, it would partially justify the dependence on $L/2$ of the number of clusters observed in Section II-D, as the clusters generally have the same weight. However, this approach is not likely to give a tight bound as the limiting inter-cluster distances observed in the semi-infinite case or in the finite case for large L is larger than 2, which is the maximal distance that can be justified by stability of the equilibrium. Moreover, the convergence of those clusters is often rapid and does not go through the meta-stable phase.

Observation 5: Typical inter-cluster distances are larger than the largest lower bound provided by stability analysis.

III. Continuous distribution of agents

To further analyze the properties of (1) and its behavior when the number of agents increases, we now consider a modified version of the model, which involves a continuum of agents. We use the interval $I = [0, 1]$ to index the agents and denote by $x_t(\alpha)$ the opinion at time t of the agent $\alpha \in I$. As an example, a uniform initial distribution of opinions is given $x_0(\alpha) = L\alpha$. At each iteration, the agent opinions are updated by

$$x_{t+1}(\alpha) = \frac{\int_{\beta \in I} a_t(\alpha, \beta) x_t(\beta) d\beta}{\int_{\beta \in I} a_t(\alpha, \beta) d\beta}, \tag{4}$$

where $a_t(\alpha, \beta) = 1$ if $|x_t(\alpha) - x_t(\beta)| \leq 1$ and 0 else. We denote by Δx_t the opinion function variation $x_{t+1} - x_t$. The relation between this system and its discrete analog ((1) or (2)) is studied in Section IV. It would be desirable to prove the convergence of (4), but at the time of writing this remains a conjecture:

Conjecture 2: The system (4) converges to an equilibrium.

Although we have no proof for this conjecture, we have a weaker result, that implies that (4) does not produce any cycle and that the opinion variations decay to 0. Before stating and proving this result, we need to introduce a few concepts. By analogy with the discrete system, we define the *adjacency operator* A_t, which maps the set of bounded measurable functions on I into itself, by

$$A_t x(\alpha) = \int_{\beta \in I} a_t(\alpha, \beta) x(\beta) d\beta,$$

and the degree function $d_t(\alpha) : I \to \Re^+$ by $d_t(\alpha) = \int_{\beta \in I} a_t(\alpha, \beta) d\beta$. Multiplying an opinion function by its degree function can be viewed as applying the operator (defined on the same set of functions as A_t)

$$D_t x(\alpha) = d_t(\alpha) x(\alpha) = \int_{\beta \in I} a_t(\alpha, \beta) x(\alpha) d\beta.$$

Finally, we define the *Laplacian operator* $L_t = D_t - A_t$. The update (4) can be rewritten, more compactly, in the form $\Delta x_t = x_{t+1} - x_t = -\frac{1}{d_t} L_t x_t$. In the sequel, we use the scalar product $\langle x, y \rangle = \int_{\alpha \in I} x(\alpha) y(\alpha) d\alpha$. The following lemmas are proved by elementary computations [2].

Lemma 1: The operators defined above are symmetric with respect to the scalar product: $\langle y, A_t x\rangle = \langle A_t y, x\rangle$, $\langle y, D_t x\rangle = \langle D_t y, x\rangle$ and $\langle y, L_t x\rangle = \langle L_t q, p\rangle$.

Lemma 2: There holds

$$\langle x, (D_t \pm A_t)x\rangle = \frac{1}{2}\int_{(\alpha,\beta)\in I^2} a(\alpha,\beta)\,(x(\alpha)\pm x(\beta))^2 .$$

Theorem 2: The system (4) does not produce cycles (except for fixed points). Moreover, the following quantity decays to 0 when $t \to \infty$

$$\int_{(\alpha,\beta)\in I^2} a_t(\alpha,\beta)\,(\Delta x_t(\alpha)+\Delta x_t(\beta))^2 . \tag{5}$$

Proof: We consider the nonnegative function

$$V(x) = \frac{1}{2}\int_{(\alpha,\beta)\in I^2} \min\left(1,(x(\alpha)-x(\beta))^2\right), \tag{6}$$

and we prove that it decreases at each iteration of (4), with

$$V(x_t)-V(x_{t+1}) \geq \frac{1}{2}\int_{(\alpha,\beta)\in I^2} a_t(\alpha,\beta)\,(\Delta x_t(\alpha)+\Delta x_t(\beta))^2 \tag{7}$$

It can be seen that when $\Delta x_t \neq 0$, V decreases strictly. This implies that the system cannot produce any non-trivial cycles. Furthermore, since V is non-negative and decreasing, it converges to a limit. Therefore $V(x_{t+1}) - V(x_t)$ tends to 0 and so does (5).

Consider an arbitrary but fixed time step t. For any time s, $V(x_s)$ can be rewritten as

$$\begin{array}{ll} & \frac{1}{2}\left(\int_{(\alpha,\beta)\in I^2, a_t(\alpha,\beta)=1} \min\left(1,(x_s(\alpha)-x_s(\beta))^2\right)\right) \\ + & \frac{1}{2}\left(\int_{(\alpha,\beta)\in I^2, a_t(\alpha,\beta)=0} \min\left(1,(x_s(\alpha)-x_s(\beta))^2\right)\right). \end{array}$$

For $s=t$, $(x_s(\alpha)-x_s(\beta))^2 \geq 1$ for all α,β such that $a_t(\alpha,\beta)=0$, and the second term above takes its maximal value $\frac{1}{2}\int_{(\alpha,\beta)\in I^2, a_t(\alpha,\beta)=0} 1$ and cannot increase between t and $t+1$. Consider now the first term. For any s, there holds

$$\begin{array}{ll} & \frac{1}{2}\left(\int_{(\alpha,\beta)\in I^2, a_t(\alpha,\beta)=1} \min\left(1,(x_s(\alpha)-x_s(\beta))^2\right)\right) \\ \leq & \frac{1}{2}\left(\int_{(\alpha,\beta)\in I^2} a_t(\alpha,\beta)(x_s(\alpha)-x_s(\beta))^2\right) = \langle x_s, L_t x_s\rangle \end{array}$$

where the inequality follows from $a(\alpha,\beta)\leq 1$ and the last equality follows from Lemma 2 and from $L_t = D_t - A_t$. For $s=t$, $(x_s(\alpha)-x_s(\beta))^2 \leq 1$ for all α,β such that $a_t(\alpha,\beta)=1$. The above inequality is thus an equality for $s=t$, and we have

$$V(x_{t+1}) - V(x_t) \leq \langle x_{t+1}, L_t x_{t+1}\rangle - \langle x_t, L_t x_t\rangle .$$

By symmetry (Lemma 1) and by the update rule (4) which implies $D_t\Delta x_t = -L_t x_t$, this becomes

$$\begin{array}{rcl} V(x_{t+1})-V(x_t) & \leq & \langle \Delta x_t, L_t \Delta x_t\rangle + 2\langle \Delta x_t, L_t x_t\rangle \\ & = & \langle \Delta x_t, L_t \Delta x_t\rangle - \langle \Delta x_t, 2D_t \Delta x_t\rangle . \end{array}$$

Since $L_t = D_t - A_t$, we have $\langle \Delta x_t, (A_t+D_t)\Delta x_t\rangle \leq V(x_t) - V(x_{t+1})$, which by Lemma 2 proves that (7) holds. ■

The following lemmas can be proved by explicit computations [2].

Lemma 3: If $x_t(\alpha) \leq x_t(\beta)$, then $x_{t+1}(\alpha) \leq x_{t+1}(\beta)$.

Lemma 4: If x_t is continuous and piecewise differentiable with a derivative bounded from below and above by positive numbers, so is x_{t+1}, provided that $x_t(1)-x_t(0)\geq 2$.

In the sequel we can thus assume without loss of generality that all x_t are nondecreasing. If x_0 satisfies the hypotheses of Lemma 4 (continuity and piecewise differentiability with positive lower and upper bounds on the derivative), so do all x_t as long as the range of opinions is no smaller than 2. However, the existence of a function x_0 satisfying the hypotheses of Lemma 4 and leading to a sequence of functions x_t such that $\max_\alpha x_t(\alpha) - \min_\alpha x_t(\alpha) \geq 2$ for all t has not been proved yet. Similarly, it is not known whether there exists a continuous initial opinion function x_0 for which we fail to obtain convergence to a "state" where all opinions are equal.

Conjecture 3: There exist continuous and piecewise differentiable initial opinion functions x_0 with positive lower and upper bounds on their derivative such that $\max_\alpha x_t(\alpha) - \min_\alpha x_t(\alpha) \geq 2$ for all t.

We now prove that under certain conditions the continuous system equilibria always satisfy the stability conditions of Theorem 1. Analogously to the discrete system, we say that a real number c is a *cluster* if $x_t(\alpha)\to c$, for all α in a positive length interval contained in I, and we call *cluster weight* the length W_c of this interval.

Theorem 3: Suppose that the initial opinion function is such that all x_t are continuous and strictly increasing. If x_t converges to a set of clusters whose distance from each other is at least 1, then any two clusters A and B are separated by at least $1+\min\left(\frac{W_A}{W_B},\frac{W_B}{W_A}\right)$.

Proof: We show that if two clusters A, B ($A<B$ without loss of generality) do not satisfy this condition, I contains a positive length interval of agents whose opinions remain between A and B but do not converge to A nor to B. But this is impossible because A and B are separated by less than 2.

Since x_0 is increasing, it follows from Lemma 3 that if $\alpha\leq\beta\leq\gamma$, $x_t(\alpha)\leq x_t(\beta)\leq x_t(\gamma)$ for all t. The set of agents converging to A and B are thus intervals, which we denote by I_A and I_B ($W_A = |I_A|$, $W_B = |I_B|$). We call their infimum and supremum respectively f_A, l_A and f_B, l_B. Consider arbitrarily small but fixed δ and ϵ. Because all agents in (f_A, l_A) have opinions converging to A, there is a time after which $|x_t(f_A+\delta)-A| < \epsilon$ and $|x_t(l_A-\delta)-A|<\epsilon$. As almost all agents outside of I_A converge to clusters distant from A by at least 1, there is also a time after which $x_t(f_A-\delta) < A-1+\epsilon$ and $x_t(l_A+\delta) > A+1-\epsilon \geq B-1+\epsilon$. Applying the same argument for B, we conclude that after a sufficiently large time, the following hold:

- The difference between I_A and the interval $I'_{A,t} = x_t^{-1}([A-\epsilon, A+\epsilon])$ of agents with an opinion close to A at time t has measure smaller than 2δ.
- The difference between I_B and the interval $I'_{B,t} = x_t^{-1}([B-\epsilon, B+\epsilon])$ of agents with an opinion close to B at time t has measure smaller than 2δ.
- The measure of the set of agents J_t with an opinion distant from A and B by at least ϵ and that can interact with agents having an opinion in $[B-1+\epsilon, A+1-\epsilon]$ is smaller than 4δ.

Consider now an agent with an opinion in $[B-1+\epsilon, A+1-\epsilon]$ (The interval of such agents is of positive measure since x_t is continuous). Its updated opinion is at most

$$x^+_{\delta,\epsilon} = \frac{(W_A-2\delta)(A+\epsilon)+(W_B+2\delta)(B+\epsilon)+4\delta(A+2)}{(W_A-2\delta)+(W_B+2\delta)+4\delta},$$

where agents of $I'_{A,t}$, $I'_{B,t}$ and J_t have the largest possible opinion $A+\epsilon$, $B+\epsilon$ and $A+2$, and were the set $I'_{B,t}$, J_t with a large opinion have the largest possible measures $W_B+2\delta$ and 4δ while the $I'_{A,t}$ with a small opinion has the smallest possible measure $W_A-2\delta$. Similarly, a lower bound on the updated opinion is

$$x^-_{\delta,\epsilon} = \frac{(W_A+2\delta)(A-\epsilon)+(W_B-2\delta)(B-\epsilon)+4\delta(B-2)}{(W_A+2\delta)+(W_B-2\delta)+4\delta}.$$

Both $x^-_{\delta,\epsilon}$ and $x^+_{\delta,\epsilon}$ tend to $\frac{W_A A+W_B B}{W_A+W_B}$ when ϵ and δ tend to 0, and since the condition of this theorem is assumed not to be satisfied, this value belongs to $[B-1, A+1]$. So, for sufficiently small ϵ and δ, $[x^-_{\delta,\epsilon}, x^+_{\delta,\epsilon}] \subseteq [B-1+\epsilon, A+1-\epsilon]$. The agent opinions in this last interval remain thus forever inside this interval and do not converge to A nor to B. ■

IV. Relation between the Discrete and Continuous Models

A discrete system can be simulated by a system involving a continuum of agents. Indeed, a vector of discrete opinions $\hat{x} \in \Re^N$ can be represented by taking a piecewise constant function x on $I=[0,1]$, with $x(\alpha) = \hat{x}(i)$ for $\alpha \in \frac{1}{N}(i-1, i)$. It follows from (4) that all $\hat{x}_t$ are constant on these intervals, and their value corresponds to the discrete opinions x_t obtained by the discrete system. Different weights can also be given to the discrete agents by varying the length of the interval on which x_0 is constant.

Before analyzing further the link between continuous and discrete systems, we prove a result on the opinion functions on which the update operator is continuous. Let $U : (Ux_t)(\alpha) = x_{t+1}(\alpha)$ be the update operator defined on the set of *nondecreasing opinion functions*. It follows from Lemma 3 that if x is nondecreasing, Ux is also nondecreasing. The operator U can thus be composed with itself arbitrarily many times ($U^{t+1}x = UU^t(x)$). We say that U is continuous at a certain opinion function x if for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any non-decreasing opinion function y, $||x-y||_\infty < \delta \Rightarrow ||Ux-Uy||_\infty < \epsilon$.

Proposition 1: Let x be a continuous piecewise differentiable opinion function on $I=[0,1]$ with positive lower and upper bounds on its derivative x'. Then U is continuous at x. As a result, if x is has also the property that for every finite $t>0$, $\max_\alpha U^t x(\alpha) - \min_\alpha U^t x(\alpha) \geq 2$, the self composition U^t is continuous at x for every finite t.

Proof: Consider such a x and let $\lfloor x' \rfloor$ be a positive lower bound on its derivative. For any α the set of agents whose opinion is within a distance 1 of $x(\alpha)$ is an interval, and we denote by $l_{\alpha,x}$ and $f_{\alpha,x}$ its supremum and infimum. It follows from the continuity and differentiability of x that for all α, $l_{\alpha,x} - f_{\alpha,x}$ is larger than some uniform bound $k(x)/\,||x'||_\infty$. The update law (4) can be rewritten as

$$Ux(\alpha) = \frac{\int_{f_{\alpha,x}}^{l_{\alpha,x}} x(\beta)d\beta}{l_{\alpha,x}-f_{\alpha,x}} \tag{8}$$

We begin by showing that if a nondecreasing opinion function y satisfies $||x-y||_\infty \leq \delta := \frac{1}{2}\epsilon\lfloor x' \rfloor$, then $|l_{\alpha,x} - l_{\alpha,y}| \leq \epsilon$. To avoid edge effects, we define x and y for $\alpha \geq 1$ by $x(\alpha) = x(1) + (\alpha-1)\lfloor x' \rfloor$ and $y(\alpha) = y(1) + (\alpha-1)\lfloor x' \rfloor$. All values $l_{\alpha,x}$ and $l_{\alpha,y}$ smaller than 1 are unaffected. Those which are equal to 1 are here overestimated, but this can only result in an overestimation of $|l_{\alpha,x} - l_{\alpha,y}|$ since $|\min(a,c) - \min(b,c)| \leq |a-b|$. Note that $x(l_{\alpha,y}) - x(l_{\alpha,x})$ can be rewritten as

$$x(l_{\alpha,y}) - (y(\alpha)+1) + 1 + y(\alpha) - x(\alpha) + x(\alpha) - x(l_{\alpha,x}). \tag{9}$$

By continuity of x and since x has been redefined so that it is unbounded, there holds $x(l_{\alpha,x}) = x(\alpha)+1$. Note that y is not necessary continuous, but due to the definition of $l_{\alpha,y}$ we have

$$y_i + 1 \in [\lim_{\beta < l_{\alpha,y}} y(\beta), \lim_{\beta > l_{\alpha,y}} y(\beta)].$$

But, again by continuity of x and because $||x-y||_\infty \leq \delta$,

$$\lim_{\beta < l_{\alpha,y}} y(\beta) \geq \lim_{\beta < l_{\alpha,y}} x(\beta) - \delta = x(l_{\alpha,y}) - \delta.$$

Similarly, $\lim_{\beta > l_{\alpha,y}} y(\beta) \leq x(l_{\alpha,y}) + \delta$, so that $|(y(\alpha)+r) - x(l_{\alpha,y})| \leq \delta$. Using this, the expression of $x(l_{\alpha,x})$ and the bound on $||x-y||_\infty$, we obtain from (9)

$$|x(l_{\alpha,y}) - x(l_{\alpha,x})| \leq 2\delta = \epsilon\lfloor x' \rfloor,$$

and so $|l_{\alpha,y} - l_{\alpha,x}| \leq \frac{1}{\lfloor x' \rfloor} \leq \epsilon$. Exactly the same results can be obtained for $|f_{\alpha,x} - f_{\alpha,y}|$. The updated value $Uy(\alpha)$ can be rewritten

$$\frac{l_{\alpha,y} - f_{\alpha,y}}{l_{\alpha,x} - f_{\alpha,x}} Ux(\alpha) + \frac{\int_{f_{\alpha,y}}^{f_{\alpha,x}} y(\beta)d\beta + \int_{l_{\alpha,x}}^{l_{\alpha,y}} y(\beta)d\beta}{l_{\alpha,y} - f_{\alpha,y}}.$$

If $||x-y||_\infty < \lfloor x' \rfloor \epsilon$, the second term of the sum has an upper bound proportional to ϵ, and the multiplicative factor of the first term is different from one by at most $\frac{2\epsilon k(x)}{||x'||_\infty}$. Since Ux is bounded, this implies that for all α, $|Uy(\alpha) - Ux(\alpha)| < K\epsilon$, which proves the continuity of U with respect to $||.||_\infty$ for such x. The results for U^t follow then directly because U preserves monotonicity and x satisfies the conditions of Conjecture 3. ■

Consider now an initial opinion function x_0 on $[0,1]$ satisfying the conditions of Conjecture 3, and let

$L = x_0(1) - x_0(0)$. This system can be approximated by a discrete one with initial opinion $\hat{x}_0 \in \Re^N$, $\hat{x}_0(i) = x_0\left(\frac{i}{N}\right)$. The discrete system is then equivalent to a continuous system where the initial opinion function $\tilde{x}_0$ is piecewise constant, with $||x_0 - \tilde{x}_0||_\infty \leq ||x'||_\infty /N$. Thus, $\tilde{x}_0$ can be made arbitrarily close to x_0. Since by Proposition 1, U^t is continuous at x_0 for any fixed t, we can have $\tilde{x}_t = U^t\tilde{x}_0$ arbitrarily close to x_t by taking $\tilde{x}_0$ sufficiently close to x_0, which can be accomplished by taking a sufficiently large N. This supports the intuition that for a large N, the continuous systems behaves approximatively as the discrete one for a certain number of time-steps. In view of Theorem 1, this suggests that the discrete system should always converge to a stable equilibrium (in the sense defined in Section II) when N is sufficiently large. However, this argument is not rigorous, because the continuity of U^t for any t does not imply the continuity of $U^\infty := \lim_{t\to\infty} U^t$. To summarize:

Fact 4: A discrete system can approximate arbitrarily well the behavior of a continuous system for a fixed number of time-steps.

The comparison between discrete and continuous systems provides a new result about discrete systems. Consider a discrete distribution $\hat{x}_0$ of N agents approximating a continuous distribution x_0 as above. Until any time step t, x_t is approximated arbitrarily well by $\hat{x}_t$ if N is sufficiently large, but x_t never reaches the equilibrium. For any t, there is thus a N above which $\hat{x}_t$ has not yet reached equilibrium. Therefore, by increasing the number N of agents in a discrete system (in a way that approximates a continuous distribution x_0), the convergence time will increase to infinity (even though it is finite for any particular finite N).

Fact 5: The finite convergence time of a discrete system tends to infinity when the number of agents grows (for some choices of the initial opinions).

V. Conclusions

In this paper, we have analyzed the equilibria of the Krause model of opinion dynamics. We have focused our attention on the inter-cluster distances, and on the experimentally observed dependence of the number of clusters on the parameters of the model. We have attempted to justify the observed inter-cluster distances; first by comparison with a semi-infinite opinion distribution, and second by introducing a notion of stability of equilibria which assesses the robustness of an equilibrium with respect to the addition of an agent with arbitrarily small "weight". We have given a necessary and sufficient condition for an equilibrium to be stable in terms of inter-cluster distances, requiring for instance any pair of identical clusters to be separated by at least 2. We have also considered a version of the model involving a continuum of agents and continuous opinion distributions, for which we have proved that the stability of the equilibrium is under certain assumptions guaranteed. Finally we have studied the relation between the discrete and the continuous models, seeking to use the results for the second on the first.

The analysis of these problems is not yet complete. We have proved that the continuous model cannot produce cycles and that the amplitude of the changes at each step decays to zero, but have not yet established convergence. It is an open question whether a continuous strictly increasing initial opinion function can converge to more than one cluster. Finally, the link between the discrete and continuous systems needs to be further studied, in order for example to prove that the discrete system equilibria are stable with a high probability when the numbers of agents increases.

Acknowledgements. We wish to thank Raphaël Jungers, Leonard Schulman and Jeff Shamma for their interest in the problems described in this paper and for some of their suggestions. Raphaël Jungers did some of the numerical simulations that we refer to in the text. Prof. Leonard Schulman suggested the connection between finite and infinite intervals. Prof. Jeff Shamma suggested looking at possible connections with stochastic stability questions.

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