

A new condition for convergence in continuous-time consensus seeking systems

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Abstract—We consider continuous-time consensus seeking systems whose time-dependent interactions are cut-balanced, in the following sense: if a group of agents influences the remaining ones, the former group is also influenced by the remaining ones by at least a proportional amount. Models involving symmetric interconnections and models in which a weighted average of the agent values is conserved are special cases. We present a result guaranteeing the convergence of every cut-balanced system, and giving a sufficient condition on the evolving interaction topology for the limit values of two agents to be the same. This condition is also necessary up to a zero-measure subset of the initial conditions. Using the fact that our convergence requires no additional condition, we show that it also applies to systems where the agent connectivity and interactions are random, or endogenous, that is, determined by the agent values. We also derive corresponding results for discrete-time systems.

I. INTRODUCTION

We consider continuous-time consensus seeking systems of the following kind: each of n agents, indexed by $i = 1, \dots, n$, maintains a value $x_i(t)$, which is a continuous function of time and evolves according to the integral equation version of

$$\frac{d}{dt}x_i(t) = \sum_{j=1}^n a_{ij}(t)(x_j(t) - x_i(t)). \quad (1)$$

Throughout we assume that each $a_{ij}(\cdot)$ is a *nonnegative* and measurable function. We introduce the following assumption which plays a central role in this paper.

Assumption 1: (Cut-balance) There exists a constant $K \geq 1$ such that for all t , and any subset S of $\{1, \dots, n\}$, we have

$$\sum_{i \in S, j \notin S} a_{ij}(t) \leq K \sum_{i \in S, j \notin S} a_{ji}(t). \quad (2)$$

Intuitively, if a group of agents influences the remaining ones, the former group is also influenced by the remaining ones by at least a proportional amount. This condition may seem hard to verify in general. But, several important particular classes of consensus-seeking systems automatically satisfy it. These include symmetric systems ($a_{ij}(t) = a_{ji}(t)$), type-symmetric systems ($a_{ij}(t) \leq Ka_{ji}(t)$), and, as will be seen later, any system whose dynamics conserve a weighted average (with positive coefficients) of the agent values.

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We present a result establishing that, under the cut-balance condition (2), and without any further assumptions, each value x_i converges to a limit. Moreover, x_i and x_j converge to the same limit if i and j belong to the same connected component of the “unbounded interactions graph,” i.e., the graph whose edges correspond to the pairs (j, i) for which $\int_0^\infty a_{ij}(\tau) d\tau$ is infinite. (As we will show, while this is a directed graph, each of its weakly connected components is also strongly connected.) Conversely, x_i and x_j generically converge to different limits if i and j belong to different connected components of that graph. (This latter result involves an additional technical assumption that $\int_0^T a_{ij}(\tau) d\tau < \infty$ for every $T < \infty$.)

Motivation for our condition comes from the fact that there are many systems in which an agent cannot influence the others without being subjected to at least a fraction of the reverse influence. This is, for example, a common assumption in numerous models of social interactions and opinion dynamics [7], [17] or physical systems.

A. Background

Systems of the form (1) have attracted considerable attention [14], [22], [24], [25], [31] (see [23], [26] for surveys), with motivation coming from decentralized coordination, data fusion [5], [32], animal flocking [8], [12], [30], and models of social behavior [3], [4], [7], [9], [16], [17].

Available results impose some connectivity conditions on the evolution of the coefficients $a_{ij}(t)$, and usually guarantee exponentially fast convergence of each agent’s value to a common limit (“consensus”). For example, Olfati-Saber and Murray [24] consider the system

$$\frac{d}{dt}x_i(t) = \sum_{j:(j,i) \in E(t)} (x_j(t) - x_i(t))$$

with a time-varying directed graph $G(t) = (\{1, \dots, n\}, E(t))$; this is a special case of the model (1), with $a_{ij}(t)$ equal to one if $(j, i) \in E(t)$, and equal to zero otherwise. They show that if the out-degree of every node is equal to its in-degree at all times, and if each graph $G(t)$ is strongly connected, then the system is average-preserving and each $x_i(t)$ converges exponentially fast to $\frac{1}{n} \sum_j x_j(0)$. They also obtain similar results for systems with arbitrary but fixed a_{ij} . Moreau [21] establishes exponential convergence to consensus under weaker conditions: he only assumes that the $a_{ij}(t)$ are uniformly bounded, and that there exist $T > 0$ and $\delta > 0$ such that the directed graph obtained by connecting i to j whenever $\int_t^{t+T} a_{ij}(\tau) d\tau > \delta$ has a rooted spanning tree,

for every t . Several extensions of such results, involving for example time delays or imperfect communications, are also available.¹

All of the above described results involve conditions that are easy to describe but difficult to ensure a priori, especially when the agent interactions are endogenously determined. This motivates the current work, which aims at an understanding of the convergence properties of the dynamical system (1) under minimal conditions. In the complete absence of any conditions, and especially in the absence of symmetry, it is well known that consensus seeking systems can fail to converge; see e.g., Ch. 6 of [1]. On the other hand, it is also known that more predictable behavior and positive results are possible in the following two cases: (i) symmetric (suitably defined) interactions, or (ii) average-preserving systems (e.g., in discrete-time models that involve doubly stochastic matrices).

B. Our contribution

Our cut-balance condition subsumes the two cases discussed above, and allows us to obtain strong convergence results. Indeed, we prove convergence (not necessarily to consensus) without any additional conditions, and then provide sufficient and (generically) necessary conditions for the limit values of any two agents to agree. In contrast, existing results show *convergence to consensus* under some fairly strong assumptions about repeated or permanent global connectivity, but offer no insight on the possible behavior when convergence to consensus fails to hold. Note though that our result is not strictly stronger than those based on repeated connectivity. In particular, our result does not apply to certain hierarchical systems where a group of nodes “follows” the others, and where convergence is easily shown under an assumption of sufficient connectivity.

The fact that our convergence result requires no assumption other than the cut-balance condition is significant because it allows us to study systems for which the evolution of $a_{ij}(t)$ is a priori unknown, possibly random or depending on the vector $x(t)$ itself. In the latter models, with endogenously determined agent interconnections, it is essentially impossible to check a priori the connectivity conditions imposed in existing results, and such results are therefore inapplicable. In contrast our results apply as long as the cut-balance condition is satisfied. The advantage of this condition is that it can be often guaranteed a priori, e.g., if the system is naturally type-symmetric.

Similar convergence results are available for the special case of discrete-time symmetric or type-symmetric systems [2], [10], [13], [15], [16], [22], though they are obtained with a different methodology. Discrete time is indeed much simpler because one can exploit the following fact: either two agents interact on a set of infinite length or they stop interacting after a certain finite time. We will indeed show

¹It is common in the literature to treat the system (1) as if the derivative existed for all t , which is not always the case. Nevertheless, such results remain correct under an appropriate reinterpretation of (1).

that such existing discrete-time results can be easily extended to the cut-balanced case.

C. Outline

The remainder of the paper is organized as follows. We present our main results in Section II. We discuss their application to systems with random interactions and endogenously determined interactions in Sections III and IV respectively. We then prove an analogous result for discrete-time systems in Section V, and end with some concluding remarks in Section VI. The proof of our main result, omitted for space reasons, is available in [11].

II. ARBITRARY TIME-DEPENDENT INTERACTIONS

We now state formally our main theorem, based on an integral formulation of the agent dynamics. The integral formulation avoids issues related to the existence of derivatives, while allowing for discontinuous coefficients $a_{ij}(t)$ and possible Zeno behaviors (i.e., a countable number of discontinuities in a finite time interval).

Without loss of generality, we assume that $a_{ii}(t) = 0$ for all t . We define a directed graph, $G = (\{1, \dots, n\}, E)$, called the *unbounded interactions graph*, by letting $(j, i) \in E$ if $\int_0^\infty a_{ij}(t) dt = \infty$. The following assumption will be in effect in some of the results.

Assumption 2: (Boundedness) For every i and j , and every $T < \infty$, $\int_0^T a_{ij}(t) dt < \infty$.

Before stating our result, we remind the reader that a directed (sub)graph is strongly connected if every two nodes i, j , are joined by a directed path, i.e., if there exists a sequence $i_0 = i, i_1, \dots, i_{p_{ij}} = j$ such that (i_k, i_{k+1}) is an arc of the graph for every k . A directed (sub)graph is weakly connected if every two nodes i, j are joined by a path that may include reverse edges: $i_0 = i, i_1, \dots, i_{p_{ij}} = j$ such that for every k , at least one of (i_k, i_{k+1}) and (i_{k+1}, i_k) is an arc of the graph. A weakly/strongly connected component is a maximal weakly/strongly connected subgraph; i.e., a subgraph that is weakly/strongly connected and that is not included in any larger weakly/strongly connected subgraph. Every directed graph admits a unique decomposition into weakly/strongly connected components.

Theorem 1: Suppose that Assumption 1 (cut-balance) holds. Let $x : \mathbb{R}^+ \rightarrow \mathbb{R}^n$ be a solution to the system of integral equations

$$x_i(t) = x_i(0) + \int_0^t \sum_{j=1}^n a_{ij}(\tau) (x_j(\tau) - x_i(\tau)) d\tau, \quad (3)$$

for $i = 1, \dots, n$. Then,

- (a) The limit $x_i^* = \lim_{t \rightarrow \infty} x_i(t)$ exists, and $x_i^* \in [\min_j x_j(0), \max_j x_j(0)]$, for all i .
- (b) Every weakly connected component of G is strongly connected.
- (c) For every i and j , we have $\int_0^\infty a_{ij}(t) |x_j(t) - x_i(t)| dt < \infty$. Furthermore, if i and j belong to the same connected component of G , then $x_i^* = x_j^*$.

If, in addition, Assumption 2 (boundedness) holds, then:

- (d) If i and j belong to a different connected component of G , then $x_i^* \neq x_j^*$, unless $x(0)$ belongs to a particular $n - 1$ dimensional sub-space of \mathbb{R}^n , determined by the functions $a_{ij}(\cdot)$.

We note that Theorem 1, proved in [11], has an analog for the case where each agent's value $x_i(t)$ is actually a multi-dimensional vector, obtained by applying Theorem 1 separately to each component.

The cut-balance condition is a rather weak assumption, but may be hard to check, especially when the interactions are not known in advance and depend on the evolution of the states. The proposition that follows provides five specific cases of cut-balanced systems that often arise naturally. Its proof, relying on simple algebraic manipulations, is available in [11]. It should however be understood that the class of cut-balanced systems is not restricted to these particular cases.

Proposition 1: A collection of nonnegative coefficients $a_{ij}(\cdot)$ that satisfies any of the following four conditions also satisfies the cut-balance condition (Assumption 1).

- (a) Symmetry: $a_{ij}(t) = a_{ji}(t)$, for all i, j, t .
- (b) Type-symmetry: There exists $K \geq 1$ such that $0 \leq K^{-1}a_{ji}(t) \leq a_{ij}(t) \leq Ka_{ji}(t)$, for all i, j, t .
- (c) Average-preserving dynamics: $\sum_j a_{ij}(t) = \sum_j a_{ji}(t)$, for all i, t .
- (d) Weighted average-preserving dynamics: There exist $w_i > 0$ such that $\sum_j w_i a_{ij}(t) = \sum_j w_j a_{ji}(t)$, for all t and i .
- (e) Bounded coefficients and set-symmetry. There exists $M \geq \alpha > 0$ such that for all i, j, t , either $a_{ij}(t) = 0$ or $a_{ij}(t) \in [\alpha, M]$; and, for any subset S of $\{1, \dots, n\}$, there exist $i \in S, j \notin S$ with $a_{ij}(t) > 0$ if and only if there exist $i' \in S, j' \notin S$ with $a_{j'i'}(t) > 0$.

Note that condition (d) remains sufficient for cut-balance if the weights w_i change with time, provided that the ratio $(\max_i w_i(t))/(\min_i w_i(t))$ remains uniformly bounded. Besides, the connectivity condition in (e) is equivalent to requiring every weakly connected component to be strongly connected in the graph obtained by connecting (i, j) if $a_{ij}(t) > 0$, for every t .

III. SYSTEMS WITH RANDOM INTERACTIONS

We give a brief discussion of systems with random interactions. Consensus seeking systems where interactions are determined by a random process have been the object of several recent studies. For example, Matei et al. [19] consider the case where the matrix of coefficients $a_{ij}(t)$ follows a (finite-state) irreducible Markov process, and is always average-preserving. They prove that the system converges almost surely to consensus for all initial conditions if and only if the union of the graphs corresponding to each of the states of the Markov chain is strongly connected. This result is extended to continuous-time systems in [18]. In [27], Tahbaz-Salehi and Jadbabaie consider discrete-time consensus-seeking systems where the interconnection is generated by an ergodic and stationary random process, without assuming that the average is preserved. They prove that the

system converges almost surely to consensus if and only if an associated average graph contains a directed spanning tree.

It turns out that convergence for the case of random interactions is a simple consequence of deterministic convergence results. In our case, Theorem 1 can be directly applied to systems where the coefficients $a_{ij}(\cdot)$ are modeled as a random process whose sample paths satisfy the cut-balance condition with probability 1 (possibly with a different constant K for different sample paths, and even in the absence of a global upper bound on K). Indeed, if this is the case, Theorem 1 implies that each $x_i(t)$ converges, with probability 1. Furthermore, if $\mathbb{P}(\int_0^\infty a_{ij}(t)dt = \infty) = 1$, then $x_i^* = x_j^*$, with probability 1. Similarly, a probabilistic analysis of the graph of unbounded interactions G can also yield an estimate of the probability of (local) consensus.

IV. SYSTEMS WITH ENDOGENOUS CONNECTIVITY

Theorem 1 dealt with the case where the coefficients $a_{ij}(t)$ are given functions of time; in particular, $x(t)$ was generated by a *linear*, albeit time-varying, differential or integral equation. We now show that Theorem 1 also applies to *nonlinear* systems where the coefficients (and the interaction topology) are endogenously determined by the vector $x(t)$ of agent values. This is possible because Theorem 1 allows for arbitrary variations of the coefficients $a_{ij}(t)$, thus encompassing the endogenous case.

Corollary 1: For every i and j , we are given a nonnegative measurable function $a_{ij} : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^+$. Let $x : \mathbb{R}^+ \rightarrow \mathbb{R}^n$ be a measurable function that satisfies the system of integral equations

$$x_i(t) = x_i(0) + \int_0^t \sum_j a_{ij}(\tau, x(\tau)) (x_j(\tau) - x_i(\tau)) d\tau \quad (4)$$

for $i = 1, \dots, n$. Suppose that there exists $K \geq 1$ such that for all x , and t , and any subset S of $\{1, \dots, n\}$, we have

$$\sum_{i \in S, j \notin S} a_{ij}(t, x) \leq K \sum_{i \in S, j \notin S} a_{ji}(t, x).$$

- (a) The limit $x_i^* = \lim_{t \rightarrow \infty} x_i(t)$ exists, and $x_i^* \in [\min_j x_j(0), \max_j x_j(0)]$.

Define a directed graph $G = (\{1, \dots, n\}, E)$ by letting $(j, i) \in E$ if and only if $\int_0^\infty a_{ij}(t, x(t)) dt = \infty$. Then:

- (b) Every weakly connected component of G is strongly connected.
- (c) If i and j belong to the same connected component of G , then $x_i^* = x_j^*$.

Proof: Let us fix a solution x to Eq. (4). For this particular function x , and for every i, j , we define a (necessarily measurable) function $\hat{a}_{ij} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by letting $\hat{a}_{ij}(t) = a_{ij}(t, x(t))$. By the assumptions of the corollary, the functions \hat{a}_{ij} satisfy the cut-balance condition (Assumption 1). Furthermore, x is also a solution to the system of (linear) integral equations

$$x_i(t) = x_i(0) + \int_0^t \hat{a}_{ij}(\tau) (x_j(\tau) - x_i(\tau)) d\tau,$$

for $i = 1, \dots, n$. The result follows by applying Theorem 1 to the latter system. ■

Note that the nonlinear system of integral equations (4) considered in Corollary 1 may have zero, one, or multiple solutions. Our result does not have any implication on the existence or uniqueness of a solution, but applies to every solution, if one exists. Naturally, Corollary 1 also holds if the coefficients $a_{ij}(x, t)$ satisfy stronger conditions such as type-symmetry or the condition $\sum_j w_j a_{ji}(t, x) = \sum_j w_i a_{ij}(t, x)$ for some positive coefficients w_i , as in Proposition 1.

We note that part (d) of Theorem 1 does not extend to the nonlinear case where the coefficients a_{ij} also depend on x . Indeed, the proof of Corollary 1 applies Theorem 1 to an auxiliary linear system, and the choice of this linear system is based on the actual solution $x(\cdot)$. Part (d) of Theorem 1 does apply to this particular linear system, and implies that x_i^* is indeed different from x_j^* whenever i and j belong to different connected components of the associated graph G , unless $x(0)$ belongs to a lower-dimensional exceptional set. However, this exceptional set is associated with the particular linear system, which is in turn determined by $x(0)$; different $x(0)$ can be associated with different exceptional sets $D(x(0))$. So, it is in principle possible that every $x(0)$ in a full-dimensional set falls in the exceptional set $D(x(0))$. This is not just a theoretical possibility, as illustrated by the four-dimensional example that follows.

Example 1: Let $n = 4$. Consider a sorted initial vector, so that $x_1(0) \leq x_2(0) \leq x_3(0) \leq x_4(0)$. Suppose that the coefficients a_{ij} have no explicit dependence on time, but are functions of x , with $a_{13}(x) = a_{31}(x) = 1$ and $a_{24}(x) = a_{42}(x) = 1$ as long $x_1 < x_2 < x_3 < x_4$. Otherwise, $a_{13}(x) = a_{31}(x) = a_{24}(x) = a_{42}(x) = 0$. All other coefficients are 0. These coefficients are symmetric, and thus cut-balanced. The corresponding system has a solution of the following form: $x_1(t), x_2(t)$ keep increasing and $x_3(t), x_4(t)$ keep decreasing, until some time t^* at which agents 2 and 3 hold the same value; after that time, all values remain constant. Thus, there is a 4-dimensional set of initial conditions for which the resulting limits satisfy $x_2^* = x_3^*$. Note that $\int_0^\infty a_{ij}(t) dt = \int_0^{t^*} a_{ij}(t) dt < \infty$, for all i, j , and the unbounded interactions graph G has no edges. Yet, despite the fact that nodes 2 and 3 belong to different strongly connected components, x_2^* and x_3^* are equal on a 4-dimensional set of initial conditions. □

Finally, we note that the condition on the graph G in Corollary 1 can be nontrivial to verify for some systems, because G depends on the evolution of x via the $a_{ij}(t, x(t))$, and the evolution of x is a priori unknown. And indeed, one may not always know whether G will be connected so that the system will converge to consensus. However, as will be seen in the application below, the first part of the Corollary guarantees the convergence of any system satisfying the cut-balance condition. And, one can then use conditions on the graph G to characterize the possible limiting states x^* .

We now point out an application of Corollary 1. We consider a nonlinear multi-agent system of a form studied in [4], [6], [9], [14], [17], [20] (often in the context of

bounded confidence models) in which the agent values evolve according to the integral equation version of

$$\frac{d}{dt}x_i(t) = \sum_{j: |x_i(t) - x_j(t)| < 1} (x_j(t) - x_i(t)). \quad (5)$$

The evolution of the interaction topology for this system is a priori unknown, because it depends on the a priori unknown evolution of x . In addition, the interaction topology can, in principle, change an infinite number of times during a finite time interval. Determining whether such a system converges can be difficult. And indeed, the convergence of an asymmetric counterpart of (5) remains open [20]. Observe now that (5) is of the form (4), with $a_{ij}(x) = 1$ if $|x_i - x_j| < 1$, and $a_{ij}(x) = 0$ otherwise. The coefficients a_{ij} are symmetric and therefore satisfy the cut-balance condition in Corollary 1. Part (a) of the corollary implies that the limit $x_i^* = \lim_{t \rightarrow \infty} x_i(t)$ exists for every i . Suppose now that for some i, j , we have $|x_i^* - x_j^*| < 1$. Then, there exists a time after which $|x_i(t) - x_j(t)| < 1$ and therefore $a_{ij}(x(t)) = 1$. As a consequence, $\int_0^\infty a_{ij}(x(t)) dt = \infty$, and Corollary 1(c) implies that $x_i^* = x_j^*$. This proves that the system converges, and that the limiting values of any two agents are either equal or separated by at least 1, a result which had been obtained by ad hoc arguments in [10].

Exactly the same argument can be made for a system that evolves according to the integral equation version of

$$\frac{d}{dt}x_i(t) = \frac{\sum_{j: |x_i(t) - x_j(t)| < r} (x_j(t) - x_i(t))}{\sum_{j: |x_i(t) - x_j(t)| < r} 1}.$$

(We let $\dot{x}_i(t) = 0$ whenever the denominator on the right-hand side is zero.) This system satisfies a type-symmetry condition with $K = N$. A variant of such a system, with a different interaction radius r_i for each i , has been studied in [14] under the assumption that the graph of interactions is strongly connected at every t .

A further variation of (5) is of the form

$$\frac{d}{dt}x_i(t) = \sum_j f(x_j(t) - x_i(t)) (x_j(t) - x_i(t)), \quad (6)$$

where f is an even nonnegative function. A multidimensional version of (6), where each x_i is a vector, is studied in [6], for the special case of a radially decreasing function f that becomes zero beyond a certain threshold. (The results in [6] also allow for a continuum of agents, which appear for example when studying discrete-agent models in the limit of a large number of agents). The system (6) is of the form (4), with $a_{ij}(x) = f(x_i - x_j)$. It satisfies a type-symmetry condition, with $K = 1$, and Corollary 1 implies convergence. Moreover, if f is bounded and is continuous except on a finite set, then for any i, j , either $x_i^* = x_j^*$, or $x_i^* - x_j^*$ belongs to the closure of the set $\{z : f(z) = 0\}$ of roots of f . To see this, Corollary 1 asserts that if $x_i^* \neq x_j^*$, then $\int_0^\infty f(x_i(t) - x_j(t)) dt < \infty$, which implies that $x_i(t) - x_j(t)$ cannot stay forever in a set on which f admits a positive lower bound.

V. DISCRETE-TIME SYSTEMS

Much of the literature on consensus-seeking processes actually concerns discrete-time systems. Typical results guarantee convergence to consensus under the assumption that the system is “sufficiently connected” on any time interval of a certain length [12], [22], [29] and sometimes provide bounds on the convergence rate. When interactions are type-symmetric, convergence to consensus is guaranteed under the weaker assumption that the system remains “sufficiently connected” after any finite time [2], [13], [22] and results 2.5.9 and 2.6.2 in [15]. One can then deduce that type-symmetric systems always converge to a limit, at which we have consensus within each of possibly many agent clusters [10], [16].

We show here how the convergence proof in [2], [10] can be extended to prove that cut-balance is also a sufficient condition for convergence in the discrete-time case, as in Theorem 1. A special case of this result asserts the convergence of systems that preserve some weighted average of the states, and thus includes a sample path version of recent results of [28] on stochastic consensus-seeking systems.

Discrete-time systems are in some sense simpler because of the absence of Zeno behaviors or unbounded sets of finite measure. However, they allow for large instantaneous variations of the agents’ values. In particular, an agent could entirely “forget” its value at time t when computing its value at time $t + 1$, leading to instabilities where agents keep switching their values. For this reason, we introduce two additional assumptions: each agent is influenced by its own value when computing its new value, and every positive coefficient is larger than some fixed positive lower bound.

Theorem 2: Let $x : \mathbb{N} \rightarrow \mathbb{R}^n$ satisfy

$$x_i(t+1) = \sum_{j=1}^n a_{ij}(t)x_j(t), \quad i = 1, \dots, n,$$

where $a_{ij}(t) \geq 0$ for all i, j , and t , and $\sum_{j=1}^n a_{ij}(t) = 1$ for all i and t . Assume that the following conditions hold.

- (a) Lower bound on positive coefficients: there exists some $\alpha > 0$ such that if $a_{ij}(t) > 0$, then $a_{ij}(t) \geq \alpha$, for all i, j , and t .
- (b) Positive diagonal coefficients: $a_{ii}(t) \geq \alpha$, for all i, t .
- (c) Cut-balance: for any nonempty proper subset S of $\{1, \dots, n\}$, there exist $i \in S, j \notin S$ with $a_{ij}(t) > 0$ if and only if there exist $i' \in S, j' \notin S$ with $a_{i'j'}(t) > 0$.

Then, the limit $x_i^* = \lim_{t \rightarrow \infty} x_i(t)$ exists, and $x_i^* \in [\min_j x_j(0), \max_j x_j(0)]$. Furthermore, consider the directed graph $G = (\{1, \dots, n\}, E)$ in which $(j, i) \in E$ if $a_{ij}(t) > 0$ infinitely often. Then, every weakly connected component of G is strongly connected, and if i and j belong to the same connected component of G , then $x_i^* = x_j^*$.

Proof: One can verify that every weakly connected component of G is strongly connected, exactly as in the proof of Theorem 1 in [11]. Consider such a connected component C . It follows from the definition of G that there exists a time t^* after which $a_{ij}(t) = a_{ji}(t) = 0$ for any $i \in C$ and $j \notin C$. Thus, the values $x_i(t)$ with $i \in C$

do not influence and are not influenced by the remaining values after time t^* . In particular, if $t^* \leq t' < t$, then $\min_{j \in C} x_j(t') \leq x_i(t) \leq \max_{j \in C} x_j(t')$ holds for all $i \in C$; furthermore, $\max_{i \in C} x_i(t)$ and $\min_{i \in C} x_i(t)$ are monotonically nonincreasing and nondecreasing, respectively.

We now show that there exists a constant $\gamma > 0$ such that for any $t' \geq t^*$, there exists a $t'' > t$ for which $\max_{i \in C} x_i(t'') - \min_{i \in C} x_i(t'') \leq \gamma(\max_{i \in C} x_i(t') - \min_{i \in C} x_i(t'))$. We assume that $|C| \geq 2$, because otherwise the claim is trivially true.

We assume that $\max_{i \in C} x_i(t') = 1$ and $\min_{i \in C} x_i(t') = 0$; the argument can be carried out for any other values by appropriate scaling and translation. For any t , let $C_k(t)$ be the set of indices $i \in C$ for which $x_i(t) \geq \alpha^k$. Clearly, $C_0(t')$ is nonempty. Consider some t and k such that $\emptyset \neq C_k(t) \neq C$. We distinguish two cases.

(i) Suppose that $a_{ij}(t) = 0$ for all $i \in C_k(t)$ and $j \in C \setminus C_k(t)$. Then, for any $i \in C_k(t)$, we have

$$x_i(t+1) = \sum_{j=1}^n a_{ij}(t)x_j(t) = \sum_{j \in C_k(t)} a_{ij}(t)x_j(t).$$

Since on the one hand we have $a_{ij}(t) = 0$ for every $j \notin C_k(t)$, and $\sum_j a_{ij}(t) = 1$, and on the other hand $x_j(t) \geq \alpha$ for all $j \in C_k(t)$, this implies that $x_i(t+1) \geq \sum_{j \in C_k(t)} a_{ij}(t)\alpha \geq \alpha^k$. Therefore, i belongs to $C_k(t+1)$ as well. So, in this case we have $C_k(t) \subseteq C_k(t+1)$.

(ii) Suppose now that $a_{ij}(t) > 0$ for some $i \in C_k(t)$ and $j \in C \setminus C_k(t)$. Then the cut-balance condition, together with $t \geq t^*$, implies that $a_{i'j'} > 0$ for at least one $i' \in C \setminus C_k(t)$ and $j' \in C_k(t)$. For this i' , we have

$$\begin{aligned} x_{i'}(t+1) &= \sum_{j=1}^n a_{i'j}(t)x_j(t) = \sum_{j \in C} a_{i'j}(t)x_j(t) \\ &\geq a_{i'j'}(t)x_{j'}(t) \geq \alpha \cdot \alpha^k = \alpha^{k+1} \end{aligned}$$

where we have used the fact that $x_j(t) \geq \min_{i \in C} x_i(t') \geq 0$, for all $j \in C$ and $t \geq t'$. Therefore, $i' \in C_{k+1}(t+1)$. Moreover, for any $i \in C_k(t)$, we have $x_i(t) = \sum_{j \in C} a_{ij}(t)x_j(t) \geq a_{ii}(t)x_i(t) \geq \alpha \cdot \alpha^k = \alpha^{k+1}$, because $a_{ii}(t) \geq \alpha$ for all i and t . Thus, if $a_{ij}(t) > 0$ for some $i \in C_k(t)$ and $j \in C \setminus C_k(t)$, then the set $C_{k+1}(t+1)$ contains $C_k(t)$ and at least one additional node.

Recall now that $C_0(t')$ is nonempty. Moreover, the definition of C as a strongly connected component of G implies that for any t and any nonempty set $S \subset C$, there exists a \hat{t} and some $i \in S, j \in S \setminus C$, such that $a_{ij}(\hat{t}) > 0$. Then, a straightforward inductive argument based on the above two cases shows the existence of a time $t'' > t'$ at which $C_{|C|-1}(t'') = C$, i.e., a time t'' at which $\min_{i \in C} x_i(t'') \geq \alpha^{|C|-1}$. Since $x_i(t)$ remains less than or equal to 1 for $i \in C$ and $t > t'$, we conclude that $\max_{i \in C} x_i(t'') - \min_{i \in C} x_i(t'') \leq (1 - \alpha^{|C|-1})(\max_{i \in C} x_i(t') - \min_{i \in C} x_i(t'))$. This inequality, together with the fact that $\max_{i \in C} x_i(t)$ and $\min_{i \in C} x_i(t)$ are respectively nonincreasing and nondecreasing after time t^* , implies the convergence of $x_i(t)$, for all $i \in C$, to a common limit. \blacksquare

Observe that part (d) of Theorem 1, convergence to generically different values for the different components of G , has no counterpart for the discrete-time case. Indeed, if $a_{ij}(1) = 1/n$ for all i, j , the system reaches global consensus after one time step, irrespective of G .

Condition (c) in Theorem 2 has a graph-theoretic interpretation. For every t , let G_t be the graph on n nodes obtained by connecting j to i if $a_{ij}(t)$ is positive. Condition (c) is satisfied if and only if for every t , every weakly connected component of G_t is strongly connected, see Proposition 1(e).

Finally, note that convergence results for discrete-time consensus seeking systems with random or endogenously determined interactions can be easily derived from Theorem 2 exactly as in Section III and Corollary 1, respectively.

VI. CONCLUDING REMARKS

In this paper, we introduced a cut-balance condition, which is a natural and perhaps the broadest possible symmetry-like assumption for consensus seeking systems. This assumption is satisfied, in particular, if the dynamics preserve a weighted average, or if no agent can influence another without incurring a proportional reverse influence. We have presented a result stating, in the absence of any additional assumptions, that the cut-balance assumption is sufficient for convergence of continuous-time consensus seeking systems. We provided a characterization of the resulting local consensus, in terms of the evolution of the interaction coefficients. We then applied our results to systems with random and endogenously determined connectivity. Similar results were also obtained for the discrete time case.

A possible extension of this work would be the generalization of our results to models involving a continuum of agents, which appear naturally when studying discrete-agent models, in the limit of a large number of agents. We discuss some of the main challenges posed by this problem in [11].

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