Convergence of cut-balanced continuous-time consensus systems[1]

Julien M. Hendrickx
Institute of Information and Communication Technologies, Electronics and Applied Mathematics
Université catholique de Louvain, Belgium
Email: julien.hendrickx@uclouvain.be

John N. Tsitsiklis
Lab. for Information and Decision Systems
Massachusetts Institute of Technology, USA
Email: jnt@mit.edu

Abstract *We introduce a cut-balance condition for continuous time consensus seeking systems, which generalizes weak forms of symmetry and of average preservation. We show that, if a system satisfies the cut balance condition, it always converges to a set of clusters, and we characterize the clusters in terms of the graph describing the infinite interactions between the agents.*

We consider continuous-time consensus seeking systems of the following form: each of n agents maintains a value $x_i(t)$ ($i = 1,\dots,n$) which is a continuous function of time and evolves according to

$$\dot{x}_i(t) = \sum_j a_{ij}(t)\left(x_j(t) - x_i(t)\right), \qquad (1)$$

with $a_{ij}(t) \geq 0$. Systems of the form (1) have attracted a considerable attention in recent years (see [2, 3] for surveys). Their study is relevant to decentralized coordination or data fusion, but also to the analysis of animal flocking and social behavior. The results available in the literature usually guarantee (exponential) convergence to a state of consensus between the agents under some persistent or intermittent connectivity conditions related to the evolution of the $a_{ij}(t)$ [1].

We make the additional assumption that the $a_{ij}(t)$ are *cut-balanced*: there exists a $K \geq 1$ such that for all t and any partition of $\{1,\dots,n\}$ into $S \cup S^c$, there holds

$$K^{-1} \sum_{i\in S, j\in S^c} a_{ji}(t) \leq \sum_{i\in S, j\in S^c} a_{ij}(t) \leq K \sum_{i\in S, j\in S^c} a_{ji}(t). \qquad (2)$$

Intuitively, no group of agents can exert an influence on the other agents without being at least proportionally influenced themselves by these other agents. It can be shown that systems satisfying this condition include symmetric systems ($a_{ij}(t) = a_{ji}(t)$), type-symmetric systems ($a_{ij}(t) \leq K a_{ji}(t)$), and any system whose dynamics preserve a weighted average with positive coefficients ($\sum_i w_i a_{ij}(t) = 0$).

Under the assumption (2), we prove that each value x_i unconditionally converges to a limit. We show moreover that x_i and x_j converge to the same limit if i and j belong to the same connected component of the so-called unbounded interactions graph, and generically to different values otherwise. By contrast, classical results in the absence of cut balance show convergence to consensus under some non-trivial condition, but do not conclude anything when the condition is not satisfied, and therefore do not apply to clustering phenomena. This aspect is significant in the study of systems for which the evolution of $a_{ij}(t)$ is a priori unknown, and in particular for systems where a_{ij} actually depends on x, as is the case in many interesting models. Checking whether a connectivity condition is satisfied would indeed in the latter case require some non trivial information about the evolution of x, which a priori forbids the use of classical convergence results.

Our main result is the following.

Theorem 1 *Let $x : \Re^+ \to \Re^n$ be a solution of (the integral version of) (1), and suppose that there exists some $K \geq 1$ such that (2) holds for all t and every subset S of of $\{1,\dots,n\}$. Then,*

i) $x_i^* = \lim_{t\to\infty} x_i(t)$ exists, and $x_i^* \in [\min_j x_j(0), \max_j x_j(0)]$.

Moreover, define the directed graph G=$(\{1,\dots,n\},E)$ by $(j,i) \in E$ if $\int_0^\infty a_{ij}(t)dt = \infty$, and suppose that $\int_I a_{ij}(t)dt$ is bounded for every finite interval I and i,j. Then, every weakly connected component of G is strongly connected. And,

ii) If i and j belong to the same connected component of G, then $x_i^* = x_j^*$,
iii) Else, $x_i^* \neq x_j^*$, unless $x(0)$ is in a particular proper subspace of $\Re^n$, determined by the functions a_{ij}.

[1] This research was supported by the National Science Foundation under grant ECCS-0701623, by the Belgian Programme on Interuniversity Attraction Poles initiated by the Belgian Federal Science Policy Office and by the Concerted Research Action (ARC) "Large Graphs and Networks" of the French Community of Belgium. Some of the results were obtained while J. Hendrickx was with the Massachusetts Institute of Technology.

References

[1] L. Moreau. Stability of continuous-time distributed consensus algorithms. In *Proceedings of the 43st IEEE Conference on Decision and Control (CDC'2004)*, volume 4, pages 3998–4003, Bahamas, December 2004.

[2] R. Olfati-Saber, J.A. Fax, and R.M. Murray. Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1):215–233, January 2007.

[3] W. Ren, R.W. Beard, and E.M. Atkins. *IEEE Control and Systems Magazine*, 27(2):71–82, April 2007.