Proof of Proposition 4 in Global analysis of firing maps, MTNS2010

A. Mauroy, J.M. Hendrickx, A. Megretski, and R. Sepulchre

July 7, 2010

In this note, we provide a full proof of the following result, which is presented as Proposition 4 in [1].

Theorem 1. *Let*

$$P(z) = a_n b_n z^n + a_{n-1} b_{n-1} z^{n-1} + \cdots + a_0 b_0.$$

If the following conditions hold

- $a_n > a_{n-1} > \cdots > a_0 > 0$,
- $b_{n-k} = b_k$, $\forall k$,
- *The sequence* $(b_0, b_1, \ldots, b_n)$ *is positive and convex, i.e.* $b_k \geq 0$ *for all* k *and* $b_k - b_{k-1} \leq b_{k+1} - b_k$ *for* $k = 1, \ldots, n-1$,

then all roots of P *are strictly in the unit-disk:* $P(z) = 0 \Rightarrow |z| < 1$.

Our proof is decomposed in three parts, presented in the three following sections.

1 Concentric circles

Suppose that starting from an initial point s_0 in the plane, one moves by a distance c, calls the arrival point s_1, then rotates by an angle θ, moves by a distance c in the new direction, calls s_2 the arrival point, and keeps repeating these operations. It is well known that all points s_k lie in that case on a same circle and are thus all at equal distance from the center g of that circle. We prove in this section that, if the distance travelled at each iteration varies, then provided that the sequence of distances is convex and increasing, the sequence of distances between s_k and g is nondecreasing. More formally, we prove the following Proposition.

Proposition 1. *Let* $(c_0, c_1, \ldots, c_m)$ *be a positive nondecreasing convex sequence, and fix* $\theta \in (0, 2\pi)$. *Let* $s_{-1} = -c_0/2$, *and* $s_k = s_{k-1} + c_k e^{ik\theta}$ *for all other* k. *Let then* $g = i\frac{c_0}{2\tan(\theta/2)}$. *There holds*

$$|s_{-1} - g| = |s_0 - g| \leq |s_1 - g| \leq \cdots \leq |s_m - g|, \tag{1}$$

and the equality holds if all c_k *are equal.*

To obtain this result, we need three lemmas. The first of them, which is about the decomposition of nonnegative nondecreasing convex sequence, can easily be proved by induction.

Lemma 1. *Let* $(c_0, c_1, \ldots, c_m)$ *be a nonnegative nondecreasing convex sequence, then there exist nonnegative coefficients* $u_1, \ldots, u_m$ *such that*

$$\begin{array}{cccc} (c_0, \; c_1, \; \ldots, \; c_m) & = & c_0 & (1, \; 1, \; 1, \; 1, \; \ldots, \; 1) \\ & + & u_1 & (0, \; 1, \; 2, \; 3, \; \ldots, \; m) \\ & + & u_2 & (0, \; 0, \; 1, \; 2, \; \ldots, \; m-1) \\ & \vdots & & \\ & + & u_m & (0, \; 0, \; 0, \; \ldots, \; 0, \; 1). \end{array} \tag{2}$$

Moreover, $c_0 + \sum_q u_q(m-q+1) = c_m$, *and* $c_0 + \sum_q u_q(m-q+2) = c_m + (c_m - c_{m-1})$.

The following two Lemmas are technical results about computations used in the proof of Proposition 1.

Lemma 2. *For every θ and integer k, there holds*

$$\Re\left(e^{-ki\theta}\left(\frac{e^{\frac{\theta-\pi}{2}i}}{2\sin(\theta/2)}+e^{i\theta}+e^{2i\theta}+\dots e^{(k-1)i\theta}\right)\right)=-1/2. \tag{3}$$

Proof. We first evaluate the first term

$$\Re\left(e^{-ki\theta}\frac{e^{\frac{\theta-\pi}{2}i}}{2\sin(\theta/2)}\right)=\Re\left(-i\frac{e^{\theta(\frac{1}{2}-k)i}}{2\sin(\theta/2)}\right)=\Im\left(\frac{e^{\theta(\frac{1}{2}-k)i}}{2\sin(\theta/2)}\right). \tag{4}$$

Consider now the second term, and remember that $z+z^2+...+z^{k-1}=\frac{z^k-z}{z-1}$. The real part of $e^{-ki\theta}\left(e^{i\theta}+e^{2i\theta}+\dots e^{(k-1)i\theta}\right)$ can thus be reexpressed as

$$\Re\left(\frac{1-e^{(1-k)i\theta}}{e^{i\theta}-1}\right)=\Re\left(\frac{e^{-\frac{1}{2}i\theta}-e^{(\frac{1}{2}-k)i\theta}}{e^{i\theta/2}-e^{-i\theta/2}}\right)=\Im\left(\frac{e^{-\frac{1}{2}i\theta}-e^{(\frac{1}{2}-k)i\theta}}{2\sin(\theta/2)}\right)$$

Together with equation (4), this implies that the first member of (3) is equal to

$$\Im\left(\frac{e^{(\frac{1}{2}-k)i\theta}}{2\sin(\theta/2)}+\frac{e^{-\frac{1}{2}i\theta}-e^{(\frac{1}{2}-k)i\theta}}{2\sin(\theta/2)}\right)=\Im\left(\frac{e^{-\frac{1}{2}i\theta}}{2\sin(\theta/2)}\right)=-\frac{1}{2}.$$

□

Lemma 3. *For every $\theta\neq 0$ and integer k, there holds*

$$\Re\left(e^{-ki\theta}\left(e^{i\theta}+2e^{2i\theta}+3e^{3i\theta}+\cdots+(k-1)e^{(k-1)i\theta}\right)\right)\geq -k/2. \tag{5}$$

Proof. The following equality can be proved using standard algebraic tools[1]

$$z+2z^2+3z^3+\cdots+(k-1)z^{k-1}=k\frac{z^k}{z-1}-z\frac{z^k-1}{(z-1)^2}.$$

The first term of equation (5) can thus be rewritten as

$$\Re\left(e^{-ki\theta}\left(k\frac{e^{ik\theta}}{e^{i\theta}-1}-e^{i\theta}\frac{e^{ik\theta}-1}{\left(e^{i\theta}-1\right)^2}\right)\right)=\Re\left(\frac{k}{e^{i\theta}-1}-e^{i\theta}\frac{1-e^{-ik\theta}}{\left(e^{i\theta}-1\right)^2}\right). \tag{6}$$

Observe now that

$$\frac{k}{e^{i\theta}-1}=\frac{ke^{-i\theta/2}}{e^{i\theta/2}-e^{-i\theta/2}}=-i\frac{ke^{-i\theta/2}}{2\sin(\theta/2)},$$

so that

$$\Re\left(\frac{k}{e^{i\theta}-1}\right)=\Im\left(\frac{ke^{-i\theta/2}}{2\sin(\theta/2)}\right)=-k/2. \tag{7}$$

The second term of (6) can be rewritten as

$$-e^{i\theta}\frac{1-e^{-ik\theta}}{\left(e^{i\theta}-1\right)^2}=-\frac{1-e^{-ik\theta}}{\left(e^{i\theta/2}-e^{-i\theta/2}\right)^2}=\frac{1-e^{-ik\theta}}{4\sin^2(\theta/2)},$$

which has a nonnegative real part. This, together with equation (7) implies the desired result. □

[1] This can be proved by recurrence, or using the observation that $1+2z+3z^2+\cdots+(k-1)z^{k-2}=\left(1+z+z^2+z^3+\cdots+z^{k-1}\right)'$.

We can now prove Proposition 1.

Proof. Observe first that $s_0 = c_0/2 = -s_{-1}$ and thus that $|s_{-1} - g| = |s_0 - g|$, as g has no real part. The inequality $|s_1 - g| \geq |s_0 - g|$ can also easily be verified.

Consider now an arbitrary $k > 1$, and let us show that $|s_k - g| \geq |s_{k-1} - g|$. The recursive definition of s_k implies that

$$|s_k - g|^2 = |s_{k-1} - g|^2 + c_k^2 + 2\Re\left(c_k e^{-ki\theta}(s_{k-1} - g)\right).$$

We therefore just need to prove that $\Re\left(e^{-ki\theta}(s_{k-1} - g)\right) \geq -\frac{1}{2}c_k$, for any given k. One can verify that $s_0 - g = \frac{c_0}{2\sin(\theta/2)} e^{i\frac{\theta-\pi}{2}}$. Together with the definition of the s_k, this implies that

$$\Re\left(e^{-ki\theta}(s_{k-1} - g)\right) = \Re\left(e^{-ki\theta}\left(\frac{c_0}{2\sin(\theta/2)} e^{i\frac{\theta-\pi}{2}} + \sum_{q=1}^{k-1} c_q e^{qi\theta}\right)\right).$$

It follows from Lemma 1 (applied to $k-1$) that this can be rewritten as

$$c_0 \Re\left(e^{-ki\theta}\left(\frac{e^{i\frac{\theta-\pi}{2}}}{2\sin(\theta/2)} + \sum_{q=1}^{k-1} e^{qi\theta}\right)\right) + \sum_{j=1}^{k-1} u_j \Re\left(e^{-ki\theta} \sum_{q=1}^{k-j} q e^{(q+j-1)i\theta}\right), \tag{8}$$

with nonnegative coefficient u_j satisfying $c_0 + \sum_{j=1}^{k-1} u_j(k+1-j) = c_{k-1} + (c_{k-1} - c_{k-2})$. We treat separately the two parts of this quantity. For the first part, it follows from Lemma 2 that

$$c_0 \Re\left(e^{-ki\theta}\left(\frac{e^{i\frac{\theta-\pi}{2}}}{2\sin(\theta/2)} + \sum_{q=1}^{k-1} e^{qi\theta}\right)\right) = -c_0/2. \tag{9}$$

For the second part, observe that

$$e^{-ki\theta} \sum_{q=1}^{k-j} q e^{(q+j-1)i\theta} = e^{-(k-j+1)i\theta} \sum_{q=1}^{k-j} q e^{qi\theta}.$$

It follows therefore from Lemma 3 that

$$\Re\left(e^{-ki\theta} \sum_{q=1}^{k-j} q e^{(q+j)i\theta}\right) \geq -\frac{k-j+1}{2}.$$

Reintroducing this and (9) in (8), we obtain

$$\Re\left(e^{-ki\theta}(s_{k-1} - g)\right) \geq -\frac{1}{2}\left(c_0 + \sum_{j=1}^{k-1} u_j(k+1-j)\right) = -\frac{1}{2}\left(c_{k-1} + (c_{k-1} - c_{k-2})\right), \tag{10}$$

where we have used the equality $c_0 + \sum_{j=1}^{k-1} u_j(k+1-j) = c_{k-1} + (c_{k-1} - c_{k-2})$. Since the sequence c_k is convex, and thus $c_k \geq c_{k-1} + (c_{k-1} - c_{k-2})$, this implies that $|s_k - g|^2 = |s_{k-1} - g|^2 + c_k^2 + 2\Re\left(c_k e^{-ki\theta}(s_{k-1} - g)\right) \geq |s_{k-1} - g|^2$ and thus that the sequence $|s_1 - g|, |s_2 - g|, \ldots, |s_m - g|$ is nondecreasing since our derivation is valid for every k.

□

Remark 1. *The convexity condition of Proposition 1 is actually also a necessary condition, in the sense that if the sequence of c_k is not convex, there exists a θ for which the inequalities 1 do not hold. Observe indeed that for $\theta = \pi$, the inequality of Lemma 3 is tight and so is thus the inequality (10). Therefore, if $c_k < c_{k-1} + (c_{k-1} - c_{k-2})$, then $|s_k - g| < |s_{k-1} - g|$.*

The following Corollary will be useful to treat polynomials of odd degrees.

Corollary 1. *Let* $(c_0, c_1, \ldots, c_m)$ *be a positive nondecreasing convex sequence, and fix* $\theta \in (0, 2\pi)$. *Let* $s'_{-1} = 0$, *and* $s'_k = s'_{k-1} + c_k e^{ik\theta}.e^{i\theta/2}$. *Let then* $g' = \frac{ic_0}{2\sin(\theta/2)}$. *There holds*

$$\left|s'_{-1} - g'\right| = |s'_0 - g'| \le |s'_1 - g'| \le \cdots \le |s'_m - g'|,$$

and the equality holds if all c_k *are equal.*

Proof. This Corollary is proved by applying a translation of $c_0/2$ followed by a rotation of $\theta/2$ to the statement of Proposition 1. It is easy to verify that this distance preserving operation sends every s_k of the Proposition 1 to the s'_k of this corollary. Moreover, it also sends g of Proposition 1 to g', as

$$e^{i\theta/2}\left(\frac{ic_0}{2\tan(\theta/2)} + \frac{c_0}{2}\right) = c_0 e^{i\theta/2}\left(\frac{i\cos(\theta/2)}{2\sin(\theta/2)} + \frac{\sin(\theta/2)}{2\sin(\theta/2)}\right) = \frac{c_0 i}{2\sin(\theta/2)} = g'.$$

□

2 Strict convex hulls and truncated polynomials

In this section, we use the notion of strict convex hull. For a finite set of points $s_1, \ldots, s_n$, we call the set $\{\sum_i \lambda_i s_i : \lambda_i > 0, \sum_i \lambda_i = 1\}$ the *strict convex hull* of $s_1, \ldots, s_n$.

Lemma 4. *Let* $s_1, \ldots, s_n \in \Re^d$ *be a set of points that are not all equal and* z *a point in the same space. If there exists* $g \in \Re^d$ *such that* $||z - g||_2 \ge ||s_k - g||_2$ *for every* $k = 1, \ldots, n$, *then* z *does not belong to the strict convex hull of* $s_1, \ldots, s_n$.

Proof. Assume without loss of generality that $g = 0$, and suppose that $z = \sum \lambda_i s_i$ with $\lambda_i > 0$. Then there holds

$$||z||_2^2 = \sum_i \lambda_i^2 ||s_i||_2^2 + \sum_{i \ne j} \lambda_i \lambda_j (s_i \cdot s_j) \le \sum_i \lambda_i^2 ||s_i||_2^2 + \sum_{i \ne j} \lambda_i \lambda_j ||s_i||_2 ||s_j||_2 \le \max_i ||s_i||_2^2$$

where the first inequality is strict unless all s_i are proportional one to each other with positive coefficients, and the second one is strict unless all s_i have the same norm. So if z is in the strict convex hull of $s_1, \ldots, s_n$, either $||z||_2 < \max_i ||s_i||_2$ or all s_i have the same norm while being all proportional to each others with positive coefficients, and are thus all equal. □

We now prove the following proposition, which translates Proposition 1 in terms of strict convex hull of the values of truncated polynomials.

Proposition 2. *Let* $b_n, b_{n-1}, \ldots, b_0$ *be a positive convex symmetric sequence as in Theorem 1. Then for any* θ, *the strict convex hull of the following points does not contain 0.*

$$\begin{array}{lclcl} s_n(\theta) & = & b_n e^{ni\theta} & & \\ s_{n-1}(\theta) & = & b_n e^{ni\theta} & + & b_{n-1} e^{(n-1)i\theta} \\ s_{n-2}(\theta) & = & b_n e^{ni\theta} & + & b_{n-1} e^{(n-1)i\theta} + b_{n-2} e^{(n-2)i\theta} \\ \vdots & & & & \end{array} \tag{11}$$

Proof. If θ is an integer multiple of 2π, all s_i are positive real numbers, and the result is immediate. Let us then fix a $\theta \in (0, 2\pi)$, and suppose first that n is even. We have then

$$(b_n, b_{n-1}, \ldots b_0) = (c_m, c_{m-1}, \ldots, c_1, c_0, c_1, \ldots c_m),$$

with $m = n/2$, and where the sequence $c_0, c_1, \ldots, c_m$ is positive, nondecreasing and convex. Let

$$g^* = c_m e^{2mi\theta} + c_{m-1} e^{(2m-1)i\theta} + \cdots + c_1 e^{(m+1)i\theta} + \frac{c_0}{2} e^{mi\theta}.$$

For every k, let then $q_k = e^{-mi\theta}(s_k - g^*)$, with the convention that $s_{n+1} = 0$. The inclusion relations are invariant under rotations and translations. Therefore, we want to prove that q_{n+1} is not in the strict convex hull of $q_0, q_1, \ldots, q_n$.

Observe that $q_{m+1} = -c_0/2$, and that $q_{m-k} = q_{m+1-k} + c_k e^{-ki\theta}$. Proposition 1 implies thus the existence of a g on the imaginary axis such that

$$|q_{m+1} - g| \leq |q_m - g| \leq |q_{m-1} - g| \leq \cdots \leq |q_0 - g| . \tag{12}$$

Moreover, observe that $q_{m+k} = -\bar{q}_{m+1-k}$, i.e., they have the same imaginary part and opposite real parts. There holds therefore $|q_{m+k} - g| = |q_{m+1-k} - g|$ since g has no real part. Together with the inequality (12), this implies that $|q_{n+1} - g| \geq |q_k - g|$ for every k. It follows then from Lemma 4 that q_{n+1} is not in the strict convex hull of the q_k, and thus that 0 is not in the strict convex hull of the s_k since the inclusion relations are invariant under rotations and translations.

Suppose now that n is odd. In that case, we can rewrite the sequence of coefficient as

$$(b_n, b_{n-1}, \dots b_0) = (c_m, c_{m-1}, \dots, c_1, c_0, c_0, c_1, \dots c_m)$$

with $m = (n-1)/2$. We define

$$g^* = c_m e^{(2m+1)i\theta} + c_{m-1} e^{(2m-1)i\theta} + \cdots + c_1 e^{(m+2)i\theta} + c_0 e^{(m+1)i\theta}$$

(observe that c_0 is not divided by 2 here), and for every k, $q_k = e^{-\left(m+\frac{1}{2}\right)i\theta}\left(s_k - g^*\right)$. Observe that $q_{m+1} = 0$, that $q_{m-k} = q_{m+1-k} + c_k e^{-ik\theta}.e^{-i\theta/2}$ and that $q_{m+1+k} = -\bar{q}_{m+1-k}$. We can then apply the same argument as in the even case, using Corollary 1 instead of Proposition 1. □

3 Proof of Theorem 1

We can now prove Theorem 1.

Proof. Fix a polynomial P, and assume without loss of generality that $a_n = 1$. Suppose, to obtain a contradiction, that $P(re^{i\theta}) = 0$ for some θ, and $r \geq 1$. Dividing $P(re^{i\theta}) = 0$ by r^n, we obtain:

$$a_n b_n e^{ni\theta} + a_{n-1} r^{-1} b_{n-1} e^{(n-1)i\theta} + \cdots + a_0 r^{-n} b_0 = 0. \tag{13}$$

Observe that since $r \geq 1$ and $1 = a_n > a_{n-1} > \dots a_0 > 0$, there holds $1 = a_n > a_{n-1} r^{-1} > \cdots > a_0 r^{-n}$. Let $\lambda_0 = a_0 r^{-n}$, and for $k = 1, \dots, n$, $\lambda_k = a_k r^{k-n} - a_{k-1} r^{k-1-n}$. Clearly, $\lambda_k \in (0,1)$ holds for every k, and $\sum_k \lambda_k = a_n = 1$. We can then rewrite equation (13) as

$$\begin{array}{rcll} 0 & = & \lambda_n & \left(b_n e^{ni\theta}\right) \\ & + & \lambda_{n-1} & \left(b_n e^{ni\theta} + b_{n-1} e^{(n-1)i\theta}\right) \\ & \vdots & & \\ & + & \lambda_0 & \left(b_n e^{ni\theta} + b_{n-1} e^{(n-1)i\theta} + \cdots + b_0\right). \end{array} \tag{14}$$

The point 0 would thus be in the strict convex hull of the points listed in equation (14), in contradiction with Proposition 2. □

Note that the convex-hull based approach of this last part of the proof can be applied to any class of polynomials of the form $\sum_k a_k b_k z^k$ with positive real coefficients a_k, b_k, and where the sequence of a_k is increasing.

References

[1] A. Mauroy, J.M. Hendrickx, A. Megretski and R. Sepulchre, *Global Analysis of Firing Maps*, Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems (MTNS2010), Budapet, Hungary, July 2010.