Compressive sensing of low-complexity signals: theory, algorithms and extensions

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Part 3
Quantized Compressed Sensing
**CS and Quantization?**

Compressed sensing theorist says:

"Linearly sample a signal at a rate function of its intrinsic dimensionality"

Information theorist and sensor designer say:

"Okay, but I need to quantize/digitize my measurements!" (e.g., in ADC)
**Scalar quantization**

Applied componentwise on $M$-dimensional vectors

$$Q_{\text{Out}}$$

**Examples:**
- uniform, resolution $\delta > 0$
  $$Q(t) = \delta(\lfloor\frac{t}{\delta}\rfloor + \frac{1}{2})$$
- or 1-bit quantization:
  $$Q(t) = \text{sign}(t) \in \{\pm 1\}$$

*Caveat*: not covered here:
- Sigma-Delta quantization for CS (see, e.g., Kramer, Saab, Guntürk, ...)
- Vector/binned quantization (see, e.g., Goyal, Pai, Nguyen, Sun, ...)
- Universal quantization (periodic) (see, e.g., Boufounos, Rane, ...)

**Pulse Code Modulation - PCM**
**Memoryless Scalar Quantization - MSQ**
QCS: former solution (Candès, Tao, ...)

1. (scalar) Quantization is like a noise

\[ q = Q[\Phi x] = \Phi x + n \]

quantization distortion (bounded)
\[ n = Q[\Phi x] - \Phi x \]
\[ \Phi_{ij} \sim \text{iid } \mathcal{N}(0, 1) \]

2. CS is robust (e.g., with basis pursuit denoise)

\[ \hat{x} = \arg\min_{u \in \mathbb{R}^N} \|u\|_1 \text{ s.t. } \|\Phi u - q\| \leq \epsilon \quad \text{(BPDN)} \]

\[ \ell_2 - \ell_1 \text{ instance optimality:} \]
If \( \|n\| \leq \epsilon \) and \( \frac{1}{\sqrt{M}} \Phi \) is RIP(\( \rho, 2K \)) with \( \rho \leq \sqrt{2} - 1 \), then

\[ \|\hat{x} - x\| \lesssim \frac{\epsilon}{\sqrt{M}} + e_0(K), \]

with \( e_0(K) = \|x - x_K\|_1/\sqrt{K} \).
QCS: former solution (Candès, Tao, ...)

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If \( \|n\| \leq \epsilon \) and \( \frac{1}{\sqrt{M}} \Phi \) is RIP(\( \rho \), 2\( K \)) with \( \rho \leq \sqrt{2} - 1 \), then

\[ \|\hat{x} - x\| \lesssim \delta + e_0(K), \]

with \( e_0(K) = \|x - x_K\|_1 / \sqrt{K} \).

Deterministic: \( \epsilon^2 \leq M\delta^2 / 4 \)

Stochastic: \( \epsilon^2 \leq M\delta^2 / 12 + c\sqrt{M} \) (w.h.p)
QCS: former solution (Candès, Tao, ...)

In short:

$$\|\hat{x} - x\| \lesssim \delta + e_0(K),$$

But quantization error doesn’t decay with $M$ !?

Solution: *be consistent!*

Enforce $Q[\Phi \hat{x}] = Q[\Phi x]$!

i.e., “consistency condition”

not guaranteed by BPDN
1-bit Compressed Sensing
Central question: 1-bit sampling?
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Signal: $s(t)$

Sampling: $s_n$
Central question: 1-bit sampling?

Signal \( s(t) \)

Sampling \( s_n \)

1-bit quantization

\[ \text{sign} (s_n) \geq 0 \]

\[ \text{sign} (s_n) \leq 0 \]
Central question: 1-bit sampling?

- Doable?
- For which “Sampling”?
- Which accuracy?

1-bit quantization

\[ \text{sign}(s_n) \]

\[ \geq 0 \]

\[ \leq 0 \]
Why 1-bit? Very Fast Quantizers!

Theoretical slope = 1/3 b/dB
Actual Slope = 1/2.3 b/dB

[FIG1] Stated number of bits versus sampling rate.

Why 1-bit? Very Fast Quantizers!

Theoretical slope = 1/3 b/dB

Actual Slope = 1/2.3 b/dB

Inverse relationship btw achievable sampling rate vs. bit depth

[FIG1] Stated number of bits versus sampling rate.

Compressed Sensing

$y = \Phi \times x$

$y$ $\Phi$ $x$

$M$ $M \times N$ $N$
1-bit Compressed Sensing

\[ q = \text{sign} \]

with:
\[
\text{sign} \ t = \begin{cases} 
  1 & \text{if } t > 0 \\
  -1 & \text{if } t \leq 0 
\end{cases}
\] component-wise

\[ \Phi \]

\[ M \times N \]

\[ x \]

\[ M \]

\[ N \]
1-bit Compressed Sensing

\[ q \overset{\text{Oversampling in } M}{=} \text{sign} \]

\[ \Phi \]

\[ x \]

\( M \)-bits! But, which information inside \( q \)?
1-bit Compressed Sensing

\[ q = \text{sign} \]

\[ \Phi \]

\[ x \]

Oversampling in \( M \) \( \rightarrow M \)-bits!

But, which information inside \( q \)?
**1-bit Compressed Sensing**

**Warning 1**: signal amplitude is lost!

\[ q = \text{sign} (\Phi (\lambda \mathbf{x})) = \text{sign} (\Phi \mathbf{x}), \quad \forall \lambda > 0 \]

\[ \Rightarrow \text{Amplitude is arbitrarily fixed} \]

Examples: \( \|\mathbf{x}\| = 1 \) or \( \|\Phi \mathbf{x}\|_1 = 1 \)
Warning 2: $\exists$ forbidden sensing!

Let $\mathbf{x}_\lambda := (1, \lambda, 0, \cdots, 0)^T \in \mathbb{R}^N$
and $\Phi \in \{\pm 1\}^{M \times N}$ (e.g., Bernoulli).

We have $\|\mathbf{x}_0 - \mathbf{x}_\lambda\| = \lambda$
but $q = \text{sign} (\Phi \mathbf{x}_0) = \text{sign} (\Phi \mathbf{x}_\lambda), \ \forall |\lambda| < 1$
$\Rightarrow$ No hope to distinguish them by increasing $M$!
Theoretical performance limits of 1-bit CS?
Lower bound: cell intersection viewpoint

\[ q = \text{sign}(\Phi x) \in \{\pm 1\}^M \]

Lower bound on any reconstruction methods such that:

1) the solution is sparse : \[ x^* \in \Sigma_K \]

2) the solution is consistent : \[ \text{sign}(\Phi x^*) = q \]
Lower bound: cell intersection viewpoint

\[ q = \text{sign}(\Phi \mathbf{x}) \in \{\pm 1\}^M \]

\[ \Rightarrow 2^M \text{ possible quantization points!} \]
Lower bound: cell intersection viewpoint

\[ q = \text{sign}(\Phi \mathbf{x}) \in \{\pm 1\}^M \]

\[ \Rightarrow 2^M \text{ possible quantization points!} \]

But, not all quantization cells occupied!

\[ \Rightarrow \text{no more than } C = 2^K \binom{N}{K} \binom{M}{K} \text{ cells!} \]
Lower bound: cell intersection viewpoint

Error of the best reconstruction method

⇔ Covering problem!

Link between accuracy and number of points

\[ S_{N-1} \cap \Sigma K \]

[Diagram of cell intersection viewpoint]
Lower bound: cell intersection viewpoint

Error of the best reconstruction method

⇔ Covering problem!

Link between accuracy and number of points

Most efficient $\epsilon$-covering of $S^{N-1} \cap \Sigma_K$ with $\epsilon$-caps

⇒ lower bound on $C$ by \( \frac{\text{vol}(S^{N-1} \cap \Sigma_K)}{\text{vol}(\epsilon\text{-cap})} \)

\[
\approx \binom{N}{K} \frac{\text{vol}(S^{K-1})}{\text{vol}(\epsilon\text{-cap})} = \binom{N}{K} \epsilon^{-K}
\]

Combining lower and upper bounds:

⇒ $\epsilon = \Omega(K/M)$

→ Lower bound on any 1-bit reconstruction error
Reaching this bound ?
Reaching this bound?

Let’s slice an orange!
Reaching this bound?

\( \mathbf{x} \) on \( S^2 \)

\( M \) vectors:

\[ \{ \varphi_i : 1 \leq i \leq M \} \Rightarrow \frac{1}{\| \varphi_i \|} \varphi_i \sim_{\text{iid}} \mathcal{U}(S^{N-1}) \]

iid Gaussian
Reaching this bound?

$x$ on $S^2$

$M$ vectors:
\[ \{ \varphi_i : 1 \leq i \leq M \} \]

iid Gaussian

1-bit Measurements

\[ \langle \varphi_1, x \rangle > 0 \]
Reaching this bound?

\( \mathbf{x} \) on \( S^2 \)

\( M \) vectors:
\[ \{ \varphi_i : 1 \leq i \leq M \} \]

iid Gaussian

1-bit Measurements

\[ \langle \varphi_1, \mathbf{x} \rangle > 0 \]
\[ \langle \varphi_2, \mathbf{x} \rangle > 0 \]
Reaching this bound?

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\( M \) vectors:

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iid Gaussian

1-bit Measurements

\[ \langle \varphi_1, x \rangle > 0 \]
\[ \langle \varphi_2, x \rangle > 0 \]
\[ \langle \varphi_3, x \rangle \leq 0 \]
Reaching this bound?

$x$ on $S^2$

$M$ vectors:

\[ \{ \varphi_i : 1 \leq i \leq M \} \]

iid Gaussian

1-bit Measurements

\[
\begin{align*}
\langle \varphi_1, x \rangle &> 0 \\
\langle \varphi_2, x \rangle &> 0 \\
\langle \varphi_3, x \rangle &\leq 0 \\
\langle \varphi_4, x \rangle &> 0
\end{align*}
\]
Reaching this bound?

$x$ on $S^2$

$M$ vectors:

$\{\varphi_i : 1 \leq i \leq M\}$

iid Gaussian

1-bit Measurements

\[ \langle \varphi_1, x \rangle > 0 \]
\[ \langle \varphi_2, x \rangle > 0 \]
\[ \langle \varphi_3, x \rangle \leq 0 \]
\[ \langle \varphi_4, x \rangle > 0 \]
\[ \langle \varphi_5, x \rangle > 0 \]
Reaching this bound?

$x$ on $S^2$

$M$ vectors:
$$\{\varphi_i : 1 \leq i \leq M\}$$

iid Gaussian

1-bit Measurements
$$\langle \varphi_1, x \rangle > 0$$
$$\langle \varphi_2, x \rangle > 0$$
$$\langle \varphi_3, x \rangle \leq 0$$
$$\langle \varphi_4, x \rangle > 0$$
$$\langle \varphi_5, x \rangle > 0$$
$$\vdots$$

Smaller and smaller when $M$ increases
$$\{u : \text{sign } (\Phi u) = \text{sign } (\Phi x)\}$$
Reaching this bound?

\( x \) on \( S^2 \)

\( M \) vectors:
\[ \{ \varphi_i : 1 \leq i \leq M \} \]

iid Gaussian

1-bit Measurements

\[
\begin{align*}
\langle \varphi_1, x \rangle &> 0 \\
\langle \varphi_2, x \rangle &> 0 \\
\langle \varphi_3, x \rangle &\leq 0 \\
\langle \varphi_4, x \rangle &> 0 \\
\langle \varphi_5, x \rangle &> 0 \\
\vdots &
\end{align*}
\]

Smaller and smaller when \( M \) increases
\[ \{ u : \text{sign} ( \Phi u ) = \text{sign} ( \Phi x ) \} \]

Lower bound on this width?
Reaching this bound?

Let $A(\cdot) := \text{sign}(\Phi \cdot)$ with $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$.

If $M = O(\epsilon^{-1} K \log N)$, then, w.h.p, for any two unit $K$-sparse vectors $x$ and $s$,

$$
A(x) = A(s) \implies \|x - s\| \leq \epsilon
$$

$$
\iff \epsilon = O\left(\frac{K}{M} \log \frac{MN}{K}\right)
$$
Reaching this bound?

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$$A(x) = A(s) \Rightarrow \|x - s\| \leq \epsilon$$

$$\iff \epsilon = O\left(\frac{K}{M} \log \frac{MN}{K}\right)$$

almost optimal
1-bit version of the RIP?
Preserving angles?
Let’s define

\[ A(u) := \text{sign} (\Phi u) \iff A_j(u) = \text{sign} (\varphi_j \cdot u) \in \{\pm 1\} \]

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Let \( u, v \in \mathbb{S}^{N-1} \) (wlog)

\[ \mathbb{P}[A_j(u) \neq A_j(v)] = ? \]
What’s known?

Let’s define

\[ A(u) := \text{sign}(\Phi u) \iff A_j(u) = \text{sign}(\varphi_j \cdot u) \in \{\pm 1\} \]

Let \( u, v \in \mathbb{S}^{N-1} \) (wlog)

\[ \mathbb{P}[A_j(u) \neq A_j(v)] = \frac{1}{\pi} \angle(u, v) = \frac{1}{\pi} \theta_{uv} \]
What’s known?

› Let’s define

\[ A(\mathbf{u}) := \text{sign} (\Phi \mathbf{u}) \iff A_j(\mathbf{u}) = \text{sign} (\varphi_j \cdot \mathbf{u}) \in \{ \pm 1 \} \]

The \( j \)th row of \( \Phi \)

Let \( \mathbf{u}, \mathbf{v} \in \mathbb{S}^{N-1} \) (wlog)

\[ \mathbb{P}[A_j(\mathbf{u}) \neq A_j(\mathbf{v})] = \frac{1}{\pi} \angle(\mathbf{u}, \mathbf{v}) \]

\[ = \frac{1}{\pi} \theta_{uv} \]

\[ A_j(\mathbf{u}) \oplus A_j(\mathbf{v}) \text{ (XOR)} \]

\[ \Rightarrow X_j = \frac{1}{2} |A_j(\mathbf{u}) - A_j(\mathbf{v})| \sim \text{Bernoulli}(\frac{\theta_{uv}}{\pi}) \in \{0, 1\} \]
Starting point: Hamming/Angle Concentration

- Metrics of interest:

\[ d_H(u, v) = \frac{1}{M} \sum_i (u_i \oplus v_i) \quad \text{(norm. Hamming)} \]
\[ d_{\text{ang}}(x, s) = \frac{1}{\pi} \arccos(\langle x, s \rangle) \quad \text{(norm. angle)} \]
Starting point: Hamming/Angle Concentration

- Metrics of interest:
  
  \[
  d_H(u, v) = \frac{1}{M} \sum_i (u_i \oplus v_i) \quad (\text{norm. Hamming})
  \]
  
  \[
  d_{\text{ang}}(x, s) = \frac{1}{\pi} \arccos(\langle x, s \rangle) \quad (\text{norm. angle})
  \]

- Known fact: if \( \Phi \sim \mathcal{N}^{M \times N}(0, 1) \) \[ e.g., \] Goemans, Williamson 1995

Let \( \Phi \sim \mathcal{N}^{M \times N}(0, 1) \), \( A(\cdot) = \text{sign} (\Phi \cdot) \in \{-1, 1\}^M \) and \( \epsilon > 0 \).

For any \( x, s \in S^{N-1} \), we have

\[
\mathbb{P}_\Phi \left[ |d_H(A(x), A(s)) - d_{\text{ang}}(x, s)| \leq \epsilon \right] \geq 1 - 2e^{-2\epsilon^2 M}.
\]

\[
\frac{1}{M} \sum_{i=1}^M X_i = \frac{1}{M} \sum_i A_i(x) \oplus A_i(s)
\]

\( \Rightarrow \) Thanks to \( A(.) \), Hamming distance concentrates around vector angles!
Binary $\epsilon$ Stable Embedding ($B_{\epsilon}SE$)

A mapping $A : \mathbb{R}^N \rightarrow \{\pm 1\}^M$ is a binary $\epsilon$-stable embedding ($B_{\epsilon}SE$) of order $K$ for sparse vectors if

$$d_{\text{ang}}(x, s) - \epsilon \leq d_H(A(x), A(s)) \leq d_{\text{ang}}(x, s) + \epsilon$$

for all $x, s \in S^{N-1}$ with $x \pm s$ $K$-sparse.

kind of “binary restricted (quasi) isometry”
Binary $\epsilon$ Stable Embedding ($B\epsilon\text{SE}$)

A mapping $A : \mathbb{R}^N \rightarrow \{\pm 1\}^M$ is a **binary $\epsilon$-stable embedding** ($B\epsilon\text{SE}$) of order $K$ for sparse vectors if

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for all $\mathbf{x}, \mathbf{s} \in S^{N-1}$ with $\mathbf{x} \pm \mathbf{s}$ $K$-sparse.

kind of “binary restricted (quasi) isometry”

- **Corollary**: for any algorithm with output $\mathbf{x}^*$ jointly $K$-sparse and consistent (i.e., $A(\mathbf{x}^*) = A(\mathbf{x})$),

  $$d_{\text{ang}}(\mathbf{x}, \mathbf{x}^*) \leq 2\epsilon!$$

- If limited binary noise, $d_{\text{ang}}$ still bounded
- If not exactly sparse signals (but almost), $d_{\text{ang}}$ still bounded
**B\(\varepsilon\)**SE existence?  **Yes!**

Let \( \Phi \sim \mathcal{N}^{M \times N}(0, 1) \), fix \( 0 \leq \eta \leq 1 \) and \( \varepsilon > 0 \). If

\[
M \geq \frac{4}{\varepsilon^2} \left( K \log(N) + 2K \log\left(\frac{50}{\varepsilon}\right) + \log\left(\frac{2}{\eta}\right)\right),
\]

then \( \Phi \) is a B\(\varepsilon\)**SE with \( \Pr > 1 - \eta \).

\[
M = O\left(\varepsilon^{-2} K \log N\right)
\]

\[
d_{\text{ang}}(x, s) - \varepsilon \leq d_H(A(x), A(s)) \leq d_{\text{ang}}(x, s) + \varepsilon
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M = O(\epsilon^{-2} K \log N)
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d_{\text{ang}}(x, s) - \epsilon \leq d_{H}(A(x), A(s)) \leq d_{\text{ang}}(x, s) + \epsilon
\]

\implies \quad \text{B}\(\varepsilon\)SE consistency “width”:

\[
\epsilon = O\left( \left( \frac{K}{M} \log \frac{MN}{K} \right)^{1/2} \right)
\]
**BεSE existence?**  Yes!

Let $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$, fix $0 \leq \eta \leq 1$ and $\epsilon > 0$. If

$$M \geq \frac{4}{\epsilon^2} \left( K \log(N) + 2K \log\left(\frac{50}{\epsilon}\right) + \log\left(\frac{2}{\eta}\right)\right),$$

then $\Phi$ is a BεSE with $\Pr > 1 - \eta$.

$$M = O(\epsilon^{-2} K \log N)$$

$$d_{\text{ang}}(x, s) - \epsilon \leq d_H(A(x), A(s)) \leq d_{\text{ang}}(x, s) + \epsilon$$

⇒

**BεSE consistency “width”:**

$$\epsilon = O\left(\left(\frac{K}{M} \log \frac{MN}{K}\right)^{1/2}\right)$$

not as optimal but stronger result!

$$d_H \leftrightarrow d_{\text{ang}}$$
Embedding low-complexity signals in a 1-bit world?
Beyond strict sparsity ...

Let $\mathcal{K} \subset S^{N-1}$ (e.g., compressible signals s.t. $\|x\|_2/\|x\|_1 \leq \sqrt{K}$)

What can we say on $d_H(A(x), A(s))$ for $x, s \in \mathcal{K}$?
Beyond strict sparsity ...

Let $\mathcal{K} \subset S^{N-1}$ (e.g., compressible signals s.t. $\|x\|_2 / \|x\|_1 \leq \sqrt{K}$)

What can we say on $d_H(A(x), A(s))$ for $x, s \in \mathcal{K}$?

Uniform tessellation: [Plan, Vershynin, 11]

$P(\# \text{ random hyperplanes btw } x \text{ and } s \propto d_{\text{ang}}(x, s))$ ?

---

Measuring the “dimension” of $\mathcal{K} \rightarrow$ Gaussian mean width:

$$w(\mathcal{K}) := \mathbb{E} \sup_{u \in \mathcal{K} - \mathcal{K}} \langle g, u \rangle, \text{ with } g_k \sim_{\text{iid}} \mathcal{N}(0, 1)$$

$$\eta = g/\|g\|$$

width in direction $\eta \in S^{N-1}$

Beyond strict sparsity ... [Plan, Vershynin]

Measuring the “dimension” of $\mathcal{K} \to$ Gaussian mean width:

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Examples of sets:

- $\mathcal{K} = \{x_i : 1 \leq i \leq |\mathcal{K}|\} \quad \{\text{finite set}\}$
- $\mathcal{K} = \Sigma_K := \{u \in \mathbb{R}^N : \|u\|_0 := |\text{supp } u| \leq K\} \quad \{k\text{-sparse signals}\}$
- $\mathcal{K} = \mathcal{C}_K := \{u \in \mathbb{R}^N : \|u\|_1 \leq \sqrt{K}, \|u\|_2 \leq 1\} \quad \{\text{“compressible” signals}\}$
- $\mathcal{K} = \mathcal{L}_r := \{U \in \mathbb{R}^{n \times n} \simeq \mathbb{R}^N : \text{rank}(U) \leq r\} \quad \{\text{rank-}r \text{ matrices}\}$
- $\overline{\mathcal{K}} = \text{conv}(\mathcal{K}) \quad \{\text{convex hull of prev. sets}\}$

Beyond strict sparsity ... [Plan, Vershynin]

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$$w(\mathcal{K}) := \mathbb{E} \sup_{u \in \mathcal{K} - \mathcal{K}} \langle g, u \rangle, \text{ with } g_k \sim_{\text{iid}} \mathcal{N}(0, 1)$$

Examples:

- $w^2(\mathcal{K}) \lesssim \log |\mathcal{K}|$ \{finite set\}
- $w^2(\mathbb{B}^N) \lesssim N$ \{\ell_2-ball\}
- $w^2(\mathcal{S} \cap \mathbb{B}^N) \lesssim \dim(\mathcal{S})$ \{subspace\}
- $w^2(\Sigma_K \cap \mathbb{B}^N) \lesssim w^2(C_K)$
  $\lesssim K \log N/K$ \{sparse/compr. signals\}
- $w^2(\mathcal{L}_r \cap \mathbb{B}^N_F) \lesssim r n$ \{rank-$r$ matrices\}

with $w(\mathcal{K}) \leq w(\mathcal{K}')$ if $\mathcal{K} \subset \mathcal{K}'$

---


Proposition  Let $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$ and $\mathcal{K} \subset \mathbb{R}^N$. Then, for some $C, c > 0,$ if
\[ M \geq C\epsilon^{-6}w^2(\mathcal{K}), \]
then, with $Pr \geq 1 - e^{-c\epsilon^2 M},$ we have
\[ d_{\text{ang}}(x, s) - \epsilon \leq d_H(A(x), A(s)) \leq d_{\text{ang}}(x, s) - \epsilon, \quad \forall x, s \in \mathcal{K}. \]
Proposition  Let $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$ and $\mathcal{K} \subset \mathbb{R}^N$. Then, for some $C, c > 0$, if

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Generalize $\mathbf{BSE}$ to more general sets. In particular, to $\mathcal{C}_K = \{u \in \mathbb{R}^N : \|u\|_2/\|u\|_1 \leq \sqrt{K}\} \supset \Sigma_K$ with $w^2(\mathcal{C}_K) \leq cK \log N/K$. 

Beyond strict sparsity ... [Plan, Vershynin]

**Proposition** Let $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$ and $\mathcal{K} \subset \mathbb{R}^N$. Then, for some $C, c > 0$, if

$$M \geq C e^{\frac{6}{c} w^2(\mathcal{K})},$$

then, with $\Pr \geq 1 - e^{-c e^2 M}$, we have

$$d_{\text{ang}}(x, s) - \epsilon \leq d_H(A(x), A(s)) \leq d_{\text{ang}}(x, s) - \epsilon, \quad \forall x, s \in \mathcal{K}.$$

Generalize $\text{BSE}$ to more general sets.

In particular, to

$$\mathcal{C}_K = \{ u \in \mathbb{R}^N : \|u\|_2/\|u\|_1 \leq \sqrt{K} \} \supset \Sigma_K$$

with $w^2(\mathcal{C}_K) \leq c K \log N/K$.

$\Rightarrow$ Extension to “1-bit Matrix Completion” possible!

i.e., $w^2(r\text{-rank } N_1 \times N_2 \text{ matrix}) \leq c r (N_1 + N_2)!$
Reconstructing a signal from its 1-bit measurements?
Dumbest 1-bit reconstruction

**Fact:** If $M = O(\epsilon^{-2} K \log N/K)$ (for $x \in \Sigma_K$ fixed, $\forall s \in \Sigma_K$)
or, if $M = O(\epsilon^{-6} K \log N/K)$ ($\forall x, s \in \Sigma_K$), then, w.h.p.,

$$\left| \frac{\sqrt{\pi}/2}{M} \langle \text{sign} (\Phi x), \Phi s \rangle - \langle x, s \rangle \right| \leq \epsilon$$  \hspace{1cm} [Plan, Vershynin, 12]
Dumbest 1-bit reconstruction

Fact: If $M = O(\epsilon^{-2} K \log N/K)$ (for $x \in \Sigma_K$ fixed, $\forall s \in \Sigma_K$)

or, if $M = O(\epsilon^{-6} K \log N/K)$ ($\forall x, s \in \Sigma_K$), then, w.h.p.,

$$|\frac{\sqrt{\pi}}{M} \langle \text{sign}(\Phi x), \Phi s \rangle - \langle x, s \rangle| \leq \epsilon$$  \[\text{Plan, Vershynin, 12}\]

Implication? [LJ, Degraux, De Vleeschouwer, 13]

Let $x \in \Sigma_K \cap S^{N-1}$ and $q = \text{sign}(\Phi x)$.

Compute

$$\hat{x} = \frac{\pi}{2M} \mathcal{H}_K(\Phi^* q)$$

Then, if previous property holds,

$$\|x - \hat{x}\| \leq 2\epsilon.$$
Initial approach [Boufounos, Baraniuk 2008]

Given $q = \text{sign} (\Phi x) =: A(x)$

$$\hat{x} = \arg\min_u \|u\|_1 \quad \text{s.t.} \quad \text{diag}(q) \Phi u > 0 \quad \text{and} \quad \|u\|_2 = 1$$

Non-convex! 2 numerical choices:
1. relax + projection on $S^{N-1}$
2. “trust region methods” → Restricted-Step Shrinkage (RSS)

Consistency constraint:
$$\{u \in \mathbb{R}^N \cap S^{N-1} : q = A(u)\}$$
$$\Leftrightarrow \{u \in \mathbb{R}^N \cap S^{N-1} : \text{diag}(q) \Phi u > 0\}$$
$$\exists x$$
Initial approach [Boufounos, Baraniuk 2008]

Given \( q = \text{sign} (\Phi x) =: A(x) \)

\[
\hat{x} = \arg \min_{u} \|u\|_1 \quad \text{s.t.} \quad \text{diag}(q) \Phi u > 0 \quad \text{and} \quad \|u\|_2 = 1
\]

(relaxed) \( \hat{x} = \arg \min_{u} \|u\|_1 + \lambda \|\text{diag}(q) \Phi u\|_2^2 \quad \text{s.t.} \quad \|u\|_2 = 1 \)

→ Solved by projected gradient descent

\[
\begin{align*}
K = 16 \\
MSE (\text{dB}) \\
0 & \quad 1000 & \quad 2000 & \quad 3000 \\
0 & \quad -10 & \quad -20 \\
\text{Classical CS} & \quad \text{1-bit CS} \\
gain brought by (almost)consistency
\end{align*}
\]
Initial approach [Boufounos, Baraniuk 2008]

Given \( q = \text{sign} (\Phi x) =: A(x) \)

\[
\hat{x} = \arg \min_{u} \|u\|_1 \quad \text{s.t.} \quad \text{diag}(q) \Phi u > 0 \quad \text{and} \quad \|u\|_2 = 1
\]

(related) \( \hat{x} = \arg \min_{u} \|u\|_1 + \lambda \|\text{diag}(q) \Phi u\|_2 \) \quad \text{s.t.} \quad \|u\|_2 = 1

\[ \rightarrow \text{Solved by projected gradient descent} \]

Can we do better?

\[
K=16
\]

Classical CS

1-bit CS

gain brought by (almost) consistency
Matching Sign Pursuit (MSP)

Iterative greedy algorithm, similar to CoSaMP [Needell, Tropp, 08]
Maintains running signal estimate and its support $T$.

**MSP iteration:**

Loop

Identify **sign violations** → $r = (\text{diag}(y) \Phi \hat{x})_-$

Compute **proxy** → $p = \Phi^T \text{diag}(y) r$

Identify **support** → $\Omega = \{\text{supp}(p|_{2K})\} \cup \{\text{supp}(\hat{x})\}$

Consistent Reconstruction over support estimate:

$$b|_\Omega = \arg \min_{u \in \mathbb{R}^N} \| (\text{diag}(y) \Phi u)_- \|_2^2 \text{ s.t. } \|u\|_2 = 1 \text{ and } u|_{\Omega^c} = 0$$

Truncate, normalize, and **update** estimate: $\hat{x} \leftarrow b|_K / \|b|_K\|_2$

P. T. Boufounos, “Greedy sparse signal reconstruction from sign measurements.”
Binary Iterative Hard Thresholding

Given \( q = A(x) \) and \( K \), set \( l = 0 \), \( x^0 = 0 \):

\[
\begin{align*}
\mathbf{a}^{l+1} &= \mathbf{x}^l + \frac{\tau}{2} \Phi^T (q - A(\mathbf{x}^l)), \\
\mathbf{x}^{l+1} &= \mathcal{H}_K(\mathbf{a}^{l+1}), \quad l \leftarrow l + 1
\end{align*}
\]

(“gradient” towards consistency)

(\( \tau > 0 \) controls gradient descent)

(proj. \( K \)-sparse signal set)

with \( \mathcal{H}_K(\mathbf{u}) = K \)-term hard thresholding

Stop when \( d_H(q, A(\mathbf{x}^{l+1})) = 0 \) or \( l = \text{max. iter.} \).
Binary Iterative Hard Thresholding

Given \( q = A(x) \) and \( K \), set \( l = 0 \), \( x^0 = 0 \):

\[
\begin{align*}
\alpha^{l+1} &= x^l + \frac{\tau}{2} \Phi^T(q - A(x^l)), \\
x'^{l+1} &= \mathcal{H}_K(\alpha^{l+1}), \quad l \leftarrow l + 1
\end{align*}
\]

(“gradient” towards consistency)  
(\( \tau > 0 \) controls gradient descent)  
(proj. \( K \)-sparse signal set)

with \( \mathcal{H}_K(u) = K \)-term hard thresholding

Stop when \( d_H(q, A(x'^{l+1})) = 0 \) or \( l = \) max. iter.

minimizes

\[
\mathcal{J}(x') = \| [\text{diag}(q)(\Phi x')]_- \|_1 \quad \text{with} \quad (\lambda)_- = (\lambda - |\lambda|)/2
\]

\[
\mathcal{J}(x') = \sum_{j=1}^{M} |(\text{sign}(\langle \varphi_j, x \rangle) \langle \varphi_j, x' \rangle)_- |
\]

\[
q_k - A(x^l)_k = 0
\]

\[
q_j - A(x^l)_j > 0
\]

(connections with ML hinge loss, 1-bit classification)
$N = 1000, \ K = 10$
Bernoulli-Gaussian model
normalized signals
1000 trials

Matching Sign pursuit (MSP)
Restricted-Step Shrinkage (RSS)
Binary Iterative Hard Thresholding (BIHT)
Binary Iterative Hard Thresholding

- Testing $B\varepsilon SE$: $d_{\text{ang}}(\mathbf{x}, \mathbf{x}^*) \leq d_H(A(\mathbf{x}), A(\mathbf{x}^*)) + \varepsilon(M)$

$$M/N = 0.7$$

$$M/N = 1.5$$
Convex Optimization \[\text{[Plan, Vershynin, 12]}\]

Let \( q = \text{sign}(\Phi x) \) for some signal \( x \in \mathcal{K} \subset B_2^N \)

**Compute** \( \hat{x} = \arg \max_{u \in \mathbb{R}^N} q^T \Phi u \quad \text{s.t.} \quad u \in \mathcal{K} \)

Convex problem if \( \mathcal{K} \) convex!
No ambiguous amplitude definition
(\( u = 0 \) avoided)

e.g., sparse, compressible, low-rank matrix

Convex Optimization  [Plan, Vershynin, 12]

Let $q = \text{sign}(\Phi x)$ for some signal $x \in \mathcal{K} \subset B_2^N$

Compute $\hat{x} = \arg \max_{u \in \mathbb{R}^N} q^T \Phi u \quad \text{s.t.} \quad u \in \mathcal{K}$

Convex problem if $\mathcal{K}$ convex!
No ambiguous amplitude definition
($u = 0$ avoided)

Remark:  (PV-L0 problem)  [Bahmani, Boufounos, Raj, 13]

$\hat{x} = \frac{1}{\|\mathcal{H}_K(\Phi^* q)\|} \mathcal{H}_K(\Phi^* q) \quad \text{if} \quad \mathcal{K} = \Sigma K$
Convex Optimization [Plan, Vershynin, 12]

Let $q = \text{sign}(\Phi \mathbf{x})$ for some signal $\mathbf{x} \in \mathcal{K} \subset B_2^N$.

Compute $\hat{\mathbf{x}} = \arg \max_{\mathbf{u} \in \mathbb{R}^N} q^T \Phi \mathbf{u}$ s.t. $\mathbf{u} \in \mathcal{K}$.

Proposition (assuming $\|\mathbf{x}\| = 1$) For some $C, c > 0$, if $M \geq C e^{-\frac{6}{c^2} w^2(\mathcal{K})}$, then, with $Pr \geq 1 - e^{-c^2 M}$, we have $\|\hat{\mathbf{x}} - \mathbf{x}\|^2 \leq \sqrt{\frac{\pi}{2}} \epsilon$. 

e.g., sparse, compressible, low-rank matrix

-2 if $\mathbf{x}$ is fixed
Convex Optimization

Let \( q = \text{sign}(\Phi x) \) for some signal \( x \in \mathcal{K} \subset B_2^N \)

Compute \( \hat{x} = \arg \max_{u \in \mathbb{R}^N} q^T \Phi u \) s.t. \( u \in \mathcal{K} \)

**Proposition** (assuming \( \|x\| = 1 \)) For some \( C, c > 0 \), if \( M \geq C \epsilon^{-6} w^2(\mathcal{K}) \), then, with \( Pr \geq 1 - e^{-c \epsilon^2 M} \), we have \( \|\hat{x} - x\|^2 \leq \sqrt{\frac{\pi}{2}} \epsilon. \)

+ Robust to noise: noise (bit flip)

Let \( q_n = \text{diag}(\eta)q \) with \( \eta_i \in \{\pm 1\}^M \), and assume \( d_H(q, q_n) \leq p \)

(under the same conditions)
\[
\|\hat{x} - x\|^2 \leq \epsilon \sqrt{\log e/\epsilon} + 11 p \sqrt{\log e/p}
\]

Note: if \( M = O(\epsilon^{-2}(p - 1/2)^{-2} K \log N/K) \) this term disappears if \( \eta_i = \pm 1 \) are iid RVs (with \( P(\eta_i = 1) = p \)
Beyond 1-bit CS?
New geometrical questions

- Properties of quantized random projections?

For a (sub)Gaussian $\Phi \in \mathbb{R}^{M \times N}$, $Q$ of resolution $\delta > 0$ and a dithering $\xi$ s.t. $\xi_i \sim_{iid} \mathcal{U}([0, \delta])$,

properties of $A(x) := Q(\Phi x + \xi)$?

Remark: Minimal impact, e.g., on CS, but dithering greatly simplifies mathematical proofs)
New geometrical questions

- Properties of quantized random projections?
  For a (sub)Gaussian $\Phi \in \mathbb{R}^{M \times N}$,
  $Q$ of resolution $\delta > 0$
  and a dithering $\xi$ s.t. $\xi_i \sim_{\text{iid}} U([0, \delta])$,
  properties of $A(x) := Q(\Phi x + \xi)$?

- Locality property? (or proximity of consistent vectors)

1. for all $x, x' \in \mathcal{K}$, $A(x) = A(x') \Rightarrow \|x - x'\| \leq \epsilon(M, N, \mathcal{K}, \delta)$?
New geometrical questions

- **Properties of quantized random projections?**
  For a (sub)Gaussian $\Phi \in \mathbb{R}^{M \times N}$,
  $Q$ of resolution $\delta > 0$
  and a dithering $\xi$ s.t. $\xi_i \sim_{\text{iid}} \mathcal{U}([0, \delta])$,
  properties of $A(x) := Q(\Phi x + \xi)$?

- **Locality property?** (or proximity of consistent vectors)

  ![Diagram](image)

  low complexity set $\mathcal{K}$

  for all $x, x' \in \mathcal{K}$, $A(x) = A(x') \Rightarrow \|x - x'\| \leq \epsilon(M, N, \mathcal{K}, \delta)$?
New geometrical questions

- Properties of quantized random projections?
  For a (sub)Gaussian $\Phi \in \mathbb{R}^{M \times N}$, $Q$ of resolution $\delta > 0$ and a dithering $\xi$ s.t. $\xi_i \sim_{\text{iid}} U([0, \delta])$, properties of $A(x) := Q(\Phi x + \xi)$?

- Locality property? (or proximity of consistent vectors)

\[1\]

1. for all $x, x' \in \mathcal{K}$, $A(x) = A(x') \Rightarrow \|x - x'\| \leq \epsilon(M, N, K, \delta)$?

+ random shift = dithering

\[\Phi = \begin{pmatrix}
\varphi_1^T \\
\vdots \\
\varphi_M^T
\end{pmatrix}\]
New geometrical questions

- Properties of quantized random projections?
  For a (sub)Gaussian $\Phi \in \mathbb{R}^{M \times N}$, $Q$ of resolution $\delta > 0$ and a dithering $\xi$ s.t. $\xi \sim \text{iid} \ U([0, \delta])$, properties of $A(x) := Q(\Phi x + \xi)$?

- Locality property? (or proximity of consistent vectors)

\[ 1 \quad \text{for all } x, x' \in \mathcal{K}, \quad A(x) = A(x') \Rightarrow \|x - x'\| \leq \epsilon(M, N, \mathcal{K}, \delta) ? \]

\[ \Phi = \begin{pmatrix} \varphi_1^T \\ \vdots \\ \varphi_M^T \end{pmatrix} \]
New geometrical questions

- Properties of quantized random projections?
  For a (sub)Gaussian $\Phi \in \mathbb{R}^{M \times N}$, $Q$ of resolution $\delta > 0$ and a dithering $\xi$ s.t. $\xi_i \sim_{iid} \mathcal{U}([0, \delta])$, properties of $A(x) := Q(\Phi x + \xi)$?

- Locality property? (or proximity of consistent vectors)

  for all $x, x' \in \mathcal{K}$, $A(x) = A(x') \Rightarrow \|x - x'\| \leq \epsilon(M, N, K, \delta)$?

Size should decay! consistency cell

for large $M$

$$\Phi = \begin{pmatrix}
\Phi_1^T \\
\vdots \\
\Phi_M^T
\end{pmatrix}$$
New geometrical questions

- Properties of quantized random projections?
  For a (sub)Gaussian $\Phi \in \mathbb{R}^{M \times N}$, 
  $Q$ of resolution $\delta > 0$ 
  and a dithering $\xi$ s.t. $\xi_i \sim_{\text{iid}} \mathcal{U}([0, \delta])$, 
  properties of $A(x) := Q(\Phi x + \xi)$?

- Locality property? (or proximity of consistent vectors)

  1. for all $x, x' \in \mathcal{K}$, $A(x) = A(x') \Rightarrow \|x - x'\| \leq \epsilon(M, N, \mathcal{K}, \delta)$?

Size should decay! (consistency cell)

and don’t forget

spiky ball!

$\ell_1$-ball in high dimension
New geometrical questions

- Properties of quantized random projections?
  For a (sub)Gaussian $\Phi \in \mathbb{R}^{M \times N}$, $Q$ of resolution $\delta > 0$ and a dithering $\xi$ s.t. $\xi_i \sim_{\text{iid}} \mathcal{U}([0, \delta])$, properties of $A(x) := Q(\Phi x + \xi)$?

- Locality property? (or proximity of consistent vectors)
  1. **for all** $x, x' \in \mathcal{K}$, $A(x) = A(x') \Rightarrow \|x - x'\| \leq \epsilon(M, N, \mathcal{K}, \delta)$?

- Quasi-isometric embedding?
  2. **for all** $x, x' \in \mathcal{K}$, $A(x) \approx A(x') \iff x \approx x'$?

- More optimal/consistent reconstruction methods?
Proximity of consistent vectors

Gaussian sensing: \( \Phi_{ij} \sim \text{iid } \mathcal{N}(0, 1), A(\cdot) := \mathcal{Q}(\Phi \cdot + \xi) \)

In all generality, provided

\[
M \gtrsim \frac{(1+\delta)^4}{\delta^2 \varepsilon^4} w^2(\mathcal{K})
\]

Then, with \( \Pr \geq 1 - 2 \exp(-\frac{c\varepsilon M}{1+\delta}) \) and uniformly for all \( \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{K} \),

\[
A(\mathbf{x}_1) = A(\mathbf{x}_2) \quad \Rightarrow \quad \|\mathbf{x}_1 - \mathbf{x}_2\| \leq \varepsilon
\]

\( \Rightarrow \varepsilon = O(M^{-1/4}) \) for consistent reconstruction \( \in \mathcal{K}! \)

Proximity of consistent vectors
Proximity of consistent vectors  [LJ 2015]

- Gaussian sensing: \( \Phi_{ij} \sim \text{iid } \mathcal{N}(0, 1), \ A(\cdot) := Q(\Phi \cdot + \xi) \)

In all generality, provided

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M \gtrsim \frac{(1+\delta)^4}{\delta^2 \epsilon^4} w^2(\mathcal{K})
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Then, with \( \Pr \geq 1 - 2 \exp(-\frac{c\epsilon M}{1+\delta}) \) and uniformly for all \( x_1, x_2 \in \mathcal{K} \),

\[
A(x_1) = A(x_2) \quad \Rightarrow \quad \|x_1 - x_2\| \leq \epsilon
\]

\( \Rightarrow \epsilon = O(M^{-1/4}) \) for consistent reconstruction \( \in \mathcal{K} \)!

For sparse vector sets

For \( \mathcal{K} = (\Psi \Sigma_K) \cap B^N \) and \( \Psi \) ONB

\[
M \gtrsim \frac{2+\delta}{\epsilon} K \log\left( \frac{N(2+\delta)^{3/2}}{K \delta \epsilon^{3/2}} \right)
\]

\( \Rightarrow \epsilon = O(M^{-1})! \)

+ Same behavior for union of subspaces & low-rank matrix!
Quantizing the RIP?

Restrained isometry Property (RIP): (as an embedding preserving distances)

\[(1 - \rho)\|u - v\|^2 \leq \frac{1}{M} \|\Phi u - \Phi v\|^2 \leq (1 + \rho)\|u - v\|^2\]

for all \(u, v \in \Sigma_K := \{u : \|u\|_0 := |\text{supp } u| \leq K\}\)
Quantizing the RIP?

Restricted isometry Property (RIP):

\[(1 - \rho)\|u - v\|^2 \leq \frac{1}{M} \|\Phi u - \Phi v\|^2 \leq (1 + \rho)\|u - v\|^2\]

for all \(u, v \in \Sigma_K := \{u : \|u\|_0 := |\text{supp } u| \leq K\}\)

Why quantizing the RIP?

- since we can ;-)
- for future algorithm guarantees
- for classification, regression in “projected” domain
- or “signal processing” in quantized CS domain
Let’s retake: for $Q(\cdot) = \delta([\frac{\cdot}{\delta}] + \frac{1}{2})$

$A(u) := Q(\Phi u + \xi)$, with $A_j(u) := Q(\varphi_j^T u + \xi_j)$

with $\Phi_{ji} \sim_{iid} \mathcal{N}(0, 1)$ and $\xi_j \sim_{iid} \mathcal{U}([0, \delta])$. 

Quantizing the RIP? for $Q(\cdot) = (b \cdot c + \frac{1}{2})^2$
**Quantizing the RIP?**

- Let’s retake: for $Q(\cdot) = \delta(\lfloor \frac{\cdot}{\delta} \rfloor + \frac{1}{2})$
  \[ A(u) := Q(\Phi u + \xi), \text{ with } A_j(u) := Q(\varphi_j^T u + \xi_j) \]
  with $\Phi_{ji} \sim_{iid} \mathcal{N}(0, 1)$ and $\xi_j \sim_{iid} \mathcal{U}([0, \delta])$.

- **Naive way:** since $|a - b| - \delta \leq |Q(a) - Q(b)| \leq |a - b| + \delta, \quad \forall a, b \in \mathbb{R}$
  \[ (1 - \rho) \|u - v\| - \delta \leq \frac{1}{\sqrt{M}} \|A(u) - A(v)\| \leq (1 + \rho) \|u - v\| + \delta, \]
  whenever $\frac{1}{\sqrt{M}} \Phi$ is RIP.
Quantizing the RIP?

Let’s retake: for $Q(\cdot) = \delta([\frac{\cdot}{\delta}] + \frac{1}{2})$

$A(u) := Q(\Phi u + \xi)$, with $A_j(u) := Q(\varphi_j^T u + \xi_j)$

with $\Phi_{ji} \sim_{iid} \mathcal{N}(0, 1)$ and $\xi_j \sim_{iid} \mathcal{U}([0, \delta])$.

Naive way: since $|a-b| - \delta \leq |Q(a) - Q(b)| \leq |a-b| + \delta$, $\forall a, b \in \mathbb{R}$

$$\begin{align*}
(1 - \rho) \|u - v\| - \delta \leq \frac{1}{\sqrt{M}} \|A(u) - A(v)\| \leq (1 + \rho) \|u - v\| + \delta,
\end{align*}$$

**multiplicative error**

With $\rho = O(\sqrt{K/M})$.

(decaying, good!)

**additive error**

(constant, weird!?)

With $\rho = O(\sqrt{K/M})$. 

Decaying additive distortion?

Let’s use another distance ($\ell_1$):

$$\frac{1}{M} \| A(u) - A(v) \|_1 = \frac{1}{M} \sum_j |A_j(u) - A_j(v)|$$

$$A(u) := Q(\Phi u + \xi)$$
$$A_j(u) := Q(\varphi_j^T u + \xi_j)$$
$$\Phi \sim \mathcal{N}^{M \times N}(0, 1), \xi \sim \mathcal{U}^M([0, \delta])$$
Decaying additive distortion?

Let’s use another distance ($\ell_1$):

$$\frac{1}{M} \| A(u) - A(v) \|_1 = \frac{1}{M} \sum_j |A_j(u) - A_j(v)|$$

Similar to random hyperplane tessellations of $\mathbb{R}^N$ [Plan, Vershynin, 13] but adapted to “hyperplane wave partition” [Thao, Vetterli, 96]
Quantizing the RIP?

- Decaying additive distortion?
- Let’s use another distance ($\ell_1$):

$$\frac{1}{M} \| A(u) - A(v) \|_1 = \frac{1}{M} \sum_j |A_j(u) - A_j(v)|$$

Similar to random hyperplane tessellations of $\mathbb{R}^N$ [Plan, Vershynin, 13] but adapted to “hyperplane wave partition” [Thao, Vetterli, 96]

\[
\Phi = \begin{pmatrix}
\varphi_1^T \\
\vdots \\
\varphi_M^T
\end{pmatrix}
\]

\[\Phi \sim \mathcal{N}^{M \times N}(0, 1), \xi \sim \mathcal{U}^M([0, \delta])\]

\[A(u) := Q(\Phi u + \xi)\]
\[A_j(u) := Q(\varphi_j^T u + \xi_j)\]

\# hyperplanes $= \sum_j |A_j(u) - A_j(v)|$
Quantizing the RIP?

Quantized Gaussian Quasi-Isometric Embedding [LJ, 2015]

Given an error $0 < \epsilon < 1$, and $\mathcal{K} \subset \mathbb{R}^N$.
If $M$ is such that

$$M \gtrsim \epsilon^{-5} w(\mathcal{K})^2,$$

then, for some $c > 0$ and for all $u, v \in \mathcal{K}$, and w.h.p., we have

$$(\sqrt{\frac{2}{\pi}} - \epsilon) \|u - v\| - c\delta \epsilon \leq \frac{1}{M} \|A(u) - A(v)\|_1 \leq (\sqrt{\frac{2}{\pi}} + \epsilon) \|u - v\| + c\delta \epsilon,$$

all distortions decay with $M$!
Quantizing the RIP?

Quantized Gaussian Quasi-Isometric Embedding [LJ, 2015]

**Given an error** $0 < \epsilon < 1$, and $\mathcal{K} \subset \mathbb{R}^N$.

If $M$ is such that

$$M \gtrsim \epsilon^{-5} w(\mathcal{K})^2,$$

then, for some $c > 0$ and for all $u, v \in \mathcal{K}$, and w.h.p., we have

$$(\sqrt{\frac{2}{\pi}} - \epsilon) \|u - v\| - c\delta\epsilon \leq \frac{1}{M} \|A(u) - A(v)\|_1 \leq (\sqrt{\frac{2}{\pi}} + \epsilon) \|u - v\| + c\delta\epsilon,$$

Remark: extension to sub-Gaussian matrices (for “not too sparse” signals)

but with a worsened multiplicative distortion [LJ 15]

\[ A(u) := Q(\Phi u + \xi) \]
\[ A_j(u) := Q(\varphi_j^T u + \xi_j) \]
\[ \Phi \sim \mathcal{N}^{M \times N}(0, 1), \ \xi \sim \mathcal{U}^M([0, \delta]) \]
Consistent Basis Pursuit for low-complexity signals
Promoting low-complexity signals

- **Atomic norm**: [Chandrasekaran 2012]
  
  Given $\mathcal{K}$, $\exists$ an atomic (convex) norm $\| \cdot \|_\#$ and a $s > 0$ s.t.
  
  $$(\mathcal{K} \cap \mathbb{B}^N) \subset \mathcal{K}_s := \{ u \in \mathbb{R}^N : \| u \|_\# \leq s, \| u \|_2 \leq 1 \}$$

- **Examples**:
  
  $\mathcal{K} = \Sigma_K := \{ u \in \mathbb{R}^N : \| u \|_0 := |\text{supp } u| \leq K \}$
  
  $\mathcal{K} = \mathcal{L}_r := \{ U \in \mathbb{R}^{n \times n} \simeq \mathbb{R}^N : \text{rank}(U) \leq r \}$

  \[ w^2(\mathcal{K}_s) \lesssim K \log N/K \]

  \[ w^2(\mathcal{K}_s) \lesssim rn \]
Consistent Basis Pursuit (CoBP)

\[ x^* \in \underset{u \in \mathbb{R}^N}{\operatorname{argmin}} \|u\|_1 \quad \text{s.t.} \quad A(u) = A(x_0), \quad u \in \mathbb{B}^N. \]

Actually, not a so new program, see e.g., [Milenkovitch, Dai, JL, Hammond, Fadili, ...]
Consistent Basis Pursuit (CoBP)

\[ x^* \in \arg\min_{u \in \mathbb{R}^N} \|u\|_{\#} \quad \text{s.t.} \quad A(u) = A(x_0), \quad u \in \mathbb{B}^N. \]

Proposition: With high probability on Gaussian \( \Phi \) and uniform \( \xi \), \( \forall x_0 \in \mathcal{K}_s \),

\[ \|x_0 - x^*\|_2 = O\left(\frac{2+\delta}{\sqrt{\delta}} \left(\frac{w(\mathcal{K}_s)^2}{M}\right)^{1/4}\right), \]

i.e., \[ \|x_0 - x^*\|_2 = O\left(M^{-1/4}\right) \] if only \( M \) varies.

Proof: use proximity of consistent low-complexity signals (4 lines)
Experiments: implementation

- Solving CoBP: Convex Optimization

\[ \mathbf{x}^* \in \arg\min_{\mathbf{u} \in \mathbb{R}^N} \| \mathbf{u} \| \# \text{ s.t. } \mathbf{A}(\mathbf{u}) = \mathbf{A}(\mathbf{x}_0), \mathbf{u} \in \mathbb{B}^N. \]

\[ \iff \arg\min_{\mathbf{u} \in \mathbb{R}^N} \| \mathbf{u} \| \# + \iota_{\text{consist.}}(\mathbf{u}) + \iota_{\mathbb{B}^N}(\mathbf{u}) \]

- Many toolboxes available
- We used a proximal algorithm, i.e.,
  Parallel Proximal Algorithm (PPXA) or CP
- & the UNLocBoX toolbox (https://lts2.epfl.ch/unlocbox)
Experiments: 1

*K*-sparse signals with Gaussian sensing

\[N = 2048, \quad K = 16,\]
\[B = 3, \quad M/K \in [8, 128]\]

20 trials per points
Experiments: 2

- Low-rank matrix and QCS (with \( A(\cdot) := Q(\Phi \cdot) \))

\[
X^* \in \arg\min_{U \in \mathbb{R}^{n \times n}} ||U||_* \quad \text{s.t.} \quad A(\text{vec}(U)) = A(\text{vec}(X_0)), \quad \text{vec}(U) \in \mathbb{B}^N.
\]
Experiments: 2

- Low-rank matrix and QCS (with $A(\cdot) := Q(\Phi \cdot)$)

$$X^* \in \arg\min \|U\| \quad \text{s.t.} \quad A(\text{vec}(U)) = A(\text{vec}(X_0)), \quad \text{vec}(U) \in \mathbb{B}^N.$$ 

Original

$N = 1024 = n^2$ ($n = 32$),
rank $r = 1$,
Complexity $< P = 64$,
$M = 16P = N$

CoBP

SNR 11 dB

BPDN

SNR 6.9 dB
Take away messages

- Associating CS and Quantization provides many interesting questions:
  - geometrically (high dim. convex geom.)
  - numerically (not totally covered here)
  - with impacts in CS sensor design
Take away messages

› Associating CS and Quantization provides many interesting questions:
  › geometrically (high dim. convex geom.)
  › numerically (not totally covered here)
  › with impacts in CS sensor design
› Beyond CS, quantifying random projections
  › leads to interesting embedding problems
  › possible impacts in dimensionality reductions
Open questions

- CoBP robustness vs pre-quantization noise?
  Does quasi-isometric embedding help?

- Quasi-isometric embeddings
  for (subsets of) Hilbert spaces [Puy, Davies, Gribonval]?

- Embeddings with other quantization schemes?
  (kernel approximation? [Recht, Boufounos, ...])

- Classification/clustering
  in the quantized domain?

\[
\mathbb{Q}(\Phi \cdot + \xi) \quad \mathbb{R}^N \quad \delta \mathbb{Z}^m
\]