Stabilizing Nonuniformly Quantized Compressed Sensing with Scalar Companders

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Abstract

In this paper we study the problem of reconstructing sparse or compressible signals from compressed sensing measurements that have undergone non-uniform scalar quantization. Previous approaches to this Quantized Compressed Sensing (QCS) problem based on Gaussian models (e.g., bounded $\ell_2$-norm) for the quantization distortion yield results that, while often acceptable, may not be fully consistent: re-measurement and quantization of the reconstructed signal do not necessarily match the initial observations. Quantization distortion instead more closely resembles heteroscedastic uniform noise, with variance depending on the observed quantization bin. Generalizing our previous work on uniform quantization, we show that for non-uniform quantizers described by the “compander” formalism, quantization distortion may be better characterized as having bounded weighted $\ell_p$-norm ($p \geq 2$), for a particular weighting. We develop a new reconstruction approach, termed Generalized Basis Pursuit DeNoise (GBPDN), which minimizes the sparsity of the reconstructed signal under this weighted $\ell_p$-norm fidelity constraint. We prove that given a budget of $B$ bits per measurement and under the oversampled QCS scenario, i.e., when the number of measurements is large compared to the signal sparsity, if the sensing matrix satisfies a proposed generalized Restricted Isometry Property, then, GBPDN provides a reconstruction error of sparse signals which decreases like $O(2^{-B}/\sqrt{p+1})$. This results in a reduction by a factor $\sqrt{p+1}$ relative to that produced by using the $\ell_2$-norm. Besides the QCS scenario, we also show that GBPDN applies straightforwardly to the related case of CS measurements corrupted by heteroscedastic Generalized Gaussian noise with provable reconstruction error reduction. Finally, we describe an efficient numerical procedure for computing GBPDN via a primal-dual convex optimization scheme, and demonstrate its effectiveness through extensive simulations.

I. INTRODUCTION

A. Problem statement

Measurement quantization is a critical step in the design and in the dissemination of new technologies implementing the Compressed Sensing (CS) paradigm. Quantization is indeed mandatory for transmitting, storing and even processing any data sensed by a CS device.

In its most popular version, CS provides uniform theoretical guarantees for stably recovering any sparse (or compressible) signal at a sensing rate proportional to the signal intrinsic dimension (i.e., its sparsity level) [1, 2]. However, except a few works [3-7], the distortion introduced by any quantization step is often poorly handled since it is often modeled as a mere Gaussian noise with bounded

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1Parts of a preliminary version of this work has been presented in SPARS11 Workshop (June 27-30, 2011 - Edinburgh, Scotland, UK), in IEEE ICIP 2011 (Sept. 11-14, 2011 - Brussels, Belgium) and in iTWIST Workshop (May 9-11, 2012 - Marseille, France).
variance. More precisely, in \cite{4, 6, 7}, the extreme case of 1-bit CS is studied, i.e., when only the signs of the measurements are sent to the decoder. In \cite{3}, an adaptation of both BPDN and the Subspace Pursuit integrates an explicit QC constraint. In \cite{5}, a model integrating additional Gaussian noise on the measurements before their quantization is analyzed and solved with a $\ell_1$-regularized maximum likelihood program.

B. Contributions

In this work, we generalize the results provided in \cite{8} to cover the case of non-uniform scalar quantization of CS measurements. We show that the theory of “Companders” \cite{9} provides an elegant framework for stabilizing the reconstruction of a sparse (or compressible) signal from quantized CS measurements. At this stage, it is worth emphasizing that in the setting studied here, the quantization scenario is not optimized to improve the CS reconstruction performance. Rather, the quantization step is assumed fixed and optimized a priori according to a common minimal distortion standpoint with respect to a source with known probability density function (pdf)\footnote{That is, a source has an absolutely continuous distribution with respect to the Lebesgue measure.}. In this case, under the High Resolution Assumption (HRA), i.e., when the bit budget of the quantizer is high and the quantization bins are narrow, compander theory describes the quantizer as acting through sequential application of a compressor, a uniform quantization, then an expander (see Section II-A for details). Our work is therefore distinct from approaches where other quantization schemes (e.g., $\Sigma\Delta$-quantization \cite{11}) are tuned to the global CS formalism or to specific CS decoding schemes (e.g., Message Passing Reconstruction \cite{10}).

Algorithms for reconstructing from quantized measurements commonly rely on mathematically describing the error induced by quantization as being bounded in some particular norm. Two natural examples of such constraints are that the $\ell_2$-norm be bounded, or that the quantization error be such that the unquantized values lie in specified, known quantization bins. One main result of this paper is to show that these two constraints are actually two extreme situations of a general class of constraints. These constraints amount to studying the quantization distortion of the CS measurements in an appropriate weighted $\ell_p$-norm. In the latter, the weights are determined from a set of $p$-optimal quantizer levels, that are computed from the observed quantized values. We draw the reader attention to the fact these weights do not depend on the original signal which is of course unknown. They are used only for signal reconstruction purposes, and are optimized with respect to the weighted norm.

Thanks to a new estimator of the weighted $\ell_p$-norm of the quantization distortion associated to these particular levels (see Lemma 3), and with the proviso that the sensing matrix obeys a generalized Restricted Isometry Property (RIP) expressed in the same norm (see (20)), we show that solving a General Basis Pursuit DeNoising program (GBPDN) – an $\ell_1$ problem constrained by a weighted $\ell_p$-norm whose radius is this distortion bound – stably recovers strictly sparse or compressible signals (see Theorem 1).

We also quantify precisely the reconstruction error of GBPDN as a function of the quantizer bit rate (under the HRA) for any value of $p$ in the weighted $\ell_p$ constraint. These results reveal a set of conflicting considerations for setting the optimal $p$. On the one hand, given a budget of $B$ bits per measurement and in the oversampled regime, i.e., when the number of measurements is high compared to what is generally prescribed by CS, the error decays as $O(2^{-B/\sqrt{p+1}})$ when $p$ increases (see Proposition 2). On the other hand, the larger $p$, the greater the number of measurements required to ensure that the generalized RIP is fulfilled (with overwhelming probability). See Proposition 4 for details.
Interestingly, it turns out that the proposed procedure can be viewed as a (variance) stabilization of the measurement distortion seen as an independent but heteroscedastic Generalized Gaussian noise, i.e., for which the variance depends on the measurement index. In this formulation, thanks to the particular weighting of the norm, each quantization bin contributes equally to the related global distortion.

C. Relation to prior work

Our work is distinctly novel in several respects. For instance, as stated above, the quantization distortion in the literature is often modeled as a mere Gaussian noise with bounded variance [3]. In [3], only uniform quantization is handled and theoretically investigated. In [5], nonuniform quantization noise and Gaussian noise on the measurements before quantization are properly dealt with using an \( \ell_1 \)-penalized maximum likelihood decoder. However, in that work, theoretical guarantees are lacking. To the best of our knowledge, this is the first work thoroughly investigating the theoretical guarantees of \( \ell_1 \) sparse recovery from nonuniformly quantized CS measurements, by introducing a new class of convex \( \ell_1 \) decoders. The way we bring the compander theory in the picture to compute the optimal weights from the quantized measurements is also an additional originality of this work.

D. Paper organization

The paper is organized as follows. In Section II we recall the theory of optimal scalar quantization seen through the compander formalism. We then explain how this point of view can help us in understanding the intrinsic constraints that quantized CS measurements must satisfy, and we introduce a new distortion measure expressed in terms of a weighted \( \ell_p \)-norm. Section III defines the GBPDN CS class of decoders, that benefits from these constraints. We show under which requirements this new decoder is provably stable before applying it to the dequantization framework. In Section IV we demonstrate that the new constraint in GBPDN can be understood as a (variance) stabilization of the quantization distortion forcing each quantization bin to contribute equally to the overall distortion error. In Section V we first describe a provably convergent primal-dual proximal splitting algorithm originating from the realm of nonsmooth convex optimization, to solve the GBPDN program. We then demonstrate the power of the proposed approach with several numerical experiments on sparse signals.

E. Notations

**General notations:** All finite space dimensions are denoted by capital letters (e.g., \( K, M, N, D \in \mathbb{N} \)), vectors (resp. matrices) are written in small (resp. capital) bold symbols. For any vector \( \mathbf{u} = (u_1, \cdots, u_D)^T \in \mathbb{R}^D \), the \( \ell_p \)-norm \((p \geq 1)\) of \( \mathbf{u} \) is \( \|\mathbf{u}\|_p = (\sum_i |u_i|^p)^{1/p} \), with \( \|\mathbf{u}\| = \|\mathbf{u}\|_2 \) and the usual adaptation \( \|\mathbf{u}\|_\infty = \max_i |u_i| \). The notation \( \|\mathbf{u}\|_0 = \#\{i : u_i \neq 0\} \) stands for the \( \ell_0 \) pseudo-norm of \( \mathbf{u} \) counting its non-zero components. The space of \( K \)-sparse vectors in the canonical basis is \( \Sigma_K = \{\mathbf{u} \in \mathbb{R}^N : \|\mathbf{u}\|_0 \leq K\} \), i.e., \( \Sigma_K \) is a union of subspaces of \( \mathbb{R}^N \) of dimension \( \leq K \). When necessary, we write \( \ell_p^D \) as the normed vector space \((\mathbb{R}^D, \|\cdot\|_p)\). The scalar product between two vectors is denoted as \( \mathbf{u}^T \mathbf{v} = \sum_i u_i v_i \). The vector \( \mathbf{1} = (1, \cdots, 1)^T \in \mathbb{R}^D \) stands for the “vector of ones”. For a matrix \( \Phi \in \mathbb{R}^{M \times N} \), \( \Phi^T \) represents its transpose. The identity matrix in \( \mathbb{R}^D \) is written \( \mathbb{I}_D \) (or simply \( \mathbb{I} \) if the space dimension is clear from the context), and given a vector \( \mathbf{u} \in \mathbb{R}^D \), \( \mathbf{U} = \text{diag}(\mathbf{u}) \) is the diagonal matrix whose diagonal entries if \( \mathbf{u} \), i.e., \( U_{ij} = u_i \delta_{ij} \) (with \( \delta_{ij} \) the Kronecker delta). Given the \( N \)-dimensional signal space \( \mathbb{R}^N \), the index set is \( |N| = \{1, \cdots, N\} \), and \( \Phi_I \in \mathbb{R}^{M \times \#I} \) is the restriction of the columns of \( \Phi \) to those indexed in the subset \( I \subset [N] \), whose cardinality is \( \#I \). Distinction
between set symbols $I, S, \ldots$ and dimensions $K, M, N, D, \ldots$ will be clear from the context. Given $x \in \mathbb{R}^N$, $x^K_\Psi$ stands for the best $K$-term $\ell_2$-approximation of $x$ in the orthonormal basis $\Psi \in \mathbb{R}^{N \times N}$, that is, $x^K_\Psi = \Psi \left( \arg \min \{ ||x - \Psi \zeta|| : \zeta \in \mathbb{R}^N, ||\zeta||_0 \leq K \} \right)$. When $\Psi = 1$, we write $x^K = x^K_1$ with $||x^K||_0 \leq K$. A random matrix $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$ is a $M \times N$ matrix with entries $\Phi_{ij} \sim_{i.i.d} \mathcal{N}(0, 1)$. The 1-D Gaussian pdf of mean $\mu \in \mathbb{R}$ and variance $\sigma^2 \in \mathbb{R}_+$ is denoted $\gamma_{\mu, \sigma}(t) := (2\pi\sigma^2)^{-1/2} \exp \left( -\frac{(t-\mu)^2}{2\sigma^2} \right)$.

For a function $f : \mathbb{R} \to \mathbb{R}$, we define $\|f\|_q := \left( \int_{\mathbb{R}} dt |f(t)|^q \right)^{1/q}$, which is either a norm or a quasi-norm for $1 \leq q < \infty$ and $0 < q < 1$ respectively. For $q = \infty$, $\|f\|_\infty := \sup_{t \in \mathbb{R}} |f(t)|$. Correspondingly, we denote by $L^q(\mathbb{R})$ the (quasi) Banach space defined by $L^q(\mathbb{R}) := \{ f : \|f\|_q < \infty \}$ for $0 < q \leq \infty$.

Landau’s asymptotic notations: This paper deals with many asymptotic results, meaning that they have to be interpreted in a high dimensional/number of bits setting. Therefore, given any particular “dimension” $D \in \mathbb{R}_+$ supposed arbitrarily large and two functions $f, g \in C^1(\mathbb{R}_+)$, we will use the family of Landau notations \cite{12}, namely,

\[
\begin{align*}
f(D) &= O(g(D)) \iff \exists C > 0, \exists D_0 > 0 : |f(D)| \leq C|g(D)|, \forall D \geq D_0, \\
f(D) &= \Theta(g(D)) \iff g(D) = O(f(D)), \\
f(D) &= \Omega(g(D)) \iff f(D) = \Theta(g(D)) \text{ and } g(D) = O(f(D)), \\
f(D) &= o(g(D)) \iff \forall \epsilon > 0, \exists D_0 > 0 : |f(D)| \leq \epsilon |g(D)|, \forall D \geq D_0, \\
f(D) &= \omega(g(D)) \iff g(D) = o(f(D)) \\
f(D) &\simeq_D g(D) \iff f(D) = g(D)(1+o(1)),
\end{align*}
\]

where all the equalities involving $O, \Omega, \Theta, o$ and $\omega$ must be understood as “one-way equality” close to a membership relation \cite{12}. Notice that the last equivalence \[\simeq_D\] is reflexive, symmetric and transitive. In practice, we will use often the property $f(D) \simeq_D g(D)$ for allowing us to “replace” $f(D)$ by $g(D)$ in any “smooth” expression involving $f(D)$ without any asymptotic impact for large value of $D$. In this paper, we introduce two new notations dealing with asymptotic quantity ordering, i.e.,

\[
\begin{align*}
f(D) &\lesssim_D g(D) \iff \exists \delta : \mathbb{R} \to \mathbb{R}_+ : f(D) + \delta(D) \simeq_D g(D), \\
f(D) &\gtrsim_D g(D) \iff -f(D) \lesssim_D -g(D),
\end{align*}
\]

with by construction $f(D) \simeq_D g(D)$ being equivalent to have jointly $f(D) \lesssim_D g(D)$ and $f(D) \gtrsim_D g(D)$. We easily show that

\[
\begin{align*}
(\forall D \in \mathbb{R}_+, f(D) \leq g(D)) \text{ and } g(D) \simeq_D h(D) &\implies f(D) \lesssim_D h(D), \\
(\forall D \in \mathbb{R}_+, f(D) \geq g(D)) \text{ and } g(D) \simeq_D h(D) &\implies f(D) \gtrsim_D h(D),
\end{align*}
\]

where for proving the first property, it suffices to take $\delta = g - f \geq 0$, and similarly for the second.

Notice that if any of the asymptotic relations above is true with respect to several large dimensions $D_1, D_2, \cdots$ we will write $\simeq_{D_1, D_2, \cdots}$ and correspondingly for $\lesssim$ and $\gtrsim$.

II. NON-UNIFORM QUANTIZATION IN COMPRESSED SENSING

Let us consider a signal $x \in \mathbb{R}^N$ to be measured. We assume that it is either strictly sparse or compressible, in a prescribed orthonormal basis $\Psi = (\Psi_1, \cdots, \Psi_N) \in \mathbb{R}^{N \times N}$. This means that the signal $x = \Psi \zeta = \sum_j \Psi_j \zeta_j$ is such that the $\ell_2^N$-approximation error $||\zeta - \zeta_K|| = ||x - x^K_\Psi||$ quickly decreases (or vanishes) as $K$ increases. For the sake of simplicity, and without loss of generality, the

\footnote{Our notation \[\simeq_D\] is also denoted by \sim in other references \cite{12}.}
sparsity basis is taken in the sequel as the standard basis, i.e., $\Psi = \mathbb{1}$, and $\zeta$ is identified with $x$. All the results can be readily extended to other orthonormal bases $\Psi \neq \mathbb{1}$.

In this paper, we are interested in compressively sensing $x \in \mathbb{R}^N$ with a given measurement matrix $\Phi \in \mathbb{R}^{M \times N}$. Each compressed sensing measurement, i.e., each component of $z = \Phi x$, undergoes a general scalar quantization. We will assume this quantization to be optimal relative to a known distribution of each entry $z_i$. For simplicity, we only consider matrices $\Phi$ that yield $z_i$ to be i.i.d. $N(0, \sigma_0^2)$ Gaussian, with pdf $\varphi_0 := \gamma_{0,\sigma_0}$. This is satisfied, for instance, if $\Phi \sim N^{M \times N}(0, 1)$, with $\sigma_0 = \|x\|_2$.

Our quantization scenario uses a $B$-bit quantizer $Q$ which has been optimized with respect to the measurement pdf $\varphi_0$ for $\mathbb{B} = 2^B = \#\Omega$ levels $\Omega = \{\omega_k : 1 \leq k \leq \mathbb{B}\}$ and thresholds $\{t_k : 1 \leq k \leq \mathbb{B} + 1\}$ with $-t_1 = t_{\mathbb{B} + 1} = +\infty$. Unlike the framework developed in [5], our sensing scenario considers that any noise corrupting the measurements before quantization is negligible compared to the quantization distortion.

Consequently, given a matrix $\Phi \in \mathbb{R}^{M \times N}$, our quantized sensing model is

$$y = Q[\Phi x] = Q[z] \in \Omega^M. \tag{1}$$

Following recent studies [3, 8, 13] in the CS literature, this work is interested in optimizing the signal reconstruction stability from $y$ under different sensing conditions, for instance, when the oversampling ratio $M/K$ is allowed to be large. Before going further into this signal sensing model, let us describe first the selected quantization framework. The latter is based on a scalar quantization of each component of the signal measurement vector.

A. Quantization, Companders and Distortion

A scalar quantizer $Q$ is defined from $\mathbb{B} = 2^B$ levels $\omega_k$ (coded by $B = \log_2 \mathbb{B}$ bits) and of $\mathbb{B} + 1$ thresholds $t_k \in \mathbb{R} \cup \{\pm \infty\} = \bar{\mathbb{R}}$, with $\omega_k < \omega_{k+1}$ and $t_k \leq \omega_k < t_{k+1}$ for all $1 \leq k \leq \mathbb{B}$ and given a bit rate $B \in \mathbb{N}$. The $k^{th}$ quantizer bin (or region) is $\mathcal{R}_k = [t_k, t_{k+1})$, with bin width $\alpha_k = t_{k+1} - t_k$.

The quantizer is then a mapping from $\mathbb{R}$ to the set of levels $\Omega = \{\omega_k : 1 \leq k \leq \mathbb{B}\}$, i.e.,

$$Q[t] = \omega_k \iff t \in \mathcal{R}_k = Q^{-1}[\omega_k].$$

An optimal scalar quantizer $Q$ with respect to a random source $Z$ in $\mathbb{R}$ of pdf $\varphi_Z$ is such that the distortion $\mathbb{E}|Z - Q[Z]|^2$ is minimized. This imposes particular choices of levels and thresholds, as for instance those determined by the Lloyd-Max Algorithm [14, 15] or by an asymptotic (with respect to $B$) companding approach [9].

Throughout this paper, we work under the HRA. This means that, given the source pdf $\varphi_Z$, the number of bits $B$ is sufficient to validate the approximation

$$\varphi_Z(t) \simeq_B \varphi_Z(\omega_k), \quad \forall t \in \mathcal{R}_k. \tag{HRA}$$

A common argument in quantization theory [9] states that, under HRA, every regular quantizer (and, a fortiori, every optimal regular quantizer) can be described by a compander (a portemanteau for “compressor” and “expander”). More precisely, we have

$$Q = G^{-1} \circ Q_\alpha \circ G,$$

$G : \mathbb{R} \rightarrow [0, 1]$ being a bijective function called the compressor, $Q_\alpha$ is a uniform quantizer of the interval $[0, 1]$ of bin width $\alpha = 2^{-B}$, and the inverse mapping $G^{-1} : [0, 1] \rightarrow \mathbb{R}$ is coined the expander.
Fig. 1: (a) Excess Kurtosis test on the distribution of the original or compressed source values in each quantizer bins (for a Gaussian source, $\sigma = 1$, $B = 2^3$ bins). The plain and the dashed curves represent excess kurtosis in the original and the compressor domain respectively. The Gaussian pdf has zero excess kurtosis, while the uniform pdf reaches the value $-1.2$ (horizontal dash-dotted line). (b) Comparing the theoretical bound $\epsilon_p$ to (the empirical mean estimate of) $\mathbb{E}[\|Q_p z - z\|_{p,w}]$. Monte-Carlo simulations (1000 trials, $B = 3, 4, 5$).

For optimal quantizers (with respect to a source pdf $\varphi_Z$), the compressor $G$ maps the thresholds $\{t_k\}$ and the levels $\{\omega_k\}$ into the values

$$t'_k := G(t_k) = (k-1)\alpha, \quad \omega'_k := G(\omega_k) = (k-1/2)\alpha.$$  

Under the HRA, the optimal $G$ is defined from $\varphi_Z$ by

$$G'(\lambda) = \frac{d}{d\lambda} G(\lambda) = \left[ \int_{\mathbb{R}} \varphi_Z^{1/3}(t) \, dt \right]^{-1} \varphi_Z^{1/3}(\lambda).$$  

We note that, for $\varphi_Z(t) = \gamma_{0,\sigma}(t) := (2\pi\sigma^2)^{-1/2} \exp(-t^2/(2\sigma^2))$ with a cumulative distribution function $\phi_Z(\lambda; \sigma^2) = \int_{-\infty}^{\lambda} \varphi_Z(t) \, dt = \frac{1}{2} \text{erfc}(-\frac{\lambda}{\sqrt{2}\sigma})$ with $\phi_Z^{-1}(\lambda' \sigma^2) = \sigma \sqrt{2} \text{erf}^{-1}(2\lambda' - 1)$, we have $G(\lambda) = \phi_Z(\lambda; 3\sigma^2)$ and $G^{-1}(\lambda') = \phi_Z^{-1}(\lambda'; 3\sigma^2)$.

The application of $G$ modifies the source $Z$ such that $G(Z) - G(Q[Z])$ behaves more like a uniformly distributed random variable over $[\alpha/2, \alpha/2]$. Fig. 1(a) illustrates this effect on the quantization of a Gaussian source $Z \sim \mathcal{N}(0, 1)$ by an optimal 3-bit quantizer $Q$ defined as above. As a simple test, the (excess) kurtosis associated to each of the two pdfs $\varphi_Z(t) | t \in R_k$ (plain curve) and $\varphi_{G(z)}(u | u \in G(R_k)) = \varphi_Z(G^{-1}(u)) | u \in G(R_k))$ (dashed curve) are shown as function of the bin index $1 \leq k \leq 2^3$. We observe that the kurtosis in the compressor domain is closer to the uniform kurtosis (of value $-1.2$) than in the original domain.

The compander formalism predicts the distortion of optimal scalar quantizer under HRA. For high bit rate $B$, the Panter and Dite formula [16] states that

$$\mathbb{E}[Z - Q[Z]]^2 \simeq \frac{2^{-2B}}{B} \int_{\mathbb{R}} G'(t)^{-2} \varphi_Z(t) \, dt = \frac{2^{-2B}}{12} \left( \int_{\mathbb{R}} \varphi_Z^{1/3}(t) \, dt \right)^3 = \frac{2^{-2B}}{12} \|\varphi_Z\|_{1/3}.  \quad (4)$$  

For instance, for uniform, Gaussian and Laplacian sources, $\frac{1}{12\sigma_Z^2} \|\varphi_Z\|_{1/3}$ equals $1$, $\sqrt{3\pi} / 2 \simeq 2.721$ and $4.5$ respectively.
Before concluding this section, it is important to notice that,

$$|G(\lambda) - G(Q[\lambda])| \leq \frac{\alpha}{2}, \quad \forall \lambda \in \mathbb{R}. \quad (5)$$

by the construction defined in (2). We describe in the next section why this last but important relation and the distortion evaluation (4) are actually two extreme cases of a general class of constraints satisfied by a quantized source \(Z\).

B. Distortion and Quantization Consistencies in Quantized Compressed Sensing

Let us consider the sensing model (1) for which the scalar quantizer \(Q\) is optimal relative to the model \(z_i \sim_{iid} \mathcal{N}(0, \sigma^2_0)\) implied by \(z = \Phi x\). This quantizer is related to the compressor \(G\) defined by (3) with respect to \(\varphi Z(t) = \varphi_0(t) := \gamma_{0, \sigma_0}(t)\).

Using the High Resolution Assumption (HRA) defined in the Section II-A, the quantization distortion is modeled in the compressor domain as

$$G(y) = G(z) + (G(Q[z]) - G(z)) = G(z) + \varepsilon,$$

with \(\varepsilon\) representing the quantization distortion. From the previous decomposition \(Q = G^{-1} \circ Q_\alpha \circ G\), we must have

$$\|\varepsilon\|_\infty = \|G(Q[z]) - G(z)\|_\infty \leq \frac{1}{2} \alpha.$$

Ideally, any estimate \(x^*\) of \(x\) (obtained from a convenient reconstruction method) should be consistent with this relation. In particular, we say that \(x^*\) satisfies the quantization consistency (QC) if \(Q[\Phi x^*] = y\). In other words, \(x^*\) satisfies QC if

$$\|G(\Phi x^*) - G(y)\|_\infty \leq \epsilon_{QC} := \frac{1}{2} \alpha. \quad (QC)$$

However, when compressed sensing measurements undergo a scalar quantization by \(Q\) as in the model (1), the reconstruction of the initial signal \(x\) is generally realized by the Basis Pursuit DeNoise (BPDN) program tuned to another consistency related to the quantizer distortion, or Distortion Consistency. In this case, the estimate \(x^*\) is provided by

$$x^* \in \text{Argmin}_u \|u\|_1 \text{ s.t. } \|y - \Phi u\| \leq \epsilon_{DC},$$

where \(\epsilon_{DC}^2 := M \frac{\sqrt{3} \pi}{2} \sigma^2_0 2^{-2B}\) is simply adjusted to the Panter-Dite formula. In other words, using the Strong Law of Large Numbers followed by the HRA and since \(z_i \sim_{iid} \mathcal{Z} \sim \mathcal{N}(0, \sigma^2_0)\),

$$\frac{1}{M} \|z - Q[z]\|^2 \sim M \mathbb{E}|Z - Q[Z]|^2 \sim B 2^{-2B} \|\varphi_0\|_{1/3}^3 = \frac{\sqrt{3} \pi}{2} \sigma^2_0 2^{-2B}, \quad (6)$$

where \(\mathcal{Z} \sim \mathcal{N}(0, \sigma^2_0)\). By definition, this distortion consistency is reported on the solution \(x^*\) in the relation

$$\|\Phi x^* - y\| \leq \epsilon_{DC}. \quad (DC)$$

However, as stated for the uniform quantization case in [8], there is no equivalence between QC and DC. In particular, the output \(x^*\) of BPDN needs not satisfy quantization consistency. This observation motivates the search for new reconstruction methods where the reconstructed signal approaches QC. Before detailing such a procedure in Section III let us first introduce a general class of parametrized constraints including DC and QC.
C. $p$-optimal levels

The QC and DC constraints correspond to two limit cases of a general class of parametric constraints related to distortion quantization measured in a weighted $\ell_p$-norm. In order to understand this, let us define from the same quantization thresholds a new category of levels for our quantization framework.

For the Gaussian pdf $\varphi_0 = \gamma_0, \sigma_0$, given a set of thresholds $\{t_k\}$, we define the $p$-optimal quantizer levels $\omega_{k,p} \in \mathbb{R}$ as

$$\omega_{k,p} := \arg\min_{\lambda \in \mathcal{R}_k} \int_{\mathcal{R}_k} |t - \lambda|^p \varphi_0(t) \, dt,$$

for $2 \leq p < \infty$, and $\omega_{k,\infty} := \frac{1}{2}(t_k + t_{k+1})$. For $p = 2$, we find the definition of the initial quantizer levels, i.e., $\omega_{k,2} = \omega_k$. In this paper, we always assume that $p$ is a positive integer but all our analysis can be extended to the positive real case. As proved in Appendix B, the $p$-optimal levels are well defined.

Lemma 1 ($p$-optimal Level Definiteness). The $p$-optimal levels $\omega_{k,p}$ are uniquely defined. Moreover, for $\sigma_0 > 0$,

$$\lim_{p \to +\infty} \omega_{k,p} = \omega_{k,\infty},$$

with $|\omega_{k,p}| = \Omega(\sqrt{p})$ for $k \in \{1, B\}$.

Using these new levels, we define $\Omega_p = \{\omega_{k,p} : 1 \leq k \leq B\}$ and the (suboptimal) quantizer family $Q_p$ (with $Q_2 = Q$) such that

$$Q_p[t] = \omega_{k,p} \iff t \in \mathcal{R}_k = Q_p^{-1}[\omega_{k,p}] = Q^{-1}[\omega_k].$$

Two important points must be explained regarding the definition of $Q_p$. First, the (re)quantization of any source $Z$ with $Q_p$ is of course possible from the knowledge of the quantized value $Q_p[Z]$. Indeed, $Q_p[Z] = Q_p[Q[Z]]$ since both quantizers share the same decision thresholds. Second, despite the sub-optimality of $Q_p$ relative to the untouched thresholds $\{t_k\}$, which corresponds to the scenario addressed in this paper where we assume the model (1) fixed, we will see later that introducing this quantizer provides improvement in the modeling of $Q_p[Z] - Z$ by a Generalized Gaussian Distribution (GGD) in each quantization bin $\mathcal{R}_k$. This point is the main source of improvement in the reconstruction method proposed in the next sections.

Remark 1. Unfortunately, there is no closed form formula for computing $\omega_{k,p}$. However, as detailed in Appendix H, they can be computed up to numerical precision using Newton’s method combined with simple numerical quadrature for the integral in (7).

The definition of the $p$-optimal levels together with the compander formalism allows us to deduce the asymptotic evolution with $B$ of several key properties of the quantization bins. Moreover, we show below that a particular threshold $T = \Theta(\sqrt{B})$ determines two regimes for this evolution related to the membership of the quantization bins to the interval $\mathcal{T} = [-T, T]$.

This is formalized in the following lemma that will play a central role in many other developments throughout the paper. To the best of our knowledge, the results contained in the lemma are new (even for the case $p = 2$) and they may be of an independent interest for characterizing Gaussian source quantization.

Lemma 2 (Asymptotic $p$-Quantization Characterization). Given the Gaussian pdf $\varphi_0$ and its associated compressor $\hat{G}$ function, choose $0 < \beta < 1$ and $p \in \mathbb{N}$, and define $T = T(B) = \sqrt{6/\sigma_0^2} (\log 2^\beta) B$,
Finally, if \( k \) estimated distortion at \( B \) decreases when \( T \) bound matches the Panter-Dite estimation (6). For \( |||\ell_p\|w\|_\varphi \) function of their width thanks to a particular weighted distortion induced by \( \varphi_0 \).

The proof of this lemma is postponed to Appendix C.

Moreover, for all \( k \) such that \( R_k \subset T \) and any \( c \in R_k \)

\[
\alpha_k := t_{k+1} - t_k = O(2^{-1-\beta}),
\]

\[
1 \leq \frac{\max(\varphi_0(t_k), \varphi_0(t_{k+1}))}{\min(\varphi_0(t_{k}), \varphi_0(t_{k+1}))} = \exp(O(B^{1/2} 2^{-1-\beta})) = 1 + O(B^{1/2} 2^{-1-\beta}),
\]

\[
\int_{R_k} |t - \omega_k|^p \varphi_0(t) \, dt \simeq_B 2^{(p+1)/2} \varphi_0(c),
\]

\[
G'(c) \simeq_B \frac{\alpha_k}{\alpha_k}
\]

Finally, if \( k \) is such that \( T(B) \in R_k \), then, writing the interval length \( L(A) = \int_A dt \) for \( A \subset \mathbb{R} \)

\[
L(R_k \cap T) = O(2^{-1-\beta}),
\]

\[
G'(\omega_k) \leq \max(G'(t_k), G'(t_{k+1})) = O(2^{-\beta}),
\]

\[
\int_{R_k} |t - \omega_k|^p \varphi_0(t) \, dt = O(B^{-(p+1)/2} 2^{-3\beta}).
\]

The proof of this lemma is postponed to Appendix C.

We now state an important result (proved in Appendix D) aiming at asymptotically estimating a specific distortion induced by \( Q_p \) on a random Gaussian vector. This distortion balances each quantization bin in \( \mathbb{R} \) of a vector \( T \).

\[\text{Lemma 3 (Asymptotic Weighted } \ell_p\text{-Distortion).} \]

Let \( z \in \mathbb{R}^M \) be a random vector where each component \( z_i \sim_{iid} \mathcal{N}(0, \sigma_i^2) \). Given the optimal compressor function \( G \) associated to \( \varphi_0 \) and the weights \( \omega = \omega(p) \) such that \( w_i(p) = G'(Q_p[z_i])^{(p-2)/p} \) for \( p \geq 2 \), we have

\[
\|Q_p[z] - z\|_{p,w} \simeq_{B,M} M^{\frac{2-n_p}{(p+1)2p}} \|\varphi_0\|_1/3 := \epsilon_p,
\]

with \( \|\varphi_0\|_1/3 = 2\pi \sigma_0^2 \alpha^{3/2} \).

This lemma provides a tight estimation for \( p = 2 \) and \( \alpha \rightarrow +\infty \). Indeed, in the first case \( \omega = 1 \) and the bound matches the Panter-Dite estimation (6). For \( p \rightarrow \infty \), we observe that \( \epsilon_\infty = 2^{-(B+1)} = \alpha/2 = \epsilon_QC \).

Fig. (b) shows how well the \( \epsilon_p \) estimates the distortion \( \|Q_p[z] - z\|_{p,w} \) for the weights and the \( p \)-optimal levels given in Lemma 2. This has been measured by averaging this quantization distortion for 1000 standard Gaussian vectors \( z \sim \mathcal{N}^M(0, 1) \) with \( M = 2^{10}, p \in \{2, \ldots, 15\} \) and \( B = 3, 4 \) and 5. We observe that the bias of \( \epsilon_p \), as measured here by the ratio \( \epsilon_p^{-1} \mathbb{E}\|Q_p[z] - z\|_{p,w} \), is rather limited and decreases when \( p \) and \( B \) increase with a maximum relative error of about 2.5\% between the true and estimated distortion at \( B = 3 \) and \( p = 2 \).

Inspired by relation (19), we introduce a new class of constraints: we say that an estimate vector \( x^* \in \mathbb{R}^N \) of a vector \( x \) sensed by the model \( \Phi \) satisfies the \( p \)-Distortion Consistency (or \( D_pC \)) if

\[
\|\Phi x^* - Q_p[y]\|_{p,w} \leq \epsilon_p,
\]

(\( D_pC \))
with the weights \( w_i(p) = G'(Q_p[y_i])^{(p-2)/p} \).

We have the following equivalences.

**Lemma 4.** Given \( y = Q[\Phi x] \), we have asymptotically in \( B \)

\[ D_2C \equiv DC \quad \text{and} \quad D_\infty C \equiv QC. \]

*Proof:* Let \( x^* \in \mathbb{R}^N \) be a vector to be tested with the DC, QC or \( D_pC \) constraints. The first equivalence for \( p = 2 \) is straightforward since \( w(2) = 1 \), \( \| \Phi x^* - Q_p[y] \|_{p,w} = \| \Phi x^* - Q[y] \|_2 \) and \( \epsilon_2^2 = \epsilon_{DC}^2 = 2^{2/3} \| \varphi_0 \|_{1/3} \).

For the second, we use the fact that \( y = Q[\Phi x] \) is fixed by the sensing model (1). Let us denote by \( k(i) \) the index of the bin to which \( Q_p[y_i] \) belongs for \( 1 \leq i \leq M \). Since \( \| \Phi x \|_\infty \) is fixed, and because relation (10) in Lemma 2 implies that the amplitude of the first or of the last \( \Theta(1) \) thresholds grow faster than \( T = \Theta(\sqrt{\beta B}) \) for \( 0 < \beta < 1 \), there is necessarily a \( B_0 \geq 0 \) such that \( -T(B) \leq t_{k(i)} \leq t_{k(i)+1} \leq T(B) \) for all \( B \geq B_0 \) and all \( 1 \leq i \leq M \).

Writing \( W_p = \text{diag}(w(p)) \), we can use the equivalence \( \| \cdot \|_\infty \leq \| \cdot \|_p \leq M^{1/p} \| \cdot \|_\infty \) and the squeeze theorem on the following limit:

\[ \lim_{p \to \infty} \| \Phi x^* - Q_p[y] \|_{p,w(p)} = \lim_{p \to \infty} \| W_p(\Phi x^* - Q_p[y]) \|_p = \lim_{p \to \infty} \| W_p(\Phi x^* - Q_p[y]) \|_\infty. \]

Moreover, since for \( B \geq B_0 \) and for all \( 1 \leq i \leq M \) the bin \( R_{k(i)} \) is finite, the limit

\[ \lim_{p \to \infty} G'(Q_p[y_i])^{(p-2)/p} \left| (\Phi x^*)_i - Q_p[y_i] \right| \]

exists and is finite. Therefore,

\[
\begin{align*}
\lim_{p \to \infty} \| \Phi x^* - Q_p[y] \|_{p,w(p)} &= \lim_{p \to \infty} \max_i G'(Q_p[y_i])^{(p-2)/p} \left| (\Phi x^*)_i - Q_p[y_i] \right| \\
&= \max_i \lim_{p \to \infty} G'(Q_p[y_i])^{(p-2)/p} \left| (\Phi x^*)_i - Q_p[y_i] \right| \\
&= \max_i G'(Q_\infty(y_i)) \left| (\Phi x^*)_i - Q_\infty(y_i) \right|.
\end{align*}
\]

For \( B \geq B_0 \), (15) provides \( G'(Q_\infty(y_i)) \approx_B \frac{\alpha}{\alpha_{k(i)}} \), so that, if we impose \( \lim_{p \to \infty} \| \Phi x^* - Q_p[y] \|_{p,w(p)} \leq \epsilon_{QC} = \alpha/2 \), we get asymptotically in \( B \)

\[ \max_i \frac{1}{\alpha_{k(i)}} \left| (\Phi x^*)_i - Q_\infty(y_i) \right| \leq \frac{1}{B}, \]

which is equivalent to imposing \( (\Phi x^*)_i \in R_{k(i)} \), i.e., the Quantization Constraint.

In summary, this suggests to define a new reconstruction procedure relying on the \( D_pQ \) distortion constraint rather than on the BPDN \( \ell_2 \) fidelity term. However, it is not straightforward to understand if introducing such general constraints still leads to provably stable reconstruction from non-uniformly quantized measurements, i.e., if we still characterize how far \( x^* \) is from the unobserved sensed signal \( x \). We tackle this problem in the following section after having precisely defined the new reconstruction procedure. The latter actually generalizes the Basis Pursuit DeQuantizer program (BPDQ [8]) to the case of non-uniform quantization.
III. DEQUANTIZING WITH GENERALIZED BASIS PURSUIT DE NOISE

The last section has provided us some weighted $\ell_{p,w}$ constraints, with appropriate weights $w$, that can be used for stabilizing the reconstruction of a signal observed through the quantized sensing model \(\ref{eq:quantized_sensing_model}\). Let us now incorporate them in a general reconstruction procedure relying on a constrained $\ell_1$ convex program before studying its stability.

A. Weighted Fidelities in Quantization and Guarantees

Given the positive weights $w$, we study the following general minimization program, coined General Basis Pursuit DeNoise (GBPDN),

$$
\Delta_{p,w}(y, \Phi, \epsilon) = \operatorname{Argmin}_{u \in \mathbb{R}^N} \|u\|_1 \text{ s.t. } \|y - \Phi u\|_{p,w} \leq \epsilon,
$$

(GBPDN($\ell_{p,w}$))

where $\|\cdot\|_{p,w}$ is the weighted $\ell_p$-norm defined in the previous section. The above convex programming problem generalizes the Basis Pursuit DeNoise (BPDN) program \(\ref{eq:BP_DN}\) to any fidelity constraints as measured by the $\ell_{p,w}$-norm, the former BPDN corresponding to $p = 2$ and $w = 1$. The Basis Pursuit DeQuantizers (BPDQ) introduced in \cite{zhang2018bpdq} are associated to $\ell_1$ measured by the $\ell_1$-norm defined in the previous section. The above convex programming problem generalizes the Basis Pursuit DeNoise (BPDN) program \(\ref{eq:BP_DN}\) to any fidelity constraints as measured by the $\ell_{p,w}$-norm, the former BPDN corresponding to $p = 2$ and $w = 1$. The Basis Pursuit DeQuantizers (BPDQ) introduced in \cite{zhang2018bpdq} are associated to $p = 1$ and $w = 1$, while the case $p = 1$ and $w = 1$ has also been covered in \cite{chen2010generalized}.

We are going to see that the stability of GBPDN($\ell_{p,w}$) is guaranteed if $\Phi$ satisfies a particular instance of the following general isometry property.

**Definition 1.** Given two normed spaces $\mathcal{X} = (\mathbb{R}^M, \|\cdot\|_{\mathcal{X}})$ and $\mathcal{Y} = (\mathbb{R}^N, \|\cdot\|_{\mathcal{Y}})$ (with $M < N$), a matrix $\Phi \in \mathbb{R}^{M \times N}$ satisfies the Restricted Isometry Property from $\mathcal{X}$ to $\mathcal{Y}$ at order $K \in \mathbb{N}$, radius $0 \leq \delta < 1$ and for a normalization $\mu = \mu(\mathcal{X}, \mathcal{Y}) > 0$, if for all $x \in \Sigma_K$,

$$
(1 - \delta)^{1/\kappa} \|x\|_{\mathcal{X}} \leq \frac{1}{\mu} \|\Phi x\|_{\mathcal{X}} \leq (1 + \delta)^{1/\kappa} \|x\|_{\mathcal{X}},
$$

(20)

$\kappa$ being an exponent function of the geometries of $\mathcal{X}, \mathcal{Y}$. Equivalently, we will write shortly that $\Phi$ is RIP$_{\mathcal{X}, \mathcal{Y}}(K, \delta, \mu)$.

We may notice that the common RIP is equivalent to\(\ref{eq:RIP}\) RIP$_{\ell_2^M, \ell_2^N}(K, \delta, 1)$ with $\kappa = 1$, while the RIP$_{p,q}$ introduced earlier in \cite{zhang2018bpdq} is equivalent to RIP$_{\ell_p^M, \ell_q^N}(K, \delta, \mu)$ with $\kappa = q$ and $\mu$ depending only on $M$, $p$ and $q$. Moreover, the RIP$_{p,K,\delta}$, defined in \cite{candes2005decoding}, is equivalent to the RIP$_{\ell_2^p, \ell_2^N}(K, \delta, \mu)$ with $\kappa = 1$, $\delta' = 2\delta/(1 - \delta)$ and $\mu = 1/(1 - \delta)$. Finally, the Restricted $p$-Isometry Property proposed in \cite{candes2008restricted} is also equivalent to the RIP$_{\ell_2^p, \ell_2^N}(K, \delta, 1)$ with $\kappa = p$.

In this work, in order to study the behavior of the GBPDN program, we are interested in the embedding induced by $\Phi$ in \cite{candes2008restricted} of $\ell_2^N$ into the normed space $\ell_{p,w}^M = (\mathbb{R}^M, \|\cdot\|_{\ell_{p,w}})$, i.e., we consider the RIP$_{\ell_2^p, \ell_2^N}(\ell_{p,w})$ property that we write shortly RIP$_{\ell_{p,w}}$.

This is motivated by the following observation. If $\Phi$ is RIP$_{p,w}(K, \delta, \mu)$, then, by definition of the weighted $\ell_{p,w}$-norm, $\Phi' = \operatorname{diag}(w)\Phi$ is RIP$_{\ell_2^p, \ell_2^N}(K, \delta, \mu)$ and the stability results proved in \cite{zhang2018bpdq} hold for the equivalent GBPDN($\ell_{p,w}$) decoder\(\ref{eq:GBPDN}\):

$$
\Delta_p(y', \Phi', \epsilon) = \operatorname{Argmin}_{u \in \mathbb{R}^N} \|u\|_1 \text{ s.t. } \|y' - \Phi' u\|_p \leq \epsilon,
$$

for $y' = \operatorname{diag}(w)y$, since $\Delta_{p,w}(y, \Phi, \epsilon) = \Delta_p(y', \Phi', \epsilon)$. Explicitly, we deduce therefore the following instance optimality for GBPDN($\ell_{p,w}$).

\(\text{Assuming the columns of } \Phi \text{ unit-norm normalized.}\)

\(\text{Previously called BPDQ in }\ref{zhang2018bpdq}.\)
Theorem 1. Let \( x \in \mathbb{R}^N \) be a signal with a \( K \)-term \( \ell_1 \)-approximation error \( e_0(K) = K^{-\frac{1}{2}} \| x - x_K \|_1 \), for \( 0 \leq K \leq N \). Let \( \Phi \) be a RIP\(_{p,w}(s, \delta, \mu) \) matrix for \( s \in \{ K, 2K, 3K \} \) and \( 2 \leq p < \infty \). Given a measurement vector \( y = \Phi x + \epsilon \) corrupted by a noise \( \epsilon \) with bounded \( \ell_{p,w} \)-norm, i.e., \( \| \epsilon \|_{p,w} \leq \epsilon \), the unique solution \( x^* = \Delta_{p,w}(y, \Phi, \epsilon) \) obeys

\[
\| x^* - x \|_2 \leq A_p e_0(K) + B_p \epsilon / \mu,
\]

for values \( A_p(\Phi, K) = \frac{2(1+C_p-\delta_{2K})}{1-\delta_{2K}-C_p} \), \( B_p(\Phi, K) = \frac{4\sqrt{1+\delta_{2K}}}{1-\delta_{2K}+C_p} \), \( C_p = O(\sqrt{(\delta_{2K}+\delta_{3K})(p-2)}) \) as \( p \to 2 \) and \( C_p = \delta_{3K} + O(p-2) \) as \( p \to 2 \).

The precise definition of \( C_p \) is given in \([8]\).

As we will see in the next sections, this theorem is particularly interesting for characterizing the impact of two kinds of measurement distortions, namely, those induced by a heteroscedastic GGD noise (see Section IV-A), or by a non-uniform scalar quantization (see Section II). In the first case, setting the weights to the inverse of the standard deviation “stabilizes” the measurement and reduces the factor \( \epsilon / \mu \) for high oversampling ratio \( M/K \). In the second case, similarly to what has been observed in the uniform quantization setting \([8]\), taking a high ratio \( M/K \) allows us to select high \( p \geq 2 \) and reduces the quantization error by a factor \( \sqrt{p+1} \).

Before detailing these two sensing scenarios, we first address the question of designing matrices satisfying the RIP\(_{p,w} \) for \( 2 \leq p < \infty \).

B. Weighted Isometric Mappings

Our goal here is to construct (random) matrices satisfying the RIP\(_{p,w} \) for \( 1 \leq p < \infty \). In order to easily quantify the conditions under which this is possible, let us introduce some properties on the weights \( w \).

As assumed before, all the weights \( w = \{ w_i : 1 \leq i \leq M \} \) are positive, for \( \| \cdot \|_{p,w} \) to be indeed a norm. Second, for any \( p \geq 1 \), we suppose that the weights have been generated by a process satisfying an appropriate moment property that we define now.

Definition 2. A weight generator \( \mathcal{W} \) is a process (random or deterministic) that associates to \( M \in \mathbb{N} \) a weight vector \( w = \mathcal{W}(M) \in \mathbb{R}^M \). This process is said to be of Converging Moments (CM) if for \( p \geq 1 \) and all \( M \geq M_0 \) for a certain \( M_0 > 0 \),

\[
\rho_p^{\min} \leq M^{-1/p} \| \mathcal{W}(M) \|_p \leq \rho_p^{\max},
\]

where \( \rho_p^{\min} > 0 \) and \( \rho_p^{\max} > 0 \) are, respectively, the biggest and the smallest values such that (22) holds.

In other words, a CM generator \( \mathcal{W} \) is such that \( \| \mathcal{W}(M) \|_p^p = \Theta(M) \). By extension, we say that the weighting vector \( w \) has the CM property, if it is generated by some CM weight generator \( \mathcal{W} \).

The CM property can be ensured if \( \lim_{M \to \infty} M^{-1/p} \| w \|_p \) exists and is nonzero. It is also ensured if the weights \( \{ w_i \}_{1 \leq i \leq M} \) are taken (with repetition) inside a finite set of positive values (of size independent of \( M \)). More generally, if \( \{ w_i \}_{1 \leq i \leq M} \) are considered as integrable iid random variables, we have \( \lim_{M \to \infty} M^{-1} \| w \|_p^p = \mathbb{E}[|w_1|^p] \) from the Strong Law of Large Numbers. Notice finally that \( \rho_p^{\min} \leq \| w \|_{\infty} = \rho_{\infty}^{\max} \) since \( \| w \|_{p} \leq M \| w \|_{\infty} \), and \( \rho_p^{\min} \geq \min_i |w_i| \).

For a weighting vector \( w \) having the CM property, we define also its weighting dynamic at moment \( p \) as the ratio

\[
\theta_p = \left( \frac{\rho_{\infty}^{\max}}{\rho_p^{\min}} \right)^2.
\]
We will see later that \( \theta_p \), which actually controls the conditioning of the weights \( w \), directly influences the number of measurements required for guaranteeing the existence of RIP\(_{p, w}\) random Gaussian matrices.

Given a weight vector \( w \), it is useful to characterize the expectation of the \( \ell_{p, w} \)-norm of a random Gaussian vector. This is the goal of the following lemma (proved in Appendix E).

**Lemma 5 (Gaussian \( \ell_{p, w} \)-Norm Expectation).** If \( \xi \sim N^M(0, 1) \) and if the weights \( w \) have the CM property, then, for \( 1 \leq p < \infty \),

\[
(1 + 2^{p+1}\theta_p^2 M^{-1})^{\frac{1}{2}} \leq \mathbb{E}\|\xi\|_{p, w} \leq (\mathbb{E}\|\xi\|_{p, w}^p)^{\frac{1}{p}} = (\mathbb{E}|g|^p)^{1/p}\|w\|_p.
\]

In particular, \( \mathbb{E}\|\xi\|_{p, w} \lesssim_M \nu_p\|w\|_p \gtrsim \nu_p M^{1/p} \rho_p^{\text{min}} \), with \( \nu_p := \mathbb{E}|g|^p = 2^{p/2}π^{-1/2}Γ(\frac{p+1}{2}). \)

With an appropriate modification of the result given in [8], we can now prove the existence of random Gaussian RIP\(_{p, w}\) matrices (see Appendix F).

**Proposition 1 (RIP\(_{p, w}\) Matrix Existence).** Let \( \Phi \sim N^{M×N}(0, 1) \) and some CM weights \( w \in \mathbb{R}^M \). Given \( p \geq 1 \) and \( 0 \leq \eta \leq 1 \), then there exists a constant \( c > 0 \) such that \( \Phi \) is RIP\(_{p, w}(K, \delta, \mu) \) with probability higher than \( 1 - \eta \) when we have jointly \( M \geq 2(2\theta_p)^p \), and

\[
M^{2/\max(2, p)} \geq c\delta^{-2}\theta_p \left( K\log[c\frac{N}{K}(1 + 12\delta^{-1})] + \log \frac{2}{\eta} \right). \tag{23}
\]

Moreover, the value \( \mu = \mu(\ell^M_{p, w}, \ell^N_2) \) in (20) is given by \( \mu = \mathbb{E}\|\xi\|_{p, w} \) for a random vector \( \xi \sim N^M(0, 1) \).

The RIP normalizing constant \( \mu \) can be bounded owing to Lemma 5.

**Remark 2.** The previous proposition leads to an interesting limit case. Let us consider \( p = 2 \) and let the weights be drawn randomly such that \( w_1 = 1 \) with probability \( q \) and \( \eta > 0 \) with probability \( 1 - q \). In that case, it is easy to compute the weighting dynamic as \( \theta_2 = (\frac{\varphi_{\text{min}}}{\varphi_{\text{max}}})^2 \approx 1/q \) when \( \eta \ll 1 \). Therefore, the sufficient condition for \( \Phi \) to be RIP\(_{2, w}\) tends to \( qM = O(K\log N/K) \) which is consistent with the fact that on average only fraction \( q \) of the \( M \) measurements significantly participate to the CS scheme, i.e., \( M' = qM \) must satisfy the common RIP requirement. This effect is studied in more details in Sec. IV-A.

C. Dequantizing Framework

Rewriting Theorem 1 under the quantization formalism defined in Lemma 3, we get a result showing that, asymptotically in \( M \) and \( B \), taking \( p \) higher than 2, yields a reconstruction error\(^6\) that decreases as \( 1/\sqrt{p + 1} \). This result (proved in Appendix G) provides also the precise behavior of the reconstruction error as a function of the bit budget \( B \).

**Proposition 2 (Dequantizing Reconstruction Error).** Given \( x \in \mathbb{R}^N \) and \( \Phi \in \mathbb{R}^{M×N} \), let us assume that each component of \( z = \Phi x \) is iid from \( Z \sim_{\text{iid}} N(0, \sigma^2_0) \). We take the corresponding optimal compressor function \( G \) defined in (3) and the \( p \)-optimal \( B \)-bits scalar quantizer \( Q_p \) as defined in [8]. Then, provided that \( \Phi \) is RIP\(_{p, w}\) and assuming that \( \delta_{2K}, \delta_{3K} \rightarrow M \) 0 in Theorem 7 the solution

\[
x^* = \Delta_{p, w}(Q_p[\Phi x], \Phi, \epsilon_p)
\]

satisfies

\[
\|x^* - x\| \lesssim_{B, M} 4c' \frac{2^{-\eta}}{\sqrt{p + 1}} + 2\epsilon_0(K).
\]

with \( c' = (9/8)(\epsilon\pi/3)^{1/2} < 1.8981 \) and for \( w \) and \( \epsilon_p \) defined in Lemma 3.

\(^6\)The noise part and not the compressibility part.
The assumption $\delta_{2K}, \delta_{3K} \to \delta_{M} > 0$ is reasonable since following the simple argument presented in \cite{8} (see Appendix B) the requirement of Proposition \cite{1} induces that both values decrease as $O(\sqrt{\log M}/M^{1/p})$ for RIP$_{p,w}$ Gaussian matrices. Section \ref{sec:5c} confirms numerically the predicted reconstruction error as a function of $p$ for various sensing scenarios.

Notice that, under HRA and for large $M$, it is possible to provide a rough estimation of the weighting dynamic $\theta_{p}$ when the weights are those provided by the $D_{p}C$ constraints. Indeed, since $w_{i}(p) = G'(Q_{p}[y_{i}])^{-p/2}$ and $G' = \gamma_{0, \sqrt{3} \sigma_{0}}$, we find

$$\|w\|_{p}^{p} = \sum_{i} G'(Q_{p}[y_{i}])^{-p/2} \simeq_{B,M} M \sum_{k} G'^{p-2}(\omega_{k,p}) p_{k}$$

$$\simeq_{B,M} M (2\pi 3 \sigma_{0}^{2}/(2-p)/2(2\pi \sigma_{0}^{2})^{-1/2} \sum_{k} \alpha_{k} \exp(-\frac{1}{2} \omega_{k,p}^{2} \frac{p+1}{3\sigma_{0}^{2}}))$$

$$\simeq_{B,M} M (2\pi 3 \sigma_{0}^{2}/(2-p)/2(2\pi \sigma_{0}^{2})^{-1/2} (2\pi 3 \sigma_{0}^{2}/p+1))^{1/2}$$

$$= M (2\pi \sigma_{0}^{2}/(2-p)/2(3-p)/2(p+1))^{-1/2},$$

where we recall that $p_{k} = \int_{R_{k}} \varphi_{0}(t) dt \simeq_{B} \varphi_{0}(c') \alpha_{k} \forall c' \in R_{k}$ (see the proof of Lemma \ref{lem:8}).

Moreover,

$$\|w\|_{\infty}^{p} = (\alpha/\alpha_{3}/2)^{p/2} = (\alpha/G^{1}(1/2 + \alpha))^{p/2} \simeq_{B} (2\pi 3 \sigma_{0}^{2}/(2-p)/2)^{p/2}.$$

Therefore, estimating $\theta_{p}^{p/2}$ with $M^{2} \|w\|_{\infty}^{2p}/\|w\|_{p}^{2p}$, we find

$$\theta_{p}^{p/2} \simeq_{B,M} \sqrt{(p+1)/3}.$$ 

Therefore, at a given $p \geq 2$, since (23) involves a minimal $M$ which evolves like $O(\theta_{p}^{p/2} (K \log N/K)^{p/2})$, considering the weighting induced by GBPDN($\ell_{p,w}$) requires to collect $\sqrt{(p+1)/3}$ times more measurements than GBPDN($\ell_{p}$), which is the price to pay for guaranteeing a bounded reconstruction error by adapting to non-uniform quantization.

**IV. DEQUANTIZING IS STABILIZING**

This section shows that, given the weights and the levels defined in Lemma \ref{lem:3}, the use of GBPDN for reconstructing $x$ from $Q[\Phi x]$ can be interpreted as stabilizing the quantization distortion seen as a heteroscedastic noise. We first start by showing how GBPDN can be associated to any heteroscedastic GGD noise stabilization before applying it to the dequantization scenario.

**A. Stabilizing Heteroscedastic GGD Noise**

The GBPDN($\ell_{p,w}$) error bound provided by (21) suggests that there is an interest to stabilize the sensing corruption by an additive heteroscedastic GGD noise during the signal reconstruction.

To understand this, let us consider the following general signal sensing model

$$y = \Phi x + \varepsilon,$$

where $\varepsilon \in \mathbb{R}^{M}$ is the noise vector contaminating the sensed signal whose entries are independent variables $\varepsilon_{i}$, each following a centered GGD($0, \alpha_{i}, p$) distribution with pdf $\propto \exp(-|t/\alpha_{i}|^{p})$, where $p > 0$ is the shape parameter (the same for all $\varepsilon_{i}$’s), and $\alpha_{i} > 0$ the scale parameter \cite{21}. It is obvious that

$$\mathbb{E}\varepsilon = 0 \quad \text{and} \quad \mathbb{E}(\varepsilon\varepsilon^{T}) = \Gamma(3/p)(\Gamma(1/p))^{-1} \text{diag}(\alpha_{1}^{2}, \cdots, \alpha_{M}^{2}).$$
Quite naturally, it is tempting to set the weights to $w_i = 1/\alpha_i$ in GBPDN($\ell_{p,w}$) since in this case, using a common Bayesian maximum a posteriori (MAP) argument, the associated constraint corresponds precisely to the negative log-likelihood of the joint pdf of $\varepsilon$.

As shown in more detail below, introducing these non-uniform weights $w_i$ leads to a reduction in the error of the reconstructed signal, relative to using constant weights. Let us restrict our analysis to strictly $K$-sparse signals $x$ in the standard basis $\Psi = 1$. We assume that there exist bounds (estimators) for the $\ell_p$ and the $\ell_{p,w}$ norms used for characterizing $\varepsilon$, i.e., we know that $\|e\|_p \simeq_M \epsilon$ and $\|e\|_{p,w} \simeq_M \epsilon_{st}$ for some $\epsilon, \epsilon_{st} > 0$ to be detailed later.

In this case, if the random matrix $\Phi \sim \mathcal{N}^{M \times N}(0,1)$ is RIP$_{p,w}(K, \delta, \mu)$ for $p \geq 2$, with $\mu = \mathbb{E}\|\xi\|_p$ for $\xi \sim \mathcal{N}^M(0,1)$, Theorem 1 asserts that

$$\|x^* - x\| \leq B_p \epsilon / \mu,$$

for $x^* = \Delta_{p,1}(y, \Phi, \epsilon)$ and $B_p \simeq_M 4$.

Conversely, for the weights to $w_i = 1/\alpha_i$, and assuming $\Phi$ being RIP$_{p,w}(K, \delta', \mu_{st})$ with $\mu_{st} = \mathbb{E}\|\xi\|_{p,w}$, we get

$$\|x^*_{st} - x\| \leq B_{p}' \epsilon_{st} / \mu_{st},$$

for $x^*_{st} = \Delta_{p,w}(y, \Phi, \epsilon)$ and $B_{p}' \simeq_M 4$.

When the number of measurements $M$ is large, using classical GGD absolute moments formula, the two bounds $\epsilon$ and $\epsilon_{st}$ can be set close to

$$\epsilon_p \simeq_M \frac{1}{M} \sum_i \mathbb{E}|\varepsilon_i|^p = \frac{1}{p} \|\alpha\|_p^p, \quad \epsilon_{st}^p \simeq_M \frac{1}{M} \sum_i w_i^p \mathbb{E}|\varepsilon_i|^p = \frac{1}{p} M,$$

Moreover, using Lemma 5

$$\mu_p \simeq_M \frac{1}{M} \sum_i \mathbb{E}|\xi_i|^p = M\mathbb{E}|g|^p, \quad \mu_{st}^p \simeq_M \mathbb{E}|g|^p \|w\|_{p}^p,$$

where $g \sim \mathcal{N}(0,1)$.

**Proposition 3.** Under the conditions defined above

$$\frac{\epsilon_{st}^p}{\mu_{st}^p} \leq_M \epsilon^p / \mu^p,$$

so that, asymptotically in $M$, GBPDN($\ell_{p,w}$) has a smaller reconstruction error compared to GBPDN($\ell_p$).

**Proof:** Let us observe that $\epsilon_{st}^p / \mu_{st}^p = \frac{1}{p} M (\mathbb{E}|g|^p \|w\|_p^p)^{-1} = \frac{1}{p} (\mathbb{E}|g|^p)^{-1} \left( \frac{1}{M} \sum_i \frac{1}{\alpha_i} \right)^{-1}$. By the Jensen inequality, we have $\left( \frac{1}{M} \sum_i \frac{1}{\alpha_i} \right)^{-1} \leq \frac{1}{M} \sum_i \alpha_i^p$, so that $\epsilon_{st}^p / \mu_{st}^p \leq \frac{1}{p} (\mathbb{E}|g|^p)^{-1} ||\alpha||_p^p / M = \epsilon^p / \mu^p$.

Similarly to what is described in Section III-C, the price to pay for this stabilization is an increase of the weighting dynamic $\theta_p = (\rho_{\text{min}}/\rho_{\text{max}})^2$ defined in Proposition 4 which in turn controls, at fixed $p$, the necessary overhead in the number of measurements $M$ for satisfying the RIP$_{p,w}(K, \delta, \mu)$ with non-constant weights. This stabilization gain will be numerically illustrated in Section IV in the case of an additive heteroscedastic Gaussian noise.

**Example.** Let us consider a simple situation where the $\alpha_i$’s take only two values, i.e., $\alpha_i \in \{1, H\}$ for some $H \geq 1$. The proportion of $\alpha_i$’s equal to $H$ is $r = \#\{i : \alpha_i = H\} / M$. In this case, stabilizing
weights are \( w_i = 1/\alpha_i \in \{1,1/H\} \), i.e., a situation that we already considered in the end of Sec. III-B.

An easy computation provides

\[
\mathcal{E} := \frac{\varphi_p}{\mu_p} \simeq \frac{1}{p} \frac{1}{\nu_p} \left( r H^p + (1 - r) \right),
\]

\[
\mathcal{E}_{st} := \frac{\varphi_p}{\mu_p} \simeq \frac{1}{p} \frac{1}{\nu_p} \left( r H^{-p} + (1 - r) \right)^{-1},
\]

so that, the “stabilization gain” with respect to an unstabilized setting can be quantified by the ratio

\[
\left( \frac{\mathcal{E}}{\mathcal{E}_{st}} \right)^{1/2} \simeq \frac{1}{M} \left( r H^{-p} + (1 - r) \right)^{1/2} \simeq \frac{1}{M,H} \left( r (1 - r) \right)^{1/2} H.
\]

We see that the stabilization provides a clear gain which increases as the measurements get very unevenly corrupted, i.e., when \( H \) is large. Interestingly, the higher \( p \) is, the less sensitive is this gain to \( r \). We also observe that the overhead in the number of measurements between the stabilized and the unstabilized situations is related to

\[
\theta_p^{1/2} \simeq \frac{1}{M} \left( r H^{-p} + (1 - r) \right)^{-1} \simeq \frac{1}{M,H} (1 - r)^{-1}.
\]

The limit case (large \( H \)) can be interpreted as ignoring \( r \) percent of the measurements in the data fidelity constraint, keeping only those for which the noise is not dominating.

B. Stabilizing Quantization Distortion

In connection with the procedure developed in Section IV-A, the weights and the \( p \)-optimal levels introduced in Lemma 3 can be interpreted as a “stabilization” of the quantization distortion as measured by the associated \( \ell_{p,w} \)-norm. This means that, asymptotically in \( M \), selecting these weights and levels, all quantization regions \( R_k \) contribute equally to this general distortion measure.

To understand this fact, we start by studying the following relation shown in the proof of Lemma 3 (see Appendix D):

\[
\| Q_p[z] - z \|_{p,w} \simeq M \sum_{k} [G'(\omega_{k,p})]^{p-2} \int_{R_k} |t - \omega_{k,p}|^p \varphi_0(t) \, dt.
\]  

(25)

Using the threshold \( T(B) = \Theta(\sqrt{B}) \) and \( T = [-T(B), T(B)] \) as defined in Lemma 2 in Appendix D shows that

\[
\| Q_p[z] - z \|_{p,w} \simeq M_B \sum_{k} [G'(\omega_{k,p})]^{p-2} \int_{R_k} |t - \omega_{k,p}|^p \varphi_0(t) \, dt,
\]  

(26)

\[
\simeq M_B \sum_{k} [G'(\omega_{k,p})]^{p-2} \frac{\alpha_k^{p+1}}{(p+1)2^p} \varphi_0(\omega_{k,p}),
\]  

(27)

using (14). However, using (15) and the relation \( G' = \varphi_0^{1/3}/\|\varphi_0\|_{1/3} \), we find \( \alpha_k^2 \varphi_0(\omega_{k,p}) \simeq_B |\omega_{k,p}|^{3}. \) Therefore, each term of the sum in (26) provides a contribution

\[
[\|G'(\omega_{k,p})\|^{p-2} \frac{\alpha_k^{p+1}}{(p+1)2^p} \varphi_0(\omega_{k,p})] \simeq_B M_B \|\varphi_0\|_{1/3} \frac{\alpha_k^{p+1}}{(p+1)2^p},
\]

which is independent of \( k! \) It means therefore that the weights \( w \) (and the \( p \)-optimal levels) are conveniently defined for stabilizing the distortion measured in the \( \ell_{p,w} \)-norm. Moreover, multiplying this value by \( M \) and by the number of bins, i.e., \( 1/\alpha = 2^B \), we recover the estimation (19).

This phenomenon is well known for \( p = 2 \) and may actually serve for defining \( G' \) itself [9]. The fact that this effect is preserved for \( p \geq 2 \) is a surprise for us.
V. Numerical Experiments

In this section, we first describe how to numerically solve the GBPDN optimization problem using a primal-dual convex optimization scheme. We then illustrate the use of GBPDN for stabilizing heteroscedastic Gaussian noise on the CS measurements, and finally apply GBPDN for reconstructing signals in the quantized CS scenario described in Section II.

A. Solving GBPDN

The optimization problem GBPDN ($\ell_p, \mathbf{w}$) is a special instance of the general form

$$\min_{\mathbf{u} \in \mathbb{R}^N} f(\mathbf{u}) + g(L\mathbf{u}),$$  \hspace{1cm} (28)

where $f$ and $g$ are closed convex functions that are not infinite everywhere (i.e., proper functions), and $L = \text{diag}(\mathbf{w})\Phi$ is a bounded linear operator, with $f(\mathbf{u}) := \|\mathbf{u}\|_1$ and $g(\mathbf{v}) := \mathbb{I}_{B^p_\epsilon}(\mathbf{v} - \mathbf{y})$ where $\mathbb{I}_{B^p_\epsilon}(\mathbf{v})$ is the indicator function of the $\ell_p$-ball $B^p_\epsilon$ centered at zero and of radius $\epsilon$, i.e., $\mathbb{I}_{B^p_\epsilon}(\mathbf{v}) = 0$ if $\mathbf{v} \in B^p_\epsilon$ and $+\infty$ otherwise. For the case of (GBPDN ($\ell_p, \mathbf{w}$)), both $f$ and $g$ are non-smooth but the associated proximity operators can be computed easily as we will show shortly. This will allow to minimize the objective in GBPDN ($\ell_p, \mathbf{w}$) by calling on proximal splitting algorithms.

Before delving into the details of the minimization splitting algorithm, we recall some results from convex analysis. The proximity operator \cite{proximity_operator} of a proper closed convex $f$ is defined as the unique solution

$$\text{prox}_f(\mathbf{u}) = \arg\min_{\mathbf{z}} \frac{1}{2}\|\mathbf{z} - \mathbf{u}\|^2 + f(\mathbf{z}).$$

If $f = \mathbb{I}_C$ for some closed convex set $C$, $\text{prox}_f$ is equivalent to the orthogonal projector onto $C$, $\text{proj}_C$. $f^*$ is the Legendre-Fenchel conjugate of $f$. For $\lambda > 0$, the proximity operator of $\lambda f^*$ can be deduced from that of $f/\lambda$ through Moreau’s identity

$$\text{prox}_{\lambda f^*}(\mathbf{u}) = \mathbf{u} - \lambda \text{prox}_{\lambda^{-1} f}(\mathbf{u}/\lambda).$$

Solving (28) with an arbitrary bounded linear operator $L$ can be achieved using primal-dual methods motivated by the classical Kuhn-Tucker theory. Starting from methods to solve saddle function problems such as the Arrow-Hurwicz method \cite{arrow_hurwicz}, this problem has received a lot of attention recently, e.g., \cite{24,26}.

In this paper, we use the relaxed Arrow-Hurwicz algorithm described in \cite{25}. Adapted to our problem, the algorithm reads:
Algorithm 1: Primal-dual scheme for solving GBPDN($\ell_p, w$).

**Inputs:** Measurements $y$, sensing matrix $\Phi$.

**Parameters:** Iteration number $N_{\text{iter}}$, $\theta \in [0, 1]$, step-sizes $\sigma > 0$ and $\tau > 0$.

**Main iteration:**
\begin{algorithmic}
\State for $t = 0$ to $N_{\text{iter}} - 1$ do
\State \hspace{1em} • Update the dual variable: $v_{t+1} = \text{prox}_{\sigma g^*} (v_t + \sigma L \tilde{u}_t)$.
\State \hspace{1em} • Update the primal variable: $u_{t+1} = \text{prox}_{\tau f} (u_t - \tau L^T v_{t+1})$.
\State \hspace{1em} • Approximate extragradient step: $\tilde{u}_{t+1} = u_{t+1} + \theta (u_{t+1} - u_t)$.
\State \end{algorithmic}

**Output:** Signal $u_{N_{\text{iter}}}$.

A sufficient condition for the sequences of Algorithm 1 to converge is to choose $\sigma$ and $\tau$ such that $\tau \sigma \|w\|_\infty \|\Phi\|_2^2 < 1$. It has been shown in [25, Theorem 1] that under this condition and for $\theta = 1$, the primal sequence $(u_t)_{t \in \mathbb{N}}$ converges to a (possibly strict) global minimizer of GBPDN($\ell_p, w$), and the average primal and dual sequences converge at the rate $O(1/t)$ on the restricted duality gap.

**Proximity operator of $f$:** For $f(u) = \|u\|_1$, $\text{prox}_{\tau f}(u)$ is the popular component-wise soft-thresholding of $u$ with threshold $\tau$.

**Proximity operator of $g$:** Recall that $g(v) = \mathbb{I}_{B_p^c}(v - y)$. Using Moreau’s identity above, and proximal calculus rules for translation and scaling, we have
\[
\text{prox}_{\sigma g^*}(v) = v - \sigma y - \text{proj}_{B_p^c}(v - \sigma y).
\]

It remains to compute the orthogonal projection $\text{proj}_{B_1^c}$ to get $\text{proj}_{B_p^c} = \sigma e \text{proj}_{B_1^c}(\cdot / (\sigma e))$. For $p = 2$ and $p = +\infty$, this projector has an easy closed form. For $2 < p < +\infty$, we used the Newton method we proposed in [8] for solving the related Karush-Kuhn-Tucker system which is reminiscent of the strategy underlying sequential quadratic programming.

**B. Gaussian Noise Stabilization Illustration**

Let us explore the impact of using non-uniform weights (e.g., stabilizing the measurement noise) for signal reconstruction when the CS measurements are corrupted by heteroscedastic Gaussian noise. This illustrates for $p = 2$ both the gain induced by stabilizing the sensing noise and the increase of measurements necessary for observing this gain.

In this illustration, the important problem dimensions are set to $N = 1024$, $K = 16$, and the oversampling factor is $M/K \in \{5, 10, \cdots , 50\}$. The $K$-sparse unit norm signals were generated independently according to a Bernoulli-Gaussian model with $K$-length support picked uniformly at random in $[N]$, and the non-zero signal entries drawn from $\mathcal{N}(0, \sigma_s^2)$ with $\sigma_s^2 \approx 1/K$.

The sensing model of each signal $x$ is
\[
y = \Phi x + \varepsilon,
\]
with $\varepsilon_i \sim_{\text{iid}} \mathcal{N}(0, \sigma_i^2)$ and $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$. The heteroscedastic behavior of $\varepsilon$ has been designed so that $\sigma_i \sim_{\text{iid}} \mathcal{U}([\sigma_0 - \delta_0, \sigma_0 + \delta_0])$ with $\sigma_0 = 0.1$ and $\delta_0 = 0.6 \sigma_0$. 
Two reconstruction methods have been tested: one with and the other without stabilizing the noise variance. In the first case, the weights have been set to $w_i = 1/\sigma_i$, while in the second $w = 1$. Since the purpose of this analysis is not focused on the design of efficient noise power estimators, $\epsilon$ and $\epsilon_{st}$ have been simply set by an oracle to $\epsilon_{st} = \|y - \Phi x\|_2$ and $\epsilon = \|y - \Phi x\|_2$.

Given the parameters above, we can compute the weighting dynamic. This one is equal to $\theta_p \simeq M M E \|w\|_2^\infty E \|w\|_2^2 = \sigma_0 + \delta_0 \sigma_0 - \delta_0 = 4$ and the average stabilization gain should be of

$$20 \log_{10} \|x - x^*\|/\|x - x_{st}^*\| \simeq M 20 \log_{10}(\epsilon\|w\|)/(\epsilon_{st}\sqrt{M}) < 2.43 \text{ dB}.$$  

Numerically, GBPDN($\ell_2, w$) and GBPDN($\ell_2$) = BPDN have been solved with the method described in Section V-B until the relative convergence of the iterates was smaller than $10^{-6}$ (with a maximum of 2000 iterations). Reconstruction results were averaged over 50 experiments.

In Fig. 2(a), the reconstruction SNR of the stabilized reconstruction is clearly superior to the un-stabilized result and this gain increases for high $M/K$. This gain is displayed in Fig. 2(b). The dashed horizontal line represents the theoretical prediction of 2.43 dB which upper-bounds the observed numerical gain.

![SNR vs. M/K](image)

**Fig. 2:** Stabilized versus normal reconstruction using GBPDN($\ell_2, w$) and BPDN respectively. (a) The reconstruction SNR using stabilized (red) and unstabilized (blue) methods. (b) Observed (blue) and theoretically predicted (dashed green) SNR gain from stabilizing.

### C. Non-Uniform Quantization

This section presents several simulations challenging the power of GBPDN for reconstructing sparse signals from non-uniformly quantized measurements when the weights and the $p$-optimal levels of Lemma 3 are combined. Several configurations have been tested for different $p \geq 2$, oversampling ratio $M/K$, number of bits $B$ and for non-uniform and uniform quantization.

For this experiment, we set the key dimensions to $N = 1024$, $K = 16$, $B = 4$, and the $K$-sparse unit norm signals have been generated as in the previous section.

The oversampling ratio was taken as $M/K \in \{10, 15, \ldots, 45\}$, the moment as $p \in \{2, 4, \ldots, 10\}$ and the matrix $\Phi$ has been drawn randomly as $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$. In the sensing, the non-uniform quantization of the measurement $\Phi x$ was defined from the companding setup, i.e., with a compressor $G$. 

![SNR vs. M/K](image)
associated to $\gamma_{0,\sigma_0}$ according to (3). The weights $w$ were computed as in Lemma 3 and the $p$-optimal levels using the numerical method described in Appendix H.

For the sake of completeness, we also compared some results to those associated a uniformly quantized CS scenario. In this case, the measurement $z = \Phi x$ are quantized as $y_i = \alpha' \lfloor z_i / \alpha' \rfloor + \alpha' / 2$, the quantization bin width $\alpha' = \alpha'(B)$ has been set by dividing regularly the interval $[-\|z\|_\infty, \|z\|_\infty]$ into the same number of bins as those used for the non-uniform quantization.

For the signal reconstruction, GBPDN was solved with the primal-dual scheme described in Section V-B until either the relative convergence tolerance on the iterates was smaller than $10^{-6}$ or a maximum number of iterations of 2000 was reached. Finally, all the reconstruction results were averaged over 50 replications of sparse signals for each combination of parameters.

Fig. 3(a) displays the evolution of the signal reconstruction quality, as measured by the SNR, as a function of the oversampling factor $M/K$. We clearly see a reconstruction quality improvement with respect to both the uniformly quantized CS scheme (dashed curve) and to increasing values of $p$ and $M/K$. This last effect is better analyzed in Fig. 3(b) where the SNR gain with respect to $p = 2$ for various values of $p$ is shown. As predicted by Proposition 2, we clearly see that, as soon as the ratio $M/K$ is large, taking higher $p$ value leads to a higher reconstruction quality than the one obtained for $p = 2$ (BPDN). Moreover, Fig. 3(b) confirms that when $p$ increases, the minimal measurement number inducing a positive SNR gain increases. For instance, to achieve a positive gain at $p = 4$, we must have $M/K \geq 15$, while at $p = 10$, $M/K$ must be higher than 20. At $p$ fixed, the reconstruction quality increased also monotonically with $M/K$.

In Fig. 4, the quantization consistency of the reconstructed signals is tested by looking at the histogram of $\alpha^{-1}(\mathcal{G}(\Phi x^*) - \mathcal{G}(y))$. We do observe that this histogram is closer to a uniform distribution for $p = 10$ than for $p = 2$, in good agreement with the “companded” quantizer definition $Q = \mathcal{G}^{-1} \circ Q_\alpha \circ \mathcal{G}$ showing that in the domain compressed by $\mathcal{G}$, this quantizer is similar to a uniform one.

As a last test, we have more thoroughly compared a uniform quantization scenario described in the experimental setup above with the BPDQ$_p$ decoder developed in [8] to the non-uniform case studied in this paper. More precisely, Fig. 5(a) shows the reconstruction SNR gain between non-uniform and uniform quantization at various $p$, i.e., $\text{SNR}(\text{GBPDN}(\ell_{p,w})) - \text{SNR}(\text{BPDQ}_p)$. We see that, at a given
Fig. 4: Testing the Quantization Consistency (QC). (a) Histogram of the components of \( \alpha^{-1} (G(\Phi x^*) - G(y)) \) for \( p = 2 \) and \( M/K = 40 \) (averaged over 100 trials). (b) Same histogram for \( p = 10 \). The QC is better respected in this case.

Fig. 5: Reconstruction gain (in dB) between non-uniform or uniform quantization at the same \( p \).

\( p \), this gain improves with \( M/K \) with the highest values obtained for \( p = 2 \). This points the fact that for \( p \neq 2 \), the quantization scheme is not optimized for reducing the \( \ell_p \)-norm distortion. This would require us to change the quantization scenario by not only optimizing the \( p \)-optimal levels but also the threshold. This will be the study of a future research.

VI. CONCLUSION

In this paper, we have shown that, when the compressive measurements of a sparse or compressible signal are non-uniformly quantized, there is a clear interest in modifying the reconstruction procedure by adapting the way it imposes the reconstructed signal to “match” the observed data. In particular, we have proved that in an oversampled scenario, replacing the common BPDN \( \ell_2 \)-norm constraint by a weighted \( \ell_p \)-norm adjusted to the non-uniform nature of the quantizer reduces the reconstruction error by a factor of \( \sqrt{p+1} \). Moreover, we showed that this improvement stems from a stabilization of the quantization distortion seen as an additive heteroscedastic GGD noise on the measurements.

In a future work, we will investigate if the quantization scheme can also be optimized with respect to the proposed reconstruction procedure, \textit{i.e.}, by adjusting the thresholds for minimizing the weighted \( \ell_p \)-distortion at a fixed bit budget.
APPENDIX A
PREPARATORY LEMMATA

This appendix contains several key lemmata that are useful for the subsequent proofs developed in the other appendices.

The first lemma will serve later to evaluate asymptotically the contribution of each quantization bin to the global quantizer distortion measured with $\ell_p, w$-norm when a Gaussian source (with pdf $\varphi_0$) is quantized.

**Lemma 6.** Given $a, b \in \mathbb{R}$ with $a < b$, $n \in \mathbb{N} \setminus \{0\}$ and a Gaussian pdf $\varphi_0 = \gamma_{0, \sigma_0}$. Let $\lambda_n$ be the (unique) minimizer of

$$\min_{\lambda \in [a, b]} \int_a^b |t - \lambda|^n \varphi_0(t) \, dt .$$

Then,

$$\int_a^b |t - \lambda_n|^n \varphi_0(t) \, dt \geq \frac{(b-a)^{n+1}}{(n+1)2^{n+1}} \left( 1 + \frac{D}{C} \right)^{(n+1)/n} \mathcal{C},$$

$$\int_a^b |t - \lambda_n|^n \varphi_0(t) \, dt \leq \frac{(b-a)^{n+1}}{(n+1)2^{n+1}} \left( 1 + \frac{D}{C} \right)^{(n+1)/n} \mathcal{D},$$

$$\frac{1}{1 + \mathbb{E}^{1/n}} \left( \frac{1 + \mathbb{E}^{1/n}}{a + b} \right) \leq \lambda_n \leq \frac{1}{1 + \mathbb{E}^{1/n}} \left( a + \mathbb{E}^{1/n} b \right),$$

with $\mathcal{C} := \min_{t \in [a, b]} \varphi_0(t)$, $\mathcal{D} := \max_{t \in [a, b]} \varphi_0(t)$ and $\mathbb{S} = \mathcal{D} / \mathcal{C}$.

**Proof:** Let us first show the upper bound \[30\]. In Lemma \[1\] and its proof, it was shown that $\lambda_n$ exists and is unique, i.e., the minimization problem is well-posed. Furthermore, $\lambda_n$ is the unique root of

$$\int_a^{\lambda_n} (\lambda_n - t)^{n-1} \varphi_0(t) \, dt = \int_{\lambda_n}^b (t - \lambda_n)^{n-1} \varphi_0(t) \, dt .$$

We can then bound each side of the above equality as follows,

$$\frac{1}{n} (\lambda_n - a)^n \mathcal{C} \leq \int_a^{\lambda_n} (\lambda_n - t)^{n-1} \varphi_0(t) \, dt \leq \frac{1}{n} (\lambda_n - a)^n \mathcal{D},$$

$$\frac{1}{n} (b - \lambda_n)^n \mathcal{C} \leq \int_{\lambda_n}^b (t - \lambda_n)^{n-1} \varphi_0(t) \, dt \leq \frac{1}{n} (b - \lambda_n)^n \mathcal{D},$$

which implies $(\lambda_n - a)^n \geq \left( \frac{\mathbb{E}}{n} \right) (b - \lambda_n)^n$ and $(b - \lambda_n)^n \geq \left( \frac{\mathbb{E}}{n} \right) (\lambda_n - a)^n$, from which we easily deduce \[31\]. Since

$$\int_a^b |t - \lambda_n|^n \varphi_0(t) \, dt = \int_a^{\lambda_n} (\lambda_n - t)^n \varphi_0(t) \, dt + \int_{\lambda_n}^b (t - \lambda_n)^n \varphi_0(t) \, dt$$

$$\leq \frac{1}{n+1} \left[ (\lambda_n - a)^{n+1} + (b - \lambda_n)^{n+1} \right] \mathcal{D},$$

using the previous inequalities provides either

$$\int_a^b |t - \lambda_n|^n \varphi_0(t) \, dt \leq \frac{1}{n+1} (\lambda_n - a)^{n+1} \left[ 1 + \left( \frac{\mathbb{D}}{\mathbb{C}} \right)^{(n+1)/n} \right] \mathcal{D},$$

or

$$\int_a^b |t - \lambda_n|^n \varphi_0(t) \, dt \leq \frac{1}{n+1} (b - \lambda_n)^{n+1} \left[ 1 + \left( \frac{\mathbb{D}}{\mathbb{C}} \right)^{(n+1)/n} \right] \mathcal{D} .$$
Taking the minimum between these two bounds and noting that \( \min(\lambda_n - a, b - \lambda_n) \leq (b - a)/2 \) gives
\[
\int_a^b |t - \lambda_n|^n \varphi_0(t) \, dt \leq \frac{(b-a)^{n+1}}{(n+1)2^{n+1}}(1 + (\frac{n}{2})^{(n+1)/n}) D.
\]
The lower bound \( (29) \) is obtained similarly.

The following lemma presents a generalization of “Q-function like” bounds for lower partial moments of a Gaussian pdf.

**Lemma 7.** Let \( \lambda > 0, n \in \mathbb{N} \) and \( \varphi = \gamma_{0,1} \). Let us define
\[
Q_n(\lambda) := \int^{+\infty}_\lambda (t - \lambda)^n \varphi(t) \, dt.
\]
Then, \( Q_n(\lambda) = \Theta(\lambda^{-(n+1)} \varphi(\lambda)) \). More precisely,
\[
\frac{n! \lambda^{n+1}}{\prod_{k=1}^{n} (\lambda^2 + k)} \varphi(\lambda) \leq Q_n(\lambda) \leq \frac{n!}{\lambda^{n+1}} \varphi(\lambda).
\]
This lemma generalizes the well known bound on \( Q = Q_0 \), namely
\[
\frac{\lambda}{\lambda^2 + 1} \varphi(\lambda) \leq Q(\lambda) \leq \frac{1}{\lambda} \varphi(\lambda).
\]

**Proof:** The proof involves integration by parts, the identities \(-\varphi'(u) = u\varphi(u)\) and
\[
\left( \frac{\varphi(u)}{u^n} \right)' = (1 + \frac{n}{u^n}) \frac{\varphi(u)}{u^{n-1}}.
\]
Therefore, the upper bound is a simple consequence of
\[
Q_n(\lambda) \leq \frac{1}{\lambda} \int^{+\infty}_\lambda (t - \lambda)^n \varphi(t) \, dt = \frac{n}{\lambda} Q_{n-1}(\lambda) \leq \cdots \leq \frac{n!}{\lambda^n} Q(\lambda) \leq \frac{n!}{\lambda^{n+1}} \varphi(\lambda).
\]
To get the lower bound, observe first that, defining \( Q_{n,k}(\lambda) := \int^{+\infty}_\lambda (t - \lambda)^n t^{-k} \varphi(t) \, dt \), we find
\[
(1 + \frac{k+1}{\lambda^2}) Q_{n,k}(\lambda) \geq \int^{+\infty}_\lambda (t - \lambda)^n (1 + \frac{k+1}{\lambda^2}) t^{-k} \varphi(t) \, dt = n Q_{n-1,k+1}(\lambda).
\]
Therefore,
\[
Q_n(\lambda) \geq \frac{n! \lambda^{2n}}{\lambda^{2n+1}} Q_{n-1,1}(\lambda) \geq \cdots \geq \frac{n! \lambda^{2n}}{\prod_{k=1}^{n} (\lambda^2 + k)} Q_{0,n}(\lambda).
\]
But \( (1 + \frac{n+1}{\lambda^2}) Q_{0,n}(\lambda) \geq \frac{\varphi(\lambda)}{\lambda^{n+1}} \), so that \( Q_n(\lambda) \geq \frac{n! \lambda^{2n+2}}{\prod_{k=1}^{n+1} (\lambda^2 + k)} \frac{\varphi(\lambda)}{\lambda^{n+1}} \), which concludes the proof.

**Appendix B**

**Proof of Lemma 4: “p-optimal Level Definiteness”**

**Proof:** For \( 2 \leq p < \infty \), \( |t - \lambda|^p \) is a continuous, coercive and strictly convex function of \( \lambda \) over \( \mathbb{R} \), and therefore so is \( \int_{\mathcal{R}_k} |t - \lambda|^p \varphi_0(t) \, dt \) since \( \varphi_0(t) > 0 \). It follows that the function \( \int_{\mathcal{R}_k} |t - \lambda|^p \varphi_0(t) \, dt \) has a unique minimizer on \( \mathbb{R} \). Moreover, this minimizer is necessarily located in \( \mathcal{R}_k \) since \( \int_{\mathcal{R}_k} |t - \lambda|^p \varphi_0(t) \, dt \) is monotonically decreasing (resp. increasing) on \( (-\infty, t_k) \) (resp. \( (t_{k+1}, +\infty) \))\(^7\). Consequently, \( \omega_{k,n} \) exists and is unique.

\(^7\)Where we used the Lebesgue dominated convergence theorem to derivate under the integration sign.
For proving the limit case \( p \to \infty \), for finite bins \( \mathcal{R}_k (k \notin \{1, \mathcal{B}\}) \) and without loss of generality for \( t_k \geq 0 \), relation (31) in Lemma 6 with \( a = t_k \) and \( b = t_{k+1} \), together with the squeeze theorem shows that

\[
\lim_{p \to +\infty} \omega_{k,p} = \lim_{p \to +\infty} \frac{1}{1 + \mathcal{S}^1/p} (\mathcal{S}^{1/p} t_k + t_{k+1}) = \lim_{p \to +\infty} \frac{1}{1 + \mathcal{S}^1/p} (t_k + \mathcal{S}^{1/p} t_{k+1}) = \omega_{k,\infty},
\]

where \( \mathcal{S} = \varphi_0(t_k)/\varphi_0(t_{k+1}) \).

For infinite bins (i.e., \( k \in \{1, \mathcal{B}\} \)) and assuming again \( t_k \geq 0 \), it follows from the beginning of the proof that \( \omega_{k,p} \) is the unique root on \([t_k, +\infty)\) of \( \mathcal{E}_p(\lambda) := \int_{t_k}^\lambda (t - \lambda)^{-p-1} \varphi_0(t) \, dt - \int_\lambda^\infty (t - \lambda)^{-p-1} \varphi_0(t) \, dt \). Let \( \tilde{\omega}_{k,p} \in [t_k, L) \) be the root of \( \mathcal{E}_p(\lambda, L) := \int_{t_k}^\lambda (t - \lambda)^{-p-1} \varphi_0(t) \, dt - \int_\lambda^L (t - \lambda)^{-p-1} \varphi_0(t) \, dt \) for some \( L \geq t_k \). We then have \( \mathcal{E}_p(\tilde{\omega}_{k,p}, p) = \int_{t_k}^{\tilde{\omega}_{k,p}} (\tilde{\omega}_{k,p} - t)^{-p-1} \varphi_0(t) \, dt - \int_{\tilde{\omega}_{k,p}}^\infty (t - \tilde{\omega}_{k,p})^{-p-1} \varphi_0(t) \, dt = - \int_L^\infty (t - \tilde{\omega}_{k,p})^{-p-1} \varphi_0(t) \, dt \leq 0 = \mathcal{E}_p(\omega_{k,p}) \), which implies that \( \omega_{k,p} \leq \omega_{k,p} \) since \( \mathcal{E}_p \) is non-decreasing for \( p \geq 1 \). However, since \( \omega_{k,p} \) is optimal on \([t_k, L]\), letting \( L = L(p) = c\sqrt{p} \), for \( c > 0 \), we have by Lemma 6 with \( a = t_k \) and \( b = L(p) \), \( \lim_{p \to +\infty} \omega_{k,p} \geq \lim_{p \to +\infty} \frac{1}{1 + \mathcal{S}^1/p} (\mathcal{S}^{1/p} t_k + c\sqrt{p}) = +\infty \) since \( \mathcal{S}^{1/p} = \exp(-t_k^2/2\sigma^2_0) \exp(c^2/2\sigma^2_0) = \Theta(1) \). This proves \( \lim_{p \to +\infty} \omega_{k,p} = +\infty = \omega_{k,\infty} \) and \( |\omega_{k,p}| = \Omega(\sqrt{p}) \) for \( k \in \{1, \mathcal{B}\} \).

**APPENDIX C**

**PROOF OF LEMMA 2: “ASYMPTOTIC p-QUANTIZATION CHARACTERIZATION”**

**Proof:** In this proof we use the quantizer symmetry to restrict the analysis to the half (positive) real line \( \mathbb{R}_+ \), on which \( \varphi_0 \) is decreasing.

Relation (9) comes from the definition of \( T(B) \) and that of \( \mathcal{G}' = \gamma_{0,\sqrt{3}\sigma_0} \). For proving (10), we can observe that \( \mathcal{G}(\lambda) = \|\varphi_0\|_1^{1/3} \int_{-\infty}^{-\lambda} \varphi_0^{-1/3}(t) \, dt = 1 - Q(\lambda/\sqrt{3}\sigma_0) \) where \( Q(t) = \frac{1}{\sqrt{2\pi}} \int_t^{+\infty} \gamma_{0,1}(u) \, du \). Since \( \frac{\lambda}{1 + \lambda^2} \gamma_{0,1}(\lambda) \leq Q(\lambda) \leq \frac{1}{\lambda} \gamma_{0,1}(\lambda) \), we obtain

\[
\frac{3\sigma^2_0}{3\sigma^2_0 + \lambda^2} \mathcal{G}'(\lambda) \leq 1 - \mathcal{G}(\lambda) \leq \frac{3\sigma^2_0}{\lambda} \mathcal{G}'(\lambda).
\]

Taking \( \lambda = T(B) \) in the last inequalities and using (9), we deduce from the quantizer definition

\[
\#\{k : \mathcal{R}_k \subset \mathcal{T}^c\} = 2 \#\{k : t_k \geq T(B)\} = 2 \alpha^{-1} (1 - \mathcal{G}(T)) = \Theta(B^{-1/2} 2^{1-\beta} B).
\]

Relation (11) is proved by noting that, if \( t_k \geq T(B) \),

\[
\int_{\mathcal{R}_k} |t - \omega_{k,p}|^p \varphi_0(t) \, dt \leq \int_{\mathcal{R}_k} (t - t_k)^p \varphi_0(t) \, dt \leq \int_{t_k}^\infty (t - t_k)^p \varphi_0(t) \, dt,
\]

where the first inequality follows from the \( p \)-optimality of \( \omega_{k,p} \in \mathcal{R}_k \). However, from Lemma 7 we know that, for \( \lambda \in \mathbb{R}_+ \),

\[
\frac{p^3 \lambda^{p+1} \sigma_{\lambda}^{2p+2}}{p^3 \lambda^{p+1} \sigma_{\lambda}^{2p+2}} \varphi_0(\lambda) \leq \sigma_0^p Q_p(\frac{\lambda}{\sigma_0}) \leq \frac{p^3 \lambda^{p+1} \sigma_{\lambda}^{2p+2}}{p^3 \lambda^{p+1} \sigma_{\lambda}^{2p+2}} \varphi_0(\lambda),
\]

with \( Q_p(\lambda) := \int_{\lambda}^\infty (t - \lambda)^p \gamma_{0,1}(t) \, dt \) and \( \sigma_0^p Q_p(\frac{\lambda}{\sigma_0}) = \int_{\lambda}^\infty (t - \lambda)^p \varphi_0(t) \, dt \).

Therefore, since \( \varphi_0 \propto (\mathcal{G}')^3 \),

\[
\int_{t_k}^\infty (t - t_k)^p \varphi_0(t) \, dt \leq \frac{p^3 \sigma_0^p (t_k+1)^{p+1}}{p^3 \sigma_0^p (t_k+1)^{p+1}} \varphi_0(t_k) \leq \frac{p^3 \sigma_0^p (t_k+1)^{p+1}}{p^3 \sigma_0^p (T_k+1)^{p+1}} \varphi_0(T) = O(B^{-(p+1)/2} 2^{-3\beta B}).
\]

Relation (12) is obtained by observing that \( \mathcal{G} \) is concave on \( \mathbb{R}_+ \). This implies \( \alpha_k \leq \alpha/\mathcal{G}'(t_{k+1}) \) and if \( k \) is such that \( 0 \leq t_{k+1} \leq T(B) \), \( \alpha_k = O(2^{-(1-\beta)B}) \). For (13), keeping the same \( k \), we note that
Lemma 8. \[ 1 \leq \frac{\varphi_0(t_k)}{\varphi_0(t_{k+1})} = \exp\left(\frac{1}{2\sigma_0^2} \alpha_k(t_k + t_{k+1})\right) \leq \exp\left(\frac{1}{3\sigma_0^2} \alpha_k t_{k+1}\right) = \exp\left(O(B^{1/2} 2^{-(1-\beta)B})\right) \] which is then arbitrarily close to 1.

For proving (14), we assume first \( p \geq 1 \). Let us consider (29) and (30) with \( a = t_k, b = t_{k+1}, e = \varphi_0(t_{k+1}) \) and \( D = \varphi_0(t_{k}) \) with \( 0 \leq t_{k+1} \leq T(B) \). From (13) we see that \( 1 \leq \frac{D}{e} = 1 + o(1) \). We show easily that this involves the equivalent relations \( e \simeq_B D, e/D \simeq_B 1 \) and \( D/e \simeq_B 1 \). Therefore, \( 1 + (\frac{D}{e})(p+1)/p \simeq_B 2 \) and \( 1 + (\frac{e}{D})(p+1)/p \simeq_B 2 \). Moreover, \( e \simeq_B \varphi_0(c) \) and \( D \simeq_B \varphi_0(t_{k+1}) \) for any \( c \in R_k \), so that (29) and (30) show finally \( \int_{R_k} |t - \omega_{k,p}|^p \varphi_0(t) dt \lesssim_B \frac{\alpha_k^{p+1}}{(p+1)^2 \omega_T^p} \varphi_0(c) \) and \( \int_{R_k} |t - \omega_{k,p}|^p \varphi_0(t) dt \gtrsim_B \frac{\alpha_k^{p+1}}{(p+1)^2 \omega_T^p} \varphi_0(c) \), which proves the relation. The case \( p = 0 \) is demonstrated similarly by observing that \( \varphi_0(t_{k+1}) \alpha_k \leq p_k := \int_{R_k} \varphi_0(t) dt \leq \varphi_0(t_k) \alpha_k \).

Let’s now turn to showing (15). From (13) and since \( G' \propto \varphi_0^{1/3} \), \( 0 \leq G'(t_k)/G'(t_{k+1}) \equiv 1 + o(1) \) so that \( G'(t_k)/G'(t_{k+1}) \simeq_B 1 \). By concavity of \( G \) on \( \mathbb{R}_+ \), we know that \( G'(t_{k+1}) \simeq_B \alpha/\alpha_k \leq G'(t_k) \). Therefore, \( 1 \leq G(t_{k+1})^{-1} \alpha/\alpha_k = 1 + o(1) \) which yields \( G'(t_{k+1}) \simeq_B \alpha/\alpha_k \). By the concavity argument again, we have \( G'(t_k) \geq \alpha/\alpha_k \), for any \( c \in R_k \) and thus \( 1 + o(1) = G(t_k)/G(t_{k+1}) \geq G'(t_k)/G'(t_{k+1}) \simeq_B 1 \). This implies \( G'(t_k) \simeq_B G'(t_{k+1}) \simeq_B \alpha/\alpha_k \).

If \( k \) is such that \( 0 \leq t_k \leq T(B) \leq t_{k+1} \), using again the concavity of \( G \) on \( \mathbb{R}_+ \), we find \( G(R_k \cap T) = T(B) - t_k \leq G(T(B)) - k \alpha \leq G(T(B)) \leq \alpha/G'(T(B)) = O(2^{-(1-\beta)B}) \), which proves (16).

For showing (17), we note that \( G'(t_k) = G'(T)(G'(t_k)/G'(T)) \). Since \( G(t_k)/G'(T) = \exp\left(\frac{T - t_k}{\beta_\sigma^2}\right)(1 - \frac{T - t_k}{T}) = \exp(\alpha) \) which is arbitrarily close to 1 (i.e., it is \( \exp(1) \)), we find \( G'(t_k) = O(2^{-\beta_B}), i.e., it inherits the behavior of \( G'(T) \).

The last relation (18) is proved similarly to (11) by appealing again to Lemma 7

\[
\int_{R_k} (t - t_k)^p \varphi_0(t) dt \leq \int_{R_k} \frac{p^2 \sigma_0^{2p+2}}{t_k^p} \varphi_0(t) dt \leq O(B^{-p+2} 2^{-3\beta_B}),
\]

where the asymptotic relation is obtained by seeing that, as soon as \( T - t_k \leq 1/2 \) (which is always possible to meet thanks to (16)),

\[
\frac{1}{t_k} = \frac{1}{t}(1 - \frac{T - t_k}{T})^{-1} \leq \frac{1}{t} (1 + 2\frac{T - t_k}{T}),
\]

and \( \varphi_0(t_k) = O(2^{-3\beta_B}) \) since \( \varphi_0 \propto (G')^3 \).

\section*{Appendix D}

\textbf{Proof of Lemma 3} “Asymptotic Weighted \( \ell_p \)-Distortion”

Before proving Lemma 3 let us show the following asymptotic equivalence.

\textbf{Lemma 8}. Let \( p \in \mathbb{N} \setminus \{0\} \) and \( \gamma > p - 3 \).

\[
\sum_{k=1}^{\infty} [G'(\omega_{k,p})]^\gamma \int_{R_k} |t - \omega_{k,p}|^p \varphi_0(t) dt \simeq_B \frac{\alpha^{2-p}}{(p+1)^{2p}} \int_{\mathbb{R}} |G'(t)|^{-p} \varphi_0(t) dt,
\]

(33)

\textbf{Proof}: Let us use the threshold \( T(B) \) defined in Lemma 2 for splitting the sum (33) in two parts, \textit{i.e.}, using the quantizer symmetry,

\[
\sum_{k=1}^{\infty} [G'(\omega_{k,p})]^\gamma \int_{R_k} |t - \omega_{k,p}|^p \varphi_0(t) dt = 2 \sum_{k \geq 0, t_k \leq T(B)} [G'(\omega_{k,p})]^\gamma \int_{R_k} |t - \omega_{k,p}|^p \varphi_0(t) dt + \mathbb{R},
\]

25
where the residual $\mathcal{R}$ reads

$$
\mathcal{R} := 2 \sum_{k: t_{k+1} \geq T(B)} [G'(\omega_{k,p})]^{\gamma} \int_{\mathcal{R}_k} |t - \omega_{k,p}|^p \varphi_0(t) \, dt,
$$

$$
= 2 [G'(\omega_{k',p})]^{\gamma} \int_{\mathcal{R}_{k'}} |t - \omega_{k',p}|^p \varphi_0(t) \, dt + 2 \sum_{k: t_k \geq T(B)} [G'(\omega_{k,p})]^{\gamma} \int_{\mathcal{R}_k} |t - \omega_{k,p}|^p \varphi_0(t) \, dt,
$$

where $k'$ is such that $t_{k'} < T(B) \leq t_{k'+1}$.

From Lemma 2, we can easily bound this residual. We know from (9), (11), (17) and (18) that, for all $k \in \{ j : \omega_{j,p} \geq t_j \geq T(B) \} \cup \{ k' \}$,

$$
[G'(\omega_{k,p})]^{\gamma} \int_{\mathcal{R}_k} |t - \omega_{k,p}|^p \varphi_0(t) \, dt = O(2^{-\beta(\gamma+3)} B^{-(p+1)/2}).
$$

However, (10) tells us that the sum in $\mathcal{R}$ is made of no more than $1 + O\left(B^{-1/2} 2^{(1-\beta)B}\right) = O\left(B^{-1/2} 2^{(1-\beta)B}\right)$ terms, so that

$$
\mathcal{R} = O\left(B^{-(p+2)/2} 2^{-(\beta(\gamma+4)-1)B}\right).
$$

Let us now study the terms for which $0 \leq t_{k+1} \leq T(B)$. Using (14) and (15) provides

$$
\sum_{k=1}^{\mathcal{R}} [G'(\omega_{k,p})]^{\gamma} \int_{\mathcal{R}_k} |t - \omega_{k,p}|^p \varphi_0(t) \, dt
\lesssim \frac{2}{B} \sum_{k: 0 \leq t_k \leq T(B)} [G'(\omega_{k,p})]^{\gamma} \frac{\alpha^k}{(p+1)^{2p}} \varphi_0(\omega_{k,p}) + \mathcal{R}
\lesssim \frac{2}{B} \frac{\alpha^p}{(p+1)^{2p}} \sum_{k: 0 \leq t_k \leq T(B)} [G'(\omega_{k,p})]^{\gamma-p} \varphi_0(\omega_{k,p}) \alpha_k + \mathcal{R}
\lesssim \frac{2}{B} \frac{2^{\frac{p}{2}}}{(p+1)^{2p}} \int_0^{T(B)} [G'(t)]^{\gamma-p} \varphi_0(t) \, dt + \mathcal{R},
$$

where, knowing that $0 \leq t_{k+1} \leq T(B)$, we have also used (14) with $p = 0$ to see that $p_k = \int_{\mathcal{R}_k} \varphi_0(t) \, dt \lesssim_B \varphi_0(c') \alpha_k$ for any $c' \in \mathcal{R}_k$.

Therefore, provided that $\beta(\gamma+4) \geq p+1$, which means that $\gamma > p-3$ since $\beta < 1$, the residual $\mathcal{R}$ decreases faster than the first term in the right-hand side of last of the last equivalence relation, so that

$$
\sum_{k=1}^{\mathcal{R}} [G'(\omega_{k,p})]^{\gamma} \int_{\mathcal{R}_k} |t - \omega_{k,p}|^p \varphi_0(t) \, dt \lesssim_B \frac{2^{\frac{p}{2}}}{(p+1)^{2p}} \int_0^{\mathcal{R}} [G'(t)]^{\gamma-p} \varphi_0(t) \, dt,
$$

since $T(B) = \Theta(B^{1/2})$ by definition.

With the three previous lemmata under our belts, we are now ready to prove Lemma 3.

**Proof of Lemma 3.** For $z_i \sim_{iid} \mathcal{N}(0, \sigma_0^2)$ with pdf $\varphi_0$, using the Strong Law of Large Numbers applied to $z_i$ conditionally on each quantization bin, we have

$$
\| Q_p[z] - z \|_{p,w}^p := \sum_{i=1}^M [G'(Q_p[z_i])]^{p-2} |z_i - Q_p[z_i]|^p,
\lesssim \frac{M}{M} \sum_{k=1}^{\mathcal{R}} [G'(\omega_{k,p})]^{p-2} \int_{\mathcal{R}_k} |t - \omega_{k,p}|^p \varphi_0(t) \, dt,
$$

where $w$ is such that the residual $\mathcal{R}$ reads

$$
\mathcal{R} := 2 \sum_{k: t_{k+1} \geq T(B)} [G'(\omega_{k,p})]^{\gamma} \int_{\mathcal{R}_k} |t - \omega_{k,p}|^p \varphi_0(t) \, dt,
where we used implicitly the quantizer symmetry in the last relation. This last relation is characterized by Lemma 8 by taking \( n = p \) and \( \gamma = p - 2 > p - 3 \), so that

\[
\|Q_p[z] - z\|_{p,w}^p \overset{M}{\sim} M \frac{2^{-p}}{(p+1)^{2p}} \int_\mathbb{R} |G'(t)|^{-2} \varphi_0(t) \, dt,
\]

\[
\overset{M}{\sim} M \frac{2^{-p}}{(p+1)^{2p}} \|\varphi_0\|_{1/3}.
\]

\[\square\]

**APPENDIX E**

**PROOF OF LEMMA 5** “GAUSSIAN \( \ell_{p,w} \)-NORM EXPECTATION”

First, the inequality \( \mathbb{E}\|\xi\|_{p,w} \leq (\mathbb{E}\|\xi\|_{p,w}^p)^{1/p} \) follows from the Jensen inequality applied on the convex function \((\cdot)^p\) on \( \mathbb{R}_+ \). Second, from our result in [8, Appendix C] it is easy to show that

\[
\mathbb{E}\|\xi\|_{p,w} \geq (\mathbb{E}\|\xi\|_{p,w}^p)^{1/p} (1 + (\mathbb{E}\|\xi\|_{p,w}^p)^{-2} \text{Var} \|\xi\|_{p,w}^p)^{\frac{1}{2} - 1}.
\]

Moreover, \( \mathbb{E}\|\xi\|_{p,w}^p = \|w\|_p^p \mathbb{E}|g|^p \), while

\[
\text{Var} \|\xi\|_{p,w}^p = \sum_i \text{Var}|w_i g|^p = \|w\|_p^{2p} \text{Var}|g|^p.
\]

Therefore, assuming CM weights,

\[
\mathbb{E}\|\xi\|_{p,w}/(\mathbb{E}\|\xi\|_{p,w}^p)^{1/p} \geq \left( 1 + (\rho_{\infty}^p/\rho_2^p)^{2p} M^{-1} (\mathbb{E}|g|^p)^{-2} \text{Var}|g|^p)^{\frac{1}{2} - 1}
\]

\[
\geq \left( 1 + 2^{p+1} \theta_p M^{-1} \right)^{\frac{1}{2} - 1},
\]

since \( \rho_{\infty}^p \leq \rho_{\infty}^p \), and \((\mathbb{E}|g|^p)^{-2} \text{Var}|g|^{2p} < 2^{p+1} [8].

**APPENDIX F**

**PROOF OF PROPOSITION 6** “RIP\(_{p,w}\) MATRIX EXISTENCE”

The proof proceeds simply by considering the Lipschitz function \( F(u) = \|u\|_{p,w} \) and the expected value \( \mu = F(\xi) \) for a random vector \( \xi \sim \mathcal{N}(M, 0, 1) \) in [8 Appendix A]. The Lipschitz constant of \( F \) is

\[
\lim_{u \to v} \frac{|F(u) - F(v)|}{\|u - v\|} = \|w\|_\infty \lambda_p,
\]

with \( \lambda_p = \max(M(2-p)/2p, 1) \) for \( p \geq 1 \). The value \( \mu = \mathbb{E}\|\xi\|_{p,w} \) can be estimated thanks to Lemma 5. Indeed, it tells us that if \( M \geq 2(2\theta_p)^p \),

\[
\mu \geq \frac{1}{2} (\mathbb{E}\|\xi\|_{p,w}^p)^{1/p} \geq \frac{1}{2} \rho_{\infty}^p \nu_p M^{1/p},
\]

with \( \nu_p = \mathbb{E}|g|^p = 2^{p/2} \pi^{-1/2} \Gamma\left(\frac{p+1}{2}\right) \).

Inserting these results in [8 Appendix A], it is easy to show that a matrix \( \Phi \sim \mathcal{N}^{M \times N}(0, 1) \) is RIP\(_{p,w}(K, \delta, \mu) \) with a probability higher than \( 1 - \eta \) if

\[
M^{2/\max(2,p)} \geq c \left( \frac{\rho_{\infty}^p}{\rho_2^p} \right)^2 \left( K \log[eN/(1 + 12\delta^{-1})] + \log \frac{2}{\eta} \right),
\]

for some constant \( c > 0. \)
APPENDIX G

DEQUANTIZING RECONSTRUCTION ERROR

Proof: From Theorem 1 it suffices to bound \(\epsilon_p/\mathbb{E}[\|\mathbf{e}\|_{p,w}]\), with \(\mathbf{e} \sim \mathcal{N}(0,1)\), when \(M\) is large and under the HRA. First, according to Lemma 5, using the Strong Law of Large Number and using the same decomposition than in the proof of Lemma 3 with the threshold \(T(B)\) (with \(\beta = (p + 1)/(p + 2)\)) and the bounds provided by Lemma 2, we find

\[
\mu^p := (\mathbb{E}[\|\mathbf{e}\|_{p,w}]^p)^p \simeq M^{3/2} \sum_{i=1}^{M} [g'(Q_p[z_i])]^{p-2} \mathbb{E}[g]^p \\
\simeq M \mathbb{E}[g]^p \sum_{k:k \geq 0} p_k [g'(\omega_{k,p})]^{p-2}.
\]

The sum in the last expression is characterized by Lemma 8 by setting inside (33) \(n = 0\) and \(\gamma = p - 2\). This provides

\[
\frac{\mu_p}{M_B} \simeq M \mathbb{E}[g]^p \int_{\mathbb{R}} [g'(t)]^{p-2} \phi_0(t) \, dt \\
\simeq M_B \mathbb{E}[g]^p \left[ \int_{\mathbb{R}} \phi_0^{1/3}(t) \right]^{2-p} \left[ \int_{\mathbb{R}} \phi_0^{(p+1)/3}(t) \, dt \right].
\]

Therefore, using the value \(\epsilon_p\) defined in Lemma 3

\[
\epsilon_{p}^p \simeq B_{,M} 2^{-p(B+1)} \mathbb{E}[g]^p \|\phi_0\|_{1/3}^{(p+1)/3} \|\phi_0\|_{(p+1)/3}^{(p+1)/3}
\]

However, for \(\alpha > 0\),

\[
\|\phi_0\|_{\alpha}^\alpha := \int_{\mathbb{R}} \phi_0^\alpha(t) \, dt = (2\pi \sigma_0^2)^{-\alpha/2} (2\pi \sigma_0^2)^{1/2} \int_{\mathbb{R}} \gamma_0 \gamma_{0,\sigma_e} \phi_0(t) \, dt = (2\pi \sigma_0^2)^{(1-\alpha)/2} \sqrt{\alpha}.
\]

Consequently, \(\|\phi_0\|_{1/3}^{(p+1)/3} = 3^{(p+1)/2} (2\pi \sigma_0^2)^{(p+1)/3}\) and \(\|\phi_0\|_{(p+1)/3}^{(p+1)/3} = (2\pi \sigma_0^2)^{(2-p)/6} \sqrt{(p+1)/3}\), so that

\[
\frac{\epsilon_p^p}{\mu_B} \simeq B_{,M} 2^{-p(B+1)} \mathbb{E}[g]^p (6\pi \sigma_0^2)^{p/2}
\]

Knowing that \(\mathbb{E}[g]^p \simeq c \sqrt{p + 1}\) with \(c = 8\sqrt{2}/(9\sqrt{\pi})\), we get

\[
\frac{\epsilon_p}{\mu_B} \lesssim B_{,M} \epsilon' 2^{-(p+1)} - \frac{2-B}{\sqrt{p+1}} \leq c' 2^{-B} \sqrt{p+1}.
\]

with \(c' = (9/8)(e\pi/3)^{1/2}\). The rest of the proof consists simply in injecting this result into (21) noting that \(A_p \simeq M 2\) and \(B_p \simeq M 4\).

APPENDIX H

COMPUTATION OF THE \(\omega_{k,p}\)

This section describes a numerical procedure for efficiently computing the \(p\)-optimal levels \(\omega_{k,p}\) of a Gaussian source \(\mathcal{N}(0,1)\) for integer \(p \geq 2\), defined by \(\omega_{k,p} := \argmin_{\lambda \in \mathbb{R}_+} E_{k,p}(\lambda)\), where \(E_{k,p}(\lambda) = \int_{t_k^{k+1}} |t - \lambda| \gamma_{0,1}(t) \, dt\). As \(E_{k,p}(\lambda)\) is strictly convex and differentiable, the desired \(\omega_{k,p}\) are the unique stationary points satisfying \(E'_{k,p}(\omega_{k,p}) = 0\).

A first approach would be to apply Newton method using first and second order derivatives of \(E_{k,p}\), which can be computed efficiently through analytical recursion relations, see e.g., [27]. However, while
this approach was appealing theoretically, the recursion relations turn out to be highly unstable in practice, and the resulting numerical method for \( \omega_{k,p} \) was numerically unreliable as soon as \( p \geq 8 \).

Instead, we compute the \( \omega_{k,p} \) by Newton method, using standard numerical quadrature for \( \mathcal{E}_{k,p} \) and \( \mathcal{E}_{k,p}'' \). We handle the semi-infinite bins by replacing \( t_1 = -\infty \) and \( t_9 = \infty \) by -39 and +39, respectively (chosen as the smallest integer \( x \) so that \( \gamma_{0,1}(x) = 0 \) when evaluated in double precision floating point arithmetic). Given quadrature weights \( c_i \), we approximate \( \mathcal{E}_{k,p} \) by 
\[
\hat{\mathcal{E}}_{k,p}(\lambda) = \sum_{i=1}^{N} c_i \gamma_{0,1}(x_i)|x_i - \lambda|^p, 
\]
with \( x_i = t_k + (i - 1)\Delta x \), where \( \Delta x = (t_{k+1} - t_k)/(N - 1) \). We then have 
\[
\hat{\mathcal{E}}_{k,p}(\lambda) = \frac{1}{\Delta x} \sum_{i=1}^{N} c_i \gamma_{0,1}(x_i)p|x_i - \lambda|^{p-1}\text{sign}(x_i - \lambda)
\]
and \( \hat{\mathcal{E}}_{k,p}''(\lambda) = \frac{1}{\Delta x^2} \sum_{i=1}^{N} c_i \gamma_{0,1}(x_i)p(p-1)|x_i - \lambda|^{p-2} \). We initialize with the midpoint for each of the finite bins, i.e., set \( \lambda_k^{(0)} = (t_k + t_{k+1})/2 \) for \( 2 \leq k \leq B - 1 \), and \( \lambda_1^{(0)} = t_2, \lambda_B^{(0)} = t_{B+1} \) for the semi-infinite bins. For each \( k \) we then iterate the Newton step 
\[
\lambda_k^{(n)} = \lambda_k^{(n-1)} - \frac{\hat{\mathcal{E}}_{k,p}'(\lambda_k^{(n-1)})}{\hat{\mathcal{E}}_{k,p}''(\lambda_k^{(n-1)})}
\]
until the convergence criterion \( |(\lambda_k^{(n)} - \lambda_k^{(n-1)})/\lambda_k^{(n)}| < 10^{-15} \) is met. We used \( c_i \) given by the fourth-order accurate Simpson’s rule, e.g., \( e = (1, 4, 2, 4, 2, 4, 1) \Delta x/3 \), which yielded empirically observed \( O(N^{-4}) \) convergence of the calculated \( w_{k,p} \). Results in this paper employed \( N = 10^4 + 1 \) quadrature points, sufficient to yield \( w_{k,p} \) accurate to machine precision.

REFERENCES