TV–L2 Refractive Index Map Reconstruction from Polar Domain Deflectometry

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OUTLINE

- Refractive Deflectometry
- Inverse Problem Formulation
- Prior conditions
- Image Reconstruction
- Results
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Deflectometry Framework

Problem:
• Reconstructing the refractive index map of transparent materials from light deflection measurements under multiple orientations.

Interests:
• (Transparent) Object surface topology.

Schlieren Deflectometry:
• “Coding light deviation in intensity variations”.

Multifocal lenses
Mathematical Model
**Mathematical Model**

First order approximation

\[ \Delta(\tau, \theta) \simeq \int_{\mathbb{R}^2} (\nabla n(\mathbf{x}) \cdot \mathbf{p}_\theta) \delta(\tau - \mathbf{x} \cdot \mathbf{p}_\theta) \, d^2 \mathbf{x} \]
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Continuous Facts

- Sensing Model

\[ \Delta(\tau, \theta) \simeq \int_{\mathbb{R}^2} (\nabla n(x) \cdot p_\theta) \delta(\tau - x \cdot p_\theta) \, d^2x \]

- Deflectometric Central Slice Theorem

\[ y(\omega, \theta) := \int_{\mathbb{R}} \Delta(\tau, \theta) e^{-i\tau \omega} \, d\tau = i\omega \hat{n}(\omega \cdot p_\theta) \]

\[ y(\omega, \theta) = i\omega \hat{n}(\omega \cdot p_\theta) \]

2-D FT of \( n \).
Sensing Operator I

\[ y(\omega, \theta) = i\omega \hat{n}(\omega p\theta) \]

Polar FT of \( n \).

Non-Equispaced FFT (NFFT) \(^\dagger\)\(^\ddagger\)

- Fast Fourier Transform for nonequispaced data defined in the Polar grid.
- Possibility to control the interpolation error.

\[ Q \in \mathbb{C}^{M \times N} \]

\( \sigma \) oversampling

m window

\(^\dagger\) M. Fenn, S. Kunis and D. Potts. (2007)

\(^\ddagger\) D. Potts and G. Steidl. (2000)
Sensing Operator II

\[ y(\omega, \theta) = i \omega \hat{n}(\omega p_{\theta}) \]

\[ D = i \text{diag}(\omega_1, \ldots, \omega_M) \in i \mathbb{R}^{M \times M} \]

\[ y(\omega, \theta) = i \omega \hat{n}(\omega p_{\theta}) \]

\[ \Phi = DQ \in \mathbb{C}^{M \times N} \]

\[ y = \Phi n + \eta \]

\[ M (N_{\theta}) < N \quad \text{ill-posed problem} \]

Observation + NFFT interpolation noise
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Sparsity prior

• Heterogeneous transparent materials with slowly varying refractive index separated by sharp interfaces.

TV and BV promote the perfect “cartoon shape” model.

“Sparse” gradient

Small Total Variation norm

\[ \| n \|_{TV} := \| \nabla n \|_{2,1} \]
Other priors

- Positive RIM
  \[ n \geq 0 \]

- The object is completely contained in the image. Pixels in the border are set to zero in order to guarantee uniqueness of the solution.
  \[ n|_{\delta \Omega} = 0 \]
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Filtered Back Projection (FBP)

\[ y = \Phi n + \eta \]

(In our notation) FBP attempts to solve

\[
\hat{n}_{est} = \left[ \arg \min_{n \in \mathbb{R}^N} \| n \|_2 \quad \text{s.t.} \quad y = \Phi n \right] \equiv \Phi^\dagger y
\]

Pseudo-inverse Operator

\[ \Phi^\dagger := \Phi^* \left( \Phi \Phi^* \right)^{-1} \]

- Does not take into account the noise in the measurements.
- Does not take into account the lack of measurements (ill-posed problem).
TV-L2 Minimization

Using the a priori information and imposing L2 data fidelity, TV-L2 attempts to solve

\[ n_{est} = \arg \min_{n \in \mathbb{R}^N} \|n\|_{TV} \quad \text{s.t.} \quad \|y - \Phi n\|_2 \leq \varepsilon, \quad n \geq 0, \quad n|_{\delta\Omega} = 0 \]

\[ n_{est} = \arg \min_{n \in \mathbb{R}^N} \|\nabla n\|_{2,1} + \nu_c(\Phi n) + \nu_{P_0}(n) \]

\( \nu_c(v) \) and \( \nu_{P_0}(v) \) are the indicator functions into the following convex sets

\[ \mathcal{C} = \{v \in \mathbb{C}^M : \|y - v\| \leq \varepsilon\} \]

\[ \mathcal{P}_0 = \{v \in \mathbb{R}^N : v_i \geq 0 \text{ if } i \in \text{int } \Omega; \quad v_i = 0 \text{ if } i \in \delta\Omega\} \]
Reconstruction Algorithm

\[ n_{est} = \arg \min_{n \in \mathbb{R}^N} \| \nabla n \|_{2,1} + \nu_C(\Phi n) + \nu_{P_0}(n) \]

Two mapping operators \( \Phi \) and \( \nabla \), where \( \Phi \Phi^* \neq I \).

Iterative Chambolle-Pock (CP)† Algorithm

\[ x_{est} = \arg \min_{x} F(Kx) + G(x) \]

Product Space Optimization

\[ x_{est} = \arg \min_{x} (F_1 F_2) \left( \begin{array}{cc} K_1 & 0 \\ 0 & K_2 \end{array} \right) \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) + H(x_1) + \nu_\pi(x_1, x_2) \]

† A. Chambolle and T. Pock. (2011)
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Deflectometry vs. Tomography I

\[ \theta = 60^\circ \]

Deflectometry

Tomography
Deflectometry vs. Tomography II

- Without noise, MSNR = 20dB

- RSNR [dB]
- $N_\theta/360$ [%]

Graphs showing comparison between Deflectometry and Tomography under different conditions.
TV-L2 vs. FBP

- No measurement noise.
- $N_\theta/360 = 100\%$.

FBP: 21.28dB

TV-L2: 68.54dB
TV-L2 vs. FBP II

- No measurement noise.
- $N_\theta/360 = 5\%$.

FBP

TV-L2

4.78dB

57.59dB
TV-L2 vs. FBP III

- $\text{MSNR} = 10\text{dB}$.
- $N_\theta/360 = 5\%$. 

FBP:

TV-L2:

- $16.84\text{dB}$
- $4.62\text{dB}$
TV-L2 vs. FBP: Noise Robustness

Without Noise

\[ \frac{N_0}{360} \% \]

\[ \text{RSNR [dB]} \]

\[ \text{MSNR} = 10\text{dB} \]

\[ \text{Without Noise} \]

- TV-L2
- FBP
Experimental Images

- Bundle of 10 fibers immersed in an optical fluid.
- $N_\theta = 60 \rightarrow N_\theta / 360 = 17\%$. 

**FBP**

**TV-L2**
Conclusion and future work

- Sparse-based reconstruction methods can be applied to Optical Deflectometry.

- Future Improvements:
  - Precondition mapping operators.
  - Light trajectory
    - Iterative estimation of the actual light ray trajectory to remove the effects of the paraxial approximation.
    - Extension to some non-linear problems.
  - Same framework valid for Phase-Contrast X-Ray Tomography, no first order approximation is needed.
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