

A Quantized Johnson Lindenstrauss Lemma: The Finding of Buffon's Needle

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November 21, 2013

Abstract

In 1733, Georges-Louis Leclerc, Comte de Buffon in France, set the ground of geometric probability theory by defining an enlightening problem: What is the probability that a needle thrown randomly on a ground made of equispaced parallel strips lies on two of them? In this work, we show that the solution to this problem, and its generalization to N dimensions, allows us to discover a quantized form of the Johnson-Lindenstrauss (JL) Lemma, *i.e.*, one that combines a linear dimensionality reduction procedure with a uniform quantization of precision $\delta > 0$. In particular, given a finite set $\mathcal{S} \subset \mathbb{R}^N$ of S points and a distortion level $\epsilon > 0$, as soon as $M > M_0 = O(\epsilon^{-2} \log S)$, we can (randomly) construct a mapping from (\mathcal{S}, ℓ_2) to $((\delta\mathbb{Z})^M, \ell_1)$ that approximately preserves the pairwise distances between the points of \mathcal{S} . Interestingly, compared to the common JL Lemma, the mapping is quasi-isometric and we observe both an additive and a multiplicative distortions on the embedded distances. These two distortions, however, decay as $O(\sqrt{\log S/M})$ when M increases. Moreover, for coarse quantization, *i.e.*, for high δ compared to the set radius, the distortion is mainly additive, while for small δ we tend to a Lipschitz isometric embedding. Finally, we show that there exists “almost” a quasi-isometric embedding of (\mathcal{S}, ℓ_2) in $((\delta\mathbb{Z})^M, \ell_2)$. This one involves a non-linear distortion of the ℓ_2 -distance in \mathcal{S} that vanishes for distant points in this set. Noticeably, the additive distortion in this case is slower and decays as $O((\log S/M)^{1/4})$.

1 Introduction

The Lemma of Johnson-Lindenstrauss (JL) [1] is a corner stone of (linear) dimensionality reduction techniques. This result, that can be seen as a direct consequence of measure concentration phenomenon [2], is at the heart of many applications in classical search methods for approximate nearest neighbors [3], high dimensional machine learning [4, 5], and in compressed sensing theory [6, 7].

In short, this lemma states that given a finite set of S points in a N -dimensional space, provided that M scales like $O(\epsilon^{-2} \log S)$ for some allowed distortion level $\epsilon > 0$, there exists a mapping projecting the elements of this set into a smaller M -dimensional space, and this does not disturb the pairwise distances of these points by more than a factor $(1 \pm \epsilon)$.

Mathematically, the classical formulation of this important lemma is as follows.

Lemma 1 (Johnson-Lindenstrauss). *Given $\epsilon \in (0, 1)$, for every set \mathcal{S} of S points in \mathbb{R}^N , if M is such that*

$$M > M_0 = O(\epsilon^{-2} \log S),$$

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then, there exists a Lipschitz mapping $\mathbf{f} : \mathbb{R}^N \rightarrow \mathbb{R}^M$ such that

$$(1 - \epsilon) \|\mathbf{u} - \mathbf{v}\|^2 \leq \|\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{v})\|^2 \leq (1 + \epsilon) \|\mathbf{u} - \mathbf{v}\|^2, \quad (1)$$

for all $\mathbf{u}, \mathbf{v} \in \mathcal{S}$.

Beyond this proof of existence, the construction of (random) Lipschitz mappings from \mathbb{R}^N to \mathbb{R}^M satisfying (1) is easy [2]. In particular, for

$$\mathbf{f}(\mathbf{u}) = \Phi \mathbf{u},$$

where $\Phi \in \mathbb{R}^{M \times N}$ is a certain random matrix (*e.g.*, whose independent entries follow identical Gaussian, Bernoulli or sub-Gaussian distributions [24]), measure concentration theory guarantees that [6]

$$\mathbb{P} [|\|\Phi(\mathbf{u} - \mathbf{v})\|^2 - \|\mathbf{u} - \mathbf{v}\|^2| \geq \epsilon \|\mathbf{u} - \mathbf{v}\|^2] \leq 2e^{-M\eta(\epsilon)},$$

where the probability is related to the generation of Φ and η is a nondecreasing function of $\epsilon \in (0, 1)$. For instance, for $\Phi \sim \mathcal{N}^{M \times N}(0, 1/M)$, *i.e.*, $\Phi \in \mathbb{R}^{M \times N}$ with $\Phi_{ij} \sim_{\text{iid}} \mathcal{N}(0, 1/M)$, $\eta(\epsilon) = \epsilon^2/2 - \epsilon^3/6 \geq \epsilon^2/3$ [6].

Proving the JL Lemma amounts then to applying a union bound on all possible pair of points \mathbf{u} and \mathbf{v} taken in \mathcal{S} . Since there is no more than $\binom{S}{2} \leq S^2/2$ such pairs, the probability that at least one pair fails to respect (1) is bounded by $2\binom{S}{2}e^{-M\eta(\epsilon)} \leq e^{2\log S - M\eta(\epsilon)}$. Therefore, as soon as $M > 2\eta(\epsilon)^{-1} \log S$, this probability can be made arbitrarily low. Moreover, generating a sequence of Φ decreases further this probability by hoping that at least one such matrix respects (1), which in the limit ensures the existence of \mathbf{f} with probability 1 in Prop. 2.

Combining such linear random mappings with a quantization procedure \mathcal{Q} (*e.g.*, uniform or non-uniform) has been recently a matter of intense research. The implicit objective of this association is to reduce the amount of bits required to encode the result of the dimensionality reduction [8], and to understand the impact of quantization on the mapping distortion. For instance, the field of 1-bit Compressed Sensing is interested in reconstructing sparse vectors from the sign of their random projections [9–13]. At the heart of this topic lies the extreme “one-bit” (or binary) mapping $\psi_{\text{bin}} : \mathbb{R}^N \rightarrow \mathcal{B}^M$ with $\mathcal{B} = \{\pm 1\}$ with

$$\psi_{\text{bin}}(\mathbf{u}) = \text{sign}(\Phi \mathbf{u})$$

for a Gaussian random matrix $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$. Thanks to ψ_{bin} , a set of vectors of \mathbb{R}^N can be mapped with a subset of the *Boolean cube* \mathcal{B}^M . For characterizing the distortion introduced by such mapping, we must define new distances: the normalized Hamming distance $d_H(\mathbf{r}, \mathbf{s}) = \frac{1}{M} \sum_i \mathbb{I}(r_i \neq s_i)$ defined for two *binary strings* $\mathbf{r}, \mathbf{s} \in \mathcal{B}^M$ and the angular distance $d_S(\mathbf{u}, \mathbf{v}) = \|\mathbf{u}\|^{-1} \|\mathbf{v}\|^{-1} \arccos \langle \mathbf{u}, \mathbf{v} \rangle$ between two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^N$. The use of d_S stems from the vector amplitude loss in the definition of ψ_{bin} . Within such a context, the following result is known (its proof is sketched in Sec. 4).

Proposition 1 ([11, 14]). *Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^N$. Fix $\epsilon > 0$ and randomly generate $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$. Then we have*

$$\mathbb{P} (|d_H(\psi_{\text{bin}}(\mathbf{u}), \psi_{\text{bin}}(\mathbf{v})) - d_S(\mathbf{u}, \mathbf{v})| \leq \epsilon) \geq 1 - 2e^{-2\epsilon^2 M}, \quad (2)$$

where the probability is with respect to the generation of Φ .

Following again a union bound argument on all pairs of a set $\mathcal{S} \subset \mathbb{R}^N$ of size S , for a fixed $\epsilon > 0$ and given $M > M_0 = O(\epsilon^{-2} \log S)$, Prop. 1 induces a certain embedding of $(\mathcal{S} \subset \mathbb{R}^N, d_S)$ in (\mathcal{B}^M, d_H) where

$$d_S(\mathbf{x}, \mathbf{s}) - \epsilon \leq d_H(\psi_{\text{bin}}(\mathbf{x}), \psi_{\text{bin}}(\mathbf{s})) \leq d_S(\mathbf{x}, \mathbf{s}) + \epsilon, \quad \forall \mathbf{u}, \mathbf{v} \in \mathcal{S}, \quad (3)$$

with high probability.

We directly notice two striking differences with the classical formulation of the JL Lemma: the use of new distance definitions of course, but more importantly, the error (ϵ) is no more multiplicative, it is now additive with respect to the angular distance d_S .

Actually, (3) shows that 1-bit quantization breaks the *isometric* property of random linear mappings. These actually become *quasi-isometric* between the metric spaces (\mathcal{S}, d_S) and $(\psi_{\text{bin}}(\mathcal{S}) \subset \mathcal{B}^M, d_H)$ in the following sense.

Definition 1 ([15]). *A function $\mathbf{h} : \mathcal{X} \rightarrow \mathcal{Y}$ is called a quasi-isometry between metric spaces (\mathcal{X}, d_X) and (\mathcal{Y}, d_Y) if there exists $C > 0$ and $D \geq 0$ such that*

$$\frac{1}{C} d_X(\mathbf{x}, \mathbf{s}) - D \leq d_Y(\mathbf{h}(\mathbf{x}), \mathbf{h}(\mathbf{s})) \leq C d_X(\mathbf{x}, \mathbf{s}) + D,$$

for $\mathbf{x}, \mathbf{s} \in \mathcal{X}$, and $E > 0$ such that $d_Y(y, \mathbf{h}(\mathbf{x})) < E$ for all $y \in \mathcal{Y}$.

This paper aims at going beyond the aforementioned binary case. We want to characterize the impact of a more general uniform quantization \mathcal{Q} of precision (or *bin width*) $\delta > 0$ on the properties of a linear dimensionality reduction procedure. In particular, our objective is to find a mapping $\psi : \mathbb{R}^N \rightarrow \mathbb{Z}_\delta^M$ with $\mathbb{Z}_\delta := \delta\mathbb{Z}$, combining a linear random projection from \mathbb{R}^N to \mathbb{R}^M with a certain $\mathcal{Q} : \mathbb{R}^M \rightarrow \mathbb{Z}_\delta^M$, for which the following general quasi-isometric relation is satisfied:

$$(1 - \epsilon)d(\mathbf{u}, \mathbf{v}) - \epsilon' \leq d'(\psi(\mathbf{u}), \psi(\mathbf{v})) \leq (1 + \epsilon)d(\mathbf{u}, \mathbf{v}) + \epsilon',$$

for all $\mathbf{u}, \mathbf{v} \in \mathcal{S}$, for some distances d and d' , and with $\epsilon, \epsilon' > 0$ decreasing with δ or M . This would generalize nicely the JL Lemma by also showing that, despite its quasi-isometric nature, the mapping is tighter when the dimensionality M increases, or that it is nearly isometric when δ vanishes.

As it will become clear in Sec. 3, we answer positively to this quest when d and d' are the ℓ_2 and ℓ_1 distances, respectively, and for $\epsilon \propto \epsilon'$. Our main result is as follows.

Proposition 2. *Let $\mathcal{S} \subset \mathbb{R}^N$ be a set of S points. Fix $0 < \epsilon < 1$ and $\delta > 0$. For $M > M_0 = O(\epsilon^{-2} \log S)$, there exist a non-linear mapping $\psi : \mathbb{R}^N \rightarrow \mathbb{Z}_\delta^M$ and two constants $c, c' > 0$ such that, for all pairs $\mathbf{u}, \mathbf{v} \in \mathcal{S}$,*

$$(1 - \epsilon) \|\mathbf{u} - \mathbf{v}\| - c\delta\epsilon \leq \frac{c'}{M} \|\psi(\mathbf{u}) - \psi(\mathbf{v})\|_1 \leq (1 + \epsilon) \|\mathbf{u} - \mathbf{v}\| + c\delta\epsilon. \quad (4)$$

In particular, this shows that there exists a quasi-isometric mapping between $(\mathcal{S} \subset \mathbb{R}^N, \ell_2)$ and $(\psi(\mathcal{S}) \subset \mathbb{Z}_\delta^M, \ell_1)$ with constants $D = c\delta\epsilon$, $C = 1/(1 - \epsilon) \geq 1 + \epsilon$ for $0 \leq \epsilon < 1$, and finite E in Def. 1. In the rest of this paper, we will forget these subtleties and abusively say that a relation such as (4) defines a quasi-isometric mapping between (\mathcal{S}, ℓ_2) and $(\mathbb{Z}_\delta^M, \ell_1)$, or equivalently, a ℓ_2/ℓ_1 quasi-isometric embedding of \mathcal{S} in \mathbb{Z}_δ^M .

We clearly see in (4) the two expected distortions: one additive of amplitude $c\delta\epsilon$ and the other multiplicative associated to an error factor $(1 \pm \epsilon)$. The additive distortion vanishes if δ tends to zero (the other not). Moreover, by inverting the relation between M and ϵ , we observe that both errors decay as $O(\sqrt{\log S/M})$ if M increases. In the case of an infinitely fine quantization ($\delta \rightarrow 0$) we recover also classical embedding results of (\mathbb{R}^N, ℓ_2) in (\mathbb{R}^M, ℓ_1) associated to measure concentration in Banach spaces [16, 17] (See Sec. 4).

Notice that Prop. 2 generalizes somehow the result obtained in [8] for universal binary schemes [10], *i.e.*, when the 1-bit quantizer is *non-regular* and has discontinuous quantization regions. The reason for this is that, despite its regularity, our quantizer can be seen as a B -bit

uniform quantizer where B should be related to $\log_2(\max_{j, \mathbf{u} \in \mathcal{S}} |(\psi(\mathbf{u}))_j|/\delta)$. Therefore we show here that the behavior of the additive distortion of binary quantized mappings discovered in [8] is also valid at a higher number of bits.

For reasons that will become clear later, the context that makes Prop. 2 possible was already defined in 1733 by Georges-Louis Leclerc, Comte de Buffon in France. In one of the volumes of his impressive work entitled “L’Histoire Naturelle¹”, this French naturalist stated and solved the following important problem [18]:

[English translation of Fig. 1(a) from [19]] “*I suppose that in a room where the floor is simply divided by parallel joints one throws a stick (N/A : later called “needle”) in the air, and that one of the players bets that the stick will not cross any of the parallels on the floor, and that the other in contrast bets that the stick will cross some of these parallels; one asks for the chances of these two players.*”

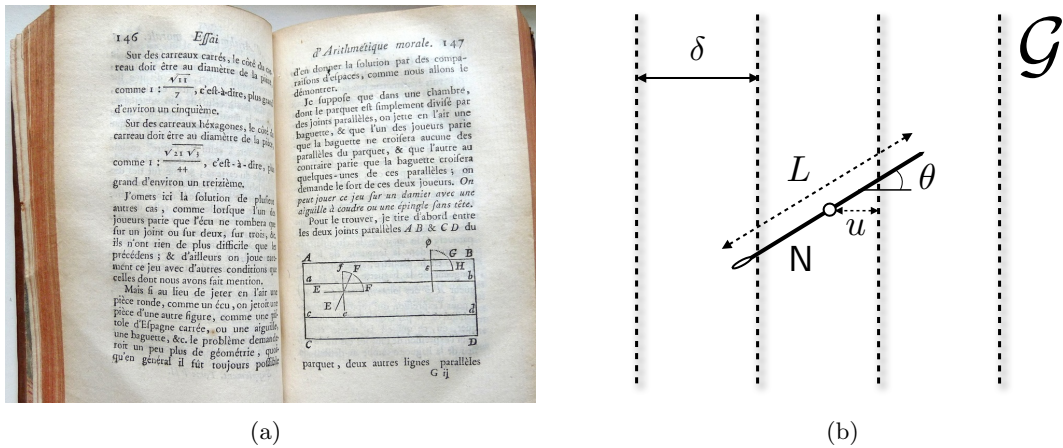


Figure 1: (a) Picture of [18, page 147] stating the initial formulation of Buffon’s needle problem (Courtesy of E. Kowalski’s blog <http://blogs.ethz.ch/kowalski/2008/09/25/buffons-needle>). (b) Scheme of Buffon’s needle problem

As explained in Sec. 2.1, the solution is astonishingly simple: for a short needle compared to the separation δ between two consecutive parallels (see Fig. 1(b)), the probability of having one intersection between the needle and the parallels is equal to the needle length times $\frac{2}{\delta\pi}$. If the needle is longer, then this probability is less easy to express but the expectation of the number of intersections (which can now be bigger than one) remains equal to this value.

This problem, and its solution published in 1777 [18], is considered as the beginning of the discipline called “geometrical probability” [20]. Moreover, the solution has also shed new light on the estimation of π , *i.e.*, by estimating the probability of intersection on a large number of throws, paving the way to the well-known stochastic (Monte Carlo) estimation methods.

In this paper, we are going to show that the analysis of Buffon’s problem, and its generalization to a N -dimensional space, allows us to specify the conditions surrounding Prop. 2. As explained in Sec. 3, the connection between the existence of a quantized embedding and Buffon’s problem is simple. Forgetting a few technicalities detailed later, *uniformly quantizing the random projections in \mathbb{R}^M of two points in \mathbb{R}^N and measuring the difference between their quantized values is fully equivalent to study the number of intersections made by the segment determined*

¹<http://www.buffon.cnrs.fr/?lang=en>

by those two points (seen as a Buffon's needle) with a parallel grid of $(N - 1)$ -dimensional hyperplanes.

As an aside to proving Prop. 2, this paper provides also, to the best of our knowledge, new results on the behavior of Buffon's needle problem in high dimensional space. For instance, we establish a few interesting bounds and asymptotic relations concerning the moments of the random variable counting the needle/grid intersections (see Sec. 2.2).

The rest of the paper is organized as follows. For the sake of clarity, Buffon's needle initial problem and its solution are first explained in Sec. 2.1 before its N -dimensional generalization developed in Sec. 2.2. The relation between this problem and the existence of a ℓ_2/ℓ_1 quantized embedding of $\mathcal{S} \subset \mathbb{R}^N$ in \mathbb{Z}_δ^M is then provided in Sec. 3. Our main result is then discussed in Sec. 4. Finally, before to conclude, we provide in Sec. 5 an extension of our analysis that provides "almost" a quasi-isometric embedding of (\mathcal{S}, ℓ_2) in $(\mathbb{Z}_\delta^M, \ell_2)$. This one must be considered with a non-linear distortion of the ℓ_2 -distance in \mathcal{S} that vanishes for large pairwise distances in this set. Noticeably, the additive distortion in this mapping decays more slowly with M , *i.e.*, as $O((\log S/M)^{1/4})$.

Conventions: Most of domain dimensions are denoted by capital roman letters, *e.g.*, M, N, \dots . Vectors, vector functions and matrices are associated to bold symbols while lowercase light letters are associated to scalar values, *e.g.*, $\Phi \in \mathbb{R}^{M \times N}$ or $\mathbf{u} \in \mathbb{R}^M$. The i^{th} component of a vector (or of a vector function) \mathbf{u} reads either u_i or $(\mathbf{u})_i$, while the vector \mathbf{u}_i may refer to the i^{th} element of a set of vectors. The set of indices in \mathbb{R}^D is $[D] = \{1, \dots, D\}$. Scalar product between two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^D$ for some dimension $D \in \mathbb{N}$ is denoted equivalently by $\mathbf{u}^T \mathbf{v} = \mathbf{u} \cdot \mathbf{v} = \langle \mathbf{u}, \mathbf{v} \rangle$. For any $p \geq 1$, the ℓ_p -norm of \mathbf{u} is $\|\mathbf{u}\|_p^p = \sum_i |u_i|^p$ with $\|\cdot\| = \|\cdot\|_2$. We will abuse the notation " ℓ_p " to either denote the ℓ_p -norm as above or the ℓ_p -distance (or metric) between two points $\mathbf{u}, \mathbf{v} \in \mathbb{R}^N$ defined by $\|\mathbf{u} - \mathbf{v}\|_p$ (*e.g.*, for defining a metric space $(\mathcal{X} \subset \mathbb{R}^D, \ell_p)$). The event indicator function \mathbb{I} is defined as $\mathbb{I}(A) = 1$ if A is verified and 0 otherwise. A uniform random variable over $\mathcal{I} \subset \mathbb{R}$ has a distribution $\mathcal{U}(\mathcal{I})$. A random matrix $\Phi \sim \mathcal{D}^{M \times N}(\Theta)$ is a $M \times N$ matrix with entries distributed as $\Phi_{ij} \sim_{\text{iid}} \mathcal{D}(\Theta)$ given the distribution parameters Θ of \mathcal{D} (*e.g.*, $\mathcal{N}^{M \times N}(0, 1)$ or $\mathcal{U}^{M \times N}([0, 1])$). A random vector in \mathbb{R}^M following $\mathcal{D}(\Theta)$ is defined by $\mathbf{v} \sim \mathcal{D}^M(\Theta)$. Given two random variables X and Y , the notation $X \sim Y$ means that X and Y have the same distribution. The probability of an event \mathcal{E} is denoted $\mathbb{P}(\mathcal{E})$. The diameter of a finite set $\mathcal{S} \subset \mathbb{R}^N$ of cardinality $|\mathcal{S}|$ is $\text{diam } \mathcal{S} = \max_{\mathbf{u}, \mathbf{v} \in \mathcal{S}} \|\mathbf{u} - \mathbf{v}\|$ and its radius is $\text{rad } \mathcal{S} = \max_{\mathbf{u} \in \mathcal{S}} \|\mathbf{u}\|$. The $(N - 1)$ -sphere in \mathbb{R}^N is $\mathbb{S}^{N-1} = \{\mathbf{x} \in \mathbb{R}^N : \|\mathbf{x}\| = 1\}$. For asymptotic relations, we use the common Landau family of notations, *i.e.*, the symbols O , Ω and Θ (their exact definition can be found in [21]). The positive thresholding function is defined by $(\lambda)_+ := \frac{1}{2}(\lambda + |\lambda|)$ for any $\lambda \in \mathbb{R}$. Given $\delta > 0$, we write $\mathbb{Z}_\delta := \delta\mathbb{Z} = \{\delta k : k \in \mathbb{Z}\}$.

2 Buffon's needle problem

2.1 Initial formulation and solution

Let us rephrase Buffon's needle problem in a more formal way. Let $\mathcal{G} \subset \mathbb{R}^2$ be a set of equispaced parallel lines in \mathbb{R}^2 , two consecutive line being separated by a distance $\delta > 0$. Let a needle \mathbf{N} of length L thrown uniformly at random on the plane \mathbb{R}^2 : its orientation θ is drawn uniformly at random on the circle $[0, 2\pi]$, while, from the δ -periodicity of \mathcal{G} , the distance u of the needle midpoint to the closest line is a uniform random variable over $[0, \delta/2]$ (see Fig. 1(b)).

The problem amounts then to compute the probability P that $\mathbf{N}(u, \theta) \cap \mathcal{G} \neq \emptyset$. As a matter of fact, this probability is easily estimated since conditionally to the knowledge of θ , there is at

least one intersection if $2u \leq L|\cos\theta|$. Therefore, we find

$$\begin{aligned} P &= \frac{1}{\pi\delta} \int_0^{2\pi} d\theta \int_0^{\delta/2} \mathbb{I}(2\min(u, \delta - u) \leq L|\cos\theta|) du \\ &= \frac{4}{\pi\delta} \int_0^{\pi/2} d\theta \int_0^{\frac{1}{2}\min(\delta, L\cos\theta)} du. \end{aligned} \quad (5)$$

We observe that if $L < \delta$, then $L\cos\theta < \delta$ and $P = \frac{2L}{\pi\delta}$, while if $L \geq \delta$, the solution reads $P = \frac{2}{\pi}\theta_1 + \frac{2L}{\pi\delta}(1 - \sin\theta_1)$ with $\cos\theta_1 = \frac{\delta}{L}$.

Notice that if $L < \delta$, only one intersection is possible and if X denotes the random variable associated to the occurrence of such an intersection, we have therefore $\mathbb{E}X = P = \frac{2L}{\pi\delta}$. Interestingly, for any $L > 0$, this expectation still keeps the same value.

Proposition 3 ([22]). *Let X be the discrete random variable counting the number of intersections of \mathcal{N} with \mathcal{G} , i.e., $X = |\{N(u, \theta) \cap \mathcal{G}\}|$ where u and θ are two random variables defined as above. Then, writing $a = L/\delta$,*

$$0 \leq X \leq [a] + 1 \quad \text{and} \quad \mathbb{E}X = \frac{2}{\pi}a.$$

Proof. We follow the spirit of the proof given in [22]. The domain of X is obvious from the problem definition. For estimating the expectation, let us observe that the needle N can always be considered as being made of two joint needles N_1 and N_2 of lengths L_1 and L_2 ($L_1 + L_2 = L$). If X_1 and X_2 are the random variable counting their respective intersection with \mathcal{G} , we have $X = X_1 + X_2$. Therefore, since $\mathbb{E}X$ necessarily depends on L through some nondecreasing function h , we find $h(L) = \mathbb{E}X = \mathbb{E}(X_1 + X_2) = \mathbb{E}X_1 + \mathbb{E}X_2 = h(L_1) + h(L_2)$. This shows that $h(L) = cL$ for some $c > 0$ independent of L . From the knowledge of $\mathbb{E}X$ for $L < \delta$, we deduce that $c = \frac{2}{\pi\delta}$. \square

Surprisingly enough, this proposition still holds if the needle is replaced by any smooth curve of length L [22]. Indeed, such curve can always be approximated by a piecewise linear contour with arbitrary small error and the proof above does not depend on a possible bending of the N_1 and N_2 . However, the distribution of X does depend on the curve shape.

Let us specify now what is known of the distribution of X in the case of a (straight) needle.

Proposition 4 ([20, pp. 72–73][23]). *Given $a = L/\delta$, define the angles $\theta_k \in [0, \pi/2]$ such that $\cos\theta_k = k/a$ for $0 \leq k \leq n$ with $n = [a]$, $\cos\theta_k = 0$ for $k < 0$ and $\cos\theta_k = 1$ for all $k > n$. The distribution of $X \in [n + 1]$ is determined by the probabilities*

$$p_k = \mathbb{P}(X = k) = \kappa_{k+1} + \kappa_{k-1} - 2\kappa_k, \quad (6)$$

with $\kappa_k = (2a \sin\theta_k/\pi) - (2k\theta_k/\pi)$.

Proof. The proof only differs from the one of [20, pp. 72–73] in notations. For $n = 0$, $p_0 = 1 - P$ with P computed in (5). For θ fixed, the conditional probability of having $n + 1$ intersections reads $a|\cos\theta| - n$ if $\theta \leq \theta_n$ and 0 otherwise. Therefore,

$$p_{n+1} = \frac{2}{\pi} \int_0^{\theta_n} (a \cos\theta - n) d\theta = \frac{2a}{\pi} \sin\theta_n - \frac{2n}{\pi} \theta_n.$$

For $1 \leq k \leq n$, there is k intersections if $\theta_{k+1} \leq \theta \leq \theta_{k-1}$. The conditional probability reads $(k + 1 - a \cos\theta)$ if $\theta_{k+1} \leq \theta \leq \theta_k$, and $(a \cos\theta - k + 1)$ if $\theta_k \leq \theta \leq \theta_{k-1}$. Therefore,

$$\begin{aligned} p_k &= \frac{2}{\pi} \int_{\theta_{k+1}}^{\theta_k} (k + 1 - a \cos\theta) d\theta + \frac{2}{\pi} \int_{\theta_k}^{\theta_{k-1}} (a \cos\theta - k + 1) d\theta \\ &= \frac{2a}{\pi} (\sin\theta_{k+1} + \sin\theta_{k-1} - 2\sin\theta_k) - \frac{2}{\pi} ((k + 1)\theta_{k+1} + (k - 1)\theta_{k-1} - 2k\theta_k). \end{aligned}$$

The rest of the proof consists in expressing these results in terms of κ_k . \square

We could continue to analyze other properties of the random variable X (e.g., characterizing its moments) but we postpone this study to the following multidimensional analysis.

2.2 N -dimensional generalization

How does Buffon's needle problem generalize in a N -dimensional space? More precisely, what phenomena do we observe on the "random throw"² of a 1-dimensional needle \mathbf{N} of length L on a infinite set \mathcal{G} of equispaced parallel hyperplanes of dimension $N - 1$ separated by a distance $\delta > 0$?

In N dimensions, the position of the needle relatively to \mathcal{G} can again be determined by its distance $u \in [0, \delta/2]$ to the closest hyperplane of \mathcal{G} , while its orientation can be characterized by a set of $(N - 1)$ angles $\{\theta, \phi_1, \phi_2, \dots, \phi_{N-3}\}$ on \mathbb{S}^{N-1} . These include the angle $\theta \in [0, \pi]$ measured between the needle and the normal vector orthogonal to all hyperplanes³, while the others range as $\phi_k \in [0, \pi]$ for $1 \leq k \leq N - 3$ and $\phi_{N-2} \in [0, 2\pi]$. We recall that in this *hyperspherical* system of coordinates, the $(N - 1)$ -sphere \mathbb{S}^{N-1} is measured by

$$\sigma(\mathbb{S}^{N-1}) = \int_0^\pi (\sin \theta)^{N-2} d\theta \left(\int_0^\pi (\sin \phi_1)^{N-3} d\phi_1 \cdots \int_0^\pi \sin \phi_{N-3} d\phi_{N-3} \int_0^{2\pi} d\phi_{N-2} \right),$$

where $\sigma(\cdot)$ denotes the rotationally invariant area measure on the $(N - 1)$ -sphere.

The first question we can ask ourselves is how the expectation of $X = |\mathbf{N} \cap \mathcal{G}|$ evolves in this multidimensional setting. Following the same argumentation than in the previous section, we must still have $\mathbb{E}X \propto a$ but what is now the proportionality factor?

Proposition 5. *In the N -dimensional Buffon's needle problem, the expected number of intersections between the needle and the hyperplanes reads*

$$\mathbb{E}X = \tau_N a, \quad \text{with} \quad \tau_N = \frac{\Gamma(\frac{N}{2})}{\sqrt{\pi} \Gamma(\frac{N+1}{2})}, \quad (7)$$

$$\tau_2 = \frac{2}{\pi} \quad \text{and} \quad \tau_3 = \frac{1}{2}.$$

Proof. As for the proof of Prop. 3, determining τ_N can be done for $L < \delta$ where $\mathbb{E}X$ matches the probability of having one intersection. In this case, following the determination of this probability for the two-dimensional case (Sec. 2.1), we can say that, conditionally to the knowledge of θ and of the $N - 2$ other angles $\{\phi_1, \dots, \phi_{N-2}\}$, there is an intersection if either $2u \leq L|\cos \theta|$. Therefore, defining $I_k := \int_0^\pi (\sin \alpha)^k d\alpha$ with $\sigma(\mathbb{S}^{N-1}) = 2I_0 \cdots I_{N-2}$ and considering the periodicity of $|\cos \theta|$, the probability P_N of having one intersection generalizes as

$$\begin{aligned} P_N &= \frac{2}{\delta \sigma(\mathbb{S}^{N-1})} \int_0^\pi (\sin \theta)^{N-2} d\theta \left(\int_0^\pi (\sin \phi_1)^{N-3} d\phi_1 \cdots \int_0^\pi \sin \phi_{N-3} d\phi_{N-3} \int_0^{2\pi} d\phi_{N-2} \right) \\ &\quad \times \int_0^{\delta/2} \mathbb{I}(2u \leq L|\cos \theta|) du \\ &= \frac{4}{\delta I_{N-2}} \int_0^{\pi/2} (\sin \theta)^{N-2} d\theta \int_0^{\delta/2} \mathbb{I}(2u \leq L \cos \theta) du \\ &= \frac{2a}{I_{N-2}} \int_0^{\pi/2} (\sin \theta)^{N-2} \cos \theta d\theta = \frac{2a}{(N-1)I_{N-2}}. \end{aligned}$$

²Assuming of course that we can throw an object in a N -dimensional space so that it stops in a fixed position of \mathbb{R}^N as it stops on the floor of the 2-dimensional formulation.

³Notice that, conversely to the two-dimensional analysis, this angle θ covers now the half circle $[0, \pi]$, the other angles guaranteeing that all orientations in \mathbb{S}^{N-1} can be obtained.

Since $I_k = \sqrt{\pi} \Gamma(\frac{k+1}{2}) / \Gamma(\frac{k}{2} + 1)$ and $\mathbb{E}X = P_N$ for $a < 1$, we find

$$\tau_N = \frac{2}{(N-1)I_{N-2}} = \frac{2\Gamma(\frac{N}{2})}{\sqrt{\pi}(N-1)\Gamma(\frac{N-1}{2})} = \frac{\Gamma(\frac{N}{2})}{\sqrt{\pi}\Gamma(\frac{N+1}{2})}.$$

The values for τ_2 and τ_3 come from the evaluations $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(3/2) = \sqrt{\pi}/2$. \square

To the best of our knowledge, τ_N was only known for the case $N = 2$ and $N = 3$ (see [20, pp. 70 and 77, respectively]). This quantity behaves as follows.

Proposition 6. *In the N -dimensional Buffon's needle problem,*

$$\frac{\sqrt{2}}{\sqrt{\pi}} (N+1)^{-\frac{1}{2}} \leq \frac{1}{a} \mathbb{E}X = \tau_N \leq \frac{\sqrt{2}}{\sqrt{\pi}} (N-1)^{-\frac{1}{2}}$$

so that $\mathbb{E}X = \Theta(a/\sqrt{N})$.

Proof. This is a direct consequence of the inequality $(\frac{2N-3}{4})^{1/2} \leq \Gamma(\frac{N}{2})/\Gamma(\frac{N-1}{2}) \leq (\frac{N-1}{2})^{1/2}$ and of the fact that $(N - \frac{3}{2})^{1/2}/(N-1) \geq 1/\sqrt{N+1}$ for $N \geq 2$. \square

We find useful to introduce right now the following general quantity which takes the value τ_N as a special case:

$$\chi_N(x) := \frac{\Gamma(x + \frac{1}{2})\Gamma(\frac{N}{2})}{\sqrt{\pi}\Gamma(\frac{N}{2} + x)}. \quad (8)$$

We can compute that $\chi_N(0) = 1$, $\chi_N(\frac{1}{2}) = \tau_N$ and $\chi_N(1) = \frac{1}{N}$. The importance of χ_N , and the notation simplification brought by its introduction, will become clear later.

Having established how the expectation of X behaves, can we go further and generalize the previous 2-dimensional distribution? A positive answer is given in the following proposition.

Proposition 7. *Given $a = L/\delta$ and the angles $\theta_k \in [0, \pi/2]$ defined in Prop. 4. The distribution of $X \in [n+1]$ is again determined by the probabilities*

$$p_k = \mathbb{P}(X = k) = \kappa_{k+1} + \kappa_{k-1} - 2\kappa_k, \quad (9)$$

with $\kappa_k = \tau_N a (\sin \theta_k)^{N-1} - k \tau_N J_N(\theta_k)$ and $J_N(\alpha) := (N-1) \int_0^\alpha (\sin \theta)^{N-2} d\theta$.

The (discrete) distribution determined by such probabilities is denoted $\text{Buffon}(a, N)$.

Proof. The proof consists in considering the hyperspherical coordinates defined in the demonstration of Prop. 4. For $k = 0$, we must only estimate $p_0 = 1 - P_N$ for any value of a . For $a < 1$, we know that $P_N = \tau_N a$, while for $a > 1$,

$$\begin{aligned} P_N &= \frac{4}{I_{N-2}\delta} \int_0^{\pi/2} (\sin \theta)^{N-2} d\theta \int_0^{\frac{1}{2} \min(\delta, L|\cos \theta|)} \mathbb{I}(2u \leq L \cos \theta) du \\ &= \frac{4}{I_{N-2}\delta} \left(\frac{\delta}{2} \int_0^{\theta_1} (\sin \theta)^{N-2} d\theta + \frac{L}{2} \int_{\theta_1}^{\pi/2} (\sin \theta)^{N-2} \cos \theta d\theta \right) \\ &= \tau_N J_N(\theta_1) + \tau_N a (1 - (\sin \theta_1)^{N-1}). \end{aligned}$$

For $k = n+1$, considering θ fixed, the conditional probability of having $n+1$ intersections reads $a|\cos \theta| - n$ if $\theta \leq \theta_n$ and 0 otherwise. Therefore,

$$p_{n+1} = \frac{2}{I_{N-2}} \int_0^{\theta_n} (a \cos \theta - n) (\sin \theta)^{N-2} d\theta = \tau_N a (\sin \theta_n)^{N-1} - \tau_N n J_N(\theta_n).$$

For $1 \leq k \leq n$, there is k intersections if $\theta_{k+1} \leq \theta \leq \theta_{k-1}$. The conditional probability reads $(k+1 - a \cos \theta)$ if $\theta_{k+1} \leq \theta \leq \theta_k$, and $(a \cos \theta - k + 1)$ if $\theta_k \leq \theta \leq \theta_{k-1}$. Therefore,

$$\begin{aligned} p_k &= \frac{2}{I_{N-2}} \int_{\theta_{k+1}}^{\theta_k} (k+1 - a \cos \theta) (\sin \theta)^{N-2} d\theta + \frac{2}{I_{N-2}} \int_{\theta_k}^{\theta_{k-1}} (a \cos \theta - k + 1) (\sin \theta)^{N-2} d\theta \\ &= \tau_N a \left((\sin \theta_{k+1})^{N-1} + (\sin \theta_{k-1})^{N-1} - 2(\sin \theta_k)^{N-1} \right) \\ &\quad - \tau_N \left((k+1) J_N(\theta_{k+1}) + (k-1) J_N(\theta_{k-1}) - 2k J_N(\theta_k) \right). \end{aligned}$$

As for Prop. 4, the rest of the proof consists in expressing these results in terms of κ_k . \square

Notice that, from a simple change of variable, the value κ_k can be conveniently rewritten as

$$\begin{aligned} \kappa_k &= \tau_N a (\sin \theta_k)^{N-1} - k \tau_N (N-1) \int_0^{\theta_k} (\sin \theta)^{N-2} d\theta \\ &= \tau_N (N-1) \int_0^{\theta_k} (\sin \theta)^{N-2} (a \cos \theta - k) d\theta \\ &= \tau_N a (N-1) \int_0^1 (1-u^2)^{\frac{N-3}{2}} \left(u - \frac{k}{a}\right)_+ du. \end{aligned} \quad (10)$$

The following proposition bounds the moments of a Buffon random variable. These will be useful later for developing our ℓ_2/ℓ_1 quantized embedding in Sec. 3.

Proposition 8. *Let $X \sim \text{Buffon}(a, N)$. If $a < 1$, for any $q \in \mathbb{N}_0$, $\mathbb{E}X^q = \tau_N a$. If $a \geq 1$, then $\mathbb{E}X^q \geq \tau_N a$ for any $q \in \mathbb{N}_0$. Moreover, for $a \geq 0$,*

$$\max(\tau_N a, \frac{1}{N} a^2) \leq \mathbb{E}X^2 \leq \tau_N a + \frac{1}{N} (a^2 - 1)_+, \quad (11)$$

and

$$|\mathbb{E}X^3 - (\tau_N a + \chi_N(\frac{3}{2}) a^3)| \leq \frac{3}{N} a^2. \quad (12)$$

For $q \geq 4$ and $a \geq 1$, the bounds are a bit more technical and reads

$$\begin{aligned} &|\mathbb{E}X^q - (\tau_N a + \chi_N(\frac{q}{2}) a^q)| \\ &\leq q \chi_N(\frac{q-1}{2}) a^{q-1} + \frac{1}{24} \binom{q}{2} \chi_N(\frac{q-2}{2}) (2a)^{q-2} + \frac{1}{12} \binom{q}{3} \chi_N(\frac{q-3}{2}) (2a)^{q-3}. \end{aligned} \quad (13)$$

For any $q \geq 2$ and any $a \geq 0$, we have finally the weaker but simple upper bound

$$\mathbb{E}X^q \leq \tau_N a + 2^{q-2} \chi_N(\frac{q}{2}) a^q + 2^{q-2} q \chi_N(\frac{q-1}{2}) a^{q-1}. \quad (14)$$

This last proposition leads to a nice asymptotic relation.

Corollary 1. *For a Buffon random variable $X \sim \text{Buffon}(a, N)$, we have asymptotically in a ,*

$$|\mathbb{E}X^q - \chi_N(\frac{q}{2}) a^q| = O(a^{q-1}).$$

Before delving in the proof of Prop. 8, we must introduce three useful lemmata.

Lemma 2. *For any sequence $\{c_k\}$*

$$\sum_{k=0}^{n+1} c_k p_k = c_0 (\kappa_{-1} - 2\kappa_0) + c_1 \kappa_0 + \sum_{k=1}^n \Delta^2(c_{k-1}) \kappa_k, \quad (15)$$

with the differencing operator Δ such that $\Delta(c_k) = c_{k+1} - c_k$.

Proof. Following [23], this is a simple consequence of the ‘‘summing by parts’’ rule for any sequences a_k and b_k , i.e., $\sum_{k=0}^{n+1} a_j \Delta(b_j) = a_{n+2} b_{n+2} - a_0 b_0 - \sum_{k=0}^{n+1} \Delta(a_k) b_{k+1}$, and the fact that $\sum_{k=0}^{n+1} c_k p_k = \sum_{k=0}^{n+1} c_k \Delta^2(\kappa_{k-1})$. \square

Lemma 3. We can compute that $\Delta^2((k-1)^2) = 2$ and $\Delta^2((k-1)^3) = 6k$, while for higher power $q \geq 4$ and $k \geq 1$,

$$|\Delta^2((k-1)^q) - q(q-1)k^{q-2}| \leq 2\binom{q}{4}(2k)^{q-4}. \quad (16)$$

A weaker bound reads

$$\Delta^2((k-1)^q) \leq 2^{q-1}\binom{q}{2}k^{q-2}. \quad (17)$$

Proof. The first two results come from the identities $\Delta^2((k-1)^2) = (k+1)^2 + (k-1)^2 - 2k^2 = 2$ and $\Delta^2((k-1)^3) = (k+1)^3 + (k-1)^3 - 2k^3 = 6k$. The last one is obtained by estimating $\Delta^2((k-1)^q)$ from a 3rd-order Taylor development of both $(k+1)^q$ and $(k-1)^q$ around k , their 4th order errors being both bounded by $\binom{q}{4}(k+1)^4 \leq \binom{q}{4}(2k)^4$. The weaker bound is obtained similarly from a first order Taylor development with a bounding of the second order error. \square

Lemma 4. The sum of κ_k is bounded as

$$\frac{1}{N}a^2 - \tau_N a \leq 2 \sum_{k=1}^n \kappa_k \leq \frac{1}{N}(a^2 - 1)_+, \quad (18)$$

while for other power $p \in \mathbb{N}_0$,

$$|(p+1)(p+2) \sum_{k=1}^n k^p \kappa_k - \chi_N(\frac{p}{2} + 1) a^{p+2}| \leq (p+2) \chi_N(\frac{p+1}{2}) a^{p+1}. \quad (19)$$

Proof. Using the alternate formulation (10) of κ_k , we find first for $p \geq 0$.

$$\sum_{k=1}^n k^p \kappa_k = \tau_N a (N-1) \int_0^1 (1-u^2)^{\frac{N-3}{2}} \sum_{k=1}^n k^p (u - \frac{k}{a})_+ du. \quad (20)$$

In the case where $p = 0$, $\frac{1}{2}(u^2 - u) \leq \sum_{k=1}^{+\infty} (u-k)_+ \leq \frac{1}{2}u^2$. This is easily observed from $u = \lfloor u \rfloor + (u - \lfloor u \rfloor) = \sum_{k=1}^{+\infty} \mathbb{I}(u \geq k) + (u - \lfloor u \rfloor)$, which integrated gives $\frac{1}{2}u^2 = \sum_{k=1}^{+\infty} (u-k)_+ + \int_0^u (v - \lfloor v \rfloor) dv$, the last integral being positive and smaller than $\frac{1}{2}u$. Therefore, for any $a > 0$,

$$\frac{1}{2}(au^2 - u) \leq \sum_{k=1}^{+\infty} (u - \frac{k}{a})_+ \leq \frac{1}{2}au^2. \quad (21)$$

Moreover, for any $s \in \mathbb{N}$ and given the definition of τ_N ,

$$\tau_N (N-1) \int_0^1 (1-u^2)^{\frac{N-3}{2}} u^s du = \tau_N \frac{N-1}{2} B(\frac{s+1}{2}, \frac{N-1}{2}) = \frac{\Gamma(\frac{s+1}{2})\Gamma(\frac{N}{2})}{\sqrt{\pi}\Gamma(\frac{N+s}{2})} = \chi_N(\frac{s}{2}), \quad (22)$$

with the ‘‘Beta’’ function $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ and χ_N defined in (8).

Therefore, using (20) combined with the lower bound of (21) and the identity $\Gamma(x)x = \Gamma(x+1)$ for any $x \in \mathbb{R}_+$, we get

$$\sum_{k=1}^n \kappa_k \geq \frac{1}{2}\chi_N(1)a^2 - \frac{1}{2}\chi_N(\frac{1}{2})a = \frac{1}{2N}a^2 - \frac{1}{2}\tau_N a.$$

Similarly, the upper bound of (21) can lead to $\sum_{k=1}^n \kappa_k \leq \frac{1}{2}\chi_N(1)a^2 = \frac{1}{2N}a^2$. A tighter bound is obtained by observing that, from (20), $\kappa_k(a=1) = 0$ and

$$\begin{aligned} \sum_{k=1}^n \frac{d}{da} \kappa_k(a) &= \tau_N (N-1) \int_0^1 (1-u^2)^{\frac{N-3}{2}} \sum_{k=1}^n u \mathbb{I}(u \geq \frac{k}{a}) du \\ &= \tau_N (N-1) \int_0^1 (1-u^2)^{\frac{N-3}{2}} u \lfloor au \rfloor du \\ &\leq \tau_N a (N-1) \int_0^1 (1-u^2)^{\frac{N-3}{2}} u^2 du = \frac{1}{N} a, \end{aligned}$$

using (22) with $s = 2$ in the last equality. Therefore,

$$\sum_{k=1}^n \kappa_k(a) = \mathbb{I}(a \geq 1) \int_1^a \sum_{k=1}^n \frac{d}{du} \kappa_k(u) du \leq \frac{1}{2N}(a^2 - 1)_+.$$

For analyzing positive power p , we rely on the fact that, for a continuous and integrable function $g : [l, m] \rightarrow \mathbb{R}$ with a unique extremum on $[l, m] \subset \mathbb{R}$,

$$\left| \sum_{k=l+1}^m g(k) - \int_l^m g(t) dt \right| \leq \max_{t \in [l, m]} |g(t)|.$$

Taking $g(t) = t^p(u - t)$ which has a unique maximum on $\frac{p}{p+1}u$ of height $(\frac{p}{p+1})^p \frac{1}{p+1} u^{p+1} \leq \frac{1}{p+1} u^{p+1}$, we find $\left| \sum_{k=1}^{\infty} k^p(u - k)_+ - \frac{1}{(p+2)(p+1)} u^{p+2} \right| \leq \frac{1}{p+1} u^{p+1}$, since $\int_0^{\infty} t^p(u - t)_+ dt = u^{p+2} B(p+1, 2) = \frac{1}{(p+2)(p+1)} u^{p+2}$. For any $a > 0$, this leads to

$$\left| (p+2)(p+1) \sum_{k=1}^{\infty} k^p(u - \frac{k}{a})_+ - a^{p+1} u^{p+2} \right| \leq (p+2) a^p u^{p+1}.$$

The result follows by inserting this last bound in (20) and reusing (22) for $s \in \{p+1, p+2\}$. \square

Notice that (19) (in Lemma 4) is probably improvable for small values of a since, as said in the proof above, $\kappa_k(1) = 0$. We note, however, that the expression is tight asymptotically in a .

Thanks to the previous lemmata, we are now ready to prove Prop. 8.

Proof of Prop. 8. If $a < 1$, then, for all $q \geq 1$, $\mathbb{E}X^q = 1^q p_1 = \tau_N a$, while if $a \geq 1$, (15) shows that $\mathbb{E}X^q = \kappa_0 + \sum_{k=1}^n \Delta^2((k-1)^q) \kappa_n \geq \tau_N a$ since $\kappa_0 = \tau_N a$.

Let us consider now more specific values of q for the case $a \geq 1$. For $q = 2$, we know from Lemmata 2 and 3 that $\mathbb{E}X^2 = \kappa_0 + 2 \sum_{k=1}^n \kappa_k$ and the upper bound follows from (18) since $\kappa_0 = \tau_N a$.

For $q = 3$, the same two lemmata provide $\mathbb{E}X^3 = \kappa_0 + 6 \sum_{k=1}^n k \kappa_k$. Moreover, from (19),

$$\left| 6 \sum_{k=1}^n k \kappa_k - \chi_N\left(\frac{3}{2}\right) a^3 \right| \leq 3 \chi_N(1) a^2,$$

which involves

$$\left| \mathbb{E}X^3 - (\tau_N a + \chi_N\left(\frac{3}{2}\right) a^3) \right| \leq 3 \chi_N(1) a^2.$$

For $q \geq 4$, the result becomes a bit technical. Again from Lemmata 2 and 3,

$$\left| \mathbb{E}X^q - (\kappa_0 + q(q-1) \sum_{k=1}^n k^{q-2} \kappa_k) \right| \leq 2^{q-3} \binom{q}{4} \sum_{k=1}^n k^{q-4} \kappa_k.$$

Using twice (19), we find

$$\begin{aligned} \left| \mathbb{E}X^q - (\kappa_0 + \chi_N\left(\frac{q}{2}\right) a^q) \right| &\leq q \chi_N\left(\frac{q-1}{2}\right) a^{q-1} + 2^{q-3} \binom{q}{4} \sum_{k=1}^n k^{q-4} \kappa_k \\ &\leq q \chi_N\left(\frac{q-1}{2}\right) a^{q-1} + \frac{2^{q-6}}{3} q(q-1) (\chi_N\left(\frac{q-2}{2}\right) a^{q-2} + (q-2) \chi_N\left(\frac{q-3}{2}\right) a^{q-3}) \\ &\leq q \chi_N\left(\frac{q-1}{2}\right) a^{q-1} + \frac{1}{24} \binom{q}{2} \chi_N\left(\frac{q-2}{2}\right) (2a)^{q-2} + \frac{1}{12} \binom{q}{3} \chi_N\left(\frac{q-3}{2}\right) (2a)^{q-3}. \end{aligned}$$

Finally, for the weak upper bound (14), we note that (19) involves

$$\binom{q}{2} \sum_{k=1}^n k^{q-2} \kappa_k \leq \frac{1}{2} \chi_N\left(\frac{q}{2}\right) a^q + \frac{1}{2} q \chi_N\left(\frac{q-1}{2}\right) a^{q-1}.$$

Using (15) and (17), we find then

$$\mathbb{E}X^q \leq \tau_N a + 2^{q-1} \binom{q}{2} \sum_{k=1}^n k^{q-2} \kappa_k \leq \tau_N a + 2^{q-2} \chi_N\left(\frac{q}{2}\right) a^q + 2^{q-2} q \chi_N\left(\frac{q-1}{2}\right) a^{q-1}.$$

\square

3 Quasi-Isometric Quantized Embedding

Buffon’s needle problem and its generalization to a N -dimensional space lead to interesting observations in the field of dimensionality reduction: it helps in understanding the impact of quantization on the classical Johnson-Lindenstrauss (JL) Lemma [1, 24].

To see this, let us consider the common uniform quantizer of bin width $\delta > 0$

$$\mathcal{Q}(\lambda) = \delta \lfloor \frac{\lambda}{\delta} \rfloor \in \mathbb{Z}_\delta, \quad (23)$$

defined componentwise when applied on vectors. Notice that we could have defined the more common *midrise* quantizer $\mathcal{Q}' : \lambda \rightarrow \delta \lfloor \lambda/\delta \rfloor + \delta/2$ with no impact on the rest of our developments.

Given a random matrix $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$ and a uniform random vector $\xi \sim \mathcal{U}^M([0, \delta])$, we define the non-linear mapping $\psi_\delta : \mathbb{R}^N \rightarrow \mathbb{Z}_\delta^M$ such that

$$\psi_\delta(\mathbf{u}) = \mathcal{Q}(\Phi \mathbf{u} + \xi), \quad (24)$$

where ξ plays a useful *dithering* role: its action randomizes the location of each unquantized component of $\Phi \mathbf{u}$ inside a quantization cell of \mathbb{R}^M [25]. Our dithered construction is similar to the one developed in [10] but our quantizer is different.

How can we interpret the action of this mapping ψ_δ ? How does it approximately preserve the distance between a pair of points $\mathbf{u}, \mathbf{v} \in \mathbb{R}^N$? Surprisingly, the answer comes from Buffon’s needle problem from the following equivalence.

Proposition 9. *Under the notations defined above, for each $j \in [M]$ and conditionally to the knowledge of $r_j = \|\varphi_j\|$, we have*

$$X_j := \frac{1}{\delta} |(\psi_\delta(\mathbf{u}))_j - (\psi_\delta(\mathbf{v}))_j| \sim_{\text{iid}} \text{Buffon}(\frac{r_j}{\delta} \|\mathbf{u} - \mathbf{v}\|, N). \quad (25)$$

Proof. Let \mathcal{G} be a grid of parallel $(N-1)$ -dimensional hyperplanes that are δ apart. Without any loss of generality, we assume them normal to the axis $\mathbf{e}_1 = (1, 0, \dots, 0)^T$ and each hyperplane corresponds to the set $\mathcal{H}_k = \{\mathbf{x} \in \mathbb{R}^N : \mathbf{e}_1^T \mathbf{x} = \delta k\}$ for $k \in \mathbb{Z}$. Let us now imagine a “needle” $\mathbf{N}(\mathbf{u}, \mathbf{v})$ whose extremities are determined by two points \mathbf{u} and \mathbf{v} somewhere in \mathbb{R}^N . Note that the parameterization of the needle with its extremities is equivalent to the one defined in Sec. 2.

Notice that the number of intersections $\mathbf{N}(\mathbf{u}, \mathbf{v})$ has with $\mathcal{G} = \cup_{k \in \mathbb{Z}} \mathcal{H}_k$ can obviously be expressed with the quantizer \mathcal{Q} as

$$\frac{1}{\delta} |\mathcal{Q}(\mathbf{e}_1^T \mathbf{u}) - \mathcal{Q}(\mathbf{e}_1^T \mathbf{v})|.$$

The reason is that, if $\mathbf{x} \in \mathbb{R}^N$ falls between $\mathcal{H}_{k(\mathbf{x})}$ and $\mathcal{H}_{k(\mathbf{x})+1}$ (the last hyperplane excluded), then $\mathcal{Q}(\mathbf{e}_1^T \mathbf{x}) = k_x \delta$ with $k_x := \lfloor \frac{\mathbf{e}_1^T \mathbf{x}}{\delta} \rfloor$. Therefore, $\frac{1}{\delta} |\mathcal{Q}(\mathbf{e}_1^T \mathbf{u}) - \mathcal{Q}(\mathbf{e}_1^T \mathbf{v})| = |k_{\mathbf{u}} - k_{\mathbf{v}}|$ is the number of hyperplanes crossing $\mathbf{N}(\mathbf{u}, \mathbf{v})$.

Let us define now a random dithering $\xi \sim \mathcal{U}([0, \delta])$ and a random rotation γ whose distribution is uniform on the rotation group $\text{SO}(N)$ of \mathbb{R}^N . This is made possible from the existence of a Haar measure on $\text{SO}(N)$ (see, e.g., [26]). From these, we can create the mapping $\mathbf{x}_{\gamma, \xi} = T_{\gamma, \xi}(\mathbf{x}) = \mathbf{R}(\gamma) \mathbf{x} + \xi \mathbf{e}_1$, where $\mathbf{R}(\gamma) \in \mathbb{R}^{N \times N}$ stands for the matrix representation of γ .

Thanks to this transformation, given two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^N$, the needle $\mathbf{N}(\mathbf{u}_{\gamma, \xi}, \mathbf{v}_{\gamma, \xi})$ of length $\|\mathbf{u}_{\gamma, \xi} - \mathbf{v}_{\gamma, \xi}\| = \|\mathbf{u} - \mathbf{v}\|$ whose extremities are defined by $\mathbf{u}_{\gamma, \xi}$ and $\mathbf{v}_{\gamma, \xi}$ is oriented

uniformly at random (conditionally to ξ) thanks to the action of γ , *i.e.*, the random vector $\mathbf{R}(\gamma)(\mathbf{u} - \mathbf{v})$ is uniform⁴ on \mathbb{S}^{N-1} .

Moreover, conditionally to γ , this needle is also positioned uniformly at random relatively to the δ -periodic grid $\mathcal{G} = \cup_{k \in \mathbb{Z}} \mathcal{H}_k$. From the action of the dithering, any fixed point $\mathbf{p} \in \mathbf{N}(\mathbf{u}_{\gamma,0}, \mathbf{v}_{\gamma,0})$ on the un-dithered needle (*e.g.*, its midpoint) has an abscisse $p_1 + \xi \sim \mathcal{U}([p_1, p_1 + \delta])$ along \mathbf{e}_1 after dithering. Therefore, from the periodicity of \mathcal{G} , the distance between $\mathbf{p} + \xi \mathbf{e}_1 \in \mathbf{N}(\mathbf{u}_{\gamma,\xi}, \mathbf{v}_{\gamma,\xi})$ and the nearest hyperplane of \mathcal{G} is distributed as $\mathcal{U}([0, \delta/2])$ conditionally to γ .

Consequently, the quantity

$$\frac{1}{\delta} |\mathcal{Q}(\mathbf{e}_1^T(\mathbf{u}_{\gamma,\xi})) - \mathcal{Q}(\mathbf{e}_1^T(\mathbf{v}_{\gamma,\xi}))|$$

counts the number of intersections between \mathcal{G} and the needle $\mathbf{N}(\mathbf{u}_{\gamma,\xi}, \mathbf{v}_{\gamma,\xi})$, which is oriented and positioned uniformly at random relatively to \mathcal{G} . In other words, we are in presence of a Buffon random variable $\text{Buffon}(\|\mathbf{u} - \mathbf{v}\|/\delta, N)$!

Moreover, for any $\mathbf{x} \in \mathbb{R}$, we have $\mathbf{e}_1^T \mathbf{R}(\gamma) \mathbf{x} = (\mathbf{R}(\gamma)^{-1} \mathbf{e}_1)^T \mathbf{x} \sim \boldsymbol{\theta}^T \mathbf{x}$ where $\boldsymbol{\theta}$ is a random vector uniformly distributed⁴ on \mathbb{S}^{N-1} . Therefore,

$$\frac{1}{\delta} |\mathcal{Q}(\mathbf{e}_1^T(\mathbf{u}_{\gamma,\xi})) - \mathcal{Q}(\mathbf{e}_1^T(\mathbf{v}_{\gamma,\xi}))| \sim \frac{1}{\delta} |\mathcal{Q}(\boldsymbol{\theta}^T \mathbf{u} + \xi) - \mathcal{Q}(\boldsymbol{\theta}^T \mathbf{v} + \xi)| \sim \text{Buffon}(\frac{1}{\delta} \|\mathbf{u} - \mathbf{v}\|, N).$$

Since any Gaussian random vector $\boldsymbol{\varphi} \sim \mathcal{N}^N(0, 1)$ can be written as $\boldsymbol{\varphi} = r \hat{\boldsymbol{\varphi}}$ with $r = \|\boldsymbol{\varphi}\|$ and $\hat{\boldsymbol{\varphi}} = \boldsymbol{\varphi}/r$ picked uniformly at random on \mathbb{S}^{N-1} , we can conclude that, conditionally to r ,

$$\frac{1}{\delta} |\mathcal{Q}(\boldsymbol{\varphi}^T \mathbf{u} + \xi) - \mathcal{Q}(\boldsymbol{\varphi}^T \mathbf{v} + \xi)| \sim \text{Buffon}(\frac{r}{\delta} \|\mathbf{u} - \mathbf{v}\|, N),$$

which, from (24), behaves exactly as the amplitude of one component of $\frac{1}{\delta}(\boldsymbol{\psi}_\delta(\mathbf{u}) - \boldsymbol{\psi}_\delta(\mathbf{v}))$.

Therefore, we can finally state that, for all $j \in [M]$ and conditionally to the knowledge of the length $r_j = \|\boldsymbol{\varphi}_j\|$,

$$X_j := \frac{1}{\delta} |(\boldsymbol{\psi}_\delta(\mathbf{u}))_j - (\boldsymbol{\psi}_\delta(\mathbf{v}))_j| \sim_{\text{iid}} \text{Buffon}(\frac{r_j}{\delta} \|\mathbf{u} - \mathbf{v}\|, N),$$

the independence of the random variables X_j resulting from the one of the rows of $\boldsymbol{\Phi}$. \square

Now that this equivalence is proved, we see how to reach the characterization of a quantized embedding determined by $\boldsymbol{\psi}_\delta$: we have to study the concentration properties of each X_j around their mean. Therefore, targeting the use of a classical concentration result due to Bernstein (explained later), we have first to analyze the moments of these random variables.

Let us start with the evaluation of their expectation. Notice that the distribution of each r_j defined in Prop. 9 is known. This one follows the radial marginal pdf ρ of a Gaussian random vector in \mathbb{R}^N of unit component variance (*i.e.*, $\mathcal{N}^N(0, 1)$), when expressed in hyperspherical coordinates. Therefore,

$$\rho(r) = \left(\int_0^\infty s^{N-1} e^{-\frac{1}{2}s^2} ds \right)^{-1} r^{N-1} e^{-\frac{1}{2}r^2} = \frac{2^{1-\frac{N}{2}}}{\Gamma(\frac{N}{2})} r^{N-1} e^{-\frac{1}{2}r^2}.$$

Simply denoting this distribution by $\|\mathcal{N}^N(0, 1)\|$, if $Z \sim \|\mathcal{N}^N(0, 1)\|$, we can also compute that, for any $q \in \mathbb{N}$,

$$\mathbb{E}Z^q = 2^{\frac{q}{2}} \frac{\Gamma(\frac{N+q}{2})}{\Gamma(\frac{N}{2})} = \frac{2^{\frac{q}{2}} \Gamma(\frac{q+1}{2})}{\sqrt{\pi} \chi_N(\frac{q}{2})}, \quad (26)$$

where χ_N was defined in (8).

This allows one to see that the expectation of each X_j is proportional to $\|\mathbf{u} - \mathbf{v}\|$ and independent of N .

⁴ This is a simple consequence of the uniqueness of the Haar measure on \mathbb{S}^{N-1} and of the fact that, given any $\mathbf{x} \in \mathbb{S}^{N-1}$, $\mathbf{R}(\gamma)\mathbf{x}$ is rotationally invariant if γ is picked uniformly at random on $\text{SO}(N)$.

Proposition 10. Let $\delta > 0$, $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$, $\xi \sim \mathcal{U}^M([0, \delta])$ and \mathcal{Q} defined as above. Given $\mathbf{u}, \mathbf{v} \in \mathbb{R}^N$ and $j \in [M]$, we have

$$\delta \mathbb{E}X_j = \mathbb{E}|\mathcal{Q}(\varphi_j^T \mathbf{u} + \xi_j) - \mathcal{Q}(\varphi_j^T \mathbf{v} + \xi_j)| = \frac{\sqrt{2}}{\sqrt{\pi}} \|\mathbf{u} - \mathbf{v}\|. \quad (27)$$

Proof. The proposition follows from the law of total expectation applied to the computation of $\mathbb{E}X_j$ with $X_j = \frac{1}{\delta} |(\psi_\delta(\mathbf{u}))_j - (\psi_\delta(\mathbf{v}))_j|$. Since, conditionally to $r = \|\varphi_j\|$, $X_j \sim \text{Buffon}(\frac{r}{\delta} \|\mathbf{u} - \mathbf{v}\|, N)$, and since $r \sim \|\mathcal{N}^N(0, 1)\|$, we have

$$\mathbb{E}X_j = \mathbb{E}(\mathbb{E}(X_j|r)) = \tau_N \mathbb{E}(\frac{r}{\delta} \|\mathbf{u} - \mathbf{v}\|) = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{1}{\delta} \|\mathbf{u} - \mathbf{v}\|.$$

□

Beyond the mere evaluation of $\mathbb{E}X_j$, we can show that, if $\|\mathbf{u} - \mathbf{v}\|$ is much larger than δ , any X_j for $j \in [M]$ behaves like the amplitude of a Gaussian random variable $\mathcal{N}(0, \|\mathbf{u} - \mathbf{v}\|^2/\delta^2)$. This fact is established hereafter from an asymptotic analysis of the moments $\mathbb{E}X_j^q$.

Proposition 11. Following the previous conventions, for any $j \in [M]$ and $\alpha = \|\mathbf{u} - \mathbf{v}\|/\delta$, we have

$$|\mathbb{E}X_j^q - \mathbb{E}|G_\alpha|^q| \leq O(\alpha^{q-1}),$$

with $G_\alpha \sim \mathcal{N}(0, \alpha^2)$ and $\mathbb{E}|G_\alpha|^q = \frac{1}{\sqrt{\pi}} 2^{\frac{q}{2}} \alpha^q \Gamma(\frac{q+1}{2}) = O(\alpha^q)$.

Proof. First notice that, if $Z \sim \|\mathcal{N}^N(0, 1)\|$, then, using (26) and a classical result on the absolute moments of a Gaussian random variable,

$$\chi_N(\frac{p}{2}) \mathbb{E}Z^p = \frac{1}{\sqrt{\pi}} 2^{\frac{p}{2}} \Gamma(\frac{p+1}{2}) = \mathbb{E}|G|^p,$$

with $p \geq 0$ and $G \sim \mathcal{N}(0, 1)$.

Let us now consider the case $q \geq 4$. Therefore, considering the random mixture $X_j \sim \text{Buffon}(r_j \alpha, N)$ with $r_j \sim \|\mathcal{N}^N(0, 1)\|$, conditionally to r_j , (13) provides

$$\begin{aligned} & |\mathbb{E}(X_j^q|r_j) - (\tau_N a + \chi_N(\frac{q}{2})a^q)| \\ & \leq q \chi_N(\frac{q-1}{2}) a^{q-1} + \frac{1}{24} \binom{q}{2} \chi_N(\frac{q-2}{2}) (2a)^{q-2} + \frac{1}{12} \binom{q}{3} \chi_N(\frac{q-3}{2}) (2a)^{q-3}, \end{aligned}$$

with $a = r_j \alpha$. From the law of total expectation, this shows that

$$\begin{aligned} & |\mathbb{E}X_j^q - (\mathbb{E}|G_\alpha| + \mathbb{E}|G_\alpha|^q)| \\ & \leq q \mathbb{E}|G_\alpha|^{q-1} + \frac{1}{24} \binom{q}{2} 2^{q-2} \mathbb{E}|G_\alpha|^{q-2} + \frac{1}{12} \binom{q}{3} 2^{q-3} \mathbb{E}|G_\alpha|^{q-3}. \end{aligned}$$

and the result follows since $\mathbb{E}|G_\alpha|^p = O(\alpha^p)$ for any $p \geq 0$. The cases $1 \leq q \leq 3$ are proved similarly from (7), (11) and (12). □

Corollary 11 shows that, for $j \in [M]$, each random variable $|(\psi_\delta(\mathbf{u}))_j - (\psi_\delta(\mathbf{v}))_j|$ asymptotically behaves like the amplitude of a Gaussian random variable of variance $\|\mathbf{u} - \mathbf{v}\|^2$ from the proximity of their moments when this variance is large. Interestingly enough, without any quantization, the random variable $|(\Phi \mathbf{u})_j - (\Phi \mathbf{v})_j|$ exactly follows this distribution for $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$. Therefore, we can expect later that the concentration properties of $\sum_j X_j$ should converge to a Gaussian concentration behavior in the same asymptotic regime.

In parallel to this asymptotic analysis, bounds on the moments of X_j can be estimated thanks to those of a Buffon random variable, as summarized in Prop. 8.

Proposition 12. *Let us define $\alpha := \|\mathbf{u} - \mathbf{v}\|/\delta$. In the conventions of Prop. 10, we have*

$$\max(\frac{\sqrt{2}}{\sqrt{\pi}} \alpha, \alpha^2) \leq \mathbb{E}X_j^2 \leq \frac{\sqrt{2}}{\sqrt{\pi}} \alpha + \alpha^2. \quad (28)$$

and, for $q > 2$,

$$\mathbb{E}X_j^q \leq \frac{\sqrt{2}}{\sqrt{\pi}} \alpha + \frac{2^{\frac{3}{2}q-2}}{\sqrt{\pi}} \alpha^q \Gamma(\frac{q+1}{2}) + \frac{2^{\frac{3}{2}q-\frac{5}{2}}}{\sqrt{\pi}} \alpha^{q-1} q \Gamma(\frac{q}{2}). \quad (29)$$

Proof. For the second moment, we start from (11) with $a = r_j \alpha$ and $r_j \sim \|\mathcal{N}^N(0, 1)\|$ to get

$$\mathbb{E} \max(\frac{1}{N} a^2, \tau_N a) \leq \mathbb{E}X_j^2 = \mathbb{E}(\mathbb{E}(X_j^2 | r_j = \|\varphi_j\|)) \leq \tau_N \mathbb{E}a + \frac{1}{N} \mathbb{E}(a^2 - 1)_+.$$

However, from (26),

$$\chi_N(\frac{q}{2}) \mathbb{E} a^q = \frac{2^{\frac{q}{2}}}{\sqrt{\pi} \delta^q} \|\mathbf{u} - \mathbf{v}\|^q \Gamma(\frac{q+1}{2}), \quad (30)$$

so that $\tau_N \mathbb{E}a = \frac{\sqrt{2}}{\sqrt{\pi}} \alpha$ and⁵ $\frac{1}{N} \mathbb{E}(a^2 - 1)_+ \leq \frac{1}{N} \mathbb{E}a^2 = \alpha^2$ which leads to

$$\max(\alpha^2, \frac{\sqrt{2}}{\sqrt{\pi}} \alpha) \leq \mathbb{E}X_j^2 \leq \frac{\sqrt{2}}{\sqrt{\pi}} \alpha + \alpha^2.$$

For higher moments, using $\mathbb{E}X_j^q = \mathbb{E}(\mathbb{E}(X_j^q | r_j))$ and following the same techniques as above, (14) and (30) provide the following upper bound

$$\mathbb{E}X_j^q \leq \frac{\sqrt{2}}{\sqrt{\pi}} \alpha + \frac{2^{\frac{3}{2}q-2}}{\sqrt{\pi}} \alpha^q \Gamma(\frac{q+1}{2}) + \frac{2^{\frac{3}{2}q-\frac{5}{2}}}{\sqrt{\pi}} \alpha^{q-1} q \Gamma(\frac{q}{2}).$$

□

In the last proposition, we can also get rid of the Γ functions by invoking the relation $\Gamma(x + \frac{1}{2}) \leq \sqrt{x} \Gamma(x)$ [27] whose recursive application provides $\Gamma(\frac{p+1}{2}) \leq 2^{\frac{1-p}{2}} \sqrt{\pi} \sqrt{p!}$. Using this we find, for $q > 2$,

$$\mathbb{E}X_j^q \leq \frac{\sqrt{2}}{\sqrt{\pi}} \alpha + 2^{q-\frac{3}{2}} \alpha^q \sqrt{q!} + 2^{q-\frac{3}{2}} \alpha^{q-1} q \sqrt{(q-1)!}. \quad (31)$$

Having delineated the behavior of the moments of each X_j , we can now study their concentration properties. This is achieved from the Bernstein inequality using a formulation from [28, p. 24] that suits the rest of our developments.

Theorem 1 (Bernstein's inequality [28]). *Let V_1, \dots, V_M be independent real valued random variables. Assume that there exist some positive numbers v and β such that*

$$\sum_{j=1}^M \mathbb{E}V_j^2 \leq v \quad (32)$$

and for all integers $k \geq 3$

$$\sum_{j=1}^M \mathbb{E}V_j^k \leq \frac{1}{2} k! \beta^{k-2} v. \quad (33)$$

Then, for every positive x ,

$$\mathbb{P}[|\sum_{j=1}^M (V_j - \mathbb{E}V_j)| \geq \sqrt{2vx} + \beta x] \leq 2e^{-x}. \quad (34)$$

⁵Bounding $\mathbb{E}(a^2 - 1)_+$ more tightly is possible but this leads later to negligible improvements in our study.

Notice that setting $x = M\epsilon^2$ in (34) with $\epsilon \geq 0$, we get:

$$\mathbb{P}\left[\left|\frac{1}{M}\sum_{j=1}^M(V_j - \mathbb{E}V_j)\right| \geq \sqrt{\frac{2}{M}}v\epsilon + \beta\epsilon^2\right] \leq 2e^{-\epsilon^2 M}. \quad (35)$$

This is the formulation that we use in the rest of this paper. From (35), we must focus our attention on the evolution of $\sqrt{2v/M}\epsilon + \beta\epsilon^2$ once v and β are adjusted to the bounds of $\mathbb{E}X_j^q$. From (28), we already know that

$$\sum_{j=1}^M \mathbb{E}X_j^2 \leq M \left(\frac{\sqrt{2}}{\sqrt{\pi}}\alpha + \alpha^2\right), \quad (36)$$

and from (31) and for $q \geq 3$,

$$\sum_{j=1}^M \mathbb{E}X_j^q \leq \frac{\sqrt{2}}{\sqrt{\pi}} M\alpha + M(2^{q-\frac{3}{2}}\alpha^q \sqrt{q!} + 2^{q-\frac{3}{2}}\alpha^{q-1} q \sqrt{(q-1)!}). \quad (37)$$

For simplifying our analysis, let us conveniently analyze two cases: a *coarse quantization* where $\alpha = \frac{1}{\delta}\|\mathbf{u} - \mathbf{v}\| < 1$ and a *fine quantization* where $\alpha \geq 1$. Under *coarse quantization* and for $q \geq 3$, (37) provides

$$\sum_{j=1}^M \mathbb{E}X_j^q \leq \frac{\sqrt{2}}{\sqrt{\pi}} M\alpha + q! M 2^{q-2} \alpha \left(1 + \frac{1}{\sqrt{3}}\right) \leq \frac{1}{2} q! 2^{q-2} M \alpha \left(\frac{\sqrt{2}}{6\sqrt{\pi}} + 2\left(1 + \frac{1}{\sqrt{3}}\right)\right),$$

while (36) leads to

$$\sum_{j=1}^M \mathbb{E}X_j^2 \leq \left(\frac{\sqrt{2}}{\sqrt{\pi}} + 1\right)M\alpha < 2M\alpha.$$

Therefore, since $\left(\frac{\sqrt{2}}{6\sqrt{\pi}} + 2\left(1 + \frac{1}{\sqrt{3}}\right)\right) < 4$, taking $v/M = 4$ and $\beta = 2$, we satisfy the two Bernstein conditions⁶. Under *fine quantization* (*i.e.*, $\alpha \geq 1$), (36) gives now $\sum_{j=1}^M \mathbb{E}X_j^2 \leq M \left(\frac{\sqrt{2}}{\sqrt{\pi}} + 1\right) \alpha^2$, and, from (31) and $q \geq 3$,

$$\begin{aligned} \sum_{j=1}^M \mathbb{E}X_j^q &\leq \frac{\sqrt{2}}{\sqrt{\pi}} M\alpha + M(2^{q-\frac{3}{2}}\alpha^q \sqrt{q!} + 2^{q-\frac{3}{2}}\alpha^{q-1} q \sqrt{(q-1)!}) \\ &\leq \frac{\sqrt{2}}{\sqrt{\pi}} M\alpha^q + M(2^{q-\frac{3}{2}}\alpha^q \sqrt{q!} + 2^{q-\frac{3}{2}}\alpha^q q \sqrt{(q-1)!}) \\ &\leq \frac{1}{2} q! (2\alpha)^{q-2} M \alpha^2 \left(\frac{\sqrt{2}}{6\sqrt{\pi}} + 2\left(1 + \frac{1}{\sqrt{3}}\right)\right) < \frac{1}{2} q! (2\alpha)^{q-2} M (2\alpha)^2, \end{aligned}$$

We see that taking $v/M = 4\alpha^2$ and $\beta = 2\alpha$ is compatible with both inequalities.

Consequently, we can state that $\sqrt{v/M} = O(1 + \alpha)$ and $\beta = O(1 + \alpha)$ around any value of α . Therefore, if $0 < \epsilon < \epsilon_0$ for some fixed value $\epsilon_0 > 0$,

$$\exists c, c' > 0 \quad \text{such that} \quad \sqrt{2v/M}\epsilon + \beta\epsilon^2 < (c + c'\alpha)\epsilon. \quad (38)$$

Let us cook now the first important result concerning our mapping ψ_δ .

Proposition 13. *Fix $\epsilon_0 > 0$, $0 < \epsilon \leq \epsilon_0$ and $\delta > 0$. There exist two values $c, c' > 0$ only depending on ϵ_0 such that, for $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$ and $\xi \sim \mathcal{U}^M([0, \delta])$, both determining the mapping ψ_δ in (24), and for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^N$,*

$$(1 - c\epsilon)\|\mathbf{u} - \mathbf{v}\| - c'\delta\epsilon \leq \frac{\sqrt{\pi}}{\sqrt{2M}} \|\psi_\delta(\mathbf{u}) - \psi_\delta(\mathbf{v})\|_1 \leq (1 + c\epsilon)\|\mathbf{u} - \mathbf{v}\| + c'\delta\epsilon. \quad (39)$$

with a probability higher than $1 - 2e^{-\epsilon^2 M}$.

⁶We could set $v/M = 4\alpha$ but we found that this tighter choice complicates the presentation of the final concentration results.

Proof. From (38) and from Theorem 1, we know that there exist two values $c, c' > 0$ such that

$$\mathbb{P}\left[\left|\frac{1}{M}\sum_{j=1}^M(X_j - \mathbb{E}X_j)\right| \geq (c + c'\alpha)\epsilon\right] \leq 2e^{-\epsilon^2 M}.$$

Therefore, since

$$X_j = \frac{1}{\delta} |(\boldsymbol{\psi}_\delta(\mathbf{u}))_j - (\boldsymbol{\psi}_\delta(\mathbf{v}))_j| = \frac{1}{\delta} |\mathcal{Q}(\boldsymbol{\varphi}_j^T \mathbf{u} + \xi_j) - \mathcal{Q}(\boldsymbol{\varphi}_j^T \mathbf{v} + \xi_j)|,$$

with $\mathbb{E}X_j = \frac{\sqrt{2}}{\sqrt{\pi}}\alpha$, we find

$$\frac{\sqrt{2}}{\sqrt{\pi}}(1 - c'\epsilon)\alpha - c\epsilon \leq \frac{1}{M}\sum_{j=1}^M X_j \leq \frac{\sqrt{2}}{\sqrt{\pi}}(1 + c'\epsilon)\alpha + c\epsilon,$$

with probability higher than $1 - 2e^{-\epsilon^2 M}$ which provides the result. \square

Finally, this last proposition provides the main result of this paper.

Proposition 1. *Let $\mathcal{S} \subset \mathbb{R}^N$ be a set of S points. Fix $0 < \epsilon < 1$ and $\delta > 0$. For $M > M_0 = O(\epsilon^{-2} \log S)$, there exist a non-linear mapping $\boldsymbol{\psi} : \mathbb{R}^N \rightarrow \mathbb{Z}_\delta^M$ and two constants $c, c' > 0$ such that, for all pairs $\mathbf{u}, \mathbf{v} \in \mathcal{S}$,*

$$(1 - \epsilon)\|\mathbf{u} - \mathbf{v}\| - c\delta\epsilon \leq \frac{c'}{M}\|\boldsymbol{\psi}(\mathbf{u}) - \boldsymbol{\psi}(\mathbf{v})\|_1 \leq (1 + \epsilon)\|\mathbf{u} - \mathbf{v}\| + c\delta\epsilon. \quad (4)$$

Proof. The proof proceeds first by simplifying (39) in Prop. 13 with the change of variable $c\epsilon \rightarrow \epsilon$ and with ϵ_0 high enough so that $0 < \epsilon < 1$ after this rescaling. Next, we follow the classical proof of the Johnson-Lindenstrauss Lemma [1, 2] already sketched in the Introduction. Given the mapping $\boldsymbol{\psi}_\delta$ associated to $\boldsymbol{\Phi} \sim \mathcal{N}^{M \times N}(0, 1)$ and $\boldsymbol{\xi}$ through (24), and considering the $\binom{S}{2} \leq S^2/2$ possible pairs of vectors in \mathcal{S} , we apply a standard union bound argument for jointly satisfying the inequality (35) for all such pairs. If $M > M_0 = 2\epsilon^{-2} \log S = O(\epsilon^{-2} \log S)$, then $2 \log S - \epsilon^2 M < 0$ and the global probability of success is higher than $1 - \exp(2 \log S - \epsilon^2 M) > 0$. Moreover, this probability can be arbitrarily boosted close to 1 by repeating the random generation of $\boldsymbol{\psi}_\delta$, considering then the event that at least one of the generated mappings will satisfy (4). This shows the existence of $\boldsymbol{\psi}$ with probability 1, in the limit of an increasingly large sequence of mappings. \square

4 Discussion

How can we analyse the quasi-isometric mapping provided by Prop. 2? It happens that there are three interesting regimes where a quantized embedding respecting (4) displays different typical behaviors.

Nearly isometric regime: Under a fine quantization scheme, *i.e.*, if

$$\delta \ll \nu_S := \min_{\mathbf{u}, \mathbf{v} \in \mathcal{S}: \mathbf{u} \neq \mathbf{v}} \|\mathbf{u} - \mathbf{v}\|, \quad (40)$$

Eq. (4) essentially provides a Lipschitz embedding of (\mathcal{S}, ℓ_2) in $(\boldsymbol{\psi}(\mathcal{S}), \ell_1)$. This makes sense since for such a fine quantization, the quantization distortion almost disappears, *i.e.*, $\mathcal{Q}(\lambda) \simeq \lambda$ for any $\lambda \gg \delta$, and it is known that, for a Gaussian random matrix $\boldsymbol{\Phi} \sim \mathcal{N}^{M \times N}(0, 1)$ and two fixed vectors \mathbf{u} and \mathbf{v} ,

$$(1 - \epsilon)\|\mathbf{u} - \mathbf{v}\| \leq \frac{\sqrt{\pi}}{\sqrt{2M}}\|\boldsymbol{\Phi}\mathbf{u} - \boldsymbol{\Phi}\mathbf{v}\|_1 \leq (1 + \epsilon)\|\mathbf{u} - \mathbf{v}\|, \quad (41)$$

with probability higher than $1 - 2e^{-\frac{1}{2}\epsilon^2 M}$.

Indeed, as explained for instance in [29, Appendix A], this is a simple consequence of the following result due to Ledoux and Talagrand.

Proposition 14 (Ledoux, Talagrand [17] (Eq. 1.6)). *If F is Lipschitz with constant $\lambda = \|F\|_{\text{Lip}}$, then, for a random vector $\zeta \in \mathbb{R}^M$ with $\zeta_i \sim_{\text{iid}} \mathcal{N}(0, 1)$ (i.e., $\zeta \sim \mathcal{N}^M(0, 1)$),*

$$\mathbb{P}[|F(\zeta) - \mu_F| > r] \leq 2e^{-\frac{1}{2}r^2\lambda^{-2}}, \quad \text{for } r > 0,$$

with $\mu_F = \mathbb{E}[F(\zeta)]$.

The Lipschitz constant of F is defined as $\|F\|_{\text{Lip}} \triangleq \sup_{\mathbf{x}, \mathbf{y} \in \mathbb{R}^M, \mathbf{x} \neq \mathbf{y}} \frac{|F(\mathbf{x}) - F(\mathbf{y})|}{\|\mathbf{x} - \mathbf{y}\|_2}$. For a Gaussian random matrix $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$, the vector $\zeta = \|\mathbf{u} - \mathbf{v}\|^{-1} \Phi(\mathbf{u} - \mathbf{v})$ is distributed as $\mathcal{N}^M(0, 1)$. Taking $F(\cdot) = \|\cdot\|_1$ with $\|F\|_{\text{Lip}} = \sqrt{M}$ and $r = M\epsilon$ with $\epsilon > 0$ provides (41) since $\mu_F = M \frac{\sqrt{2}}{\sqrt{\pi}}$.

As explained before, the result (41) is easily extendable (from a union bound argument) to the embedding of a finite set \mathcal{S} of S points in \mathbb{R}^N provided $M > M_0 = O(\epsilon^2 \log S)$. Noticeably, Prop. 2 converges exactly to this isometric mapping if $\delta \ll \nu_S$.

Quasi-isometric binary regime: In the case where δ is greater than the diameter $\text{diam } \mathcal{S}$, i.e., the greatest distance between any pair of points in this set, then the quantization distortion dominates and the quantized embedding reduces to a quasi-isometric embedding. Indeed, for such a situation, we reach

$$\|\mathbf{u} - \mathbf{v}\| - (1+c)\delta\epsilon \leq \frac{c'}{M} \|\psi(\mathbf{u}) - \psi(\mathbf{v})\|_1 \leq \|\mathbf{u} - \mathbf{v}\| + (1+c)\delta\epsilon,$$

since then $\|\mathbf{u} - \mathbf{v}\| \leq \delta$. This is reminiscent of the observations made in [11, 14] about the embedding properties of “binarized” random projections. As explained in the Introduction, given $\mathbf{u}, \mathbf{v} \in \mathbb{R}^N$ and $\epsilon > 0$, if we randomly generate Φ as $\mathcal{N}^{M \times N}(0, 1)$, then, from (2),

$$d_S(\mathbf{u}, \mathbf{v}) - \epsilon \leq d_H(\text{sign}(\Phi\mathbf{u}), \text{sign}(\Phi\mathbf{v})) \leq d_S(\mathbf{u}, \mathbf{v}) + \epsilon,$$

with a probability higher than $1 - 2e^{-2\epsilon^2 M}$. In short this result amounts to first showing that the signs of $\varphi_j^T \mathbf{u}$ and $\varphi_j^T \mathbf{v}$ differ with a probability $d_S(\mathbf{u}, \mathbf{v})$ for any $j \in [M]$, and second to observing that the sum of all such signs collected at every j (as performed in the Hamming distance d_H) behaves as a Binomial random variable of M trials and probability $d_S(\mathbf{u}, \mathbf{v})$. This kind of random variable is known to concentrate quickly around its mean $d_S(\mathbf{u}, \mathbf{v})$ from a simple application of the Chernoff-Hoeffding inequality [11].

In this considered case, the 1-bit sign quantization of the random projections is not strictly equivalent to our quantization scheme defined in (24), i.e., there is no *dithering*. This absence imposes the definition of other distances d_S and d_H , while in our case the dither allows one to recover an Euclidean (ℓ_2) distance in \mathbb{R}^N rather than the angular one. However, for both kind of quantizations, we do observe the same quasi-isometric behavior with a dominant additive distortion ϵ .

High measurement regime: This is possibly the most interesting regime since it displays some “bless of dimensionality” for tightening the two quasi-isometric distortions as M increases. It was formerly observed in [30, p. 3] that for a scalar uniform quantizer \mathcal{Q} such as ours, if $M > M_0 = O(\epsilon^2 \log M)$, the JL Lemma induces *a priori* a quasi-isometric mapping with a

much looser additive distortion. Indeed, given $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$ (with this prescribed M), for any points $\mathbf{u}, \mathbf{v} \in \mathcal{S}$ we have

$$(1 - \epsilon) \|\mathbf{u} - \mathbf{v}\| - c\delta \leq \frac{1}{\sqrt{M}} \|\mathcal{Q}(\Phi\mathbf{u}) - \mathcal{Q}(\Phi\mathbf{v})\| \leq (1 + \epsilon) \|\mathbf{u} - \mathbf{v}\| + c\delta,$$

for some $c > 0$. For our uniform quantizer \mathcal{Q} and any dithering $\xi \in \mathbb{R}^M$, this is easily obtained from the relation $|\mathcal{Q}(\lambda) - \lambda| \leq \lambda/2$ and from

$$\|\Phi\mathbf{u} - \Phi\mathbf{v}\|^2 - \frac{1}{2} M\delta^2 \leq \|\mathcal{Q}(\Phi\mathbf{u} + \xi) - \mathcal{Q}(\Phi\mathbf{v} + \xi)\|^2 \leq \|\Phi\mathbf{u} - \Phi\mathbf{v}\|^2 + \frac{1}{2} M\delta^2,$$

with $\|\Phi\mathbf{u} - \Phi\mathbf{v}\|$ close to $\|\mathbf{u} - \mathbf{v}\|$ up to a distortion factors $(1 \pm \epsilon)$ by the JL Lemma. Notice that taking the square root of this inequality for lowering the power 2 is not a problem since $(a - b) \leq (a^2 - b^2)^{1/2}$ if $a > b > 0$ and $(a^2 + b^2)^{1/2} < a + b$ for any $a, b > 0$.

Similarly, readily introducing the quantization in the ℓ_2/ℓ_1 isometric embedding explained in (41) has also the same impact since

$$\|\Phi\mathbf{u} - \Phi\mathbf{v}\|_1 - \frac{1}{2} M\delta \leq \|\mathcal{Q}(\Phi\mathbf{u} + \xi) - \mathcal{Q}(\Phi\mathbf{v} + \xi)\|_1 \leq \|\Phi\mathbf{u} - \Phi\mathbf{v}\|_1 + \frac{1}{2} M\delta.$$

In both situations, the additive error induced by the quantization is constant with M . As expressed by Prop. 2 and Prop. 13, our analysis shows that there exists a mapping for which the same error scales actually as $O(\delta/\sqrt{M})$, *i.e.*, the finding of Buffon's needle helped us to reduce that distortion by a factor \sqrt{M} !

5 Towards an ℓ_2/ℓ_2 quantized embedding

We could ask ourselves if there exists another form of the quantized embedding given in Prop. 2, one that involves only the use of ℓ_2 -distances for both a set $\mathcal{S} \subset \mathbb{R}^N$ and its image in \mathbb{Z}_δ^M . The expected asymptotic case would be obvious: in the limit where δ vanishes, the standard JL Lemma should be recovered.

Unfortunately, such a perfect result seems hard to reach with the mathematical tools developed in this work. Instead, we are able to show the existence of a mapping ψ that is "close" to this situation in the sense that the ℓ_2 -distance in \mathbb{R}^N is actually distorted by a non-linear function whose action is mostly perceptible when δ is high with respect to the pairwise distance of the embedded points. Noticeably, the additive distortion of the mapping decays also more slowly with M , *i.e.*, like $O((\log S/M)^{1/4})$, than for the ℓ_2/ℓ_1 quasi-isometric mapping of Prop. 2.

Proposition 15. *Let $\mathcal{S} \subset \mathbb{R}^N$ be a set of $S = |\mathcal{S}|$ points and fix $0 < \epsilon < 1$. For $M > M_0 = O(\frac{1}{\epsilon^2} \log S)$, there exist a non-linear mapping $\psi : \mathbb{R}^N \rightarrow \mathbb{Z}_\delta^M$ and one constant $c > 0$ such that, for all pairs $\mathbf{u}, \mathbf{v} \in \mathcal{S}$,*

$$(1 - \epsilon) g_\delta(\|\mathbf{u} - \mathbf{v}\|) - c\delta\sqrt{\epsilon} \leq \frac{1}{\sqrt{M}} \|\psi(\mathbf{u}) - \psi(\mathbf{v})\| \leq (1 + \epsilon) g_\delta(\|\mathbf{u} - \mathbf{v}\|) + c\delta\sqrt{\epsilon}, \quad (42)$$

for a certain non-linear function $g_\delta(\lambda)$ such that $|g_\delta(\lambda) - \lambda| = O(\sqrt{\delta\lambda})$ for $\lambda \gg \delta$ and $|g_\delta(\lambda) - (\sqrt{2}\lambda/\sqrt{\pi})^{1/2}| = O(\lambda)$ for $\lambda < \delta$.

For reasons that will become clear below, the function g_δ is actually defined by

$$g_\delta(\lambda) := \delta g\left(\frac{\lambda}{\delta}\right), \quad g(\lambda) := (\mathbb{E}X_\lambda^2)^{1/2}, \quad (43)$$

with the random mixture $X_\lambda \sim \text{Buffon}(r\lambda, N)$ and $r \sim \|\mathcal{N}^N(0, 1)\|$. Using (28), we know that $\max(\frac{\sqrt{2}}{\sqrt{\pi}}\lambda, \lambda^2) \leq g^2(\lambda) \leq \frac{\sqrt{2}}{\sqrt{\pi}}\lambda + \lambda^2$, which provides the asymptotic properties of g_δ from

$$\max\left(\left(\frac{\sqrt{2}}{\sqrt{\pi}}\delta\lambda\right)^{1/2}, \lambda\right) \leq g_\delta(\lambda) \leq \left(\frac{\sqrt{2}}{\sqrt{\pi}}\delta\lambda\right)^{1/2} + \lambda.$$

Because of the action of g_δ , ψ in Prop. 15 does not provides a ℓ_2/ℓ_2 quasi-isometric of \mathcal{S} in \mathbb{Z}_δ^M . We are only close to this situation if the smallest pairwise distance $\nu_{\mathcal{S}}$ in \mathcal{S} defined in (40) is large compared to δ .

Strictly speaking, we cannot even say that the mapping ψ in Prop. 15 generates a quasi-isometric embedding between (\mathcal{S}, d_δ) and $(\psi(\mathcal{S}), \ell_2)$ with the function $d_\delta(\mathbf{u}, \mathbf{v}) = g_\delta(\|\mathbf{u} - \mathbf{v}\|)$. Indeed, it is not sure if d_δ is actually a distance and, therefore, (\mathcal{S}, d_δ) is not a metric space, which prevents us to match the basic requirements of Def. 1. Nevertheless, the asymptotic behavior of g_δ shows that such a quasi-isometry is not far when the pairwise distances between points of \mathcal{S} are big compared to δ .

However, we see that an ‘‘almost’’ ℓ_2/ℓ_2 quantized embedding exists between a finite set $\mathcal{S} \subset \mathbb{R}^N$ and its image in \mathbb{Z}_δ^M with multiplicative and additive embedding errors decaying as $O(\sqrt{\log S/M})$ and $O((\log S/M)^{1/4})$, respectively. This constitutes a striking difference with the ℓ_2/ℓ_1 quasi-isometric embedding of Prop. 2 where both kind of errors decay as $O(\sqrt{\log S/M})$.

On a more practical side, we may be interested in using Prop. 15 for some numerical applications. As explained in the Prop. 16 at the end of this section, a random construction of ψ is simply provided by (24) but unfortunately there is no known closed-form expression for g_δ . We know only its quadratic and linear asymptotic behaviors for large or small argument, respectively. Despite the absence of explicit formula, it is probably possible to estimate numerically g_δ from (43). This could be done in two steps. First, by integrating numerically the second moment of a Buffon random variable $\text{Buffon}(a, N)$ and fitting the result with a polynomial function of a with the desired level of accuracy in a certain range of values. Second, since $a \sim \|\mathcal{N}^N(0, 1)\|$, by applying the law of total expectation to each term of this polynomial in a using (26).

Let us finish this section by proving Prop. 15. The developments are quite similar to those presented in Sec. 3. They begin with the following result.

Proposition 16. *Fix $\epsilon_0 > 0$, $0 < \epsilon \leq 1$ and $\delta > 0$. There exist two values $c, c' > 0$ only depending on ϵ_0 such that, for $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$ and $\xi \sim \mathcal{U}^M([0, \delta])$, both determining ψ_δ in (24), and for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^N$,*

$$(1 - c\epsilon) g_\delta^2(\|\mathbf{u} - \mathbf{v}\|) - c'\delta^2\epsilon \leq \frac{1}{M} \|\psi_\delta(\mathbf{u}) - \psi_\delta(\mathbf{v})\|^2 \leq (1 + c\epsilon) g_\delta^2(\|\mathbf{u} - \mathbf{v}\|) + c'\delta^2\epsilon, \quad (44)$$

with a probability higher than $1 - 2e^{-\epsilon^2 M}$,

Proof. The proof requires to consider the moments of the random variable $\tilde{X}_j = X_j^2$ with X_j defined by (25) and, as for Sec. 3, to find reasonably small values for v and β for fulfilling (32) and (33) in Theorem 1 with $V_j = \tilde{X}_j$. Notice that by definition of the function g above and by the equivalence (25), we have

$$\frac{1}{M} \sum_{j=1}^M \mathbb{E} \tilde{X}_j = g^2(\alpha) = \frac{1}{\delta^2} g_\delta^2(\|\mathbf{u} - \mathbf{v}\|), \quad (45)$$

for $\alpha = \|\mathbf{u} - \mathbf{v}\|/\delta$. Moreover, (28) provides

$$\max\left(\frac{\sqrt{2}}{\sqrt{\pi}} \alpha, \alpha^2\right) \leq g^2(\alpha) \leq \frac{\sqrt{2}}{\sqrt{\pi}} \alpha + \alpha^2, \quad (46)$$

For the q -moments of \tilde{X}_j with $q \geq 2$, we know from (29) that

$$\begin{aligned} \mathbb{E} \tilde{X}_j^q &\leq \frac{\sqrt{2}}{\sqrt{\pi}} \alpha + \frac{2^{3q-2}}{\sqrt{\pi}} \alpha^{2q} \Gamma(q + \frac{1}{2}) + \frac{2^{3q-\frac{5}{2}}}{\sqrt{\pi}} \alpha^{2q-1} 2q \Gamma(q) \\ &\leq \frac{\sqrt{2}}{\sqrt{\pi}} \alpha + \frac{1}{2^{5/2}\sqrt{\pi}} (2\sqrt{2}\alpha)^{2q} q! + \frac{1}{\sqrt{\pi}} (2\sqrt{2}\alpha)^{2q-1} q! \\ &= \frac{\sqrt{2}}{\sqrt{\pi}} \alpha + \frac{q!}{2\sqrt{\pi}} (2\sqrt{2}\alpha)^{2q-1} (\alpha + 2), \end{aligned} \quad (47)$$

using $\Gamma(q + \frac{1}{2}) \leq \sqrt{q} \Gamma(q) \leq q!/\sqrt{2}$ for $q \geq 2$. For coarse quantization, *i.e.*, $\alpha < 1$, (47) provides

$$\begin{aligned} \mathbb{E} \tilde{X}_j^q &\leq \frac{\sqrt{2}}{\sqrt{\pi}} \alpha + \frac{q!}{2\sqrt{\pi}} (2\sqrt{2})^{2q-4} (2\sqrt{2})^3 3\alpha \\ &\leq \frac{\sqrt{2}}{\sqrt{\pi}} \alpha + \frac{\sqrt{2}}{\sqrt{\pi}} q! 8^{q-2} 24\alpha \\ &\leq \frac{1}{2} 8^{q-2} q! (1 + 96) \frac{\sqrt{2}}{2\sqrt{\pi}} \alpha < \frac{1}{2} 8^{q-2} q! 40\alpha \end{aligned}$$

We can thus select $v/M = 40$ and $\beta_{\text{cq}} = 8$. For fine quantization and $\alpha > 1$, starting again from (47), a similar development provides

$$\begin{aligned} \mathbb{E} \tilde{X}_j^q &\leq \frac{\sqrt{2}}{\sqrt{\pi}} \alpha + \frac{q!}{2\sqrt{\pi}} (2\sqrt{2}\alpha)^{2q-4} (2\sqrt{2})^3 3\alpha^4 \\ &\leq \frac{\sqrt{2}}{\sqrt{\pi}} \alpha + \frac{\sqrt{2}}{\sqrt{\pi}} q! (8\alpha^2)^{q-2} 24\alpha^4 \\ &\leq \frac{1}{2} (8\alpha^2)^{q-2} q! (1 + 96) \frac{\sqrt{2}}{2\sqrt{\pi}} \alpha^4 < \frac{1}{2} (8\alpha^2)^{q-2} q! 40\alpha^4, \end{aligned}$$

promoting the values $v/M = 40\alpha^4$ and $\beta = 8\alpha^2$.

Consequently, gathering both quantization scenarios, we have $\sqrt{2v/M} = O(1 + \alpha^2)$ and $\beta = O(1 + \alpha^2)$ around any value of $\alpha \geq 0$. Therefore, if $0 < \epsilon < \epsilon_0$, there exist two values $c, c' > 0$ only depending on ϵ_0 such that

$$\sqrt{2v/M} \epsilon + \beta \epsilon^2 \leq (c + c' \alpha^2) \epsilon.$$

Applying Theorem 1 for this bound allows one to state that

$$\mathbb{P} \left[\left| \frac{1}{M} \sum_{j=1}^M (\tilde{X}_j - \mathbb{E} \tilde{X}_j) \right| \geq (c + c' \alpha^2) \epsilon \right] \leq 2e^{-c^2 M},$$

or equivalently, using (45), that

$$\left| \frac{\delta^2}{M} \sum_{j=1}^M \tilde{X}_j - g_\delta(\|\mathbf{u} - \mathbf{v}\|) \right| \leq (c\delta^2 + c' \|\mathbf{u} - \mathbf{v}\|^2) \epsilon,$$

with a probability higher than $1 - e^{-c^2 M}$. Finally, using (46), we see that with the same probability

$$(1 - c'\epsilon) g_\delta^2(\|\mathbf{u} - \mathbf{v}\|) - c\delta^2 \epsilon \leq \frac{\delta^2}{M} \sum_{j=1}^M \tilde{X}_j \leq (1 + c'\epsilon) g_\delta^2(\|\mathbf{u} - \mathbf{v}\|) + c\delta^2 \epsilon.$$

□

Given Prop. 16, the proof of Prop. 15 is highly similar to the one of Prop. 2.

Proof of Prop. 15. We first note that (44) in Prop. 16 is equivalent to

$$(1 - c\epsilon) g_\delta(\|\mathbf{u} - \mathbf{v}\|) - \delta\sqrt{c'\epsilon} \leq \frac{1}{\sqrt{M}} \|\boldsymbol{\psi}_\delta(\mathbf{u}) - \boldsymbol{\psi}_\delta(\mathbf{v})\| \leq (1 + c\epsilon) g_\delta(\|\mathbf{u} - \mathbf{v}\|) + \delta\sqrt{c'\epsilon}, \quad (48)$$

using again the fact that $(a - b) \leq (a^2 - b^2)^{1/2}$ if $a > b > 0$ and $(a^2 + b^2)^{1/2} < a + b$ for any $a, b > 0$, and also the inequalities $\sqrt{1 - c\epsilon} \geq 1 - c\epsilon$ and $\sqrt{1 + c\epsilon} \leq 1 + c\epsilon$. The rest of the proof is similar to the one of Prop. 2 in Sec. 3 and we omit it to avoid repetitive explanations. □

6 Conclusion

In this paper, we were interested in studying the behavior of the JL Lemma when this one is combined with a uniform quantization procedure of bin width $\delta > 0$. The main result of our study is the existence of a (randomly constructible) ℓ_2/ℓ_1 quasi-isometric mapping between a set $\mathcal{S} \subset \mathbb{R}^M$ and \mathbb{Z}_δ^M . Our proof relies on generalizing the well-known Buffon’s needle problem to a N -dimensional space and in finding an equivalence between this context and the quantization of randomly projected pair of points. The final observation of our analysis is that such a mapping displays an additive and a multiplicative distortions of the pairwise distances of points in this set. The two distortions vanishes like $O(\sqrt{\log S/M})$ as the dimension M increases, while the additive distortion additionally scales like δ . As an aside, we have also obtained several interesting results concerning the generalization of Buffon’s needle problem in N dimensions, delineating the behavior of the moments of the related random variable $\text{Buffon}(a, N)$. We have concluded our study by showing that there exists almost an ℓ_2/ℓ_2 embedding of $\mathcal{S} \subset \mathbb{R}^M$ in \mathbb{Z}_δ^M displaying a quasi-isometric behavior. However, this mapping induces a non-linear distortion of the ℓ_2 -distances in \mathcal{S} and, compared to the ℓ_2/ℓ_1 embedding described above, the additive distortion decays more slowly as $O((\log S/M)^{1/4})$.

We acknowledge the fact that there may exist other quantization schemes (*e.g.*, non-regular) that, when combined with random linear mappings, lead to faster distortion decays (*e.g.*, exponential). For instance, in [10], it is shown that if two randomly projected vectors lead to equal quantized projections according to a non-regular quantizer, *i.e.*, if their distance is 0 in this projected domain, their true distance must decrease exponentially with the projected space dimension M . The Locally Sensitive Hashing (LSH) methods introduced in [31] for reaching fast approximate nearest neighbors search is another form of efficient quantized dimensionality reduction that approximately preserves distances between embedded points. Knowing if such results can be extended to provide quasi-isometric mappings with faster decaying distortions than $O(\sqrt{\log S/M})$ leads to interesting open questions.

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